Take-home exam 1

Dynamical Systems

Student: Males-Araujo Yorlan, yorlan.males@yachaytech.edu.ec
Lecturer: Mario Cosenza, mcosenza@yachaytech.edu.ec

> **Problem 1: Classification of fixed points**
>
> A particle of mass $m = 1$ is moving in the potential $V(x) = -(1/2)x^2 + (1/4)x^4$. Find and classify the fixed points (node, saddle, focus) according to their stability.

In this problem, we need to use the Hamiltonian formalism. As such, we first write

$$H(x, p) = T + V = \frac{p^2}{2} - \frac{1}{2}x^2 + \frac{1}{4}x^4,$$

which is used to find the equations of motion:

$$\begin{aligned} \dot{x} &= +\frac{\partial H}{\partial p} = p, \\ \dot{p} &= -\frac{\partial H}{\partial x} = x - x^3 = x(1 - x^2). \end{aligned}$$

The fixed points are obtained by setting them to zero and solving

$$\begin{aligned} 0 &= p, \\ 0 &= x(1 - x^2) \implies x = 0, \pm 1. \end{aligned}$$

They combine to give us the fixed points

$$\begin{aligned} \mathbf{x}_0^* &= (0, 0), \\ \mathbf{x}_1^* &= (1, 0), \\ \mathbf{x}_2^* &= (-1, 0). \end{aligned}$$

In order to classify the fixed points, we need to calculate the Jacobian matrix. In our case,

$$J = \begin{bmatrix} \dfrac{\partial \dot{x}}{\partial x} & \dfrac{\partial \dot{x}}{\partial p} \\ \dfrac{\partial \dot{p}}{\partial x} & \dfrac{\partial \dot{p}}{\partial p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{bmatrix}.$$

By evaluating the Jacobian at the fixed points, we obtain the eigenvalues for each

$$\begin{aligned} J(\mathbf{x}_0^*) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \implies \lambda^2 - 1 = 0 \implies \lambda = \pm 1 \\ J(\mathbf{x}_1^*) &= \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \implies \lambda^2 + 2 = 0 \implies \lambda = \pm i\sqrt{2}. \end{aligned}$$

Since $J(\mathbf{x}_1^*) = J(\mathbf{x}_2^*)$, we have them all and we can proceed with the classification:

(1) $\mathbf{x}_0^*$ is a saddle because it has **real eigenvalues with opposite signs.**

(2) $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$ are limit cycles because their **eigenvalues are purely imaginary.**

All in all,

$$\boxed{\begin{aligned} \mathbf{x}_0^* &= (0,0) \rightarrow \text{ saddle,} \\ \mathbf{x}_1^* &= (1,0) \rightarrow \text{ limit cycle,} \\ \mathbf{x}_2^* &= (-1,0) \rightarrow \text{ limit cycle.} \end{aligned}}$$

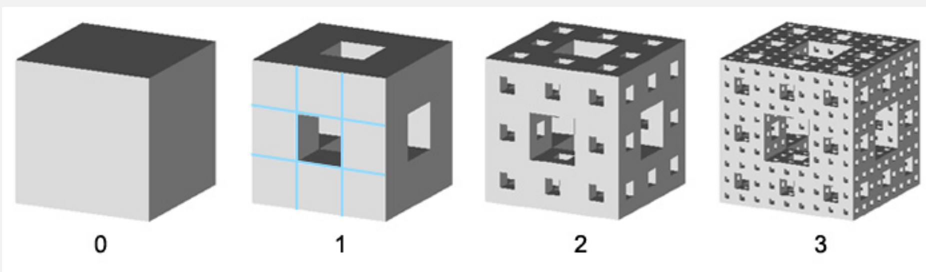
Problem 2: Hopf bifurcation

Consider the system $\ddot{x} + \lambda(x^2 - 1)\dot{x} + x - a = 0$. Find the curves on the space of parameters (λ, a) where a Hopf bifurcation occurs.

The second solution goes here.

Problem 3: Fractal dimension

Calculate the fractal dimension of the following object shown at three successive levels of construction.



We observe that the object has been broken in 20 pieces, and each piece would need to be scaled by a factor of 3 to obtain the original object. Then,

$$\boxed{D = \frac{\log N}{\log \epsilon} = \frac{\log 20}{\log 3} \approx 2.7268,}$$

which is less than 3, as expected.

Problem 4: Sensitivity and analytical solution

Consider the map $x_{n+1} = f(x_n) = (2x_n^{2/3} - 1)^3$, for $x_n \in [-1, 1]$.

(a) Show, by iterating two close initial conditions, that this map is chaotic.

(b) Show that $x_n = \cos^3(2^n \cos^{-1}(x_0^{1/3}))$ is a solution $\forall n$.

