Project 46: The Pendulum

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SUSTC

Abstract:
The simple pendulum has been the object of much mathematical study. A device as simple as the pendulum may not seem to warrant much mathematical attention, but the study of its motion was begun by some of the greatest scientists of the 17th century. The scientist Christian Huygens studied the pendulum extensively from a geometric standpoint, and Issac Newton's theory of gravitation made it possible to study the pendulum from an analytic standpoint, which is what we will do in this project.

Keywords:
Pendulum，Pendulum Equation, Linear Approximation, Spring Pendulum, Lissajous

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1. Introduction

The pendulum is a familiar object. The pendulum allows one to keep time with great accuracy. Its most common appearance is in old-fashioned clocks that, even in this day of quartz timepieces and atomic clocks, remain quite popular. Much of the pendulum's fascination comes from the well known regularity of its swing and thus its link to the fundamental natural force of gravity. The development of pendulum clocks made sea travel safer during a time when new worlds in the Americas were just being developed and discovered. The compound pendulum is still studied today because of its complicated "chaotic" Behaviour.

The history of the physics of the pendulum stretches back to the early moments of modern science itself. We might begin with the story, perhaps apocryphal, of Galileo's observation of the swinging chandeliers in the cathedral at Pisa. By using his own heart rate as a clock, Galileo presumably made the quantitative observation that, for a given pendulum, the time or period of a swing was independent of the amplitude of the pendulum's displacement. Like many other seminal observations in science, this one was only an approximation of reality. Yet it had the main ingredients of the scientific enterprise; observation, analysis, and conclusion. Galileo was one of the first of the modern scientists, and the pendulum was among the first objects of scientific enquiry.

A simple pendulum is a small mass suspended from a light rigid rod. Ideally the mass will be small enough so that it will not deform the rod, but the rod should be light so that we can neglect its mass. (We will consider what happens if the rod stretches later in the project.) If we displace the bob slightly and let it go, the pendulum will start swinging. We will assume at first that there is no friction. In this idealized situation, once the pendulum is started, it will never stop. The only force acting on the pendulum is the force of gravity.

2. Derivation of the Pendulum Equation

The equation of the Simple Pendulum is

$$\frac{d^2\theta}{dt^2}+\frac{g}{L}\sin\theta=0$$

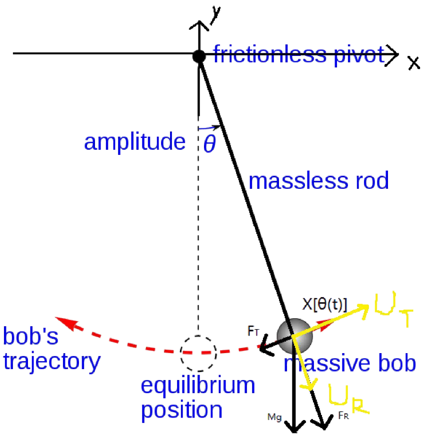

Figure 46.1: The Pendulum

The force acting on the pendulum can be broken into two components, one in the direction of the rod and the other in the direction of the pendulum's motion. If the rod does not stretch, the component directed along the rod plays no part in the pendulum's motion because it is counterbalanced by the force in the rod. The component in the tangential direction causes the pendulum to move. If we impose a coordinate system centered at the point of suspension of the pendulum and let $X[\theta[t]]$ denote the vector giving the position of the pendulum bob, then Figure 46.1 illustrates the force of gravity on the pendulum and the resolution of this force into two components $\boldsymbol{F_R}$ and $\boldsymbol{F_T}$.

We will use the vector form of Newton's law, $\boldsymbol{F}=m\boldsymbol{A}$, to determine the equation of motion of the pendulum. We assume that the origin of our fixed inertial coordinate system is at the pivot point of the pendulum shown in the figure. The axes have their usual orientation.

Radial and Tangential Position

Now we express the position vector $X[\theta[t]]$ of the mass in terms of the displacement angle θ, given that the length of the pendulum is L. Write **X** in terms of the unit vector, we get

$$U_R[\theta]=\begin{bmatrix}+\sin\theta\\-\cos\theta\end{bmatrix}$$

If the vector $\boldsymbol{U_R}$ points in the direction of the rod, then it has two components along with x & y axises. Because it's a unit vector, $\boldsymbol{U_R}$ can be written as above.

