Implied Volotility and Convergence order of Binomial Tree Methods

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1. IMPLIED VOLATILITY

1.1. Introducing Implied Volatility.

Recall BS option pricing model, option value is a function of (K, S, σ, τ, r). among these parameters, σ is hard to observe, while others are easy to get. on the other hand, the option price V, which BS formula tried to explain, is also easy to get in the market. so we can these infomation and solve σ, which is called implied volatility.

1.2. Introducing Volatility Smile, Volatility Term Structure and Volatility Surface.

For any option traded in the market, one can use its market price and solve the implied volatility. Graphing implied volatilities against strike prices K for a given expiry yields a “smile".[13] For the option with equity as underlying, this smile is skewed, which means only left downward sloping part appears.

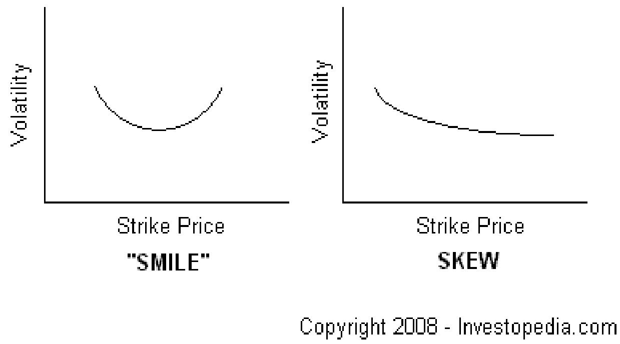

Figure 1. Volatility Smile and Skew

Volatility term structure show the relation between the implied volatility and the time to maturity τ. usually, implied volatility is a decreasing function of τ, since implied volatility can't be negative, we can imaging volatility term structure is also convex. Plotting the implied volatility with respect to τ and K yields volatility surface.

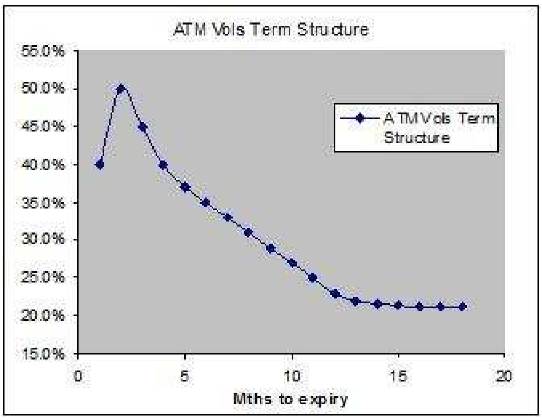

Figure 2. Volatility Term Structure

1.3. Data source and related data issue.

1. option price V: I choose the SPX pm settled option[4], this is an index option and the underlying is S&P500 index. Further it is European style option so that BS formula can be used in this case. I found it is really hard to find an European sytle option in the stock market, almost all option are American style, only few index option is European style. One may ask why not use warrant? Yes, warrant is very similar to option, the good news about warrant is that there are a lot of European style warrant, while the dilution issue stops us from applying BS formula directly. sometimes the real market price is delayed and I use the average of bid and ask as the approximation of current market price.
2. risk free rate r: I choose LIBOR dollar rate as the benchmark of risk free interest rate, It may differs from the textbook which usually use US government bond's spot rate as risk free. This is because in recent years bond rate is frequently manipulated by government for regulation purpose, especially short term bond like T-bills. While the LIBOR rate[5] is determined by market and in short term those world's best borrowers can hardly default. also we should notice risk free rate is dependent on the concerning time period, options with different expiry have differnt risk free rate.

1.4. Code and Result.

the R code is as follows:

```
setwd("F:\\compFinance\\prjt2")
load("spx.RData")
x = spx
library(quantmod)
library(RQuantLib)
Sys.setlocale(,"English")
#x = getOptionChain("^SPXPM",NULL)

libor = c(0.14775,0.19095,0.22810,
          0.2605,0.2929,0.32530,
          0.360183,0.3950667,0.4299500,
          0.4648333,0.4997167,0.53460)*0.01
dvd = 1.46*0.01
vix = 0.1235
SPXpm = 1881.93
underlying = SPXpm*100
AugOption <- x$'Jul 2014'
mty <- 11/72
vCall = (AugOption$calls$Bid + AugOption$calls$Ask)/2*100
vPut = (AugOption$puts$Bid + AugOption$puts$Ask)/2*100
callLen = length(vCall)
putLen = length(vPut)
sumCall = list(K = AugOption$calls$Strike[1:(callLen-1)]*100,V =
vCall[1:(callLen-1)],
               sigma = vector("numeric",callLen-1))
sumPut = list(K = AugOption$puts$Strike[2:putLen]*100,V = vPut[2:putLen],
               sigma = vector("numeric",putLen-1))
for(i in 1:(callLen-1)){
  #print(i)
  sumCall$sigma[i] = EuropeanOptionImpliedVolatility(type="call",sumCall$V[i],
underlying,
                     sumCall$K[i], dividendYield=dvd, riskFreeRate=libor[2],
maturity=mty,
                     volatility=vix)$impliedVol
 # print(sumCall$K[i])
}
```

```
for(i in 1:(putLen-1)){
 # print(i)
  sumPut$sigma[i] = EuropeanOptionImpliedVolatility(type="put",sumPut$V[i],
underlying,
        sumPut$K[i], dividendYield=dvd, riskFreeRate=libor[2],maturity=mty,
        volatility=vix)$impliedVol
}
plot(sumCall$K,sumCall$sigma,xlim=range(sumCall$K,sumPut$K),
ylim=range(sumCall$sigma,sumPut$sigma),col="red",xlab="Strike Price",
ylab="Implied Volatility")
lines(sumCall$K,sumCall$sigma,col="red")
points(sumPut$K,sumPut$sigma,col="green")
lines(sumPut$K,sumPut$sigma,col="green")
legend(180000,0.25,legend=c("sigma by call","sigma by put"),col=c("red",
"green"),
       lwd=c(2,2),lty=c(1,1),bty="o",cex=0.8)
```

This yields the implied volatility of both call and put on SPX on Jul 2014.

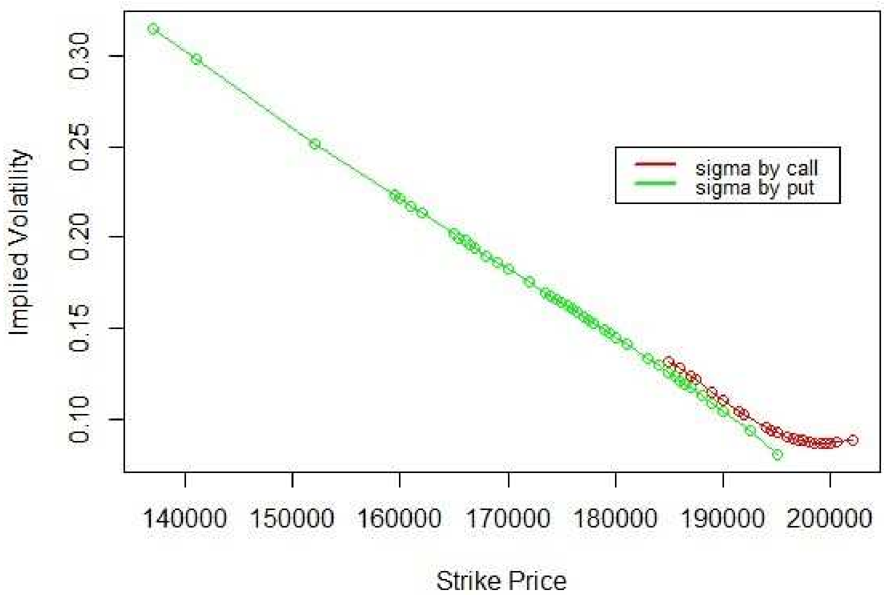

Figure 3. Volatility Smile

1.5. Discussion: Explanation of Implied Volatility Gap.

Due to call put parity, the implied volatiliy should be same for both call and put, If not the arbitrage exist. but in Figure 3. we can see implied volatility for call is always slightly above the put. this could be explained by the daily settlemen mechanism in derivative market. Since high spot interest rate and hight daily return are correlated, when the call holder receive the money in daily settlement he is more likely to reinvest at a hight interest rate, when he have to replenish the margin account, It is very possible interest rate is low and he has a low cost to fund these money. similar discussion can conclude daily settlement favors call holders and punish put holders. when the implied volatility gap exist, this ensure the call holders receive relatively less and pay more.

