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Homotopy Perturbation Method for Solving Systems of Nonlinear Coupled Equations

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Abstract

In recent years, many more of the numerical methods were used to solve a wide range of mathematical, physical and engineering problems linear and nonlinear. In this article, we shall use the homotopy perturbation method (HPM) to solve some systems of partial differential equations viz. the systems of coupled Burgers' equations in one- and two- dimensions and the system of Laplace's equation. This article, confirms the power, simplicity and efficiency of the method compared with more of the other methods, also confirms that this method is a suitable method for solving any partial differential equations or systems of partial differential equations as well.

Keywords: Homotopy perturbation method; Coupled Burgers' equations; Laplace's equation; Partial differential equations

1 Introduction

The HPM, proposed first by Ji-Huan He [1,2], for solving differential and integral equations, linear and nonlinear, has been the subject of extensive analytical and numerical studies. The method, which is a coupling of the traditional perturbation method and homotopy in topology, deforms continuously to a simple problem which is easily solved. This method, which does not require a small parameter in an equation, has a significant advantage in that provides an analytical approximate solution to a wide range of linear and nonlinear problems in applied sciences. The HPM is applied to Volterra's integro-differential equation [3], nonlinear oscillators [4], bifurcation of nonlinear problems [5], bifurcation of delay-differential equations [6], nonlinear wave equations [7], boundary value problems [8] and to other fields [9-17]. The HPM yields a very rapid convergence of the solution series in most cases, usually only a few iterations leading to very accurate solutions. Thus, He's HPM is a universal one which can solve various kinds of linear and nonlinear equations.

The aim of this article is to extend the application of the He's HPM to solve linear and nonlinear systems of partial differential equations such as the systems of coupled Burgers' equations in one- and two- dimension and the system of Laplace's equation. The obtained results confirms the power, simple and easy of the method to implement.

In the following we shall illustrate the HPM introduced by He [1,2] and a modification algorithm of the HPM introduced by Momani [18].

2 Homotopy perturbation method

To achieve our goal, we consider the following nonlinear differential equation [1-18]:

$$L(u) + N(u) = f(r), \quad r \in \Omega, \tag{1}$$

with the boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{2}$$

where L is a linear operator, N is a nonlinear operator, B is a boundary operator, Γ is the boundary of the domain Ω and $f(r)$ is a known analytic function.

By the homotopy perturbation technique [1,2], He construct a homotopy: $v(r,p) : \Omega \times [0,1] \to R$ which satisfies:

$$\mathcal{H}(v,p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[L(v) + N(v) - f(r)\right] = 0, \tag{3}$$

or

$$\mathcal{H}(v,p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - f(r)\right] = 0, \tag{4}$$

where $r \in \Omega,\ p \in [0,1]$ is an impeding parameter and u_0 is an initial approximation which satisfies the boundary conditions. Obviously, from Eqs. (3) and (4), we have:

$$\mathcal{H}(v,0) = L(v) - L(u_0) = 0, \tag{5}$$

$$\mathcal{H}(v,1) = L(v) + N(v) - f(r) = 0. \tag{6}$$

The changing process of p from zero to unity is just of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, $L(v) - L(u_0)$ and $L(v) + N(v) - f(r)$ are called homotopic. The basic assumption is that the solution of Eqs. (3) and (4) can be expressed as a power series in p :

$$v = v_0 + pv_1 + p^2 v_2 + \ldots \,. \tag{7}$$

The approximate solution of Eq. (1), therefore, can be readily obtained:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots \,. \tag{8}$$

The convergence of the series (8) has been proved in [1,2].

Alternative framework

To illustrate the modification algorithm of the HPM, consider the following nonlinear partial differential equation with time derivative of any order [18]:

$$D_t^n u(x,t) = L(u, u_x, u_{xx}) + N(u, u_x, u_{xx}) + f(x,t), \quad t > 0, \tag{9}$$

where L is a linear operator, N is a nonlinear operator and f is a known analytic function, subject to the initial conditions:

$$\frac{\partial^m}{\partial t^m} u(x,0) = h_m(x), \quad m = 0,1,2,...,n-1. \tag{10}$$

