Take-home exam 3

Dynamical Systems

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Yachay Tech, June 1, 2025.

Problem 1: Bifurcation diagram and Lyapunov exponent

Consider the following map $f(x_n) \in [0, 1]$,

$$x_{n+1} = f(x_n) = x_n + \omega - \frac{1}{2\pi}\sin(2\pi x_n) \bmod 1.$$

(a) Plot the bifurcation diagram of x_n vesurs ω. [1]

(b) Calculate the Lyapunov exponent as a function of ω. [1]

(a) Reusing code from the previous exam, we obtained the bifurcation diagram for $\omega \in [0, 2]$.

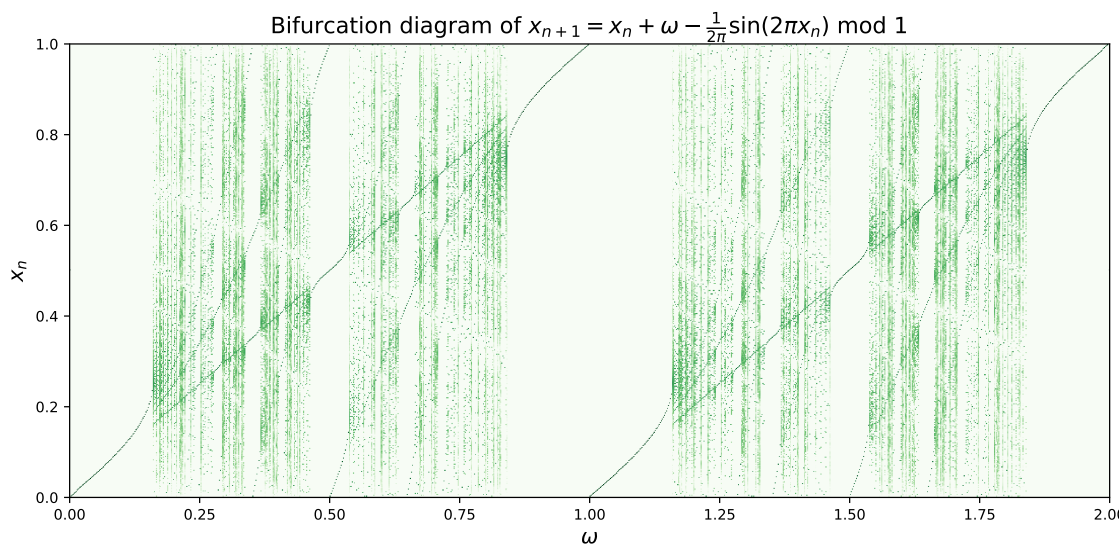

Figure 1: Long term behavior of the map $f(x_n)$ as a function of ω.

We observe the diagram repeats itself after $\omega = 1$.

(b) Again, recycling code, we calculated the Lyapunov exponent as a function of ω, and we show the figure below.

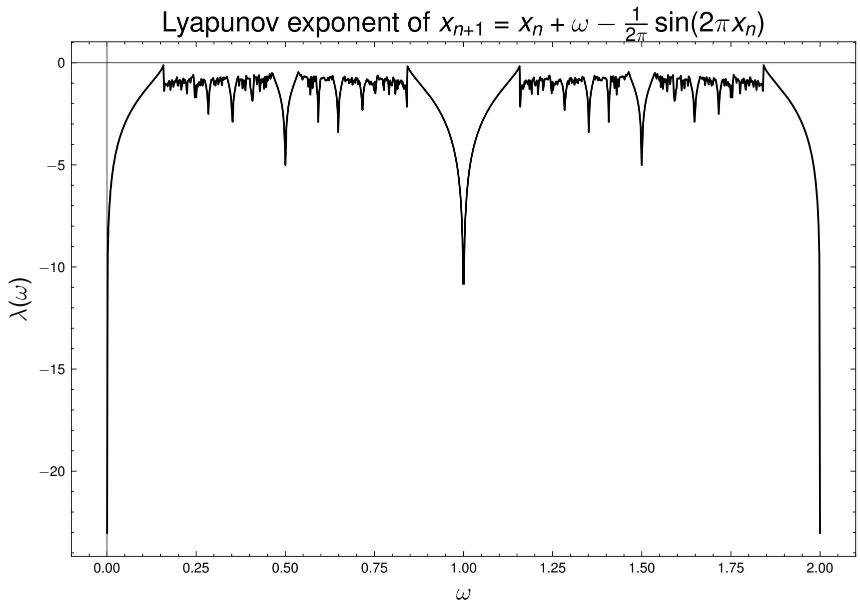

Figure 2: Behavior of the map as a function of ω.