2. CONVERGENCE OF BINOMIAL METHOD FOR OPTION VALUATION

2.1. Introduction.

We discusse several different versions of the binomial model as it may be used for option pricing. As introduced in the course there are primarily three parameters p, u and d that need to be calculated to use the binomial model.

Binomial models (and there are several) are arguably the simplest techniques used for option pricing. The mathematics behind the models is relatively easy to understand and (at least in their basic form) they are not difficult to implement.

We discuss the general mathematical concepts behind the binomial model with particular attention paid to the original binomial model formulation by Cox, Ross and Rubinstein (CRR).

However, there are many other versions of the binomial model. Each of the approaches has its advantages and disadvantages for pricing different types of options. However, they all involve a similar three step process.

1. Calculate potential future prices of the underlying asset(s) at expiry (and possibly at intermediate points in time too).
2. Calculate the payoff of the option at expiry for each of the potential underlying prices.
3. Discount the payoffs back to today to determine the option price today.

There are many different approaches to calculating values for p, u and d. We will discuss some of then , which are developed by,

- Cox-Ross-Rubinstein[1]: This is the method most people think of when discussing the binomial model, and the one discussed in this tutorial.
- Jarrow-Rudd[8]: This is commonly called the equal-probability model.
- Tian[12]: This is commonly called the moment matching model.
- Leisen-Reimer[11]: This uses a completely different approach to all the other methods, relying on **approximating the normal distrbution used in the Black-Scholes model**.

Three equations are required to be able to uniquely specify values for the three parameters of the binomial model. Two of these equations arise from the expectation that over a small period of time the binomial model should behave in the same way as an asset in a risk neutral world.

$$\frac{\mathrm{d}S}{S} = r\,\mathrm{d}t + \sigma \mathrm{d}W$$

Then use Ito Lemma, we can show that

$$E(S(\Delta t)) = e^{r\Delta t} S(0), E(S^2(\Delta t)) = e^{(2r+\sigma^2)\Delta t} S^2(0)$$

This leads to the equation (Matching Return)

$$p\,u + (1-p)\,d = e^{r\Delta t} \tag{1}$$

which ensures that over the small period of time Δt the expected return of the binomial model matches the expected return in a risk-neutral world, and the equation (Matching Variance),

$$pu^2 + (1-p)d^2 = e^{(\sigma^2+2r)\Delta t} \tag{2}$$

which ensures that the variance matches.

For CRR, JR, and TIAN, we show the definitions as follows:

Definition 1. (CRR,JR and Tian)

(Cox-Ross-Rubenstein model)

$$u = 1/d \Leftrightarrow \begin{cases} u = \exp\left\{\sigma\sqrt{\Delta t}\right\} \\ d = 1/u \end{cases}, p = \frac{\exp\{r\Delta t\} - d}{u - d} \text{ (together equantion 1, 2)}$$

(Jarrow-Rudd model)

$$p = \frac{1}{2} \Leftrightarrow \begin{cases} u = \exp\left\{\mu'\Delta t + \sigma\sqrt{\Delta t}\right\} \\ d = \exp\left\{\mu'\Delta t - \sigma\sqrt{\Delta t}\right\} \end{cases}, \text{where } \mu' = r - \frac{1}{2}\sigma^2$$

(Tian model)

$$pu^3 + (1-p)d^3 = e^{(3r+3\sigma^2)\Delta t} \Leftrightarrow \begin{cases} u = \frac{MV}{2}\left(V + 1 + \sqrt{V^2 + 2V - 3}\right) \\ d = \frac{MV}{2}\left(V + 1 - \sqrt{V^2 + 2V - 3}\right) \end{cases}, \text{where} \begin{cases} M = \exp(r\Delta t) \\ V = \exp(\sigma^2\Delta t) \end{cases}$$

Remark 2. Actually, $u = 1/d$ cannot get CRR directly, so is $p = 1/2$.

When, $u = 1/d$, together with equation 1 and 2, we have

$$\begin{aligned} u &= A + \sqrt{A^2 - 1} \\ d &= A - \sqrt{A^2 - 1} \\ A &= \frac{1}{2}\left(e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t}\right) \end{aligned}$$

However, using Taylor expansion in terms of $\sqrt{\Delta t}$, we can get CRR.

Similarly, $p = 1/2$, we can get

$$\begin{aligned} u &= e^{r\Delta t}\left(1 + \sqrt{e^{\sigma^2\Delta t} - 1}\right) \\ d &= e^{r\Delta t}\left(1 - \sqrt{e^{\sigma^2\Delta t} - 1}\right) \end{aligned}$$

Then by Taylor expansion, we have JR tree.

2.2. CRR, JR and Tian for European Option converge with order 1.

We try to prove that the binomial tree methods given by CRR, JR and Tian are linear convergent to Black-Schole formula. Before do that, we should review an essential work given by Leisen and Reimer[11]. We use his notation which given as follows:

Notation 3. *$X_t^{s,x}$ shall be the solution of the stochastic differential equation:*

$$X_t^{s,x} = x + \int_s^t rX_{t'}^{s,x}\mathrm{d}t' + \int_s^t \sigma X_{t'}^{s,x}\mathrm{d}W_{t'}$$

The probability measure is denoted by P_W.

P_B is the probability measure of discrete process Y_n of $X_t^{s,x}$, then

$$e_n := e^{-rT}\left|E_W\left[f\left(X_T^{0,S}\right)\right] - E_B[f(Y_n)]\right|$$

Notation 4. *$c(t, S) := e^{-r(T-t)}E_W\left[f\left(X_T^{t,S}\right)\right]$ is the* BS *price of a call option with stock value S at time t and maturity at time T.*

The time axis will be discretized in steps of length $\Delta t = \frac{T}{n}$. The discrete points will be denoted by $t_i := i \cdot \Delta t$.

We use the abbrevation $X_{k+1} := X_{t_{k+1}}^{t_k, Y_k}$.

The information structure is modelled by $\mathcal{A}_k = \sigma(Y_i | j \leqslant k)$.

Notation 5.

$$c_1(t,S) := \frac{\partial c}{\partial S}(t,S) \quad \tilde{c}_1(t,S) := S\,\frac{\partial c}{\partial S}(t,S)$$
$$c_2(t,S) := \frac{\partial^2 c}{\partial S^2}(t,S) \quad \tilde{c}_2(t,S) := S^2\,\frac{\partial^2 c}{\partial S^2}(t,S)$$
$$c_3(t,S) := \frac{\partial^3 c}{\partial S^3}(t,S) \quad \tilde{c}_3(t,S) := S^3\,\frac{\partial^3 c}{\partial S^3}(t,S)$$
$$c_4(t,S) := \frac{\partial^4 c}{\partial S^4}(t,S) \quad \tilde{c}_4(t,S) := S^4\frac{\partial^4 c}{\partial S^4}(t,S)$$

$$R_3(t,z_1,z_0) := \int_{z_0}^{z_1} (z_1 - S)^3 c_4(t,S)\mathrm{d}S, \forall z_0, z_1 \in \mathbb{R}^+, \forall t \in [0,T]$$
$$d_1 = \frac{\ln S/K + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$$

