Take-home exam 1

Dynamical Systems

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> **Problem 1: Classification of fixed points**
>
> A particle of mass $m = 1$ is moving in the potential $V(x) = -(1/2)x^2 + (1/4)x^4$. Find and classify the fixed points (node, saddle, focus) according to their stability.

We need to use the Hamiltonian formalism for this problem. As such, we first write

$$H(x, p) = T + V = \frac{p^2}{2} - \frac{1}{2}x^2 + \frac{1}{4}x^4,$$

which is used to find the equations of motion

$$\begin{aligned} \dot{x} &= +\frac{\partial H}{\partial p} = p, \\ \dot{p} &= -\frac{\partial H}{\partial x} = x - x^3 = x(1 - x^2). \end{aligned}$$

The fixed points are obtained by setting them to zero and solving

$$\begin{aligned} 0 &= p^*, \\ 0 &= x(1 - x^2) \implies x^* = 0, \pm 1. \end{aligned}$$

They combine to give us

$$\begin{aligned} \mathbf{x}_0^* &= (0, 0), \\ \mathbf{x}_1^* &= (1, 0), \\ \mathbf{x}_2^* &= (-1, 0). \end{aligned}$$

In order to do the classification, we need to calculate the Jacobian matrix. In our case,

$$J = \begin{bmatrix} \dfrac{\partial \dot{x}}{\partial x} & \dfrac{\partial \dot{x}}{\partial p} \\ \dfrac{\partial \dot{p}}{\partial x} & \dfrac{\partial \dot{p}}{\partial p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{bmatrix}.$$

By evaluating the Jacobian at the points found, we obtain the eigenvalues for each

$$\begin{aligned} J(\mathbf{x}_0^*) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \implies \lambda^2 - 1 = 0 \implies \lambda = \pm 1 \\ J(\mathbf{x}_1^*) &= \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \implies \lambda^2 + 2 = 0 \implies \lambda = \pm i\sqrt{2}. \end{aligned}$$

Since $J(\mathbf{x}_1^*) = J(\mathbf{x}_2^*)$, we have them all and we can proceed with the classification:

- $\mathbf{x}_0^*$ is a saddle because it has real eigenvalues with opposite sign.
- $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$ are limit cycles because their eigenvalues are purely imaginary.

All in all,

$$\boxed{\begin{aligned}\mathbf{x}_0^* &= (0,0) \to \text{ saddle,} \\ \mathbf{x}_1^* &= (1,0) \to \text{ limit cycle,} \\ \mathbf{x}_2^* &= (-1,0) \to \text{ limit cycle.}\end{aligned}}$$

Problem 2: Hopf bifurcation

Consider the system $\ddot{x} + \lambda(x^2 - 1)\dot{x} + x - a = 0$. Find the curves on the space of parameters (λ, a) where a Hopf bifurcation occurs.

We start by rewriting the equation as a system of first-order ODEs. We define

$$\mathbf{S}(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix},$$

so that

$$\frac{d\mathbf{S}}{dt} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\lambda(x_1^2 - 1)x_2 - x_1 + a \end{bmatrix}.$$

We have to look for the fixed points now. The first equation yields $x_2^* = 0$, and we can plug it into the second one:

$$0 = -x_1^* + a \implies x_1^* = a.$$

So the fixed points are $\mathbf{x}^* = (a, 0)$. We can proceed with the Jacobian matrix,

$$J = \begin{bmatrix} 0 & 1 \\ -2\lambda x_1 x_2 - 1 & -\lambda(x_1^2 - 1) \end{bmatrix},$$

and evaluate it at the fixed point

$$J(\mathbf{x}^*) = \begin{bmatrix} 0 & 1 \\ -1 & -\lambda(a^2 - 1) \end{bmatrix}.$$

Then, the characteristic equation is

$$\omega^2 + \lambda(a^2 - 1)\omega + 1 = 0,$$

whose roots are

$$\omega_{1,2} = \frac{-\lambda(a^2 - 1) \pm i\sqrt{4 - \lambda^2(a^2 - 1)^2}}{2},$$

assuming $4 - \lambda^2(a^2 - 1)^2 > 0$, so that $\omega_1 = \omega_2^*$. Furthermore, a Hopf bifurcation occurs when the eigenvalues cross the imaginary axis, which happens when the real part of the eigenvalues changes sign. In this system, this occurs when