It seems to match with the bifurcation diagram. Moreover, it appears to be negative for the most part of the interval, except at some points where it is equal to zero. This indicates that the map is mostly stable.

Problem 2: Two-dimensional map analysis

Consider the two-dimensional map

$$\begin{aligned} x_{n+1} &= 1 - a|x_n| + y_n, \\ y_{n+1} &= bx_n \end{aligned}$$

where a and b are real parameters.

(a) By using an initial condition close to the origin, plot the attractor for $a = 1.7$ and $b = 0.5$. Is is a strange attractor? [1]

(b) Plot the bifurcation diagram of x_n as a function of $b \in [-0.7, 0.7]$, with fixed value $a = 1.5$. [1]

(c) Calculate the Lyapunov exponents for this map as functions the parameters a and b. [1]

(d) Calculate the fractal dimension of the the attractor for $a = 1.7$ and $b = 0.5$. [1]

(a) Setting functions for each variable (plugging y_n in x_{n+1}), we plotted the attractor for the given parameters. Of course, we excluded the first iterations to avoid the transient behavior. The attractor is shown in Figure 3.

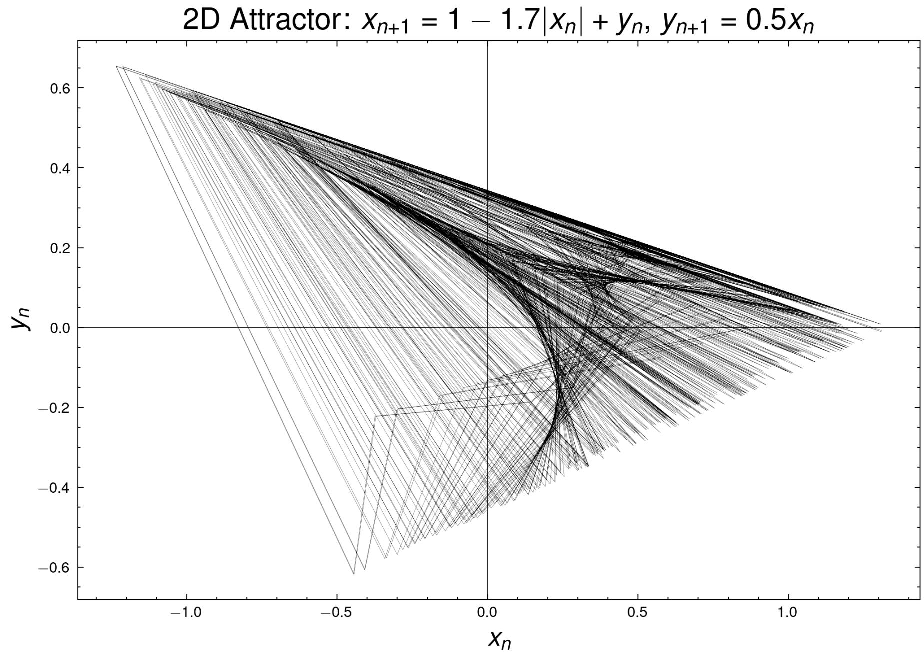

Figure 3: Attractor of the system for $a = 1.7$ and $b = 0.5$.

We can see that the attractor is indeed strange and looks like a fractal. It's awesome.

(b) The results of the calculation is shown in Figure 4. It has a funny yet intriguing shape.

