Take-home exam 2

Dynamical Systems

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> **Problem 1: Superestable orbit**
>
> Let r_n^* be the parameter value of a quadratic map $f_r(x)$ for which there exists a superstable orbit of period 2^n.
>
> (a) Consider an arbitrary function $h(r_n)$ evaluated at r_n^*. Show that, for $n \gg 1$, $h(r_n)$ satisfies the relation [1]:
> $$\frac{[h(r_n^*) - h(r_\infty)]\delta^n}{h'(r_\infty)} = \text{cte}.$$
>
> (b) Show that for $n \gg 1$, [1]
> $$(r_n^* - r_\infty)\delta^n \propto \frac{\delta^2}{\delta - 1}.$$

(a) We start by expanding the function $h(r_n^*)$ around r_∞:

$$h(r_n^*) = h(r_\infty) + h'(r_\infty)(r_n^* - r_\infty) + O((r_n^* - r_\infty)^2).$$

For large n, we can neglect higher-order terms:

$$h(r_n^*) - h(r_\infty) \approx h'(r_\infty)(r_n^* - r_\infty).$$

Now, we can use the relation for the superstable orbit:

$$r_n^* - r_\infty = C\delta^{-n},$$

where C is a constant. Substituting this into the previous equation gives us

$$h(r_n^*) - h(r_\infty) \approx h'(r_\infty)C\delta^{-n},$$

and rearranging yields

$$\boxed{\frac{[h(r_n^*) - h(r_\infty)]\delta^n}{h'(r_\infty)} = C}.$$

(b) We can express $r_n^* - r_\infty$ as a telescopic series,

$$r_n^* - r_\infty = (r_n^* - r_{n+1}^*) + (r_{n+1}^* - r_{n+2}^*) + \ldots = \sum_{j=0}^{\infty}(r_{n+j}^* - r_{n+j+1}^*).$$

Let $\Delta_n = r_{n+j}^* - r_{n+j+1}^*$, so that $\Delta_{n+1} = \Delta_n/\delta$. We can then write

$$r_n^* - r_\infty = \Delta_n \sum_{j=0}^{\infty} \frac{1}{\delta^j} = \Delta_n \frac{\delta}{\delta - 1},$$

using geometric series properties. The last step is given by $\Delta_n = \Delta_1/\delta^{n-1}$, where Δ_1 is a constant. Thus,

$$r_n^* - r_\infty = \frac{\Delta_1}{\delta^{n-1}} \frac{\delta}{\delta - 1} \implies \boxed{(r_n^* - r_\infty)\delta^n \propto \frac{\delta^2}{\delta - 1}}.$$

Problem 2: Schwarz derivative

Consider the logarithmic map $x_{n+1} = f(x_n) = b + \log|x_n|$.

(a) Calculate the Schwarz derivative of this map. [1]

(b) Show that the Lyapunov exponent for this map can be expressed as [1]

$$\lambda = b - \frac{1}{n}\sum_{j=1}^{n} x_j.$$

(a) The Schwarz derivative is given by

$$Sf(x) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

So we start by calculating all the derivatives,

$$\begin{aligned} f'(x) &= \frac{1}{x}, \\ f''(x) &= -\frac{1}{x^2}, \\ f'''(x) &= \frac{2}{x^3}, \end{aligned}$$

substituting them in the relation, and simplifying we get

$$\begin{aligned} Sf(x) &= \frac{2}{x^3}\frac{x}{1} - \frac{3}{2}\left(-\frac{1}{x^2}\frac{x}{1}\right)^2 \\ &= \frac{2}{x^2} - \frac{3}{2}\left(-\frac{1}{x}\right)^2 \\ &= \frac{2}{x^2} - \frac{3}{2x^2} \\ &= \frac{1}{2x^2}. \end{aligned}$$

In conclusion, the Schwarz derivative is *always positive*, and it is given by

$$\boxed{Sf(x) = \frac{1}{2x^2}.}$$

(b) The Lyapunov exponent is given by

$$\lambda = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log|f'(x_j)|.$$

Since we already have the expression for $f'(x)$, we can just substitute it and simplify again:

$$\begin{aligned}\lambda &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left|\frac{1}{x_j}\right| \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log |x_j|.\end{aligned}$$

Now, we can use the map to express $\log |x_j|$ in terms of b and x_{j-1}:

$$\begin{aligned}\lambda &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log |x_j| \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} (b - x_{j+1}) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (b - x_i).\end{aligned}$$

Using properties of summation and limits, we can get b out:

$$\begin{aligned}\lambda &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} b - \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i \\ &= \lim_{n\to\infty} \frac{nb}{n} - \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i \\ &= b - \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i.\end{aligned}$$