Notation 6. $$M_2(t) := \frac{Ke^{-rt}}{\sqrt{2\pi}\sigma}, M_3(t) := \frac{Ke^{-rt}}{\sqrt{2\pi}\sigma^2}(1+\sigma), M_4(t) := 4\frac{e^{(2r+3\sigma^2)t}}{\sqrt{2\pi}K^2\sigma^2}$$

Notation 7. *Moments, general monments and pseudo-moments*

$$\begin{aligned}
\bar{m}_n^1 &:= E_B\left[\frac{Y_{k+1}}{Y_k}|\mathcal{A}_k\right] - E_W\left[\frac{X_{k+1}}{Y_k}|\mathcal{A}_k\right]\\
\bar{m}_n^2 &:= E_B\left[\left(\frac{Y_{k+1}}{Y_k}\right)^2|\mathcal{A}_k\right] - E_W\left[\left(\frac{X_{k+1}}{Y_k}\right)^2|\mathcal{A}_k\right]\\
\bar{m}_n^3 &:= E_B\left[\left(\frac{Y_{k+1}}{Y_k}\right)^3|\mathcal{A}_k\right] - E_W\left[\left(\frac{X_{k+1}}{Y_k}\right)^3|\mathcal{A}_k\right]\\
m_n^1 &:= E_B\left[\frac{Y_{k+1}}{Y_k} - 1|\mathcal{A}_k\right] - E_W\left[\frac{X_{k+1}}{Y_k} - 1|\mathcal{A}_k\right]\\
m_n^2 &:= E_B\left[\left(\frac{Y_{k+1}}{Y_k} - 1\right)^2|\mathcal{A}_k\right] - E_W\left[\left(\frac{X_{k+1}}{Y_k} - 1\right)^2|\mathcal{A}_k\right]\\
m_n^3 &:= E_B\left[\left(\frac{Y_{k+1}}{Y_k} - 1\right)^3|\mathcal{A}_k\right] - E_W\left[\left(\frac{X_{k+1}}{Y_k} - 1\right)^3|\mathcal{A}_k\right]\\
p_n &:= E_B\left[\left(\ln\frac{Y_{k+1}}{Y_k}\right)\left(\frac{Y_{k+1}}{Y_k} - 1\right)^3\middle|\mathcal{A}_k\right]\\
\bar{p}_n &:= E_W\left[\left(\ln\frac{X_{k+1}}{Y_k}\right)\left(\frac{X_{k+1}}{Y_k} - 1\right)^3\middle|\mathcal{A}_k\right]
\end{aligned}$$

Here we use intervals $[Y_k, Y_{k+1}]$ *if* $Y_{k+1} \geqslant Y_k$ *and* $[Y_{k+1}, Y_k]$ *if* $Y_k > Y_{k+1}$.

Theorem 8. *[11] Let* $\{Y_0^n, \ldots, Y_n^n\}$ *be a lattice approach with* $Y_0^n = Y_0(\forall n)$*, let* e_n *be the error in the price of a European call option that is*

$$e_n := e^{-rT}\left|E_W\left[f\left(X_T^{0,S}\right)|\mathcal{A}_0\right] - E_B[f(Y_n)|\mathcal{A}_0]\right|$$

Then there exists a constant, only depending on S_0, K, r, σ, T *such that*

$$e_n \leqslant k((m_n^2 + m_n^3 + p_n)n + 1/n)$$

Proof. (Sketch) Using $f(Y_n) = c(T, Y_n)$ and riskless property of BS price, we have

$$e_n = \left|E_B\left[\sum_{k=0}^{n-1} e^{-rt_{k+1}}\{E_B[c(t_{k+1}, Y_{k+1}) - c(t_{k+1}, Y_k)|\mathcal{A}_k] - E_W[c(t_{k+1}, X_{k+1}) - c(t_{k+1}, Y_k)|\mathcal{A}_k]\}|\mathcal{A}_0\right]\right|$$

for $k = n-1$, by lemmata [16,17], it is evaluated separatedly as $O\left(\frac{1}{n}\right)$. The other time points are evaluated by a Taylor expansion(dispite of coefficient of series) around Y_k. This yields:

$$
\begin{aligned}
e_n \leqslant\ & O(1/n) + |E_B[\Sigma_{k=0}^{n-2} e^{-rt_{k+1}}\{E_B[c_1(t_{k+1}, Y_k)(Y_{k+1} - Y_k) + c_2(t_{k+1}, Y_k)(Y_{k+1} - Y_k)^2 \\
& + c_3(t_{k+1}, Y_k)(Y_{k+1} - Y_k)^3 + R_3(t_{k+1}, Y_{k+1}, Y_k)|\mathcal{A}_k] \\
& - E_W[c_1(t_{k+1}, Y_k)(X_{k+1} - Y_k) + c_2(t_{k+1}, Y_k)(X_{k+1} - Y_k)^2 \\
& + c_3(t_{k+1}, Y_k)(X_{k+1} - Y_k)^3 + R_3(t_{k+1}, X_{k+1}, Y_k)|\mathcal{A}_k]\ \}|\mathcal{A}_0\ ]|
\end{aligned}
$$

Since

$$
\begin{aligned}
E_B[c_1(t_{k+1}, Y_k)(Y_{k+1} - Y_k)|\mathcal{A}_k] &= \tilde{c}_1(t_{k+1}, Y_k) E_B\left[\frac{Y_{k+1}}{Y_k} - 1 \middle| \mathcal{A}_k\right] \\
E_W[c_1(t_{k+1}, Y_k)(X_{k+1} - Y_k)|\mathcal{A}_k] &= \tilde{c}_1(t_{k+1}, Y_k) E_W\left[\frac{X_{k+1}}{Y_k} - 1 \middle| \mathcal{A}_k\right]
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
e_n \leqslant\ & O(1/n) + |\Sigma e^{-rt_{k+1}} E_B[\tilde{c}_1(t_{k+1}, Y_k)|\mathcal{A}_0]| \cdot |m_n^1| & (3)\\
& + |\Sigma e^{-rt_{k+1}} E_B[\tilde{c}_2(t_{k+1}, Y_k)|\mathcal{A}_0]| \cdot |m_n^2| & (4)\\
& + |\Sigma e^{-rt_{k+1}} E_B[\tilde{c}_3(t_{k+1}, Y_k)|\mathcal{A}_0]| \cdot |m_n^3| & (5)\\
& + |E_B[\Sigma e^{-rt_{k+1}}\{E_B[R_3(Y_{k+1})|\mathcal{A}_k] + E_W[R_3(X_{k+1})|\mathcal{A}_k]\}|\mathcal{A}_0]| & (6)
\end{aligned}
$$

□

Now our task is to valuate these parts separately, we extract some important lemmata.

Lemma 9.

$$
\begin{aligned}
|\tilde{c}_2(t, S)| &\leqslant \frac{M_2(T-t)}{\sqrt{T-t}} \\
|\tilde{c}_3(t, S)| &\leqslant \frac{M_3(T-t)}{T-t} \\
|c_4(t, S)| &\leqslant \frac{M_4(T-t)}{T-t}
\end{aligned}
$$

Lemma 10. $|R_3(t, z_1, z_0)| \leqslant \left(\ln \frac{z_1}{z_0}\right)(z_1 - z_0)^3 \max_{S \in [z_0, z_1]} |c_4(t, S)|$.