Then we do some operation of $\boldsymbol{U_R}$ following:

$$\frac{dU_R}{d\theta}=\frac{d\begin{bmatrix}+\sin\theta\\-\cos\theta\end{bmatrix}}{d\theta}=\frac{\begin{bmatrix}d\sin\theta\\-d\cos\theta\end{bmatrix}}{d\theta}=\frac{\begin{bmatrix}\cos\theta d\theta\\\sin\theta d\theta\end{bmatrix}}{d\theta}=\frac{d\theta\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}}{d\theta}=\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}=U_T$$

$$\frac{d^2U_R}{d\theta^2}=\frac{d}{d\theta}\frac{dU_R}{d\theta}=\frac{d}{d\theta}\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}=\frac{\begin{bmatrix}d\cos\theta\\d\sin\theta\end{bmatrix}}{d\theta}=\begin{bmatrix}-\sin\theta\\\cos\theta\end{bmatrix}=-U_R$$

We can prove that $\boldsymbol{U_T}$ is perpendicular to the rod actually. So $\boldsymbol{U_T}$ points tangent to the motion of the rod.

Proof:

$$U_RU_T=(\sin\theta,-\cos\theta)(\cos\theta,\sin\theta)=\sin\theta\cos\theta-\sin\theta\cos\theta=0$$

Thus $\boldsymbol{U_T}$ is perpendicular to $\boldsymbol{U_R}$.

Then we can sketch the pair $\boldsymbol{U_R}$ and $\boldsymbol{U_T}$ for such as $\theta=0,\pm\frac{\pi}{6},-\frac{\pi}{4},\pm\frac{\pi}{3},\pi,-\frac{4\pi}{3}$ with drawing $\boldsymbol{U_T}$ with its tail at the tip of $\boldsymbol{U_R}$ just like the picture above.

The next step in finding the equations of motion of the pendulum is to use some basic vector geometry. The gravity vector is simple in *x*-*y* coordinates, but we need to express it in radial and tangential components.

The magnitude of the gravitational force on the bob of mass *m* is *mg* and it acts down. The constant *g* is the universal acceleration due to gravity *g=9.8* in mkgs units. As a vector, the force due to gravity is

$$F_G=mg\begin{bmatrix}0\\-1\end{bmatrix}=\begin{bmatrix}0\\-mg\end{bmatrix}$$

Radial and Tangential Gravity

Decompose this vector $\mathbf{F}_G$ into the two vectors $\mathbf{F}_R$, the radial component of gravity, and $\mathbf{F}_T$, the tangential component of gravity. Express both vectors in terms of the vectors $U_R[\theta]$ and $U_T[\theta]$ defined above. Now we compute our decomposition

$$F_G=\begin{bmatrix}0\\-mg\end{bmatrix}=mg\begin{bmatrix}0\\-1\end{bmatrix}=mg\begin{bmatrix}-\cos\theta\sin\theta+\cos\theta\sin\theta\\-\sin^2\theta-\cos^2\theta\end{bmatrix}=mg\begin{bmatrix}-\cos\theta\sin\theta\\-\sin^2\theta\end{bmatrix}+mg\begin{bmatrix}\cos\theta\sin\theta\\-\cos^2\theta\end{bmatrix}$$
$$=-mg\sin\theta\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}+mg\cos\theta\begin{bmatrix}\sin\theta\\-\cos\theta\end{bmatrix}=-mg\sin\theta U_T+mg\cos\theta U_R=-F_TU_T+F_RU_R=F_T+F_R$$

The derivative of a general position vector with respect to *t* is the associated velocity vector. The second derivative of the position vector with respect to *t* is the acceleration vector.