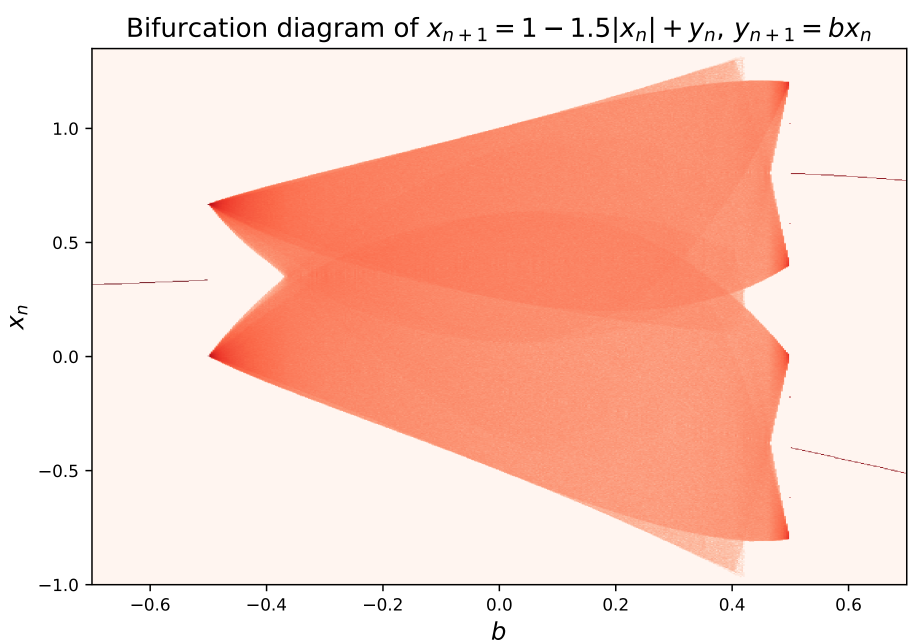

Figure 4: Long term values of the map as a function of b with fixed $a = 1.5$.

(c) We calculated the Lyapunov exponent for both parameters a and b, and plotted the results using a heatmap, where colors convey information about the magnitude of the exponent. The results are shown in Figure 5.

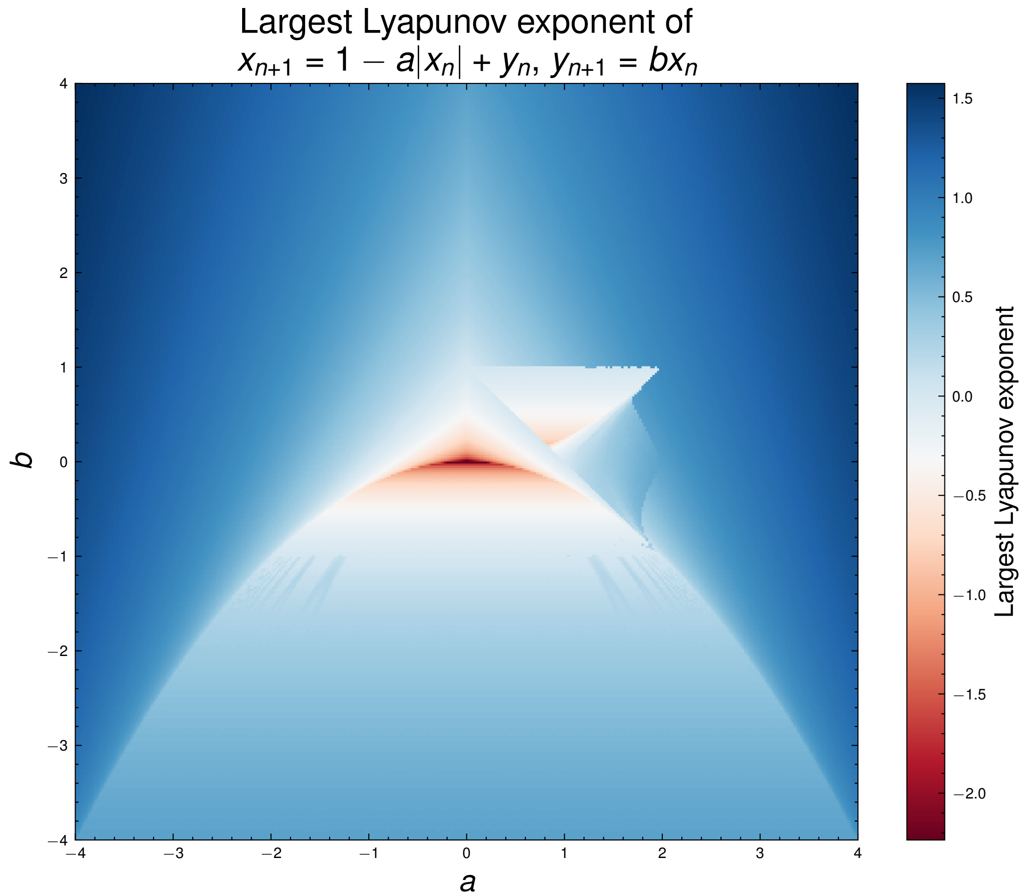

Figure 5: Lyapunov exponent heatmap of the two-dimensional map.