Lemma 11.

$$
\sum_{k=0}^{n-2} e^{-rt_{k+1}} E_B[|\tilde{c}_3(t_{k+1}, Y_k)|\,|\mathcal{A}_0] \leqslant \left(\frac{16\sigma M_3}{\sqrt{pq}} + M_3\right) \cdot n, \text{where } M_3 = M_3(0)
$$

Lemma 12.

$$
\sum_{k=0}^{n-2} e^{-rt_{k+1}} E_B[E_B[|R_3(t, Y_{k+1}, Y_k)||\mathcal{A}_k]\,|\mathcal{A}_0] \leqslant \left(\frac{24\sigma M_4}{\sqrt{pq}} + M_4 E_B[Y_k^3]\right) \cdot n\,, \text{where } M_4 = M_4(0)
$$

Lemma 13.

$$
\sum_{k=0}^{n-2} e^{-rt_{k+1}} E_B[E_W[|R_3(t, X_{k+1}, Y_k)||\mathcal{A}_k]\,|\mathcal{A}_0] \leqslant \frac{24\sigma M_4}{\sqrt{pq}}(2Y_0)^3 \bar{p}_n + E_B[Y_k^3]\left(\frac{4M_4 e^{4|m|}}{\sqrt{2\pi}(T-t)}(\Delta t)^4 \sqrt{\ln \frac{1}{\Delta t}} + M_4 \bar{p}_n\right)
$$

where $M_4 = M_4(0)$, $m = \frac{T}{n}\left(r - \frac{\sigma^2}{2}\right)$.

Lemma 14. $$E_B[Y_k^3|\mathcal{A}_0] \leqslant \text{const}$$

Lemma 15. $$\bar{p}_n = O(\Delta t^2)$$

The next two lemmata are using in canseling $k = n-1$ term:

Lemma 16.

$$|E_B[E_W[f(X_n) - f(Y_{n-1})|\mathcal{A}_{n-1}]|\mathcal{A}_0]| = O(1/n)$$

Lemma 17.

$$|E_B[E_B[f(Y_n) - f(Y_{n-1})|\mathcal{A}_{n-1}]|\mathcal{A}_0]| = O(1/n)$$

From these Lemma, we finished proof.

Corollary 18. *CRR, JR, Tian Binomial methods converges with order 1.*

Proof. *We just give a proof of* CRR*, others are simlilar to it through prvious theorem.*

To prove CRR converges with order , we only need to verify: $\bar{m}_n^2 = O(\frac{1}{n^2})$,$\bar{m}_n^3 = O(\frac{1}{n^2})$,$p_n = O(\frac{1}{n^2})$.

Since $pu + qd = e^{r\Delta t}$,*we have*

$$pu^2 + qud = ue^{r\Delta t}, pud + qd^2 = de^{r\Delta t}$$

Hence

$$pu^2 + qd^2 = e^{r\Delta t}(u+d) - 1$$

Then

$$\bar{m}_n^2 = \left|pu^2 + qd^2 - e^{2r\Delta t + \sigma^2\Delta t}\right| = e^{r\Delta t}\left|u + d - e^{-r\Delta t} - e^{r\Delta t + \sigma^2\Delta t}\right| = e^{r\Delta t}O(1/n^2) = O(1/n^2)$$

Similarly, we have $\bar{m}_3^2 = O(1/n^2)$.

Also we have $u - 1 = O(1/\sqrt{n})$, $d - 1 = O(1/\sqrt{n})$ *from series expansion of exp function. Since* $\ln u = -\ln d = \sigma\sqrt{\Delta t}$, *we get* $p_n = O(1/n^2)$. □

We have some image results for these three trees:

