Project 34: Lanchester's Combat Models

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SUSTC

Abstract:

Different mathematical models explain the main features of combats, among them are models referring to the attrition of the forces involved. Lanchester, the first man who analyses the main features of battles, declared his first combat model in his 1916 book *Aircraft in Warfare* published during World War I. This project investigates some of Lanchester's work on modeling combat.

Keywords:

Lanchester Equations, Combat Models, The Square Law, The Combat Invariant, Guerrilla Combat, Operational Losses, Eigenvectors, Eigenvalues

1.Who was Lanchester?

Frederick William Lanchester was an English inventor, who, in 1892, developed a theory of aerodynamics, but was persuaded not to publish such outlandish theories for fear of ruining his reputation and future career as a serious engineer.

By 1896 he had built with his brother the first petrol car in England. He went on to produce a cars with a number of "firsts" with a mid mounted engine, disc brakes, a water cooled engine and a system of twin balancing shafts (that are used today on modern designs), the crankshaft damper, fuel injection, turbochargers, steering wheels, the accelerator pedal, detachable wire wheels, stamped steel pistons, piston rings, hollow connecting rods, the torsional vibration damper, the harmonic balancer, and tinted glass.

Not content with this, on the outbreak of the First World War, he turned his brain to the war effort, designing engines, and developing theories of flight, which included the design for aircraft which remains the basis for almost all aircraft design to this day. He also developed theories for predicting the outcome of aerial combats.

His theories have been taken up by the USA, and renamed Operational Research, and were used against the Japanese in the Second World War, especially in encounters between aircraft carrier fleets in the Pacific. The Japanese went on to adopt his theories to overrun many US and European Industries in world trade by applying his theories to their Marketing, and Operational decisions.

2.The Basic Combat Model

Suppose there are two forces which represented by Blue and Red numbering x[t] and y[t] who fight each other. Lanchester assumed that if the forces were fighting a conventional war then the combat loss rates would be proportional to the total number of enemy forces. Expressed as a set of differential equations, this becomes

$$\frac{dx}{dt} = -ay$$
$$\frac{dy}{dt} = -bx$$

where a and b are positive constants representing the effectiveness of the y and x forces. Initial conditions would represent the initial strength of each force.

Now we explain why a and b are such means. Figure 1a shows the causal loop diagram of the simplest form of Lanchester´s Law indicating the aggregated causes of attrition of combat forces. Figure 1b is the corresponding stock and flow model. The attrition rate is equal to the number of the forces remaining on the opposing side multiplied by their respective effectiveness.

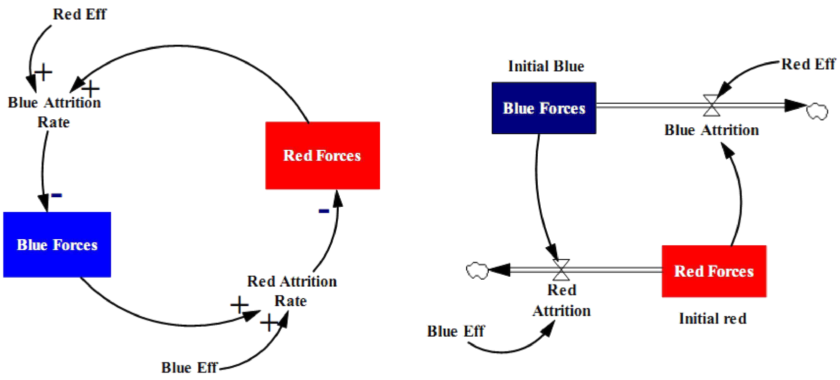
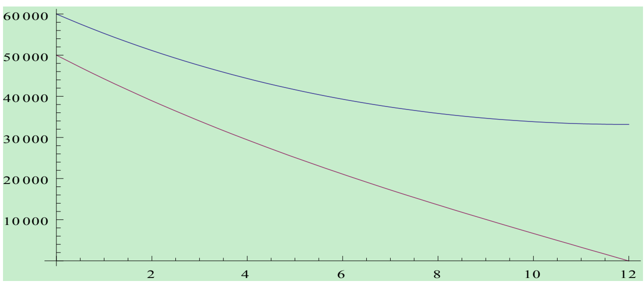