In view of the homotopy technique, we can construct the following homotopy:

$$\frac{\partial^n u}{\partial t^n} - L(u, u_x, u_{xx}) - f(x,t) = p\left[\frac{\partial^n u}{\partial t^n} + N(u, u_x, u_{xx}) - D_t^n u\right], \tag{11}$$

or

$$\frac{\partial^n u}{\partial t^n} - f(x,t) = p\left[\frac{\partial^n u}{\partial t^n} + L(u, u_x, u_{xx}) + N(u, u_x, u_{xx}) - D_t^n u\right], \tag{12}$$

where $p \in [0,1]$. The homotopy parameter p always changes from zero to unity. When $p = 0$, Eq. (11) becomes the linearized equation:

$$\frac{\partial^n u}{\partial t^n} = L(u, u_x, u_{xx}) + f(x,t), \tag{13}$$

and Eq. (12) becomes the linearized equation:

$$\frac{\partial^n u}{\partial t^n} = f(x,t), \tag{14}$$

and when $p = 1$, Eq. (11) or Eq. (12) turns out to be the original differential Eq. (9). The basic assumption is that the solution of Eq. (11) or Eq. (12) can be written as a power series in p :

$$u = u_0 + pu_1 + p^2 u_2 + \dots \, . \tag{15}$$

Finally, we approximate the solution $u(x,t)$ by:

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t). \tag{16}$$

For illustrating the HPM in solving systems of partial differential equations, linear and nonlinear, we consider the following three systems of coupled Burgers' equations and Laplace's equation.

3 Applications

Problem 3.1. As a first application is the following one-dimensional coupled Burgers' equations:

$$u_t - u_{xx} - 2uu_x + (uv)_x = 0, \tag{17a}$$

$$v_t - v_{xx} - 2vv_x + (uv)_x = 0, \tag{17b}$$

with the initial conditions:

$$u(x,0) = \cos x, \quad v(x,0) = \cos x. \tag{17c}$$

In view of Eq. (12), the homotopy for Eqs. (17a) and (17b) are:

$$\frac{\partial u}{\partial t}=p\left[\frac{\partial u}{\partial t}+u_{xx}+2uu_x-(uv)_x-D_t u\right], \tag{18a}$$

$$\frac{\partial v}{\partial t}=p\left[\frac{\partial v}{\partial t}+v_{xx}+2vv_x-(uv)_x-D_t v\right]. \tag{18b}$$

As above, the basic assumption is that the solutions of Eqs. (17a) and (17b) can be written as a power series in p:

$$u=u_0+pu_1+p^2u_2+\dots, \tag{19a}$$

$$v=v_0+pv_1+p^2v_2+\dots. \tag{19b}$$

Therefore, substituting (19) and the initial conditions (17c) into (18) and equating the terms with identical powers of p, we can obtain the following set of linear partial differential equations:

$$\frac{\partial u_0}{\partial t}=0, \quad u_0(x,0)=\cos x,$$

$$\frac{\partial v_0}{\partial t}=0, \quad v_0(x,0)=\cos x,$$

$$\frac{\partial u_1}{\partial t}=\frac{\partial u_0}{\partial t}+(u_0)_{xx}+2u_0(u_0)_x-u_0(v_0)_x-v_0(u_0)_x-D_t u_0, \quad u_1(x,0)=0,$$

$$\frac{\partial v_1}{\partial t}=\frac{\partial v_0}{\partial t}+(v_0)_{xx}+2v_0(v_0)_x-u_0(v_0)_x-v_0(u_0)_x-D_t v_0, \quad v_1(x,0)=0,$$

$$\frac{\partial u_2}{\partial t}=\frac{\partial u_1}{\partial t}+(u_1)_{xx}+2u_0(u_1)_x+2u_1(u_0)_x-u_0(v_1)_x-u_1(v_0)_x-$$

$$v_1(u_0)_x-v_0(u_1)_x-D_t u_1, \quad u_2(x,0)=0,$$

$$\frac{\partial v_2}{\partial t} = \frac{\partial v_1}{\partial t} + (v_1)_{xx} + 2v_0(v_1)_x + 2v_1(v_0)_x - u_0(v_1)_x - u_1(v_0)_x -$$

$$v_1(u_0)_x - v_0(u_1)_x - D_t v_1, \quad v_2(x,0) = 0,$$

$$\frac{\partial u_3}{\partial t} = \frac{\partial u_2}{\partial t} + (u_2)_{xx} + 2u_0(u_2)_x + 2u_1(u_1)_x + 2u_2(u_0)_x - u_0(v_2)_x - u_1(v_1)_x -$$

$$u_2(v_0)_x - v_0(u_2)_x - v_1(u_1)_x - v_2(u_0)_x - D_t u_2, \quad u_3(x,0) = 0,$$

$$\frac{\partial v_3}{\partial t} = \frac{\partial v_2}{\partial t} + (v_2)_{xx} + 2v_0(v_2)_x + 2v_1(v_1)_x + 2v_2(v_0)_x - u_0(v_2)_x - u_1(v_1)_x -$$

$$u_2(v_0)_x - v_0(u_2)_x - v_1(u_1)_x - v_2(u_0)_x - D_t v_2, \quad v_3(x,0) = 0,$$