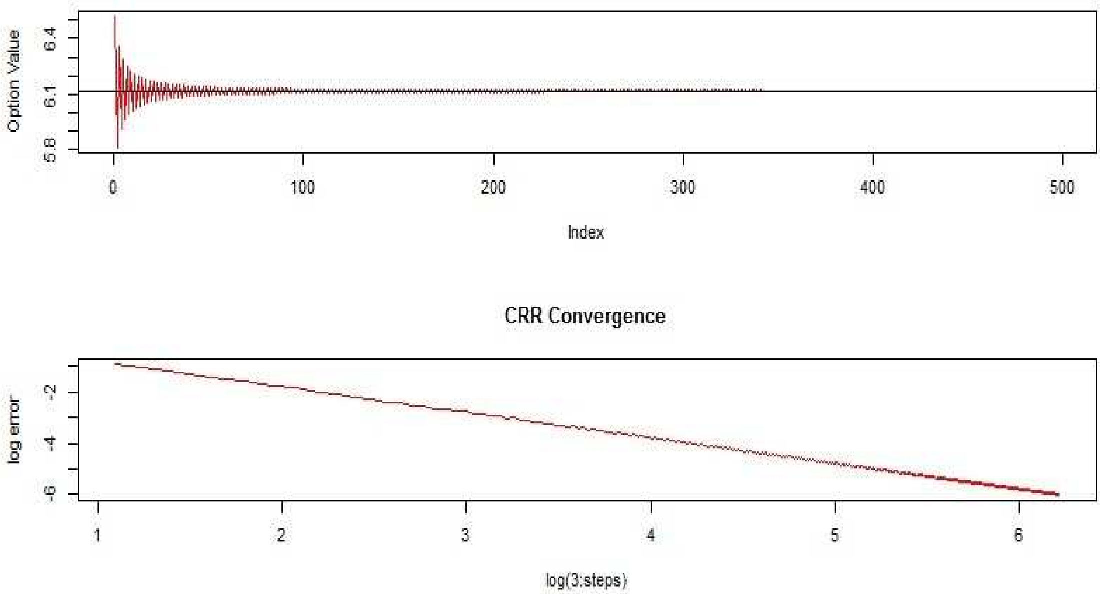

Figure 4. CRR Convergence

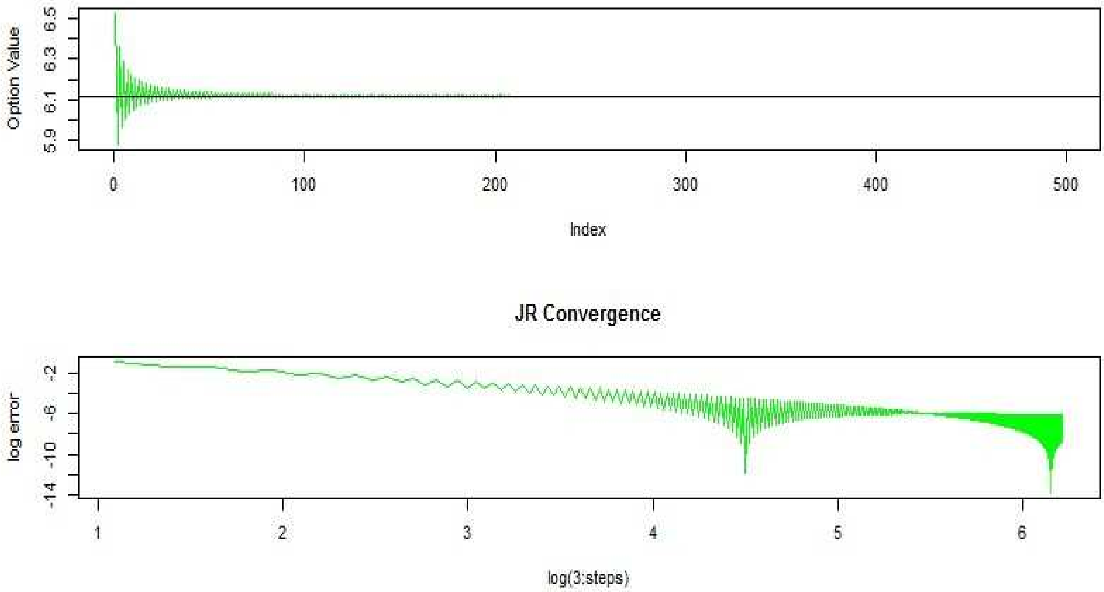

Figure 5. JR Convergence

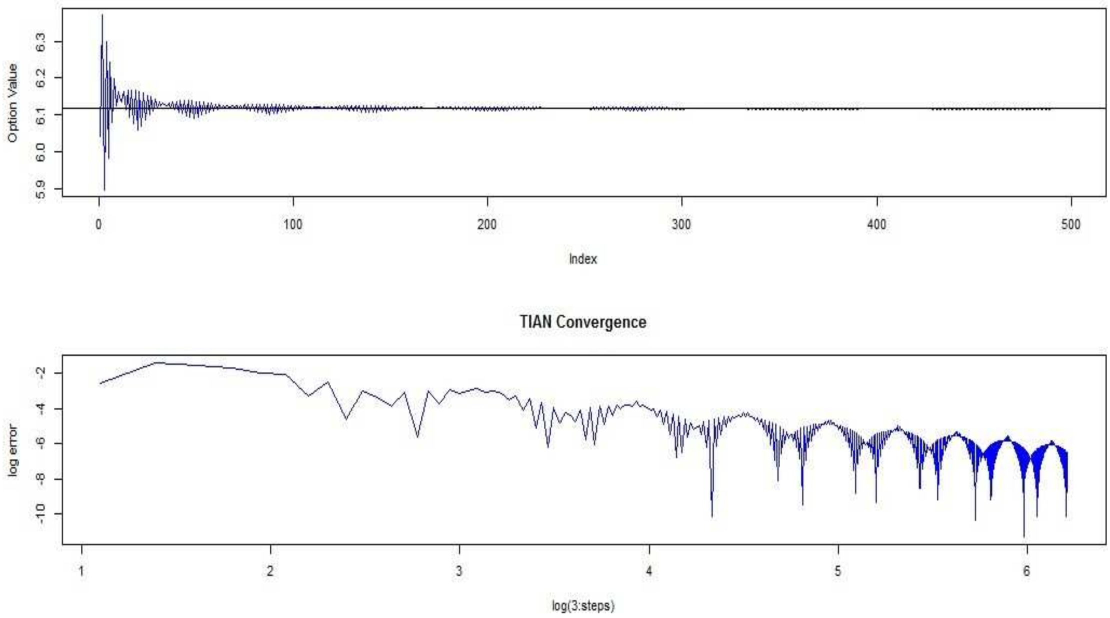

Figure 6. TIAN Convergence

More precisely, for example For CRR tree, we have[3]

Corollary 19. (CRR model) *If the underlying asset is such that* $u = e^{\sigma/\sqrt{n}}$ *and* $d = e^{-\sigma/\sqrt{n}}$, *then the price at* $t = 0$ *of a call option with value* $(u^j d^{n-j} - K)^+$ *at time* $T = n\Delta t = 1$ *satisfies:*

$$C(n,S) = \mathrm{BS} + \frac{1}{n}\left\{-\sqrt{2/\pi}Ke^{-r}e^{-\frac{1}{2}(\sigma/2-(r-\ln K)/\sigma)^2}\left(\sigma S(S-1) + \frac{\sigma^4 + 12(\sigma^2+r^2) + 8r\ln K + 4(\ln K)^2}{96\sigma}\right)\right\} + O\left(\frac{1}{n\sqrt{n}}\right)$$

2.3. Higher order Binomial Method for European Option Valuation.

The model given by Leisen and Reimer[11] giving more accurate approximation of the option value compared to the other models. This model has an important advantage against the other models. The model have **quadratic converges** in the number of time steps, while the other models have a linear convergence. That's why the accuracy is much better. The idea is that we don't fix the strick overall, we fix the center of the tree. This implies that the strike is always contained in the binomial tree grid.

However, Leisen and Reimer haven't prove that order. Mark S. Joshi has proved that in [6] using the asympototic expansion of binomial coeeficient. What's more, Joshi[7] has created any high order convergent Binomial Method using this way.

An asymptotic expansion describes the asymptotic behavior of a function in terms of a sequence of **gauge functions**. The definition was introduced by Poincaré (1886), and it provides a solid mathematical foundation for the use of many divergent series. Here we'd like to introduce asympototic expansion[2], which is adapted from the lecture notes of prof. John Hunter.

2.3.1. Asympototic expansions.

Definition 20. (Asympototic expansions) *A sequence of functions* φ_n: $\mathbb{R}\backslash 0 \to \mathbb{R}$, *where* $n = 0, 1, 2, \ldots$, *is an asymptotic sequence as* $x \to 0$ *if for each* $n = 0, 1, 2, \ldots$ *we have*

$$\varphi_{n+1} = o(\varphi_n) \text{ as } x \to 0$$

We call φ_n *a* ***gauge function****. If* $\{\varphi_n\}$ *is an* ***asymptotic sequence*** *and* $f: \mathbb{R}\backslash 0 \to \mathbb{R}$ *is a function, we write*

$$f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x) \text{ as } x \to 0 \tag{7}$$

if for each $N = 0, 1, 2, \ldots$ *we have*

$$f(x) - \sum_{n=0}^{\infty} a_n \varphi_n(x) = o(\varphi_N) \text{ as } x \to 0$$

We call (7) the ***asymptotic expansion*** *of f with respect to* $\{\varphi_n\}$ as $x \to 0$.

Example 21. The function log sin x has an asymptotic expansion as $x \to 0^+$ with respect to the asymptotic sequence $\{\log x, x^2, x^4, \ldots\}$:

$$\log \sin x \sim \log x + \frac{1}{6}x^2 + \ldots \text{ as } x \to 0^+$$

Asymptotic power series,

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^n \text{ as } x \to 0$$

are among the most common and useful asymptotic expansions.

2.3.2. Leisen-Reimer Tree.

Now we first consider one improved method given by Leisen and Reimer:

Definition 22. (Leisen-Reimer Tree)

$$\begin{cases} u = e^{r\Delta t}\frac{p'}{p} \\ d = e^{r\Delta t}\frac{1-p'}{1-p} \end{cases}, \begin{cases} p' = h^{-1}(d_1) \\ p = h^{-1}(d_2) \end{cases}$$

where $h^{-1}(*)$ is a discrete approximation to the cummulative distribution function for a normal distribution.

There are several ways this can be calculated. One suggested by Leisen and Reimer is to use, the Peizer-Pratt method to invert the binomial distribution(n=2j+1):

$$h^{-1}(z) = \frac{1}{2} \mp \left[\frac{1}{4} - \frac{1}{4}\exp\left\{-\left(\frac{z}{n+1/3}\right)^2\left(n+\frac{1}{6}\right)\right\}\right]^{1/2}$$

where n is the number of time points in the model (including times 0 and T) which must be odd, and d_1 and d_2 are their usual definitions from the Black-Scholes formulation.

Maybe you can get an intuitive improved convergence of LR method in the following graph:

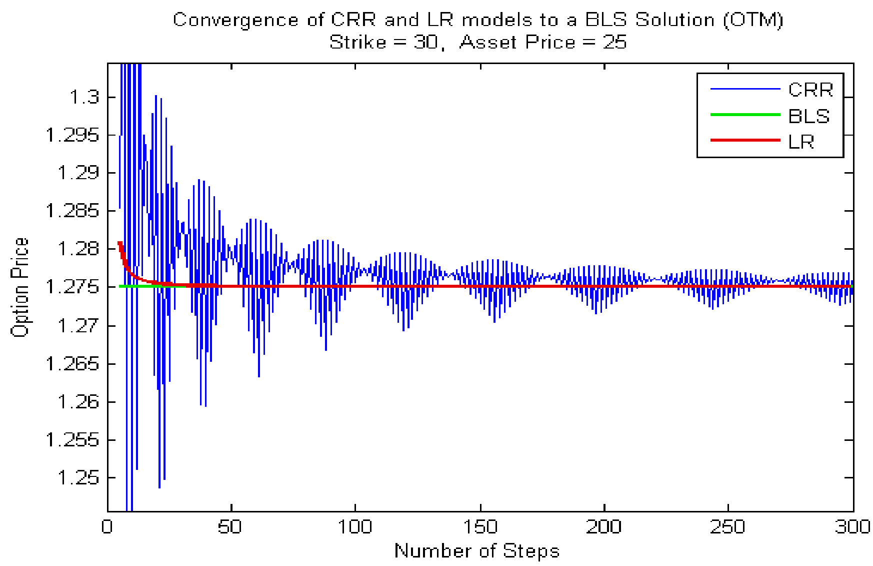

Figure 7. Convergence of LR

2.4. Convergence of Ammerican Put Options.

The theorem 8 is also true for Ammerican put options, however there is also an lower bound for convergence[10].