(a) In python, we coded this simple function that implements the map and returns the evolution of an initial condition:

```python
# Simple function
def map_evolution(x0: float, iter: int) -> list:
    """
    Computes the evolution of an initial condition.

    Parameters
    ----------
    x0 : float
        Initial condition.
    iter : int
        Number of iterations

    Returns
    -------
    x : list
        Evolution of the initial condition.
    """
    # Add the first element
    x = [x0]

    # Iteration process
    for i in range(iter + 1):

        # Compute next element
        next = (2 * x[i] - 1)**3

        # Save it
        x.append(next)

    return x

```

Then we just chose two *extremely* close initial conditions and found the evolution of both:

```python
# Initial conditions
x0 = 0.9999999998
x1 = 0.9999999999

# Iterate!
x0_results = map_evolution(x0, iter = 14)
x1_results = map_evolution(x1, iter = 14)

```

The results can be seen in figure 1. They evolve together for the first 10 iterations, but, in the 11th iteration, they start to separate. It takes no time for them to do so, even though they were very close at the beginning. $\boxed{\text{This is a clear sign of chaos.}}$

(b) We'll do it by induction, starting by plugging the expression in the map

$$x_{n+1} = [2(\cos^3(2^n \cos^{-1}(x_0^{1/3}))^{2/3} - 1]^3.$$

Then, some straightforward algebra gives us

$$\begin{aligned} x_{n+1} &= [2\cos^2(2^n \cos^{-1}(x_0^{1/3})) - 1]^3 \\ &= [2\cos^2(2 \cdot 2^{n-1} \cos^{-1}(x_0^{1/3})) - 1]^3. \end{aligned}$$

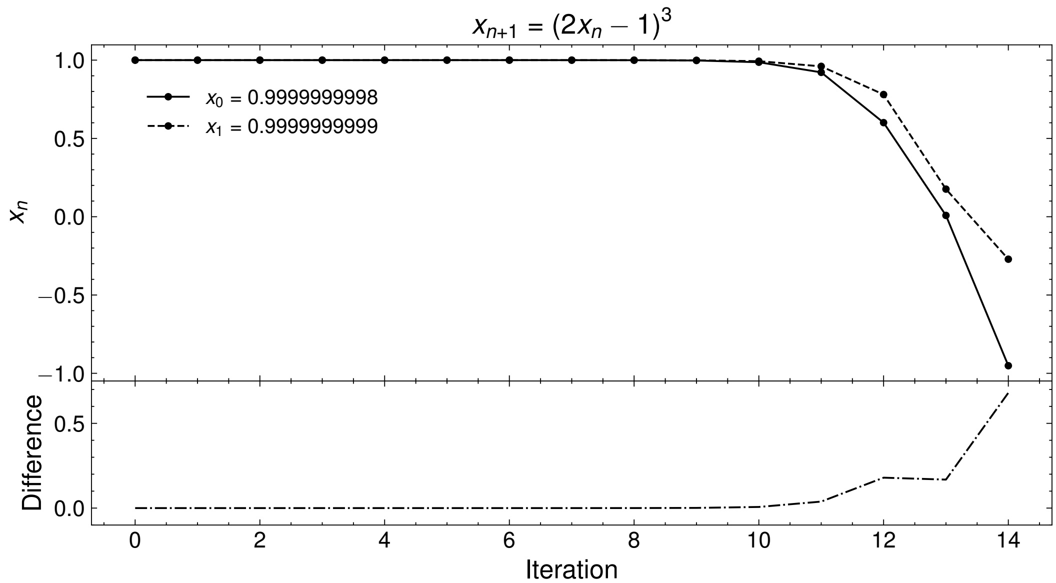


Figure 1: Evolution of two close initial conditions, plus the difference between them.

We can now apply the identity $\cos^2(2x) = (1 + \cos(4x))/2$, resulting in

$$\begin{aligned} x_{n+1} &= \left[2 \cdot \frac{1 + \cos\left(2^2 \cdot 2^{n-1} \cos^{-1}(x_0^{1/3})\right)}{2} - 1\right]^3 \\ &= \left[\cos\left(2^{n+1} \cos^{-1}(x_0^{1/3})\right)\right]^3 \\ &= \cos^3(2^{n+1} \cos^{-1}(x_0^{1/3})). \end{aligned}$$

Thus, the expression holds for $n + 1$. To finish, let's verify the base case

$$x_0 = \cos^3(2^0 \cos^{-1}(x_0^{1/3})) = \cos^3(\cos^{-1}(x_0^{1/3})) = x_0^{3/3} = x_0.$$

$\boxed{\text{Therefore, } x_n = \cos^3(2^n \cos^{-1}(x_0^{1/3})) \text{ is a solution } \forall n.}$

Problem 5: Bifurcation diagram and Lyapunov exponent

Consider the map $x_{n+1} = f(x_n) = \sin^2(r \arcsin\sqrt{x_n})$, for $x_n \in [0, 1]$.

(a) Obtain the bifurcation diagram of x_n as a function of r, for $r \in [1, 4]$.

(b) Calculate the Lyapunov exponent as a function of r, for $r \in [1, 4]$.

(a) Once passed the transient regime, we recorded the stable x_n points for r in the specified range. For plotting, thanks to the helpful description by PAR-commonswiki (2005), we were able to get a good-looking diagram using a matrix to store the values of x_n and r, which allowed us to control the resolution in a simple yet effective manner. The result is shown in figure 2.