Figure: 1a 1b

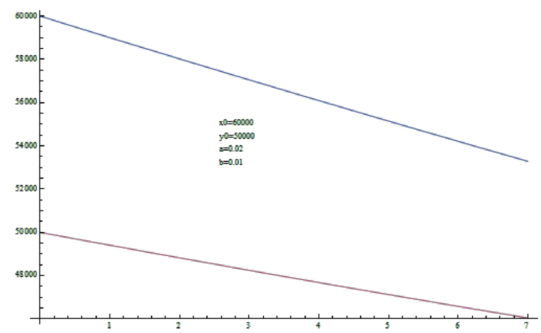
Now, Let's do some real examples about this model. Assume two forces have equal effectiveness, $a=b=0.1$, and different initial strengths. One force starts at 60,000 and the other at 50,000. After 12 units of time, the smaller force is annihilated and the larger force loses about 25,000 men.

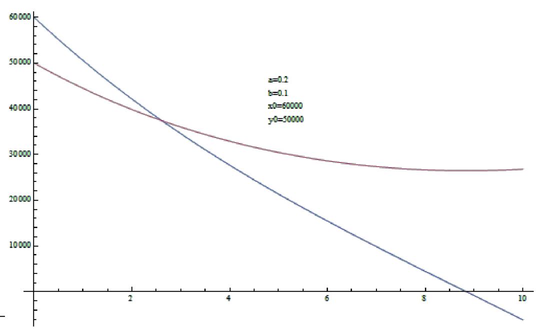
We solve it in Mathematica:

> Dsolve[{partial_t x[t]==-0.1 y[t], partial_t y[t]==-0.1 x[t], x[0]==60000, y[0]==50000,{x[t],y[t]},t]



And what's more,





From above examples, we can explicitly draw the following conclusion:

The initial strength of the forces are more important than the effectiveness of the opposite forces when the coefficients are small, while the opposite result reaches if the coefficients are not very small. But how to estimate? We do the calculation below in **Mathematica**:

```
In[28]:= DSolve[{∂t x[t] == -a y[t], ∂t y[t] == -b x[t], x[0] == c, y[0] == d},
          {x[t], y[t]}, t]
```

$$\text{Out[28]=}\ \left\{\left\{x[t] \to \frac{e^{-\sqrt{a}\sqrt{b}\,t}\left(\sqrt{b}\,c+\sqrt{a}\,d+\sqrt{b}\,c\,e^{2\sqrt{a}\sqrt{b}\,t}-\sqrt{a}\,d\,e^{2\sqrt{a}\sqrt{b}\,t}\right)}{2\sqrt{b}},\right.\right.$$

$$\left.\left. y[t] \to \frac{e^{-\sqrt{a}\sqrt{b}\,t}\left(\sqrt{b}\,c+\sqrt{a}\,d-\sqrt{b}\,c\,e^{2\sqrt{a}\sqrt{b}\,t}+\sqrt{a}\,d\,e^{2\sqrt{a}\sqrt{b}\,t}\right)}{2\sqrt{a}}\right\}\right\}$$