Damien Lamberton[9] establish some error estimates for the CRR binomial approximation of American put prices in the Black-Scholes model. He prove that if P is the American put price and P_n its n-step binomial approximation, there exist positive constants c and C such that $-c/n^{2/3} < P_n - P < C/n^{3/4}$.

For the Leisen-Reimer tree, Leisen [10] has shown that American puts have first order convergence, whilst the models of Jarrow and Rudd and Tian have convergence between order $1/2$ and 1.

Theorem 23. *Let $(\bar{R}_n)_{n\in\mathbb{N}}$ be a sequence of lattices and m_n^2, m_n^2, p_n its respective (pseudo-)moments. Then there exixts a constant $k_1(S_0, K, r, \sigma, T)$ such that*

$$P^a(0,S_0) - P_n^a(0,S_0) \geqslant k_1\left\{n(m_n^2 + m_n^3 + p_n) + \frac{1}{\sqrt{n}}\right\}$$

If $(\bar{R}_n)_{n\in\mathbb{N}}$ is constructed according to Leisen-Reimer lattice approach, then we have stronger results that there exists a constant $k_1(S_o, K, r, \sigma, T)$ such that

$$P^a(0,S_0) - P_n^a(0,S_0) \geqslant k_1\left\{n(m_n^2 + m_n^3 + p_n) + \frac{1}{n}\right\}$$

Corollary 24. *American put option prices calculated using the lattice approach of Leisen-Reimer converge with order one.*

Corollary 25. *American put option prices calculated using the lattice approaches of JR and Tian converge with order one from above and order 1/2 from below.*

2.5. Sawtooth and periodic humps.

For a European option, when we increase n and plot Error$_n$ against n we see the following shape

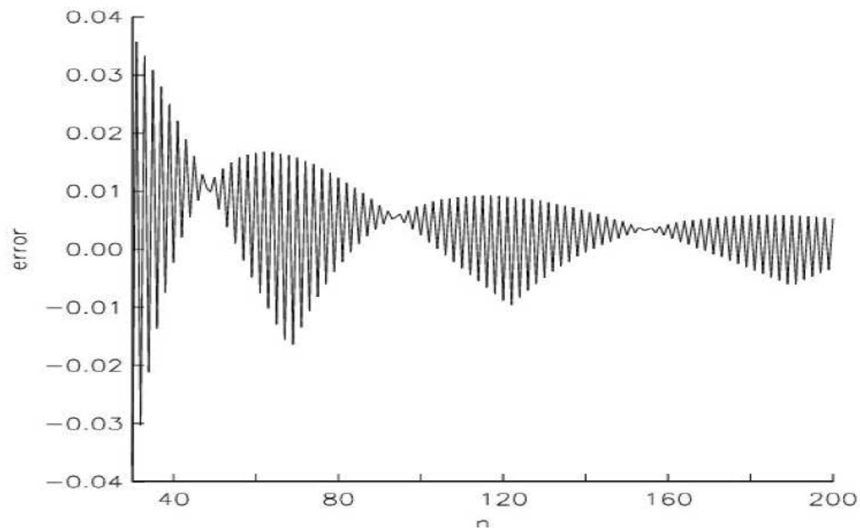

Figure 8. Sawtooth Effect

We see two distinct features, the first is a sawtoothing and the second is periodic humps. The sawtoothing is known as the 'odd-even effect' where as you move from say i steps to $i+1$ steps the change in V_n is very large. The following explains the odd-even effect:

The binomial approximation to the normal is depicted for lattices with 6 and 7 steps. The red shading denotes which nodes contribute value to the option price if K = 100.

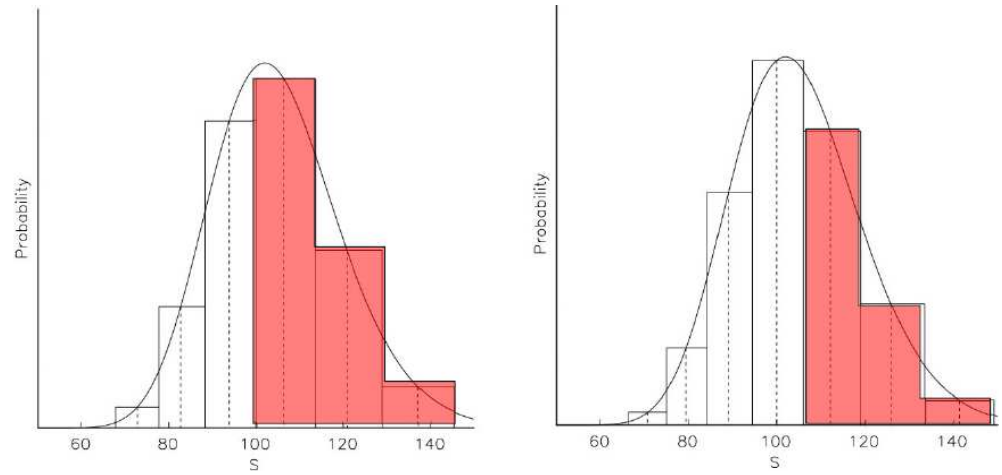

Figure 9. odd-even effect

Periodic humps are connected to the position of the binomial nodes. Let L be denoted by

$$\Lambda = \frac{S_k - K}{S_k - S_{k-1}}$$

where S_k is the closest node above the exercise price and S_{k-1} below. The plots of Λ against the error from the binomial lattice as follows: The dashed lines here denote the error and the solid lines the corresponding value of Λ. (Only even numbers of steps were considered.)

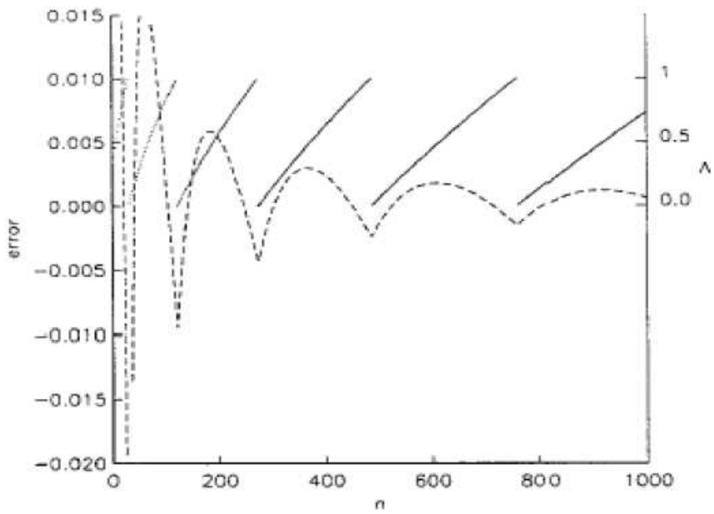

Figure 10. periodic humps effect

3. ZNJS

3.1. Financial report on ZNJS.

每股指标	14-03-31	13-12-31	13-09-30	13-06-30	13-03-31	12-12-31	12-09-30	12-06-30	12-03-31
基本每股收益(元)	0.1600	0.5202	0.1000	0.2776	0.1400	0.2765	0.2067	0.2800	0.1200
每股净资产(元)	6.7854	6.6247	5.9586	5.8586	5.6810	5.5432	5.2600	5.0500	4.7700
每股公积金(元)	0.6829	0.6829	0.5369	0.5369	0.5369	0.5369	0.5297	0.5297	0.5297
每股未分配利润(元)	4.8927	4.7320	4.2394	4.1394	3.9618	3.8240	3.5619	3.3552	3.0752
每股经营现金流(元)	-2.1870	-3.0612	-0.5784	0.3188	-0.2580	-0.6044	0.1485	0.0429	0.1800

