DENOISE METHODS IN SIGNAL PROCESS

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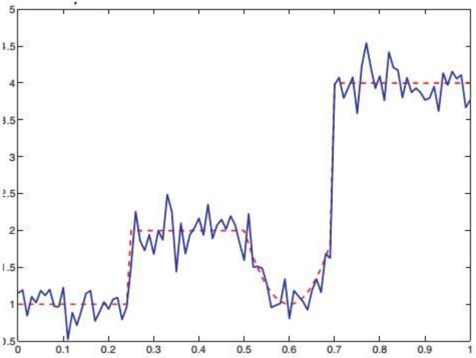
Outline

Mathematical Expression

- An original signal $u(t)$ on a time interval
- Discrete equi-distance data
- Eliminate errors

Graphic Explanation



Outline

Method of Tikhonov Regularization

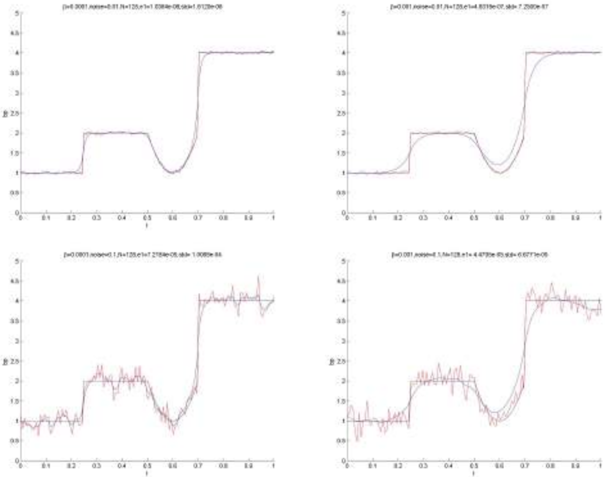
$$\min\left(\phi_\beta\left(u\right)\right) = \min(||Gu - d||_2^2 + \beta^2\left\|u\right\|_2^2) \tag{1}$$

$$m_\alpha = \left(G^*G + \beta^2 I\right)^{-1} G^* d \tag{2}$$

Ror 1D TR:

$$\min \phi_\beta\left(u\right) = \frac{h}{2}\sum_{i=1}^{N}\frac{1}{2}[(u_i - b_i)^2 + (u_{i-1} - b_{i-1})^2] + \frac{\beta h}{2}\sum_{i=1}^{N}\left(\frac{u_i - u_{i-1}}{h}\right)^2 \tag{3}$$

Results of TR



Outline

Method of Total Variation Regularization

$$L = \begin{pmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & \cdots & \cdots & & \\ & & & -1 & 1 & \\ & & & & -1 & 1 \end{pmatrix}$$

$$\min \|Gm - d\|_2^2 + \alpha \|Lm\|_1 \tag{4}$$

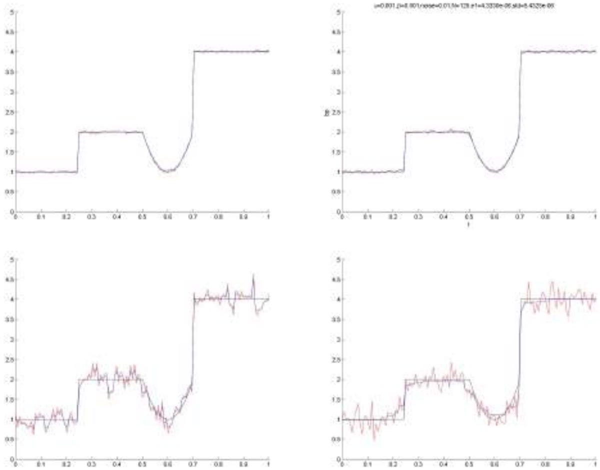
$$|x| \approx \sqrt{x^2 + \varepsilon} \tag{5}$$

Method of Total Variation Regularization

$$\min \phi_2(\mathbf{u}) =$$

$$\frac{h}{2}\sum_{i=1}^{N}\frac{1}{2}[(u_i - b_i)^2 + (u_{i-1} - b_{i-1})^2] + \frac{\beta h}{2}\sum_{i=1}^{N}\sqrt{\left(\frac{u_i - u_{i-1}}{h}\right)^2 + \varepsilon}$$

Results of TVR



Outline

2D TR

(a) Original picture

(b) Noisy picture

Figure: The pictures before and after the TV minimization

2D TR

Figure: TV denoising picture eps=0.01

Method of 2D TVR

$$E = \int_{\Omega} (|\nabla I| + \frac{1}{2}\lambda (I - I_0)) dxdy \tag{6}$$

$$E_{\Phi} = \int_{\Omega} (\Phi |\nabla I| + \frac{1}{2}\lambda (I - I_0)) dxdy \tag{7}$$