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Consequently, the first few components of the homotopy perturbation solution for Eq. (17) are derived in the following form:

$$\begin{aligned} u_0(x,t) &= \cos x, \\ v_0(x,t) &= \cos x, \end{aligned}$$

$$\begin{aligned} u_1(x,t) &= -\cos x.t, \\ v_1(x,t) &= -\cos x.t, \end{aligned}$$

$$\begin{aligned} u_2(x,t) &= \cos x.\frac{t^2}{2}, \\ v_2(x,t) &= \cos x.\frac{t^2}{2}, \end{aligned}$$

$$
\begin{aligned}
u_3(x,t) &= -\cos x.\frac{t^3}{6}, \\
v_3(x,t) &= -\cos x.\frac{t^3}{6},
\end{aligned}
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and so on, in the same manner the rest of components can be obtained. The *n-term* approximate solution for Eq. (17) is given by:

$$
u(x,t) = \sum_{i=0}^{n-1} u_i(x,t) = \cos x \left[1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + ...\right], \tag{20a}
$$

$$
v(x,t) = \sum_{i=0}^{n-1} v_i(x,t) = \cos x \left[1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + ...\right]. \tag{20b}
$$

In closed form this gives the solution:

$$
u(x,t) = \cos x. \exp(-t), \tag{21a}
$$

$$
v(x,t) = \cos x. \exp(-t), \tag{21b}
$$

which is the exact solution for the one-dimensional coupled Burgers' equations (17).

Problem 3.2. As a second application is the following two-dimensional coupled Burgers' equations:

$$
u_t - \nabla^2 u - 2u\nabla u + (uv)_x + (uv)_y = 0, \tag{22a}
$$

$$
v_t - \nabla^2 v - 2v\nabla v + (uv)_x + (uv)_y = 0, \tag{22b}
$$

with the initial conditions:

$$u(x,y,0)=\cos(x+y),\quad v(x,y,0)=\cos(x+y), \tag{22c}$$

In view of Eq. (12), the homotopy for Eqs. (22a) and (22b) are:

$$\frac{\partial u}{\partial t}=p\left[\frac{\partial u}{\partial t}+\nabla^2 u+2u\nabla u-(uv)_x-(uv)_y-D_t u\right], \tag{23a}$$

$$\frac{\partial v}{\partial t}=p\left[\frac{\partial v}{\partial t}+\nabla^2 v+2v\nabla v-(uv)_x-(uv)_y-D_t v\right], \tag{23b}$$

As the above problem, substituting (19) and the initial conditions (22c) into (23) and equating the terms with identical powers of $p,$ we can obtain the following set of linear partial differential equations:

$$\frac{\partial u_0}{\partial t}=0,\quad u_0(x,y,0)=\cos(x+y),$$

$$\frac{\partial v_0}{\partial t}=0,\quad v_0(x,y,0)=\cos(x+y),$$

$$\frac{\partial u_1}{\partial t}=\frac{\partial u_0}{\partial t}+\nabla^2 u_0+2u_0\nabla u_0-u_0\nabla v_0-v_0\nabla u_0-D_t u_0,\quad u_1(x,y,0)=0,$$

$$\frac{\partial v_1}{\partial t}=\frac{\partial v_0}{\partial t}+\nabla^2 v_0+2v_0\nabla v_0-u_0\nabla v_0-v_0\nabla u_0-D_t v_0,\quad v_1(x,y,0)=0,$$

$$\frac{\partial u_2}{\partial t}=\frac{\partial u_1}{\partial t}+\nabla^2 u_1+2u_0\nabla u_1+2u_1\nabla u_0-u_0\nabla v_1-$$

$$u_1\nabla v_0-v_0\nabla u_1-v_1\nabla u_0-D_t u_1,\quad u_2(x,y,0)=0,$$

$$\frac{\partial v_2}{\partial t}=\frac{\partial v_1}{\partial t}+\nabla^2 v_1+2v_0\nabla v_1+2v_1\nabla v_0-u_0\nabla v_1-$$