Figure 11. Report 1

成长能力指标	14-03-31	13-12-31	13-09-30	13-06-30	13-03-31	12-12-31	12-09-30	12-06-30	12-03-31
营业收入(元)	44.6亿	71.5亿	38.7亿	45.8亿	27.3亿	48.4亿	30.6亿	32.7亿	18.6亿
毛利润(元)	7.25亿	16.9亿	7.48亿	11.4亿	6.57亿	13.8亿	7.28亿	8.96亿	4.98亿
归属净利润(元)	1.88亿	6.07亿	1.17亿	3.24亿	1.61亿	3.23亿	2.41亿	3.27亿	1.41亿
扣非净利润(元)	1.88亿	5.80亿	1.13亿	3.24亿	1.56亿	3.18亿	2.34亿	3.19亿	1.45亿
营业收入同比增长(%)	63.14	47.50	26.46	39.98	46.94	10.11	0.82	14.48	-7.56
归属净利润同比增长(%)	16.65	88.14	-51.62	-0.87	14.24	10.80	5.90	51.64	-28.60
扣非净利润同比增长(%)	20.60	82.49	-51.78	1.39	7.93	-3.93	16.81	48.99	-26.01
营业收入环比增长(%)	-37.63	84.87	-15.65	67.73	-43.61	58.49	-6.63	76.06	-57.74
归属净利润环比增长(%)	-69.09	420.32	-63.98	101.40	-50.15	33.79	-26.20	132.12	-51.65
扣非净利润环比增长(%)	-67.51	414.12	-65.19	107.43	-50.84	35.86	-26.82	120.80	-56.24

Figure 12. Report 2

盈利能力指标	14-03-31	13-12-31	13-09-30	13-06-30	13-03-31	12-12-31	12-09-30	12-06-30	12-03-31
摊薄净资产收益率(%)	1.78	7.75	1.32	3.76	2.44	4.46	3.22	4.37	1.97
摊薄总资产收益率(%)	0.27	1.17	0.20	0.61	0.43	0.77	0.58	0.82	0.40
毛利率(%)	16.27	23.62	19.34	24.93	24.04	28.50	23.83	27.38	26.78
净利率(%)	4.21	8.50	3.02	7.07	5.89	6.66	7.90	9.99	7.58

Figure 13. Report 3

This above table shows the financial conditions of the company seasonally. We can easily notice that the profit(and net income or so) has seasonality. So we should analyze the financial data yearly but not seasonally. We can see it more clearly as below.

Figure 14. Payoff per share

3.1.1. Four Principles in investing.

- Safety is measured not by specific lien or other contractual rights, but by the ability of the issuer to meet all of its obligations.
- This ability should be measured under conditions of depression rather than prosperity.
- Deficient safety cannot be compensated for by an abnormally high coupon rate.
- The selection of all bonds for investment should be subject to rules of exclusion and to specific quantitative tests corresponding to those prescribed by statute to govern investments of saving banks.

Here we introduce the concept of ROE (Return On Equity) ROE = (Net Profit/Equity). This quantity can reflect the earning power of a company, so as a investor we always expect it to be higher.

3.1.2. Comparison in the industry.

杜邦分析比较

排名	代码	简称	ROE(%)				净利率(%)			
			3年平均	11A	12A	13A	3年平均	11A	12A	13A
28	000961	中南建设	16.24	17.16	15.94	15.63	7.37	7.58	7.92	6.60
行业平均			12.58	13.06	12.34	12.33	10.13	0.94	15.38	14.08
行业中值			8.43	9.65	8.15	8.97	11.51	12.97	11.27	10.59
1	000863	三湘股份	27.74	25.27	31.86	26.10	22.00	25.76	22.96	17.29
2	000537	广宇发展	26.38	31.57	23.98	23.59	19.62	18.87	15.73	24.26
3	002146	荣盛发展	25.89	24.97	26.19	26.52	15.75	16.13	15.95	15.16
4	600067	冠城大通	25.16	25.01	23.22	27.24	12.45	8.54	13.28	15.53
5	000540	中天城投	24.89	24.37	18.53	31.79	14.12	15.77	12.34	14.27

Figure 15. DuPont analysis

Since ZNJS ranks very well in the table, within the consideration of financial perspective, its ROE performs better than the average of the whole industry. We can conclude that this company have a relatively good earning power *in the industry.*

成长性比较

排名	代码	简称	基本每股收益增长率(%)						营业收入增长率(%)					
			3年复合	13A	TTM	14E	15E	16E	3年复合	13A	TTM	14E	15E	16E
33	000961	中南建设	20.15	17.17	2.22	24.15	26.78	10.18	23.55	40.60	9.41	18.73	22.75	29.42
行业平均			25.53	14.42	-0.63	31.76	26.01	20.97	22.93	27.59	1.28	23.00	26.27	21.57
行业中值			25.94	8.55	-0.34	29.74	25.97	18.65	21.71	20.74	1.10	23.97	23.76	20.53
1	600094	大名城	78.06	35.19	-4.11	118.94	87.38	37.60	69.88	75.36	0.06	85.38	93.95	36.35
2	600175	美都控股	70.26	66.03	-61.80	193.15	48.10	13.69	1.42	73.26	0.82	-3.82	17.68	-7.83
3	000718	苏宁环球	57.80	-43.55	-52.61	170.48	35.27	7.39	--	52.53	-19.65	24.18	27.62	--
4	000732	泰禾集团	56.98	111.54	1.47	71.08	58.78	42.42	52.01	135.46	-0.42	47.75	56.61	51.80
5	000671	阳 光 城	53.62	16.66	14.60	89.84	47.26	29.68	45.17	36.53	14.40	60.92	42.18	33.72

Figure 16. Comparison of growth

This table makes us have a better understanding on the future tendency of the development of the right company and the whole industry. Clearly the company has a poor growing ability, the growth expectation decreases sharply. While the industrial average has the same appearance, which shows that this problem comes out from the intrinsic property of the real estate industry.

3.1.3. Problems.

Figure 17. Performance in Market

This graph shows that ZNJS mainly follows the tendency of the whole real estate industry. But one can notice that, whenever the industry perform bad, ZNJS perform only worse. This tell us that, ZNJS is not a good security to invest. *Because its growing ability should be measured under conditions of depression rather than prosperity*. Which means a 'not bad' performance in depression is much more worthy than a 'good' performance in prosperity.

We now can conclude that ZNJS relies highly on the industry performance. Its invest value is nearly completely dependent on its real estate intrinsic property.

百分比报表

指标 2014-03-31	金额(元)	占比
总资产	703亿	100%
流动资产	650亿	92.46%
货币资金	63.1亿	8.98%
应收账款	59.2亿	8.42%
存货	426亿	60.59%
预付账款	49.4亿	7.04%
非流动资产	53.0亿	7.54%
固定资产	16.2亿	2.30%
无形资产	5.92亿	0.84%
长期待摊费用	17.7亿	2.52%
金融资产	254万	0.00%
总负债金额	595亿	100%
流动负债	456亿	76.59%
非流动负债	139亿	23.41%

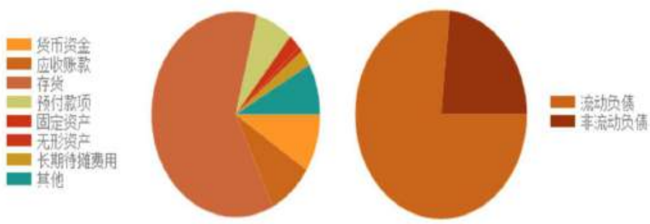

Figure 18. Percentages

We should be careful about the 'inventory' section. Over 60% inventory, this tells us that 2/3 of its properties are not been sold. Maybe this is good when the housing price growing constantly, but for now, this unsold properties can only be more risky like a bomb.