Finally, for large n, we can write the expression as:

$$\boxed{\lambda = b - \frac{1}{n} \sum_{j=1}^{n} x_i.}$$

> **Problem 3: Bifurcation diagram**
>
> Consider the following map for $x_n \in [0, 1]$, depending on two real parameters s and c,
>
> $$x_{n+1} = f(x_n) = |\tanh[s(x_n - c)]|.$$
>
> Obtain the bifurcation diagram of x_n as a function of $c \in [0, 1]$ with fixed value $s = 1.3$. [1]

We reused the code from the previous take-home exam to get the bifurcation diagram. The only thing we needed to change was the map function, which is now:

```python
# Map
def tanh_map(x, c):
    return abs(np.tanh(1.3 * (x - c))
```

The result shows *inverse period-doubling* for $c \in [0, 1]$, and it indicates that the Schwarz derivative should be positive. The bifurcation diagram is shown in Figure 1.

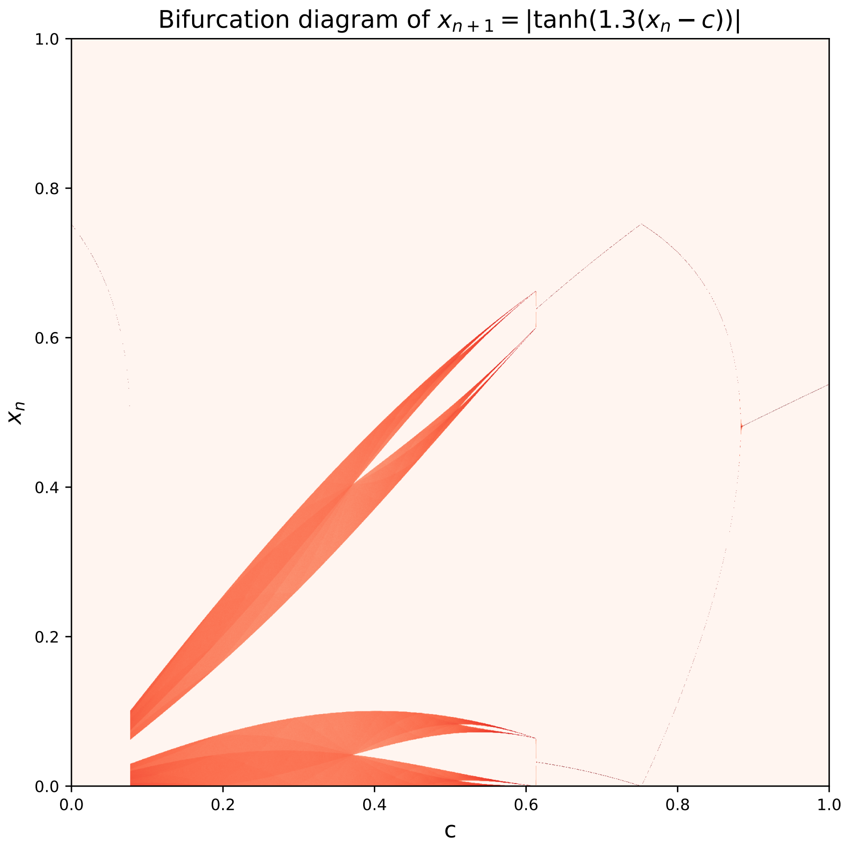

Figure 1: Inverse period-doubling for $s = 1.3$.

Problem 4: Lyapunov exponent

Consider the following map for $x_n \in [0, 1]$:

$$x_{n+1} = f(x_n) = \begin{cases} (1+\epsilon)x_n + x_n^2, & \text{if } x_n < x^*, \\ 1 - x_n, & \text{if } x_n \geq x^*, \end{cases}$$

where x^* is the solution of $(1+\epsilon)x_n + x_n^2 = 1$.

(a) Calculate the Lyapunov exponent versus $\epsilon \in [-0.4, 0.4]$. [1]

(b) Describe the route to chaos exhibited by this map on such interval. [1]

(a) Again, we can use the same code as before but with some modifications. The map function and its derivative are given by:

```python
# Map
def piecewise_map(x, eps, x_star):
    if x < x_star:
```

```
            return (1 + eps) * x + x**2
        else:
            return 1 - x

    # Derivative
    def derivative(x, eps, x_star):
        if x < x_star:
            return 1 + eps + 2 * x
        else:
            return -1

```

where x_star has been previously calculated for each ϵ. The result is shown in Figure 2. The