We may find it impossible if we let both x[t]=0 & y[t]=0, because let x[t]=0, we have

$$\sqrt{b}c+\sqrt{a}d+\sqrt{b}ce^{2\sqrt{a}\sqrt{b}t}-\sqrt{a}de^{2\sqrt{a}\sqrt{b}t}=0$$

$$t=\frac{\ln\frac{\sqrt{b}c+\sqrt{a}d}{-\sqrt{b}c+\sqrt{a}d}}{2\sqrt{ab}}$$

If we let y[t]=0, then we have

$$\sqrt{b}c+\sqrt{a}d-\sqrt{b}ce^{2\sqrt{a}\sqrt{b}t}+\sqrt{a}de^{2\sqrt{a}\sqrt{b}t}=0$$

$$t=\frac{\ln\frac{\sqrt{b}c+\sqrt{a}d}{\sqrt{b}c-\sqrt{a}d}}{2\sqrt{ab}}$$

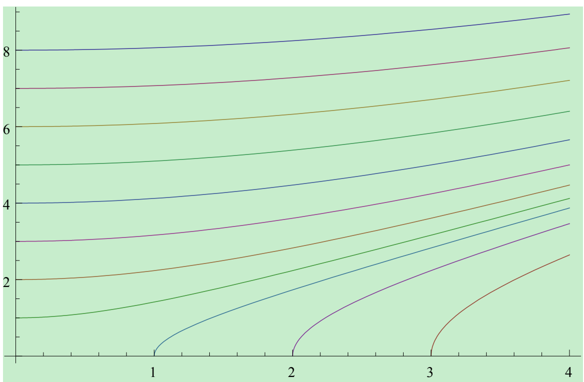
However, one of $\frac{\sqrt{b}c+\sqrt{a}d}{\sqrt{b}c-\sqrt{a}d},\frac{\sqrt{b}c+\sqrt{a}d}{-\sqrt{b}c+\sqrt{a}d}$ must be a negative value, which doesn't have a meaningful logarithm. In fact, the negative one must win the battle, and the others must die out.

There may be another condition that $\sqrt{b}c-\sqrt{a}d=0$. At this situation, we can not only say there is no one win the war for both of forces are approach to 0, but also say the success of the forces exist random for x[t] & y[t] get more and more smaller. We can imagine there are two man leave in the battle for a long time, then if they have the same attack force and defence force, the victory belongs to the one god prefers.

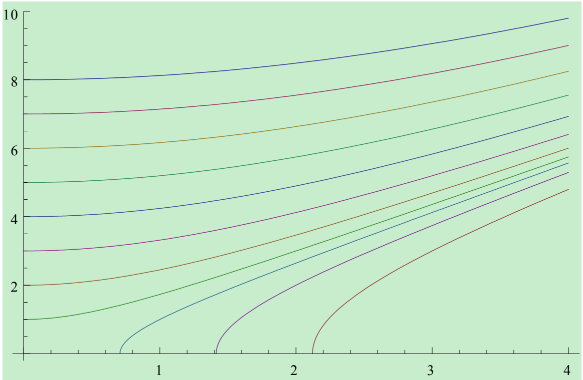
By study this situation, we also can find that at the point which when one of the forces value get 0 the numerical results fail to make sense in the combat model. What happens if the two forces are of equal effectiveness and equal strength? we get much closer to our next work.

Flow Experiments:

(1)a=b



(2)a≠b



3. The Principle of Concentration

If one force can concentrate its entire strength on a portion of an opposing army and destroy it, it may then be able to attack the remaining portion of the opposing army with its remaining strength and destroy it. Lanchester believed air power would be so effective because of this. An air force can be deployed in full force against select portions of an opposing army.

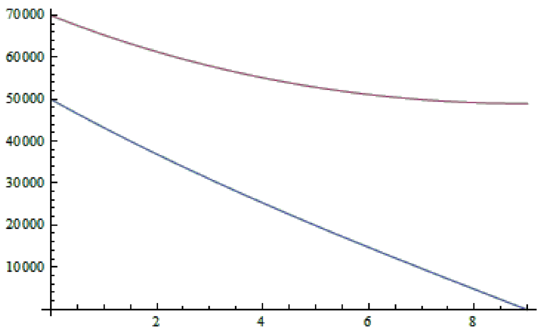
• Divide and Conquer

1) First consider two forces *x* and *y* of equal effectiveness. Suppose the *x*-force starts with 50,000 men and the *y*-force with 70,000 men. We use the **Mathematica** program to determine the outcome of the battle if all 50,000 *x*-troops faced all 70,000 *y*-troops.(with a=b=0.1)