$$\lambda(a^2 - 1) = 0 \implies a^2 = 1.$$

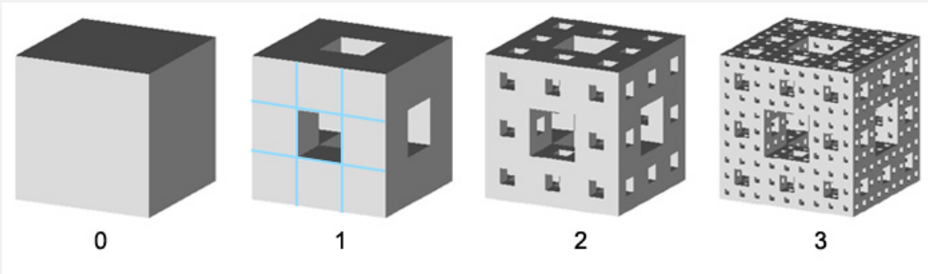
We ommit the case when $\lambda = 0$ because it would take $\dot{x}$ away from the ODE. Therefore, the bifurcation happens at the curves

$$\boxed{\begin{aligned} a &= 1, \\ a &= -1, \end{aligned}}$$

when limited by $4 - \lambda^2(a^2 - 1)^2 > 0$.

Problem 3: Fractal dimension

Calculate the fractal dimension of the following object shown at three successive levels of construction.



We observe that the object has been broken into 20 pieces, and each piece would need to be scaled by a factor of 3 to obtain the original object. Then,

$$\boxed{D = \frac{\log N}{\log \epsilon} = \frac{\log 20}{\log 3} \approx 2.7268,}$$

which is less than 3, as expected.

Problem 4: Sensitivity and analytical solution

Consider the map $x_{n+1} = f(x_n) = (2x_n^{2/3} - 1)^3$, for $x_n \in [-1, 1]$.

(a) Show, by iterating two close initial conditions, that this map is chaotic.

(b) Show that $x_n = \cos^3(2^n \cos^{-1}(x_0^{1/3}))$ is a solution $\forall n$.

(a) In python, we coded this simple function that implements the map and returns the evolution given an initial condition:

```python
# Simple function
def map(x0: float, iter: int) -> list:
    """
    Computes the evolution of an initial condition.

    Parameters
    ----------
    x0 : float
        Initial condition.
    iter : int
        Number of iterations
```

```
        Returns
        -------
        x : list
            Evolution of the initial condition.
        """
        # Add the first element
        x = [x0]

        # Iteration process
        for i in range(iter + 1):

            # Compute next element
            next = (2 * x[i] - 1)**3

            # Save it
            x.append(next)

        return x
```

Then we just chose two *extremely* close initial conditions (denoted x_0 and x_1), and found the map evolutions:

```
    # Initial conditions
    x0 = 0.9999999998
    x1 = 0.9999999999

    # Iterate!
    x0_evol = map(x0, iter = 14)
    x1_evol = map(x1, iter = 14)
```

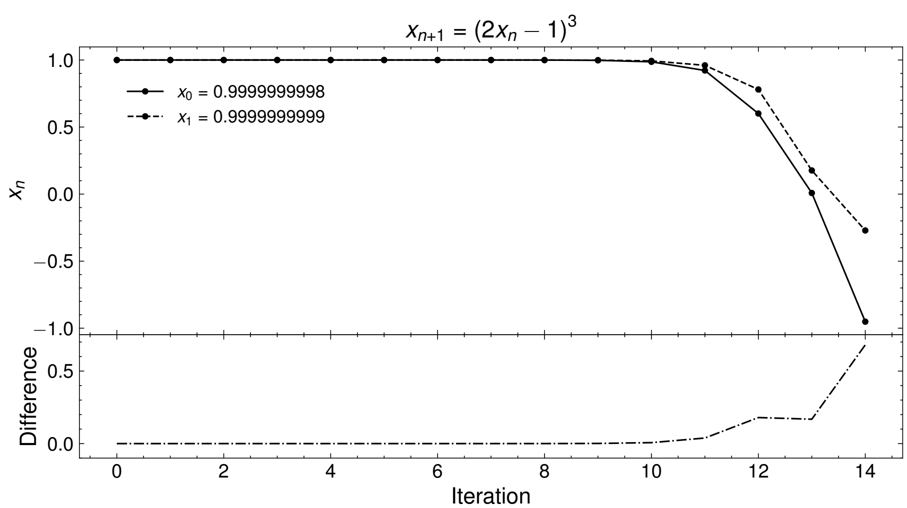


Figure 1: Evolution of two close initial conditions, plus the difference between them.