And moreover, here are 76.59% of Current Liabilities, added on the above large inventory, this company can easily be in debt and become bankruptcy in depression. It could have good performance in prosperity, but the high risk is not worth it while. This is because *'Deficient safety cannot be compensated for by an abnormally high coupon rate'.*

3.2. Prediction.

All of the analysis above can be used in the prediction. Since we have concluded that ZNJS's performance highly depends on real estate industry. Though the government says that there is no danger of collapse, but the whole market appears negative attitude. And compare with the 2008 financial crisis, there are too many similar symptoms in the market. I personally argue that the real estate might not collapse, but wouldn't continue its prosperity as in the last 10 years. It's hard to predict the tendency next two months, but the conclusion in the long term must be negative.

3.3. Determination of the option price of ZNJS.

Here we want to first determine the european call option price, since by B-S formula

$$C(S,t) = S\,N(d_1) - E\,e^{-r(T-t)}\,N(d_2) \tag{8}$$

where

$$d_1 = \frac{\ln S/E + (r + \frac{1}{2}\sigma^2)\,(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln S/E + (r - \frac{1}{2}\sigma^2)\,(T-t)}{\sigma\sqrt{T-t}}$$

Since S, E, T, t, and r are all known, the only unknown parameter is the volatility σ.

Then how to evaluate the volatility from the historical data?

Suppose that we have the stock price data every trading day and denote as S_i. Further assume that the stock price follows the geometric Brownian motion, that is

$$S_{i+1} = S_i \exp\{(\mu - \frac{1}{2}\sigma^2)\,(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}\,Z_i\} \tag{9}$$

where Z_i follows standard normal distribution.

Take the log of both side, here we have

$$\ln S_{i+1} - \ln S_i = (\mu - \frac{1}{2}\sigma^2)\,(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}\,Z_i \tag{10}$$

Now let's do some simplifications, denote $D_i = \ln S_{i+1} - \ln S_i$ and $\Delta\,t = t_{i+1} - t_i$ then the equation becomes

$$D_i = (\mu - \frac{1}{2}\sigma^2)\,\Delta\,t + \sigma\sqrt{\Delta\,t}\,Z_i \tag{11}$$

We can evaluate the variance of D_i using MatLab, say d^2, and the corresponding evaluated volatility of S_i must be

$$\hat{\sigma}^2 = \frac{d^2}{\Delta\,t} \tag{12}$$

3.3.1. Evaluation.

For simplicity, we claim there are 250 trading days per year. And the data we use is taken daily, so the quantity $\Delta\,t = 1/250$ presumed. To assure the consistency, we only use the open price of each day.

We get the variance $d^2 = 9.0877 \times 10^{-4}$, then $\hat{\sigma}^2 = 0.2272$, and then $\hat{\sigma} = 0.4766$. Then substitute it into the pricing formula we can get the answer.

3.3.2. Application.

Today's stock price $S = 6.96$, take the interest rate of 10-year treasury bonds as the risk-free interest rate, $r = 4.1348\%$, if we want to sell call options expire in 3 months, with strike price 8 yuan, then the option price can be calculated:

$$d_1 = \frac{\ln S/E + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln 6.96/8 + (0.0413 + \frac{1}{2}0.2272)\,3/12}{0.4766\,\sqrt{3/12}} = -0.4219$$

$$d_2 = \frac{\ln S/E + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln 6.96/8 + (0.0413 - \frac{1}{2}0.2272)\,3/12}{0.4766\,\sqrt{3/12}} = -0.6602$$

and then

$$C(S,t) = S\,N(d_1) - E\,e^{-r(T-t)}\,N(d_2) = 6.96\,N(-0.4219) - 8 \times e^{-0.0413\times 3/12}\,N(-0.6602) = 0.3268$$

4. 沪港通

沪港通是指上海证券交易所和香港联合交易所允许两地投资者通过当地证券公司（或经纪商）买卖规定范围内的对方交易所上市的股票，是沪港股票市场交易互联互通机制。

4.1. 组成部分.

沪港通包括沪股通和港股通两部分：
沪股通，是指投资者委托香港经纪商，经由香港联合交易所设立的证券交易服务公司，向上海证券交易所进行申报（买卖盘传递），买卖规定范围内的上海证券交易所上市的股票；
港股通，是指投资者委托内地证券公司，经由上海证券交易所设立的证券交易服务公司，向香港联合交易所进行申报（买卖盘传递），买卖规定范围内的香港联合交易所上市的股票.

4.2. 实施范围.

试点初期，沪股通的股票范围是上海证券交易所上证180指数、上证380指数的成份股，以及上海证券交易所上市的A+H股公司股票。港股通的股票范围是香港联合交易所恒生综合大型股指数、恒生综合中型股指数的成份股和同时在香港联合交易所、上海证券交易所上市的A+H股公司股票。
双方可根据试点情况对投资标的范围进行调整。

4.3. 试点条件.

试点初期，香港证监会要求参与港股通的境内投资者仅限于机构投资者，及证券帐户及资金帐户余额合计不低于50万元的个人投资者。

4.4. 影响.

沪港通是中国资本市场对外开放的重要内容，有利于加强两地资本市场联系，推动资本市场双向开放，具有以下三方面积极意义：
（一）有利于通过一项全新的合作机制增强我国资本市场的综合实力。沪港通可以深化交流合作，扩大两地投资者的投资渠道，提升市场竞争力。
（二）有利于巩固上海和香港两个金融中心的地位。沪港通有助于提高上海及香港两地市场对国际投资者的吸引力，有利于改善上海市场的投资者结构，进一步推进上海国际金融中心建设；同时有利于香港发展成为内地投资者重要的境外投资市场，巩固和提升香港的国际金融中心地位。
（三）有利于推动人民币国际化，支持香港发展成为离岸人民币业务中心。沪港通既可方便内地投资者直接使用人民币投资香港市场，也可增加境外人民币资金的投资渠道，便利人民币在两地的有序流动。

极端情况可能暂停

港交所股本证券与定息产品及货币联席主管陈秉强2014年5月8日表示，一旦出现极端情况，“沪港通”的机制可以暂停。
陈秉强在2014年5月8日出席沪港通传媒研讨会时还表示，沪港通在5月至8月期间举行参与者推介会，并会要求参与者提交意向书，9月份将会邀请参与者进行市场演习，以确保市场有足够的准备。陈秉强指出，“沪港通”的机制下，总额度为3000亿元人民币，相对于A股市值，只有1%左右，预计不会对A股构成大波动。不过，一旦出现极端情况，必要时“沪港通”的机制可以暂停。但他指出，暂停“沪港通”机制是最后的防线，在极端的情况下才会使用，如果有个别股票亦涉及极端操作的话，亦可因应当时情况关闭交易通道。
在交易方面，沪港通只能在二级市场进行交易，在现行机制下，额度利用按先到先得原则，此外，只有买盘才受额度监控，投资者在任何时候，不论额度水准，都可输入卖盘。在两地假期交易安排上，如果款项交收日香港市场不开市，沪股通在此前一个交易日也将不开放交易。
港交所表示，根据与两地银行的沟通进展，将进一步提升沪港通模式，希望未来可以实现各市场投资者均可完全按照对方市场的交易日进行交易。

陈秉强表示，互联互通有助于人民币资金双向流动，完善资金池的循环，预计人民币兑换业务的深度将会增加。至于离岸人民币资金池是否会相应增大，他表示，这将视乎流入流出相对比例，可能会增加离岸人民币兑换市场的规模，而对离岸人民币资金池的影响也将是正面的。
陈秉强认为，沪股通政策推行后，制度成本会逐渐减少，对人民币发展有正面影响及创造有利条件，相信会有相关人民币的定价产品出现。
陈秉强表示，目前银行与个人每日人民币兑换上限的限制不适用于券商，因为券商不属于银行体系，其人民币兑换不会影响人民币资金池的定额。

LIST OF FIGURES

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