$$u_1\nabla v_0 - v_0\nabla u_1 - v_1\nabla u_0 - D_t v_1, \quad v_2(x,y,0)=0,$$

$$\frac{\partial u_3}{\partial t} = \frac{\partial u_2}{\partial t} + \nabla^2 u_2 + 2u_0\nabla u_2 + 2u_1\nabla u_1 + 2u_2\nabla u_0 - u_0\nabla v_2 - u_1\nabla v_1 -$$

$$u_2\nabla v_0 - v_0\nabla u_2 - v_1\nabla u_1 - v_2\nabla u_0 - D_t u_2, \quad u_3(x,y,0)=0,$$

$$\frac{\partial v_3}{\partial t} = \frac{\partial v_2}{\partial t} + \nabla^2 v_2 + 2v_0\nabla v_2 + 2v_1\nabla v_1 + 2v_2\nabla v_0 - u_0\nabla v_2 - u_1\nabla v_1 -$$

$$u_2\nabla v_0 - v_0\nabla u_2 - v_1\nabla u_1 - v_2\nabla u_0 - D_t v_2, \quad v_3(x,y,0)=0.$$

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Consequently, the first few components of the homotopy perturbation solution for Eq. (22) are derived in the following form:

$$\begin{aligned} u_0(x,y,t) &= \cos(x+y), \\ v_0(x,y,t) &= \cos(x+y), \end{aligned}$$

$$\begin{aligned} u_1(x,y,t) &= -2\cos(x+y).t, \\ v_1(x,y,t) &= -2\cos(x+y).t, \end{aligned}$$

$$\begin{aligned} u_2(x,y,t) &= 2\cos(x+y).t^2, \\ v_2(x,y,t) &= 2\cos(x+y).t^2, \end{aligned}$$

$$\begin{aligned} u_3(x,y,t) &= -\frac{4}{3}\cos(x+y).t^3, \\ v_3(x,y,t) &= -\frac{4}{3}\cos(x+y).t^3, \end{aligned}$$

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and so on, in the same manner the rest of components can be obtained. The *n-term* approximate solution for Eq. (22) is given by:

$$u(x,y,t) = \sum_{i=0}^{n-1} u_i(x,y,t) = \cos(x+y)\left[1 - (2t) + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + ...\right], \tag{24a}$$

$$v(x,y,t) = \sum_{i=0}^{n-1} v_i(x,y,t) = \cos(x+y)\left[1 - (2t) + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + ...\right]. \tag{24b}$$

In closed form this gives the solution:

$$u(x,y,t) = \cos(x+y)\exp(-2t), \tag{25a}$$

$$v(x,y,t) = \cos(x+y)\exp(-2t), \tag{25b}$$

which is the exact solution for the two-dimensional coupled Burgers' equations (22).

Problem 3.3. Finally, the third application is the system of Laplace's equation.

Many physical processes are governed by Laplace's equation:

$$\nabla^2 u = 0, \tag{26}$$

this equation can be replaced by a system of first-order equations. In this case, let u and v represent the unknown variables. We require that:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{27a}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{27b}$$

which is the famous Cauchy-Riemann equations (Churchill, 1960). These equations are extensively used in conformally mapping one region onto another [19].

Remark: It should be noted that some differences exist in solving Laplace's equation and Cauchy-Riemann equations. A solution of the Cauchy-Riemann equations is a solution of Laplace's equation but the converse is not necessarily true.

Now, consider the system of Laplace's equation:

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \tag{28a}$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0, \tag{28b}$$

with the initial conditions:

$$u(0,y) = \cos y, \quad v(0,y) = \sin y. \tag{28c}$$

In view of Eq. (12), the homotopy for Eqs. (28a) and (28b) are:

$$\frac{\partial u}{\partial x} = p\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - D_x u\right], \tag{29a}$$

$$\frac{\partial v}{\partial x} = p\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - D_x v\right]. \tag{29b}$$