Euler-Lagrange Equation

$$F = div(\Phi' \frac{\nabla I}{|\nabla I|}) + \lambda (I_0 - I) = 0 \tag{8}$$

$$I_t = F, \;\; I|_{t=0} = I_0$$

Method of 2D TVR

$$\begin{aligned} &\min_I \int_\Omega \Phi(|\nabla I|)dxdy \\ &subject\ to\ \frac{1}{|\Omega|}\int_\Omega (I - I_0)dxdy = \sigma^2 \end{aligned} \tag{9}$$

$$\lambda = \frac{1}{\sigma^2|\Omega|}\int_\Omega div(\Phi' \frac{\nabla I}{|\nabla I|})(I - I_0)dxdy \tag{10}$$

Result of 2D TVR

(a) TV denoising picture eps=0.01

(b) TV denoising picture eps=0.1

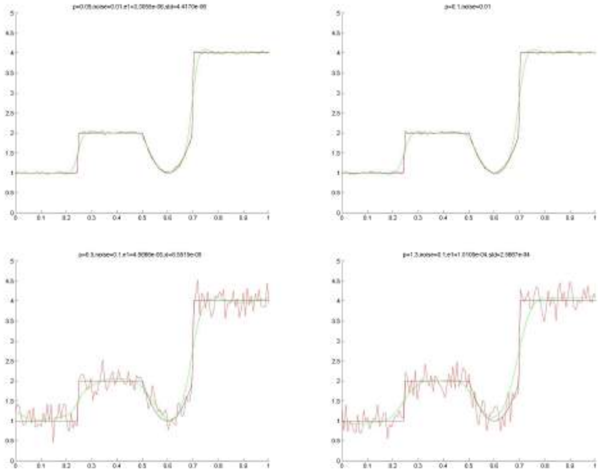
Figure: The compare of two results with different eps

Outline

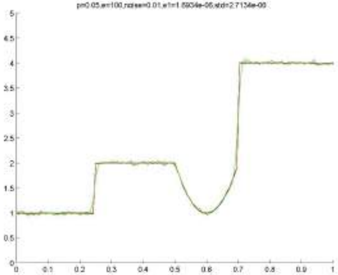
1-order to 2-order

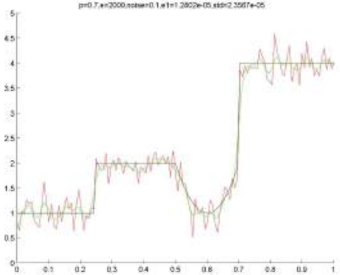
- First order differential quotient in TR & TVR
- Regularization terms measure roughly
- substitute by the 2-order different quotient

2-order TR



2-order TVR


5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1


5
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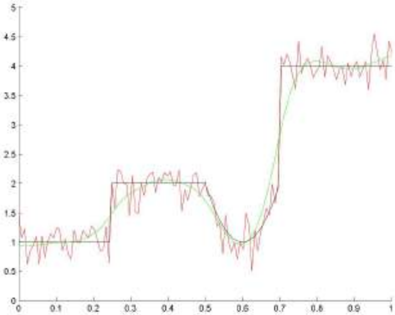
2
1.5
1
0.5
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

More for 2-order TVR

- Geting very smooth curve by increasing β



Outline

1. Background
2. Methods and Matlab
   - Tikhonov Regularization
   - Total Variation Regularization
   - Two Dimension TR & TVR
   - Second Order TR & TVR
   - Optimization with Constrains
   - Discrete Fourier Transformation
   - Average Smoothing Method

General Model

$$\begin{array}{lll} \min & f(x), & x \in \mathbb{R}^n \\ st & c_i(x) = 0 & i \in E\{1,2,\ldots,l\}, \\ & c_i(x) \leq 0 & i \in I\{l+1, l+2, \ldots, l+m\}. \end{array}$$

Karush-Kuhn-Tucjer Condition

Theorem

assume $f(x), c_i(x)$*have continuous first order derivative, and*x^**is the local solution , and* $\nabla c_i(x)(i \in (E \cup I^*))$*are linear independent, then exist constant* $\lambda^* = (\lambda_1^*, \lambda_2^*, ..., \lambda_{l+m}^*)$*s.t.*

$$\begin{aligned}
\nabla_x L(x^*, \lambda^*) &= \nabla f(x^*) + \sum_{i=1}^{l+m} \lambda_i^* \nabla c_i(x^*) = 0 \\
c_i(x^*) = 0 &, \quad i \in E \\
c_i(x^*) \leq 0 &, \quad i \in I \\
\lambda_i^* \geq 0 &, \quad i \in I \\
\lambda_i^* c_i(x^*) = 0 &, \quad i \in I
\end{aligned}$$