Radial and Tangential Acceleration

Now we show that the velocity of the position $X[\theta[t]]$ is given by

$$\frac{dX}{dt}=\frac{d[LU_R[\theta[t]]]}{dt}=L\frac{d\theta[t]}{dt}U_T\left[\theta[t]\right]$$

We can verify this as follow by writing $\boldsymbol{X}$ in sine-cosine components.

$$\frac{d\overrightarrow{X}}{dt}=\frac{d\overrightarrow{X}[\theta[t]]}{dt}=\frac{d}{dt}L\overrightarrow{U_R}=L\frac{d\overrightarrow{U_R}}{dt}=L\frac{\overrightarrow{U_T}\cdot d\theta}{dt}=L\frac{d\theta[t]}{dt}\overrightarrow{U_T}\left[\theta[t]\right]$$

Calculate the second derivative of the position vector $X[\theta[t]]$ with respect to t and express the result in terms of the unit vectors $\boldsymbol{U_R}$ and $\boldsymbol{U_T}$. You will need to use the chain rule and product rule in symbolic form, since the angle θ depends on t and we do not have an explicit formula for $\theta[t]$. Write our answer in the form as follow:

$$A=\frac{d^2X}{dt^2}=\frac{d}{dt}\frac{dX}{dt}=\frac{d}{dt}\left[L\frac{d\theta[t]}{dt}U_T\left[\theta[t]\right]\right]=L\left(\frac{d}{dt}\frac{d\theta}{dt}\right)U_T+L\frac{d\theta}{dt}\left(\frac{d}{dt}U_T\right)$$

$$=L\frac{d^2\theta}{dt^2}U_T-L\frac{d\theta}{dt}\frac{U_R d\theta}{dt}=A_TU_T-L\left(\frac{d\theta}{dt}\right)^2U_R=A_TU_T+A_RU_R=\overrightarrow{A_T}+\overrightarrow{A_R}$$

We can now use $\boldsymbol{F=mA}$ to determine the equation of motion of the pendulum. We first need to note that the rod from which the bob is suspended provides a force that counteracts the radial component of gravity and prevents any acceleration in the radial direction. Hence when we equate $\boldsymbol{F}$ and $m\boldsymbol{A}$, the radial component of the acceleration is exactly balanced by the force of tension in the rod and the two expressions cancel. Thus to determine the equation of motion of the pendulum we need only equate the expressions m times the tangential component of acceleration and the tangential component of gravity, $m\boldsymbol{A}_T=\boldsymbol{F}_T$.

Combining Results

Equating these two expressions and using the formulas above we can derive the pendulum equation:

By Newton Second law in component: $m\boldsymbol{A}_T=\boldsymbol{F}_T$ Thus

$$mL\frac{d^2\theta}{dt^2}U_T=mA_T=F_T=-mg\sin\theta U_T$$

Both sides cancel the m and U_T, we get

$$L\frac{d^2\theta}{dt^2}=-g\sin\theta$$

i.e.

$$\frac{d^2\theta}{dt^2}+\frac{g}{L}\sin\theta=0$$

We can see that no matter how much height of the bob the equation of motion for the pendulum are the same if other conditions is same. Physically, the equation of motion for the pendulum does not depend on mass.

3. Numerical Solutions of the Pendulum Equation

In order to solve second-order differential equations numerically, we must introduce a phase variable. If we let $\phi=\frac{d\theta}{dt}$, then the pendulum equation can be written as the system of differential equations:

$$\phi = \frac{d\theta}{dt}$$
$$\frac{d\phi}{dt} = -\frac{g}{L}\sin\theta$$

This system can then be solved by the computer program **Mathematica**. Also, this system is autonomous. That means that we can use the **Flow2D** program to study the behavior of the pendulum.

Equilibria

the nadir of the system is stable while the roof is unstable because it has gravity in fact if we put the pendulum near the top we will find that the bob will be not stable for if there exists a small change it will disobey the trajectory.

Figure 46.2 illustrates the phase plane trajectories of solutions to the system of differential equations for various initial conditions. The displacement angle θ is plotted on the horizontal axis, and the angular velocity ϕ is plotted on the vertical axis.

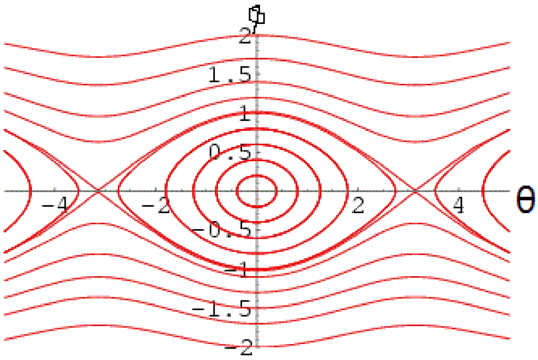

Figure 46.2: Phase Diagram for the Pendulum Equation

An Invariant for the Pendulum

Now we calculate an invariant for the pendulum motion by forming the ratio

$$\frac{d\phi}{d\theta} = \frac{\frac{d\phi}{dt}}{\frac{d\theta}{dt}} = \frac{-\frac{g}{L}\sin\theta}{\phi} = -\frac{g}{\phi L}\sin\theta$$