As the above problems, substituting (19) and the initial conditions (28c) into (29) and equating the terms with identical powers of p, we can obtain the following set of linear partial differential equations:

$$\frac{\partial u_0}{\partial x} = 0, \quad u_0(0,y) = \cos y,$$

$$\frac{\partial v_0}{\partial x} = 0, \quad v_0(0,y) = \sin y,$$

$$\frac{\partial u_1}{\partial x} = \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} - D_x u_0, \quad u_1(0,y) = 0,$$

$$\frac{\partial v_1}{\partial x} = \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} - D_x v_0, \quad v_1(0,y) = 0,$$

$$\frac{\partial u_2}{\partial x} = \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} - D_x u_1, \quad u_2(0,y) = 0,$$

$$\frac{\partial v_2}{\partial x} = \frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y} - D_x v_1, \quad v_2(0,y) = 0,$$

$$\frac{\partial u_3}{\partial x} = \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} - D_x u_2, \quad u_3(0,y) = 0,$$

$$\frac{\partial v_3}{\partial x} = \frac{\partial v_2}{\partial x} - \frac{\partial u_2}{\partial y} - D_x v_2, \quad v_3(0,y) = 0,$$

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Consequently, the first few components of the homotopy perturbation solution for Eq. (28) are derived in the following form:

$$\begin{aligned} u_0(x,y) &= \cos y, \\ v_0(x,y) &= \sin y, \end{aligned}$$

$$\begin{aligned} u_1(x,y) &= \cos y.x, \\ v_1(x,y) &= \sin y.x, \end{aligned}$$

$$\begin{aligned} u_2(x,y) &= \cos y.\frac{x^2}{2}, \\ v_2(x,y) &= \sin y.\frac{x^2}{2}, \end{aligned}$$

$$\begin{aligned} u_3(x,y) &= \cos y.\frac{x^3}{6}, \\ v_3(x,y) &= \sin y.\frac{x^3}{6}, \end{aligned}$$

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and so on, in the same manner the rest of components can be obtained. The *n-term* approximate solution for Eq. (28) is given by:

$$u(x,y) = \sum_{i=0}^{n-1} u_i(x,y) = \cos y \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right], \tag{30a}$$

$$v(x,y) = \sum_{i=0}^{n-1} v_i(x,y) = \sin y \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right]. \tag{30b}$$

In closed form this gives the solution:

$$u(x,y) = \exp(x)\cos y, \tag{31a}$$

$$v(x,y) = \exp(x)\sin y, \tag{31b}$$

which is the exact solution for the system of Laplace's equation (28).

4 Conclusion

The modified homotopy perturbation method suggested in this article is an efficient method for obtaining the exact solution of the coupled Burgers' equations and the system of Laplace's equation. Therefore, this method is a powerful mathematical tool to solve any system of partial differential equations linear and nonlinear, it is also a promising method to solv other linear and nonlinear partial differential equations.

References

[1] J.H. He, Comput. Methods Appl. Mech. Engrg. 178 (1999) 257.
[2] J.H. He, Int. J. Non-Linear Mech. 35(1) (2000) 37.
[3] M. El-Shahed, Int. J. Nonlin. Sci. Numer. Simul. 6(2) (2005) 163.
[4] J.H. He, Appl. Math. Comput. 151 (2004) 287.
[5] J.H. He, Int. J. Nonlin. Sci. Numer. Simul. 6(2) (2005) 207.
[6] J.H. He, Phys. Lett. A 374(4-6) (2005) 228.
[7] J.H. He, Chaos Solitons Fractals 26(3) (2005) 695.

[8] J.H. He, Phys. Lett. A 350(1-2) (2006) 87.

[9] J.H. He, Appl. Math. Comput. 135 (2003) 73.

[10] J.H. He, Appl. Math. Comput. 156 (2004) 527.

[11] J.H. He, Appl. Math. Comput. 156 (2004) 591.

[12] J.H. He, Chaos Solitons Fractals 26(3) (2005) 827.

[13] J.H. He, Int. J. Mod. Phys. B 20(10) (2006) 1141.

[14] A. Siddiqui, R. Mahmood, Q. Ghori, Int. J. Nonlin. Sci Numer. Simul. 7(1) (2006) 7.

[15] A. Siddiqui, M, Ahmed, Q. Ghori, Int. J. Nonlin. Sci. Numer. Simul. 7(1) (2006) 15.

[16] S. Abbasbandy, Appl. Math. Comput. 172 (2006) 485.

[17] S. Abbasbandy, Appl. Math. Comput. 173 (2006) 493.

[18] S. Momani, Z. odibat, Phys. Lett. A 365 (2007) 345.

[19] D.A. Anderson, J.C. Tannehill, R.H. Pletcher, Computational Fluid Mechanics and Heat Transfer, Hemisphere Publishing Corporation, Washington New York London, McGraw-Hill Book Company, (1984).

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