For our Problem

- $N(x) : x \to n \; x \in \mathbb{R}^n, n \geq 0$

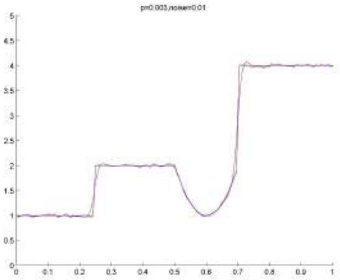
$$\begin{aligned} \min \quad & \|u-b\|_2 \quad u \in \mathbb{R}^n \\ st \quad & N(u) \leq \varepsilon \end{aligned}$$

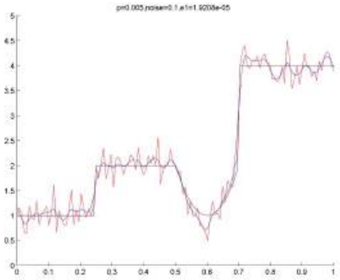
$$N_1(u) = \sum_{i=1}^{n} (u_{i+2} - 2u_i + u_{i-1})^2$$

KKT condition:

$$\nabla_{u,\lambda^*} L(u,\lambda^*) = \nabla(\|u-b\|_2 + \sum_{i=1}^{l+m} \lambda_i^* (N(u) - \varepsilon)) = 0$$

Optimization with Constrains





Outline

Method of DFT

$$S_n(x) = \frac{a_o}{2} + a_n \cos nx + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx)$$

Find continuous least square approximation, minimize the error term

$$E(S_m) = \sum_{j=0}^{2m-1} (y_i - S_m(x_j))^2$$

Method of DFT

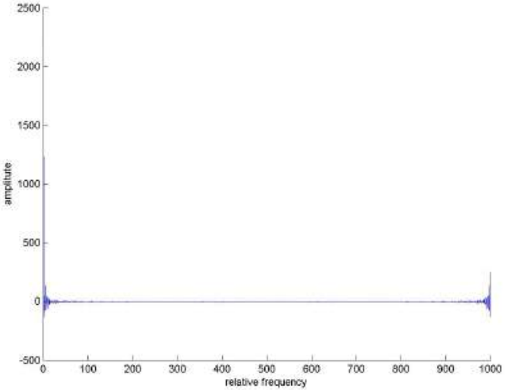
the coefficient is

$$a_k = \frac{1}{m}\sum_{j=0}^{2m-1} y_j \cos kx_j, k = 0.1\ldots.., m,$$

$$b_k = \frac{1}{m}\sum_{j=0}^{2m-1} y_j \sin kx_j, k = 0.1\ldots.., m-1$$

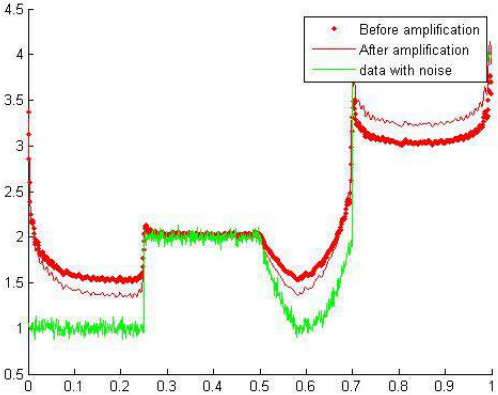
DFT

- signal transform intensities of different frequencies



DFT

- Discard the high frequency region

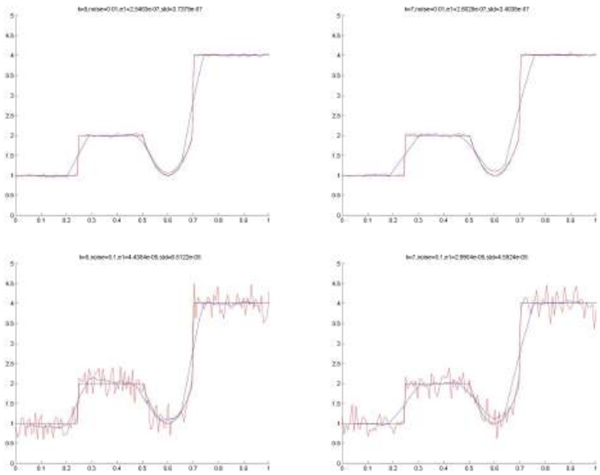


Outline

Average Smoothing Method

- For data jumps intensivly
- Average of data in Neighbourhood
- Smooth the signal (If noise is i.d.d)

Result of Average Smoothing Method



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