In[1]:= DSolve[{∂_tx[t] == -0.1 y[t], ∂_ty[t] == -0.1 x[t], x[0] == 50000, y[0] == 70000}, {x[t], y[t]}, t]

{{x[t] → $e^{-0.1`t}$ (60000.` - 10000.` $e^{0.2`t}$), y[t] → $e^{-0.1`t}$ (60000.` + 10000.` $e^{0.2`t}$)}}

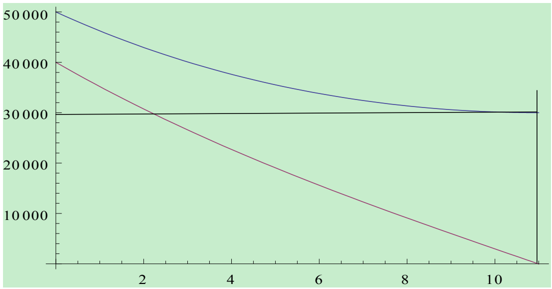
In[3]:= Plot[{$e^{-0.1`t}$ (60000.` - 10000.` $e^{0.2`t}$), $e^{-0.1`t}$ (60000.` + 10000.` $e^{0.2`t}$)}, {t, 0, 9}]

Out[3]=


2) Let the x and y forces have equal effectiveness(a=b=0) and let the x-force start at 50,000 and the y-force start at 70,000. Suppose now, however, that the x-force can concentrate its entire strength on 40,000 opposing troops before facing the remaining 30,000 y-force men in a second battle.

```
In[4]:= DSolve[{∂t x[t] == -0.1 y[t], ∂t y[t] == -0.1 x[t], x[0] == 50000,
          y[0] == 40000}, {x[t], y[t]}, t]
Out[4]= {{x[t] → e^(-0.1 t) (45000. + 5000. e^(0.2 t)), y[t] → e^(-0.1 t) (45000. - 5000. e^(0.2 t))}}

In[19]:= Plot[{e^(-0.1` t) (45000.` + 5000.` e^(0.2` t)), e^(-0.1` t) (45000.` - 5000.` e^(0.2` t))},
          {t, 0, 11}]
```




```
In[13]:= NSolve[e^(-0.1 t) (45000 - 5000 e^(0.2 t)) == 0, t]
         Solve::ifun :
           Inverse functions are being used by Solve, so some solutions may not be found; use
              Reduce for complete solution information. >>
         {{t → 10.986122886681096`}}

In[20]:= e^(-0.1` 10.986122886681096`) (45000.` + 5000.` e^(0.2` × 10.986122886681096`))
Out[20]= 30000.
```

After the first battle, the strength of two forces become the same.

```
In[21]:= DSolve[{∂t x[t] == -0.1 y[t], ∂t y[t] == -0.1 x[t], x[0] == 30000,
          y[0] == 30000}, {x[t], y[t]}, t]
Out[21]= {{x[t] → 30000. e^(-0.1 t), y[t] → 30000. e^(-0.1 t)}}
```

Two forces may be internecine after the second combat.

3) Suppose now that an x-force of 50,000 troops is outnumbered by 120,000 y-troops. Assuming that the x-force can strike a portion of the y-force in full strength and then face the other portion with its remaining men, is there any way for the x-force to win the battle? Justify our answer.

Now we list a few examples following (a=b=0.1):

x[0]	y[01]	Result1V	y[02]	ResultALLV	Result
50000	30000	X[1]=39910	90000	Y[2]=79930	Y wins
50000	40000	X[1]=30000	80000	Y[2]=74162	Y wins
50000	50000	X[1]=Y[1]=0	70000	Y[2]=70000	Y wins
50000	60000	Y[1]=33162	60000	Y[2]=60000	Y wins
50000	70000	Y[1]=48990	50000	Y[2]=50000	Y wins
50000	80000	Y[1]=62450	40000	Y[2]=40000	Y wins

It would be convenient to have a simple analytical method to use to compute answers to questions like those in the exercise above. Computer simulations can probably give us the answers we need, but there is an easier way.