The results can be seen in figure 1. They evolve together for the first 10 iterations, but, in the 11th iteration, they start to separate. It takes little time for them to do so, even though they were very close at the beginning. $\boxed{\text{This is a clear sign of chaos.}}$

(b) We'll do it by induction, starting by plugging the expression in the map

$$x_{n+1} = [2(\cos^3(2^n \cos^{-1}(x_0^{1/3}))^{2/3} - 1]^3.$$

Then, some straightforward algebra gives us

$$\begin{aligned} x_{n+1} &= [2\cos^2(2^n \cos^{-1}(x_0^{1/3})) - 1]^3 \\ &= [2\cos^2(2 \cdot 2^{n-1} \cos^{-1}(x_0^{1/3})) - 1]^3. \end{aligned}$$

We can now apply the identity $\cos^2(2x) = (1 + \cos(4x))/2$, resulting in

$$\begin{aligned} x_{n+1} &= \left[2 \cdot \frac{1 + \cos\left(2^2 \cdot 2^{n-1} \cos^{-1}(x_0^{1/3})\right)}{2} - 1\right]^3 \\ &= \left[\cos\left(2^{n+1} \cos^{-1}(x_0^{1/3})\right)\right]^3 \\ &= \cos^3(2^{n+1} \cos^{-1}(x_0^{1/3})). \end{aligned}$$

Thus, the expression holds for $n+1$. To finish, let's verify the base case

$$x_0 = \cos^3(2^0 \cos^{-1}(x_0^{1/3})) = \cos^3(\cos^{-1}(x_0^{1/3})) = x_0^{3/3} = x_0.$$

$\boxed{\text{Therefore, } x_n = \cos^3(2^n \cos^{-1}(x_0^{1/3})) \text{ is a solution } \forall n.}$

Problem 5: Bifurcation diagram and Lyapunov exponent

Consider the map $x_{n+1} = f(x_n) = \sin^2(r \arcsin \sqrt{x_n})$, for $x_n \in [0, 1]$.

(a) Obtain the bifurcation diagram of x_n as a function of r, for $r \in [1, 4]$.

(b) Calculate the Lyapunov exponent as a function of r, for $r \in [1, 4]$.

(a) Once passed the transient regime, we recorded the data points for r in the specified range. For plotting, thanks to the helpful description by PAR-commonswiki (2005), we were able to get a good-looking diagram using a matrix to store the values of x_n and r, which allowed us to control the resolution in a simple yet effective manner. The result is shown in figure 2.

It clearly shows transition to chaos by *period doubling*, and, interesting aspect, sinusoidal-like curves, indicating points that are not being visited.

(b) To calculate the Lyapunov exponent, we used the formula

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(x_i)|,$$

and, in our case, we have

$$f'(x) = \frac{r \sin(2r \arcsin \sqrt{x})}{2\sqrt{x(1-x)}}.$$

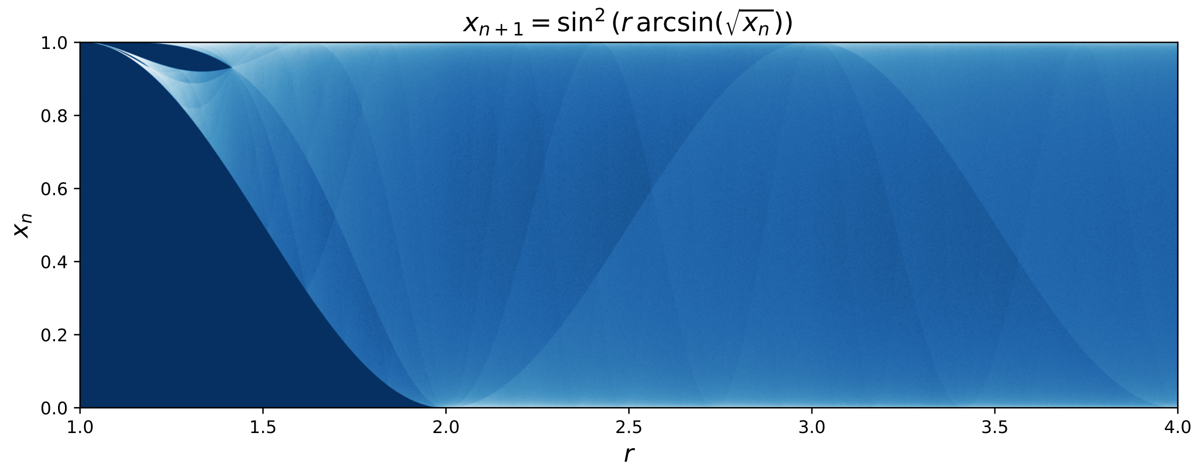


Figure 2: Bifurcation diagram for $r \in [1, 4]$.