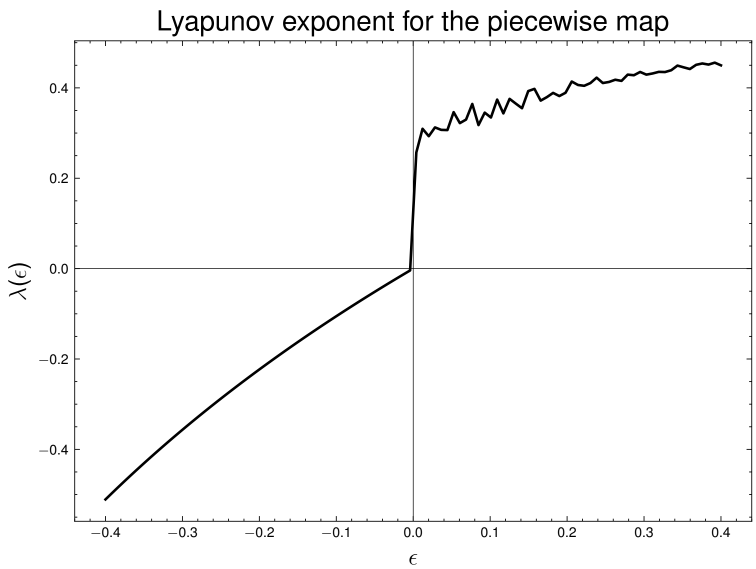

Figure 2: Evolution of the piecewise map for $\epsilon \in [-0.4, 0.4]$.

Lyapunov exponent is mostly negative for $\epsilon < 0$ and positive for $\epsilon > 0$, which suggests the map is chaotic for $\epsilon > 0$. The transition to chaos is not smooth at all: there's a sudden and rapid change at $\epsilon \approx 0$.

(b) Of all the routes to chaos we have studied, the one that adjusts better to the behavior of this map is *intermittency* because of the very rapid transition from periodic orbits to chaos. The Lyapunov exponent goes from negative to positive extremely quickly.

Problem 5: Not period-doubling

Consider the following map for $x_n \in [0, 1]$,

$$x_{n+1} = f(x_n) = \frac{1 - b^{x_n(1-x_n)}}{1 - b^{1/4}}$$

(a) Is this map unimodal for $b \in (0, 1)$? [1]

(b) Plot the bifurcation diagram of x_n on the interval $b \in (0, 1)$ and $x_n \in [0, 1]$. Why does this map not show period doubling? [1]

(c) Find the Lyapunov exponent for $b \in (0, 1)$. [1]

(a) One way to see if the map is unimodal is to plot $f(x_n)$ versus x_n for different values of b, and then *visually* check if the function has one maximum or minimum. We did it for many b's and the results are shown in Figure 3.

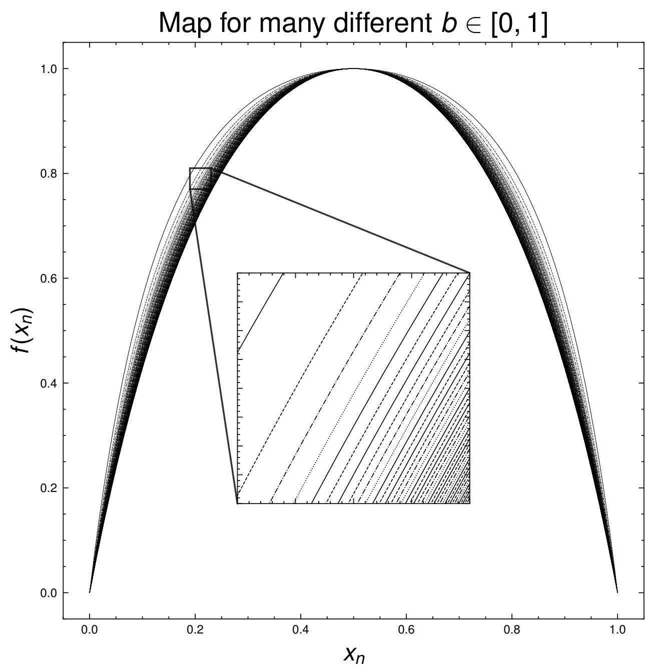

Figure 3: Plots of $f(x_n) = \frac{1-b^{x_n(1-x_n)}}{1-b^{1/4}}$ for $b \in (0, 1)$.

There is only one maximum, which suggests the map in unimodal. However, a more rigorous way to check it is to calculate the derivative of the map and see if it has only *one root*. So, the derivative is given by

$$f'(x) = \frac{(1-2x)b^{x(1-x)}\log(b)}{(1-b^{1/4})}$$

and it effectively has one root only: $x = 1/2$. Thus, $\boxed{\text{the map is indeed unimodal.}}$

(b) As always, reusing code but changing the map:

```python
# Map
def frac_map(x, b):
    num = 1 - b**(x * (1 - x))
    den = 1 - b**(1/4) + 1e-10 # Avoid division by zero
    return num/den

```

We obtained the results shown in Figure 4.

The map is chaotic from start to finish for $b \in (0, 1)$, and it also covers the full range of $x_n \in [0, 1]$. It's quite interesting to see this kind of behavior.