4.The Square Law

Consider the case where the two forces have differing combat effectiveness. (In particular, we will use *a=0.0106* and *b=0.0544* for experiments. These coefficients have been used to model the course of the battle of Iwo Jima in World War II.) The outcome of the battle in this case is not as clear as with the battles above. A force of superior numbers may still lose to a smaller force if the combat effectiveness of the smaller force is high. We can still determine the outcome of a battle, however, by using the differential equations above to derive an integral invariant.

$$\frac{dx}{dt} = -ay$$
$$\frac{dy}{dt} = -bx$$

we can take the quotient of $\frac{dy}{dt}$ and $\frac{dx}{dt}$ to obtain

$$\frac{dy}{dx} = \frac{bx}{ay}$$

Now we can separate variables and integrate to get a relationship between x and y.

$$\frac{dy}{dx} = \frac{bx}{ay} \Leftrightarrow aydy = bxdx \Rightarrow \int aydy = \int bxdx \Leftrightarrow \frac{ay^2}{2} = \frac{bx^2}{2} + C \Leftrightarrow y = \sqrt{\frac{b}{a}x^2 + C}$$

The initial values of x and y will determine the value of the arbitrary constant introduced by the integration.

Substitute x[0] & y[0], we get $C = y_0^2 - \frac{b}{a}x_0^2$,thus we have

$$a(y^2 - y_0^2) = b(x^2 - x_0^2) \qquad (1)$$

The curves of the integral invariants are like the one we ploted in Flow.
For given values of a and b and initial strengths of the x and y forces, we can use the integral invariant to determine who will win the battle by sign of C.

If C>0, then y will win for when x=0, y≠0

If C=0, then y=b/a x, x&y will be on deuce

If C<0, then x will win for y=0,x≠0

Now we can beautify our results.

- The Square Law

Complete the following sentences and derive the square law:

-
- 1. If $bx_0^2 < ay_0^2$, then y will win

- 2. If $bx_0^2=ay_0^2$, then x & y will be on deuce
- *3.* If$bx_0^2>ay_0^2$, then x will win

Consider the specific case of a=0.0106 and *b=0.0544*. We use the **Mathematica** program to solve the differential equations for several initial conditions.

X0	Y0	bx0^2-ay0^2	result
60000	50000	$1.6934*10^8$	X wins
20000	50000	$-4.74*10^6$	Y wins
22071	50000	0	on deuce

The square law can be describe by the flow very well.

5. Guerrilla Combat

Lanchester's combat equations have been modified to model combat in which one force is a conventional force and one force is a guerrilla force. The modification takes into account the fact that the fighting effectiveness of a guerrilla force is due to its ability to stay hidden. Thus, while losses to the conventional force are still proportional to the number of guerrillas, the losses to the guerrilla forces are proportional to both the number of the conventional force and its own numbers. Large numbers of guerrillas cannot stay hidden and, therefore, derive little advantage from guerrilla combat.

If *x* is a guerrilla force and *y* a conventional force, the modified combat equations are:

$$\frac{dx}{dt}=-axy$$
$$\frac{dy}{dt}=-bx$$

Where here again *a* and *b* are constants representing the combat effectiveness of each force.Just as above, we can calculate an integral invariant to determine how solutions to the differential equation behave. Take the ratio of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ to obtain $\frac{dy}{dx}$ and integrate to obtain an expression involving x and *y* and an arbitrary constant.

