Divided Difference and Derivatives

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Definition of divided difference

$$f[x_0] = f(x_0)$$

$$f[x_0, \cdots, x_k] = \frac{f[x_1, \cdots, x_k] - f[x_0, \cdots, x_{k-1}]}{x_k - x_0}$$

Theorem:
Let the function f be defined and has k bounded derivatives on inverval $[a, b]$ and let $x_0, x_1, \cdots, x_n$ be distinct $n + 1$ points in $[a, b]$. Then there exists some point $\xi \in [a, b]$ such that

$$f[x_0, x_1, \cdots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

.

Proof:

Interpolation polynomial $p(x)$ in Newton's form

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, \cdots, x_n](x - x_0) \cdots (x - x_{n-1})$$

Define

$$g(x) := f(x) - p(x)$$

Obviously $x_0, x_1, \cdots, x_n$ are $n + 1$ roots of $g(x)$.

By Rolle's Theorem, there exist $\xi_1^{(1)} \in [x_0, x_1]$,$\xi_2^{(1)} \in [x_1, x_2]$,$\cdots$, $\xi_n^{(1)} \in [x_{n-1}, x_n]$ such that

$$g'(\xi_1^{(1)}) = 0, g'(\xi_2^{(1)}) = 0, \cdots, g'(\xi_n^{(1)}) = 0$$

Again by Rolle's Theorem, there exist $\xi_1^{(2)} \in [\xi_1^{(1)}, \xi_2^{(1)}]$, $\cdots$, $\xi_{n-1}^{(2)} \in [\xi_{n-1}^{(1)}, \xi_n^{(1)}]$ such that

$$g''(\xi_1^{(2)}) = 0, g''(\xi_2^{(2)}) = 0, \cdots, g''(\xi_{n-1}^{(2)}) = 0$$

Keep using Rolle's Theorem, we finally get that there exists $\xi_1^{(n)} \in [\xi_1^{(n-1)}, \xi_2^{(n-1)}]$ such that

$$g^{(n)}(\xi_1^{(n)}) = 0$$

that is

$$\begin{aligned} & f^{(n)}(\xi_1^{(n)}) - f[x_0, x_1, \cdots, x_n]n! = 0 \\ \Rightarrow \quad & \frac{f^{(n)}(\xi_1^{(n)})}{n!} = f[x_0, x_1, \cdots, x_n] \end{aligned}$$