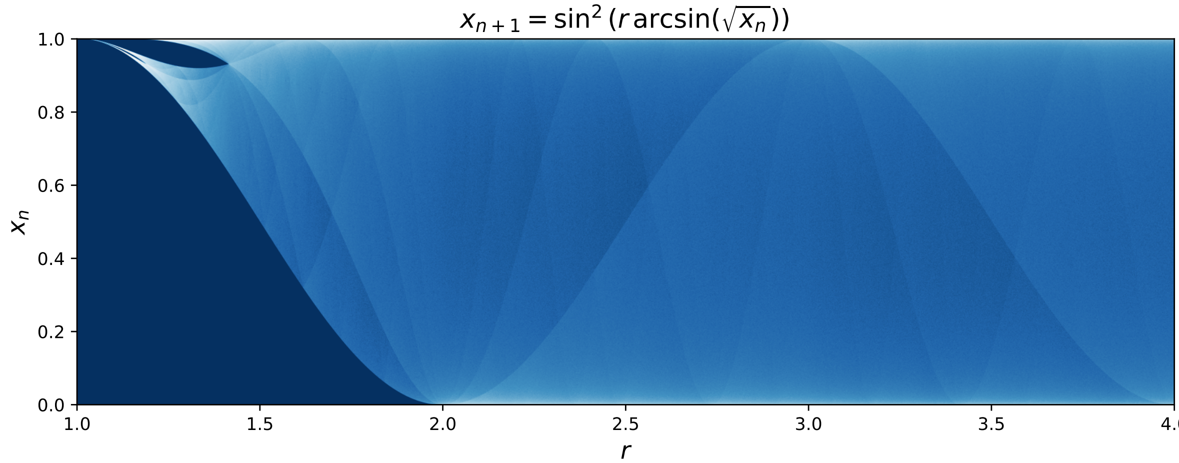


Figure 2: Bifurcation diagram for $r \in [1, 4]$.

It clearly shows transition to chaos by *period doubling*, and, interesting aspect, sinusoidal-like curves where there are no or few stable points as r increases.

(b) To calculate the Lyapunov exponent, we used the formula

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(x_i)|,$$

and, in our case, we have

$$f'(x) = \frac{r \sin\left(2r \arcsin \sqrt{x}\right)}{2\sqrt{x(1-x)}}.$$

In the implementation, we added small offsets both in the logarithm and derivative denominator to avoid division-by-zero errors. The results are shown in figure 3.

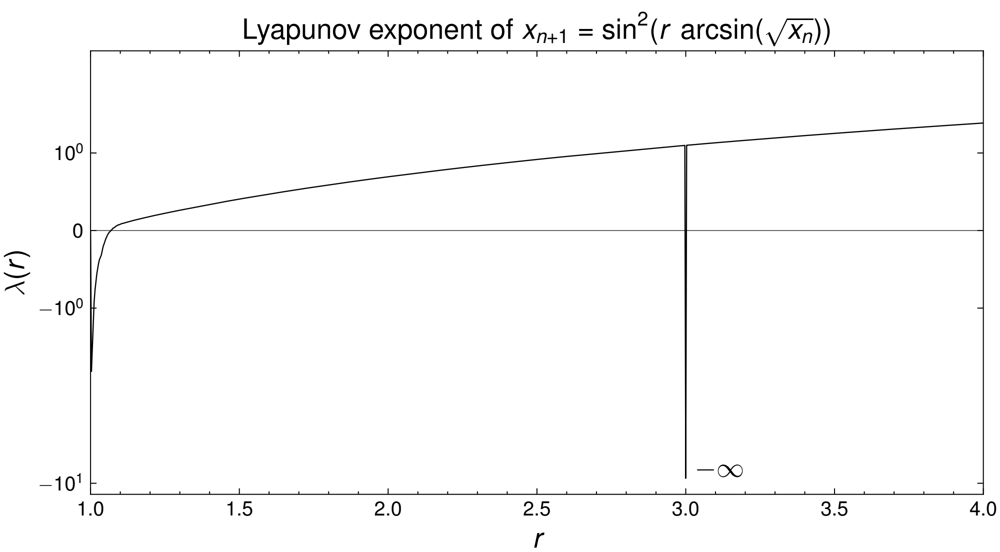


Figure 3: Lyapunov exponent for $r \in [1, 4]$.

Most of it was expected just by looking at the bifurcation diagram: negative in the beginning and rapidly increasing to positive. However, it is of special attention what happens at $r = 3.0$, where the Lyapunov exponent goes from a positive value to $-\infty$, which means the system reaches a superstable point! After that, it immediatly goes back to the trend it was following in the positive region. It's strange but wonderful.

Problem 6: Phase space

The evolution of a system is described by the following equation:

$$\dddot{x} + a\ddot{x} + \dot{x} - |x| + 1 = 0, \text{ for } a > 0.$$

(a) Find the fixed points of this system.

(b) Plot the attractor of this system in its phase space for $a = 0.6$. Is it strange?

(c) Show that this system is not chaotic for $a = 0.68$.

Solution goes here.

References

PAR-commonswiki (2005). File:tentmap bifurcationdiagram.png. [Online; accessed 27-March-2025].