By the primary equations, we get

$$\frac{dy}{dx}=\frac{b}{ay}$$

Then

$$ydy=\frac{b}{a}dx \Rightarrow \int ydy=\int\frac{b}{a}dx \Leftrightarrow \frac{y^2}{2}=\frac{b}{a}x+C \Leftrightarrow y=\sqrt{\frac{2b}{a}x+C}$$

Substitute x[0] & y[0], we get $C=y_0^2-\frac{2b}{a}x_0$, then we have

$$a(y-y_0)^2=2b(x-x_0)$$

Derive another combat law:

- 1. If $ay_0^2 < 2bx_0$, then the guerrilla force wins.
- 2. If $ay_0^2 = 2bx_0$, then the forces annihilate each other.
- 3. If $ay_0^2 > 2bx_0$*, the conventional force wins.*

Let a=0.0000025 and b=0.1. We use the **Mathematica** program to solve the differential equations for several initial conditions.

X0	Y0	ay0^2-2bx0	result
200	5000	22.5	Y wins
350	5000	−7.5	X wins
312.5	5000	0	on deuce

Consider the case in which both forces are guerrilla forces. Then the combat equations *become:*

$$\frac{dx}{dt} = -axy$$
$$\frac{dy}{dt} = -bxy$$

By the primary equations, we get

$$\frac{dy}{dx} = \frac{b}{a}$$

Then

$$\int \frac{dy}{dx}dx = \int \frac{b}{a}dx \Leftrightarrow y = \frac{b}{a}x + C$$

Substitute x[0] & y[0], we get $C = y_0 - \frac{b}{a}x_0$, then we have

$$a(y - y_0) = b(x - x_0)$$

We can derive another combat law called linear law:

- 1. If $ay_0 < bx_0$, then the guerrilla force x wins.
- 2. If $ay_0 = bx_0$, then the forces annihilate each other.
- 3. If $ay_0 > bx_0$*,* then the guerrilla force y wins.

Similarly, we can use Mathematica to do examples to corroborate our results.

6. Operational Losses (Optional)

Our final modifications of Lanchester's combat model will take into account operational losses. These are losses that occur through accidents, "friendly" fire, or disease. We will assume that the rate of these losses is proportional to the size of the force. The new combat model then takes the form:

$$\frac{dx}{dt} = -ex - ay$$
$$\frac{dy}{dt} = -bx - fy$$

Where *a* and *b* are the usual combat effectiveness constants and *e* and *f* are operational loss rates. It seems plausible that these should be small in comparison to *a* and *b*. This is a linear system. Suppose that a, *b*, *e*, and *f* are positive. We can show that the characteristic roots of the linear dynamical system above are real and of opposite sign.

In this instance, it is not possible to find an integral invariant as above. Notice, however, that the expressions describing rates of change are linear in *x* and *y*. If this is the case, it can be shown that if *e* and *f* are small compared to *a* and *b* then the solutions to these equations behave exactly like the solutions to the combat model without operational losses. We showed above that if the initial conditions are on a certain line through the origin, then the forces mutually annihilate each other. A unit vector that is drawn in the direction of this line is called a *characteristic vector* (*eigenvektor* in German). The key to extending the combat law to this case is in finding an invariant direction or unit vector in the direction.

The computer can find these vectors and enable you to derive a combat law for the model with operational losses. To do this we first write the differential equation in matrix form.

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -e & -a \\ -b & -f \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For example, in ***Mathematica*** we can define a matrix in the computer as a list of lists,

```
In[51]:= m = {{-e, -a}, {-b, -f}} // MatrixForm

Out[51]//MatrixForm=
  ( -e  -a )
  ( -b  -f )
```

The command **Eigenvectors[*m*]** will then return two unit vectors. One will be in the wrong quadrant, but the other will give the direction of the line along which the initial conditions must lie in order for the two forces to mutually annihilate each other.

For example, if *a=b=0.1* and *e=f=0.001*, then the two eigenvectors are given in the list of lists,

```
In[55]:= m = {{-0.001, -0.1}, {-0.1, -0.001}}

Out[55]=
      {{-0.001, -0.1}, {-0.1, -0.001}}

In[57]:= Eigenvectors[m] // MatrixForm

Out[57]//MatrixForm=
  (  0.707107  0.707107 )
  ( -0.707107  0.707107 )
```

The second list is the vector

$$E_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

which points along the line "*y=x*" in the first quadrant.