canceling *dt*, separating variables, and integrating. The we get

$$\phi d\phi = -\frac{g}{L}\sin\theta d\theta \Rightarrow \int \phi d\phi = \int -\frac{g}{L}\sin\theta d\theta \Leftrightarrow \frac{\phi^2}{2} = \frac{g}{L}\cos\theta + C$$

Thus

$$E\left[\phi[t],\theta[t]\right] = \frac{L}{2}\phi^2[t] + g\left(1-\cos\left(\theta[t]\right)\right) = cL + g = constant$$

Make a contour plot of the function $E[\phi,\theta]$.

In Mathematica

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Plot3D[1/2 y^2+10(1-Cos[x]), {x,-10,10}, {y,-10,10}]
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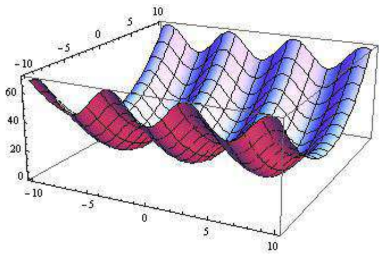
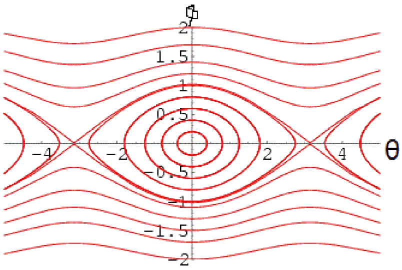
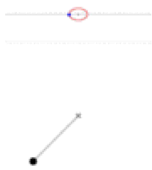
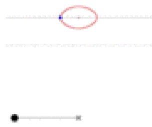
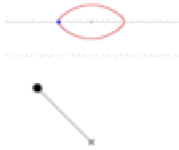
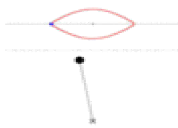
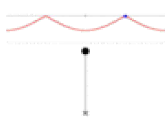


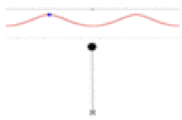
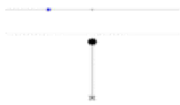
What's more,we have get phase figure 46.2 above and moving to here.



As you can see from the phase plane there are two types of motion that the pendulum displays, the motion corresponding to the closed flow trajectories and that corresponding to the oscillating flow curves. physically if the motion that do **part circular motion** is occurring when the flow trajectories are closed curves. The motion that no place its velocity is 0 occurring when the flow trajectories oscillate without closing on themselves i.e. the bob do **circular motion**.

We conclude several situation below by tables:

Initial angle of 0°	 Initial angle of 45°	 Initial angle of 90°
 Initial angle of 135°	 Initial angle of 170°	 just barely enough energy for a full swing

 Pendulum with enough energy for a full swing	 Initial angle of 180°, unstable equilibrium.	

We can also try some calculations with various lengths. A 9.8 m pendulum is several stories high and oscillates rather slowly. It would be difficult to build such a large pendulum and have it swing through a full rotation.

The Period of the Pendulum

We would like to have a formula for the period, so we imagine the situation where we release the pendulum from rest at an angle $\alpha < \pi/2$. For the time period while θ is a decreasing function, say one-quarter oscillation, $0 \le \theta \le \alpha$, we could invert the function $\theta[t]$, $t = t[\theta]$ and note that $\frac{1}{\frac{d\theta}{dt}} = \frac{dt}{d\theta} = \frac{1}{\phi}$.) Now we proceed with some more clever tricks.