(c) We were expecting a positive Lyapunov exponent because of the previous part, and that's exactly what we got. In order to do it, we had first to compute and code the derivative of the map:

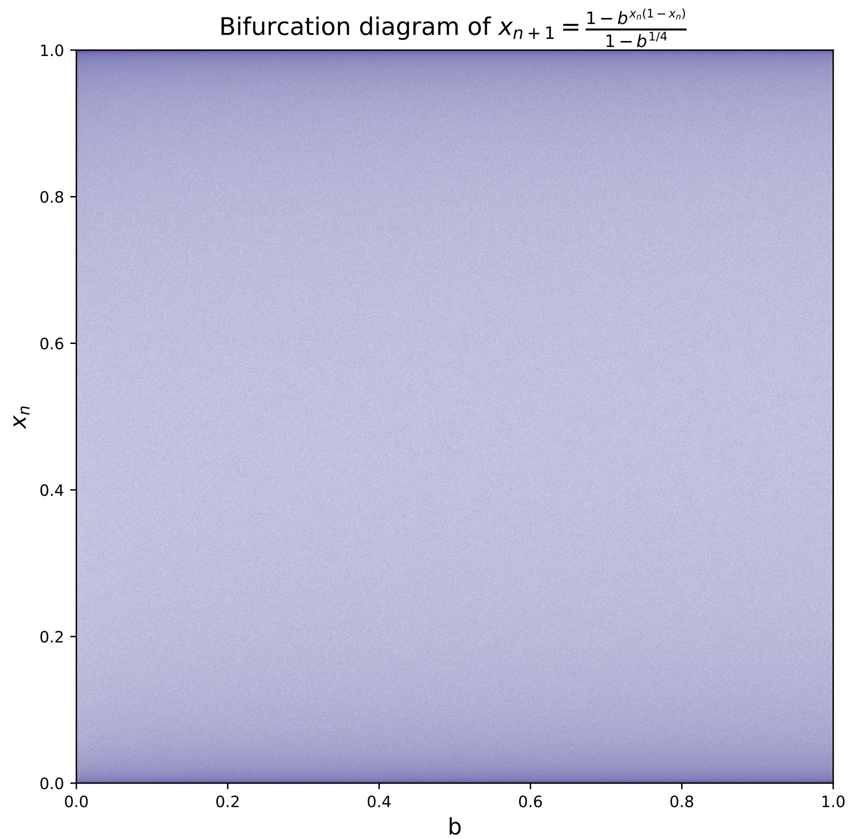

Figure 4: Pure chaos for $b \in (0, 1)$.

```
    # Derivative
    def frac_derivative(x, b):
        num = (1 - 2 * x) * b**(x * (1 - x)) * np.log(b)
        den = - (1 - b**(1/4)) + 1e-10 # Again, avoid division by zero
        return num/den

```

Then, we ran the code and obtained the results shown in Figure 5.

It is indeed positive for all $b \in (0, 1)$, which goes hand in hand with its bifurcation diagram.

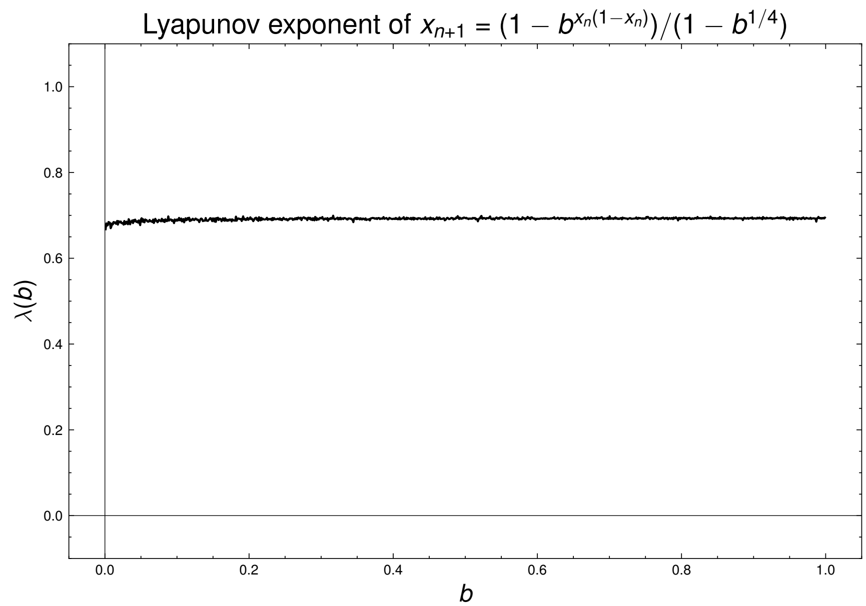

Figure 5: Lyapunov exponent for $b \in (0, 1)$.