The computer also has a command **Eigenvalues[*m*]** that returns the characteristic roots of the linear dynamical system. These is the root of following equation:

$$\begin{vmatrix} -e-r & -a \\ -b & -f-r \end{vmatrix} = (e+r)(f+r) - ab = r^2 + (e+f)r - ab = 0$$

The roots are approximately $r_1=0.099$ and $r_2=-0.101$ in this case. The single command **Eigensystem[*m*]** returns the eigenvalues and eigenvectors. Eigenvectors and eigenvalues are related by the equation

$$mE_2 = r_2 E_2$$

or

$$\begin{bmatrix} -e & -a \\ -b & -f \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = r_2 \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

In particular,

$$\begin{bmatrix} -1/1000 & -1/10 \\ -1/10 & -1/1000 \end{bmatrix} \begin{bmatrix} x_0 \\ x_0 \end{bmatrix} = -0.101 \begin{bmatrix} x_0 \\ x_0 \end{bmatrix}$$

for any constant x_0. In this case,

$$\begin{bmatrix} x[t] \\ y[t] \end{bmatrix} = ke^{-0.101t} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = e^{-0.101t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

is a solution of the differential equations for any initial vector

$$k \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

In other words, if $x_0=y_0$, then $x[t]=y[t]=x_0e^{-0.101t}$

1. Compute the matrix product

$$\begin{bmatrix} -1/1000 & -1/10 \\ -1/10 & -1/1000 \end{bmatrix} \begin{bmatrix} k \\ k \end{bmatrix} = \begin{bmatrix} -0.101k \\ -0.101k \end{bmatrix}$$

for any constant k.

2. Show that the matrix equation

$$mE_2 = \begin{bmatrix} -1/1000 & -1/10 \\ -1/10 & -1/1000 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -1/1000 & -1/10 \\ -1/10 & -1/1000 \end{bmatrix} k \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -1/1000 & -1/10 \\ -1/10 & -1/1000 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} k \\ k \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1/1000 & -1/10 \\ -1/10 & -1/1000 \end{bmatrix} \begin{bmatrix} k \\ k \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -0.101k \\ -0.101k \end{bmatrix} = -0.101 \times \frac{1}{\sqrt{2}} \begin{bmatrix} k \\ k \end{bmatrix} = r_2 E_2$$

holds with $r_2=-0.101$ and $E_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

3. Compute the derivatives of the components of the vector below and verify the equation

$$\frac{d}{dt}\begin{bmatrix} x_0 e^{r_2 t} \\ y_0 e^{r_2 t} \end{bmatrix} = \begin{bmatrix} \frac{d}{dt} x_0 e^{r_2 t} \\ \frac{d}{dt} y_0 e^{r_2 t} \end{bmatrix} = \begin{bmatrix} x_0 r_2 e^{r_2 t} \\ y_0 r_2 e^{r_2 t} \end{bmatrix} = r_2 \begin{bmatrix} x_0 e^{r_2 t} \\ y_0 e^{r_2 t} \end{bmatrix}$$

4. When $x_0=y_0$, we use the computations above to show that

$$\begin{bmatrix} -1/1000 & -1/10 \\ -1/10 & -1/1000 \end{bmatrix} X = \begin{bmatrix} -1/1000 & -1/10 \\ -1/10 & -1/1000 \end{bmatrix} \begin{bmatrix} x_0 e^{r_2 t} \\ x_0 e^{r_2 t} \end{bmatrix} = \begin{bmatrix} -1/1000 & -1/10 \\ -1/10 & -1/1000 \end{bmatrix} \begin{bmatrix} x_0 \\ x_0 \end{bmatrix} e^{r_2 t}$$

$$= \begin{bmatrix} -1/1000 & -1/10 \\ -1/10 & -1/1000 \end{bmatrix} \begin{bmatrix} k \\ k \end{bmatrix} e^{r_2 t} \frac{1}{\sqrt{2}} = r_2 \begin{bmatrix} k \\ k \end{bmatrix} e^{r_2 t} \frac{1}{\sqrt{2}} = r_2 \begin{bmatrix} x_0 \\ x_0 \end{bmatrix} e^{r_2 t} = r_2 X = \frac{dX}{dt}$$

