Take-home exam 2

Dynamical Systems

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Problem 1: Superestable orbit

Let r_n^* be the parameter value of a quadratic map $f_r(x)$ for which there exists a superstable orbit of period 2^n.

(a) Consider an arbitrary function $h(r_n)$ evaluated at r_n^*. Show that, for $n \gg 1$, $h(r_n)$ satisfies the relation [1]:

$$\frac{[h(r_n^*) - h(r_\infty)]\delta^n}{h'(r_\infty)} = \text{cte}.$$

(b) Show that for $n \gg 1$, [1]

$$(r_n^* - r_\infty)\delta^n \propto \frac{\delta^2}{\delta - 1}.$$

Solution here.

Problem 2: Schwarz derivative

Consider the logarithmic map $x_{n+1} = f(x_n) = b + \log|x_n|$.

(a) Calculate the Schwarz derivative of this map. [1]

(b) Show that the Lyapunov exponent for this map can be expressed as [1]

$$\lambda = b - \frac{1}{n}\sum_{j=1}^{n} x_j.$$

Solution here.

Problem 3: Bifurcation diagram

Consider the following map for $x_n \in [0, 1]$, depending on two real parameters s and c,

$$x_{n+1} = f(x_n) = |\tanh[s(x_n - c)]|.$$

Obtain the bifurcation diagram of x_n as a function of $c \in [0, 1]$ with fixed value $s = 1.3$. [1]

Solution here.

Problem 4: Lyapunov exponent

Consider the following map for $x_n \in [0, 1]$:

$$x_{n+1} = f(x_n) = \begin{cases} (1+\epsilon)x_n + x_n^2, & \text{if } x_n < x^*, \\ 1 - x_n, & \text{if } x_n \geq x^*, \end{cases}$$

where x^* is the solution of $(1+\epsilon)x_n + x_n^2 = 1$.

(a) Calculate the Lyapunov exponent versus $\epsilon \in [-0.4, 0.4]$. [1]

(b) Describe the route to chaos exhibited by this map on such interval. [1]

Solution here.

Problem 5: Not period-doubling

Consider the following map for $x_n \in [0, 1]$,

$$x_{n+1} = f(x_n) = \frac{1 - b^{x_n(1-x_n)}}{1 - b^{1/4}}$$

(a) Is this map unimodal for $b \in (0, 1)$? [1]

(b) Plot the bifurcation diagram of x_n on the interval $b \in (0, 1)$ and $x_n \in [0, 1]$. Why does this map not show period doubling? [1]

(c) Find the Lyapunov exponent for $b \in (0, 1)$. [1]

Solution here.

References