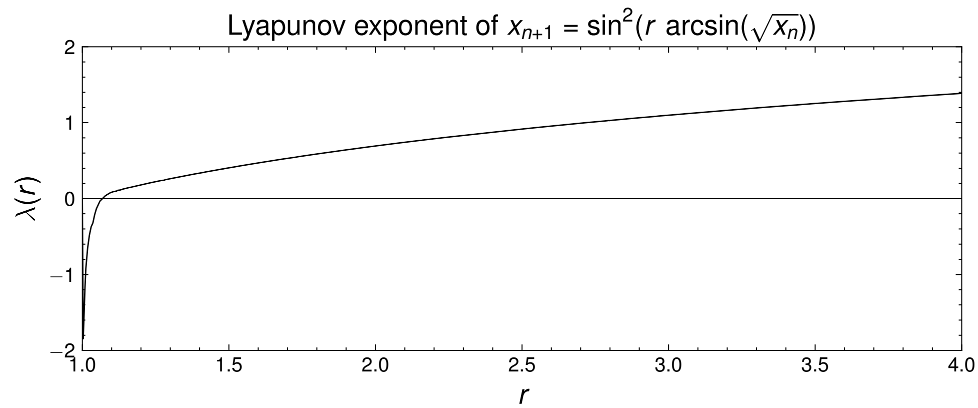


Figure 3: Lyapunov exponent for $r \in [1, 4]$.

In the implementation, we added small offsets both in the logarithm and derivative denominator to avoid division-by-zero errors. The results are shown in figure 3.

We could have forseen the results just by looking at the bifurcation diagram. That is, negative for a very small range of r at the beginning, and then increasingly positive for the rest of it.

Note: When solving the problem, we came across an interesting finding: the map is *not* chaotic for $r = 3.0$ and $x_0 = 0.250$. This is shown in figure 4, where we plotted the non-transient evolution of such conditions. The Lyapunov exponent for this case goes to negative infinity!

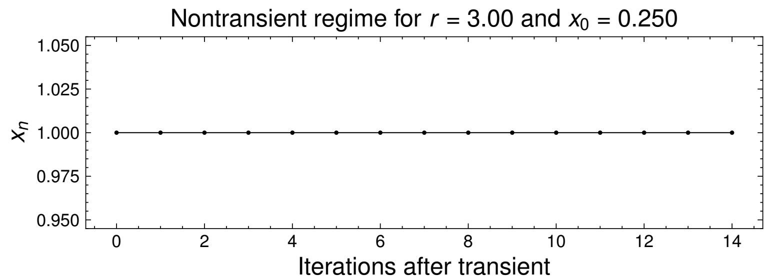


Figure 4: Strange non-chaotic behaviour for specific conditions.

Problem 6: Phase space

The evolution of a system is described by the following equation:

$$\dddot{x} + a\ddot{x} + \dot{x} - |x| + 1 = 0, \text{ for } a > 0.$$

(a) Find the fixed points of this system.

(b) Plot the attractor of this system in its phase space for $a = 0.6$. Is it strange?

(c) Show that this system is not chaotic for $a = 0.68$.

(a) We first need to convert the third-order ODE into a system of three first-order ODEs. We do this by defining the state vector

$$\mathbf{S}(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix},$$

and writing the system of ODEs as

$$\frac{d\mathbf{S}}{dt} = \begin{bmatrix} x_2 \\ x_3 \\ -ax_3 - x_2 + |x_1| - 1 \end{bmatrix}.$$

Similarly as before, we set each equation to zero and solve for the fixed points:

$$\begin{aligned} 0 &= x_2, \\ 0 &= x_3, \\ 0 &= -ax_3 - x_2 + |x_1| - 1. \end{aligned}$$

Now, plugging the first two in the third one gives us

$$0 = |x_1| - 1 \implies x_1 = \pm 1.$$

Thus, the fixed points are

$$\boxed{\begin{aligned} \mathbf{x}_0^* &= (1, 0, 0), \\ \mathbf{x}_1^* &= (-1, 0, 0). \end{aligned}}$$