5. *The* condition on r_2 is needed so that the equations

$$\begin{bmatrix} x[t] \\ y[t] \end{bmatrix} = \begin{bmatrix} x_0 e^{r_2 t} \\ y_0 e^{r_2 t} \end{bmatrix} = e^{r_2 t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

are parametric equations for a ray point toward the origin.

In other words, the equations trace the line segment starting at (x_0, y_0) and ending at $(0,0)$ for $0 \le t \le \infty$.

The conditions:

In general, if we have an eigenvector-eigenvalue pair for a matrix ***m***,

$$mE_2 = r_2 E_2$$

then the vector

$$X[t] = ke^{r_2 t} E_2$$

satisfies two conditions:

$$mX = m(ke^{r_2 t} E_2) = ke^{r_2 t} mE_2 = ke^{r_2 t} r_2 E_2 = r_2 X$$

and

$$\frac{dX}{dt} = \frac{dke^{r_2 t}}{dt} E_2 = r_2 ke^{r_2 t} E_2 = r_2 X$$

so that

$$\frac{dX}{dt} = mX$$

If the eigenvalue r_2 is negative, then the function ***X[t]*** is a parametric form of the segment from (x_0, y_0) to $(0,0)$. In other words, the solution tends to the origin along a line in the direction of $\boldsymbol{E_2}$. Consider the model above for two forces of equal combat effectiveness,a=b=0.1, and operational loss rates e=0.00001 and f=0.0001. We Use the Eigensystem command to find the eigenvector-eigenvalue pairs of the matrix above. Here is the results from **Mathematica**:

```
In[58]:= m = {{-0.00001, -0.1}, {-0.1, -0.0001}}
Out[58]=
{{-0.00001, -0.1}, {-0.1, -0.0001}}

In[61]:= Eigensystem[m]
Out[61]=
{{-0.100055, 0.099945},
 {{0.706948, 0.707266}, {-0.707266, 0.706948}}}
```

For $E_2 = \begin{bmatrix} 0.706948 \\ 0.707266 \end{bmatrix}$ & $r_2 = -0.100055$

So if $\frac{x_0}{y_0} > \frac{0.706948}{0.707266}$,then force x wins

if $\frac{x_0}{y_0} = \frac{0.706948}{0.707266}$,then the forces annihilate each other.

if $\frac{x_0}{y_0} < \frac{0.706948}{0.707266}$,then the forces y wins

7. Conclusions

We had reviewed several combat models, coming from different sources, but adapting to employ system dynamics concepts and tools. The objectives of the models are a recurrent subject in the system dynamist world and had been treated in different studies. The combat models objectives are not as different as the general models of any science, but it would not be a complete treatment of this subject if we not recall again. A combat model is a fair representation of reality observed in each battle. It helps in tactical planning, even having a limited power to foresee battle details. To be useful must be able to implement and produce credible results. It must be built upon assumption grounded in sound tactical theory. The model should allow to the user the opportunity to vary inputs concerning the allocation and deployment of the platforms. One of the normal weaknesses of modeling is trying to include as much details as possible, sometimes absolutely unnecessary, which only obscure the main issues. In a real battle there are so many unforeseen factors that probably are useless trying to include in a model many details, and loosing the main aspects. The concept of “bounded rationality” in the decisions is applicable as well in a battle environment, as the commander takes his decision based upon in few pieces of information and normally with time pressures.

References:

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