We took, as reviewed in class, the largest Lyapunov exponent. The results are stunning; it seems to display some kind of structure. We can also see that it is mostly positive.

(d) To calculate the fractal dimension, we used the Kaplan-Yorke formula. Since the map contains an absolute value, we had to consider both cases: $x > 0$ and $x < 0$. However, we found the same result for both cases, so we'll just show the case $x > 0$.

For $a = 1.7$ and $b = 0.5$, the Jacobian matrix is given by

$$J = \begin{pmatrix} -1.7 & 1 \\ 0.5 & 0 \end{pmatrix},$$

so that the characteristic equation is

$$\mu^2 + 1.7\mu - 0.5 = 0.$$

The roots of this equation are

$$\mu_1 \approx 0.25567,$$
$$\mu_2 \approx -1.95567.$$

With these, we can obtain the Lyapunov exponents via $e^{\lambda} = \mu$. We have

$$\lambda_1 = \ln(0.25567) \approx -1.3638,$$
$$\lambda_2 = \ln(-1.95567) \approx 0.6707,$$

where we took the real part of the argument in the second logarithm.

Since $\lambda_1 < 0$ and $\lambda_2 > 0$, the fractal dimension of the attractor for the given parameters is

$$\boxed{D_f = 1 + \frac{\lambda_2}{|\lambda_1|} = 1 + \frac{0.6707}{1.3638} \approx 1.49.}$$

Problem 3: Escape and repeller

Consider the map

$$x_{n+1} = \frac{a}{2}\left(1 - |1 - 2x_n|\right).$$

(a) Show that x_n escapes the interval $[0, 1]$ for $a = 2 + \epsilon$. [1]

(b) Calculate the fractal dimension of the repeller set as a function of ϵ. [1]

(a) This can be done by just iterating the map for $a = 2 + \epsilon$ and checking if the values of x_n escape the interval $[0, 1]$. A result of this is shown in Figure 6.

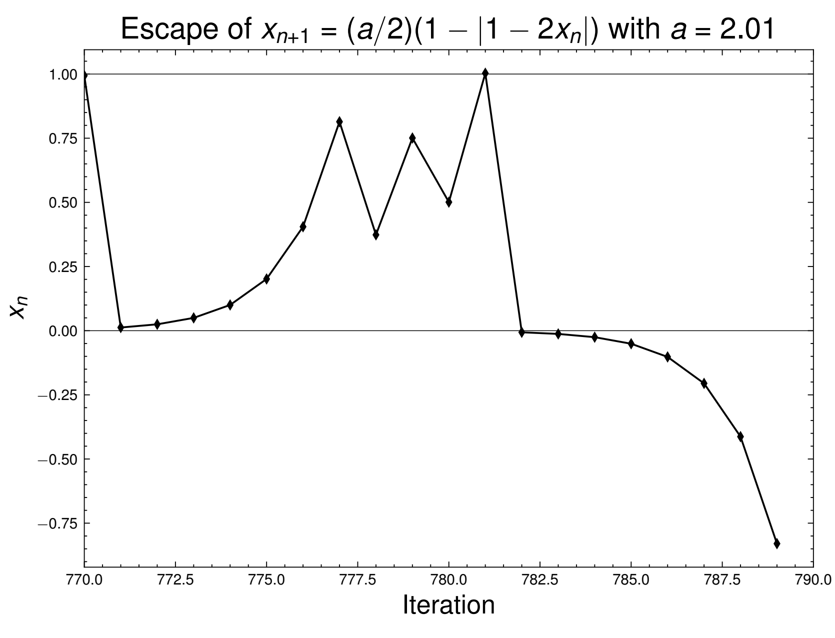

Figure 6: Escape of the map for $a = 2 + \epsilon$.

The map does escape the interval if $a > 2$.

(b) The fractal dimension of the repeller set was calculated using the escape rate method. For each value of ϵ, we sampled initial points across $[0, 1]$ and determined which points remained bounded after many iterations. The fractal dimension was then computed using the formula

$$D = 1 - \frac{\gamma}{\ln(a)}$$

where γ is the escape rate. We plot the fractal dimension as a function of ϵ in Figure 10.