(b) Rewriting the ODE as a system of first-order ODEs is also useful for solving it *numerically*. In python, we started by creating a function that implements such system:

```python
# Little function
def system(t: float, S: np.ndarray, a: float) -> np.ndarray:
    """
    System of first-order ODEs.

    Parameters
    ----------
    t : float
        Time.
    S : np.ndarray
        State vector.
    a : float
        Parameter.

    Returns
    -------
    np.ndarray
        Derivative of the state vector.
    """
    # Extract variables
    x1, x2, x3 = S

    # Define the equations
    dx1_dt = x2
    dx2_dt = x3
    dx3_dt = -a * x3 - x2 + abs(x1) - 1

    return [dx1_dt, dx2_dt, dx3_dt]
```

After doing that, we called the `solve_ivp` function from the `scipy` library to solve the system of equations:

```python
    # Set a
    a = 0.60

    # Time span and evaluation
    t_span = (0, 1000)
    t_eval = np.linspace(t_span[0], t_span[1], 15000)

    # Initial conditions
    S0 = [0.1, 0.1, 0.1]

    # And solve the system!
    sol = solve_ivp(slope, t_span, S0, t_eval, method='RK45', args=(a,))

    # Extract the solutions
    x1 = sol.y[0]
    x2 = sol.y[1]
    x3 = sol.y[2]
```

Finally, we plotted the results in phase space, shown in figure 5. For visualization purposes, we didn't include the first part of the evolution, and also added a small set of axes indicating the orientation. I think it came out pretty well.

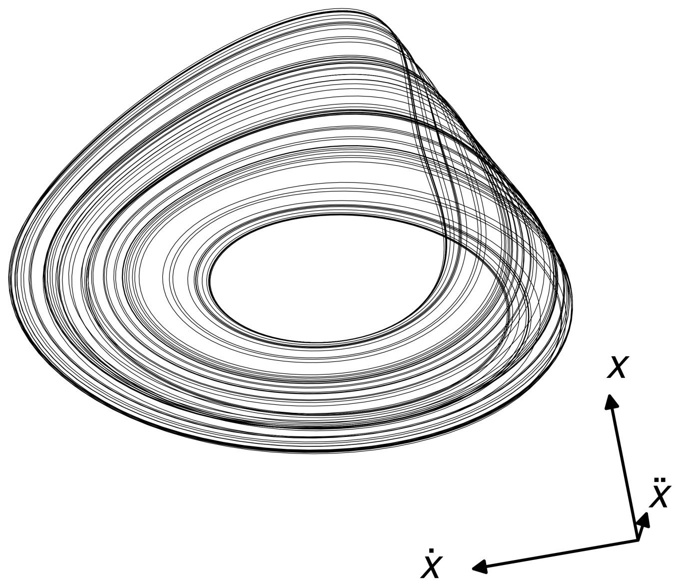


Figure 5: Good view of the system evolution in phase space for $a = 0.60$.

To some degree, it resembles Rossler's attractor. $\boxed{\text{Answering, it is indeed of } \textit{strange} \text{ type.}}$

(c) Reusing the same code, we set $a = 0.68$ and the plot is shown in figure 6.

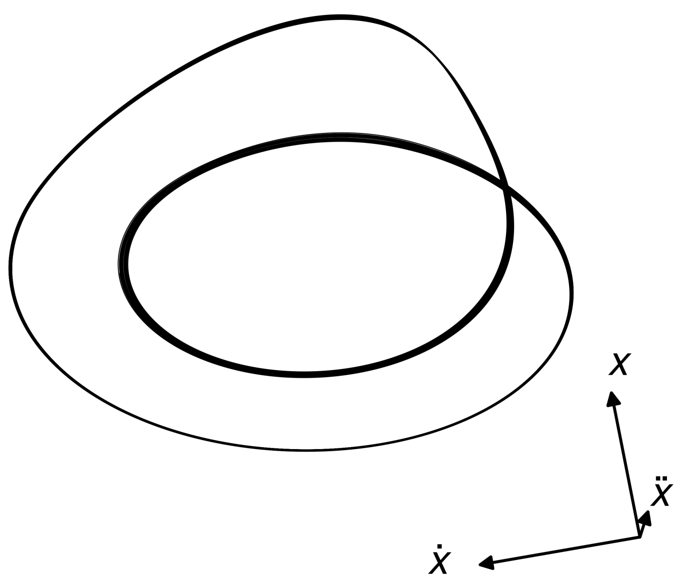


Figure 6: Non-chaotic system evolution in phase space for $a = 0.68$.

The result is quite different from the previous: it looks like a *closed curve* in phase space. An evolution like this indicates the system exhibits stable oscillatory behaviour. In other words, $\boxed{\text{it is not chaotic when } a = 0.68.}$

References

PAR-commonswiki (2005). Tentmapbifurcationdiagram.png. [Accessed 27-March-2025].