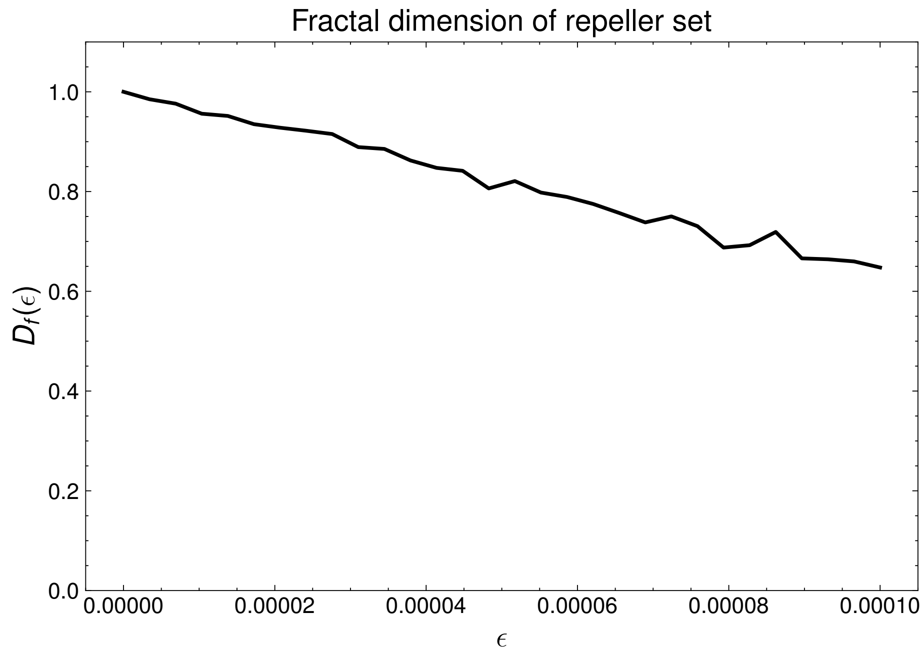

Figure 7: Repeller's fractal dimension.

The fractal dimension decreases as ϵ increases.

Problem 4: GOY method

The logarithmic map

$$x_{n+1} = b + \ln|x_n|$$

exhibits robuts chaos in the parameter interval $b \in [0, 1]$.

(a) Calculate the unstable fixed point for $b \in [0, 1]$. [1]

(b) Use the GOY method for controlling chaos to stabilize the fixed point at $b = 0$. Show the time series of x_n before and after the control is applied. [1]

(a) The fixed point is found by solving the equation

$$x^* = \ln|x^*| \implies x^* - \ln|x^*| = 0.$$

We did it numerically, and the result is shown below.

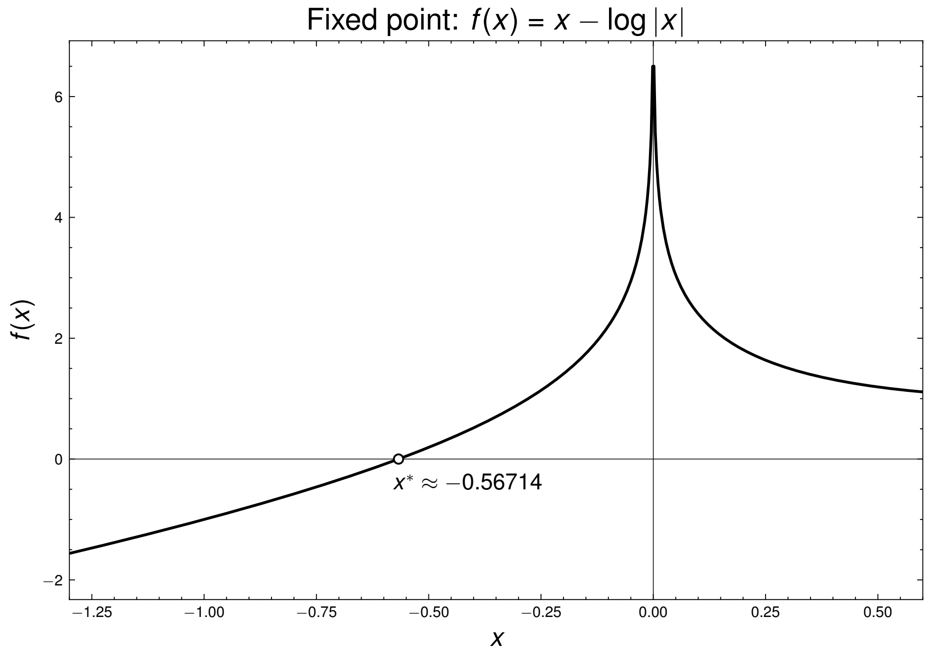

Figure 8: Fixed point found.

It was also found to be *unstable* since the absolute value of the derivative evaluated at the point is less than 1.

(b) Since the point is unstable, we can apply the GOY method.

First, to verify the point was actually unstable, we plotted the time series given some initial condition. This is shown in Figure 9.

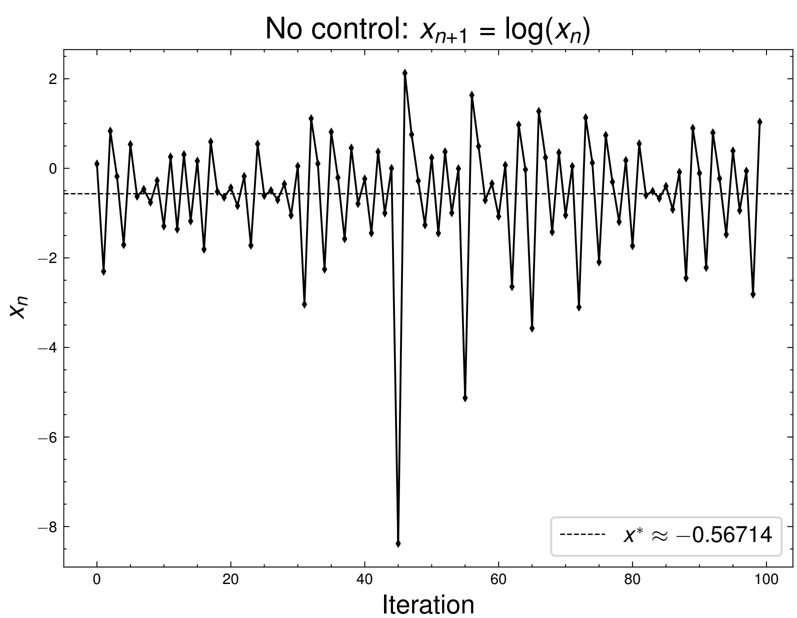

Figure 9: Not controlled map.

We observe the evolution kind of oscillates around the unstable fixed point in short intervals.

After doing that, we implemented the GOY method to keep the evolution close to the fixed point by providing small kicks when it tries to go further from it. By doing this, we were able to control chaos. The final results are shown below.

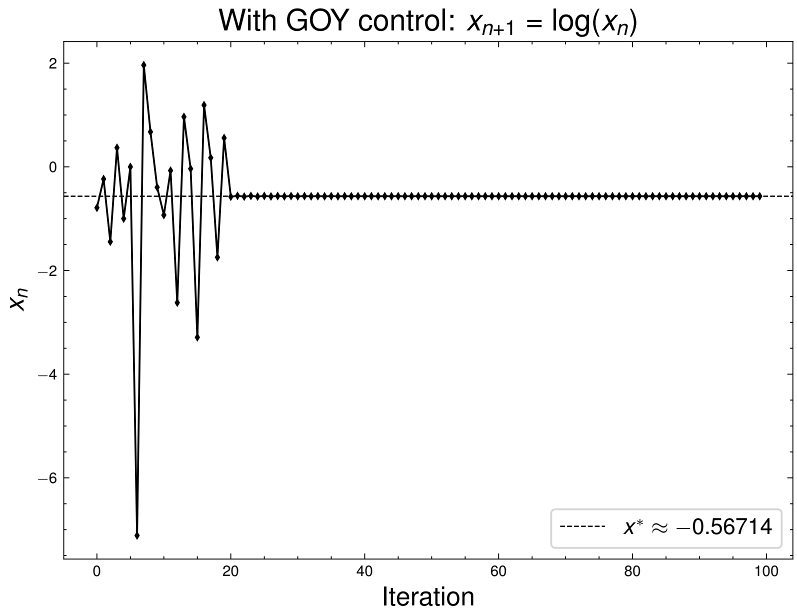

Figure 10: Chaos being controlled around the unstable fixed point.

Once the map gets close to the point, the method switches on and it does let it leave. It turned out to be great.

Thank you for the course and everything, prof. Mario. Truly life-changing.