Then we compute the derivative

$$\frac{d\left(\phi^2\right)}{d\theta} = 2\phi\frac{d\phi}{d\theta} = 2\phi\frac{d\phi}{dt}\frac{dt}{d\theta} = 2\phi\frac{d\phi}{dt}\frac{1}{\phi} = 2\frac{d\phi}{dt} = 2\frac{d^2\theta}{dt^2}$$

Use the pendulum equation and integrate with respect to θ

$$\frac{d\left(\phi^2\right)}{d\theta} = -2\frac{g}{L}\sin\theta$$

$$\phi^2 = 2\frac{g}{L}\int_\alpha^\theta(-\sin\theta)d\theta = 2\frac{g}{L}[\cos\theta - \cos\alpha]$$

$$thus\ \frac{d\theta}{dt} = \phi = -\sqrt{\frac{2g}{L}}\sqrt{\cos\theta - \cos\alpha}$$

Finally, separate variables and integrate to the bottom of the swing, one-quarter period,

$$dt = -\sqrt{\frac{L}{2g}}\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}$$

$$T = 4\int_\alpha^0 dt[\theta] = -4\sqrt{\frac{L}{2g}}\int_\alpha^0\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}$$

This integrand is discontinuous at $\theta = \alpha$ and may cause your computer trouble. We do some trig,

$$\cos\theta = 1 - 2\sin^2\left(\tfrac{\theta}{2}\right)$$
$$\cos\alpha = 1 - 2\sin^2\left(\tfrac{\alpha}{2}\right)$$

Now

$$-\frac{1}{\sqrt{2}}\int_{\alpha}^{0}\frac{d\theta}{\sqrt{\cos\theta-\cos\alpha}} = -\int_{\alpha}^{0}\frac{1}{\sqrt{\sin^2\left(\frac{\alpha}{2}\right)-\sin^2\left(\frac{\theta}{2}\right)}}\frac{d\theta}{2} = \int_{0}^{\frac{\pi}{2}}\frac{1}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\psi}}d\psi$$

with the change of variables $\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\sin\psi$ and differentials $\cos\frac{\theta}{2}\frac{d\theta}{2} = \sin\frac{\alpha}{2}\cos\psi d\psi$, with $\psi = 0 when \theta = 0 \,\&\, \psi = \frac{\pi}{2} when \sin\frac{\theta}{2} = \sin\frac{\alpha}{2}$, $using\cos\psi = \sqrt{1-\sin^2\psi}$, we get finally,

$$T(\alpha) = 4\sqrt{\frac{L}{g}}\int_{0}^{\frac{\pi}{2}}\frac{1}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\psi}}d\psi = 8\sqrt{\frac{1}{1-\sin^2\left(\frac{\alpha}{2}\right)}}\sqrt{\frac{L}{g}}EllipticF\left[\frac{\pi}{4},-\frac{2\sin^2\frac{\alpha}{2}}{1-\sin^2\frac{\alpha}{2}}\right]$$

This is an "elliptic integral of the first kind," or in ***Mathematica*** jargon, "EllipticF." The integral cannot be expressed in terms of elementary functions, but the computer can work with this expression perfectly well.

4. Linear Approximation to the Pendulum Equation

If the displacement angle θ is small, then $\sin\theta \approx \theta$ and we can approximate the pendulum equation by the simpler differential equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

This is a second-order linear constant-coefficient differential equation and can be solved explicitly for given initial conditions.

The Linearized Pendulum's Period

We can show that the period of the linearized pendulum is a constant $2\pi\sqrt{\frac{L}{g}}$.
The true period of the pendulum differs from this amount more and more as we increase the initial release angle.

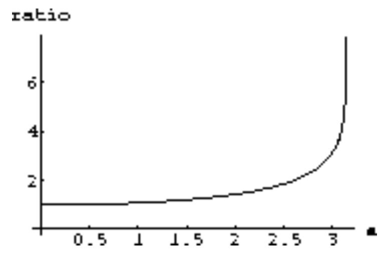


Figure 46.3: The Ratio $T[\alpha]/(2\pi\sqrt{\frac{L}{g}})$

Given initial values $\theta=\theta_0\ \&\ \frac{d\theta}{dt}=\phi_0$, we solve the linearized pendulum equation symbolically.

its characteristic equation *is*

$$\lambda^2+\frac{g}{L}=0$$

$$solution\ \ \theta=C\sin\left(\sqrt{\frac{g}{L}}t+\varphi_0\right)$$

Thus T= $2\pi/\sqrt{\frac{g}{L}}=2\pi\sqrt{\frac{L}{g}}$

if $\phi_0=0$,then $\theta=C\sin\left(\sqrt{\frac{g}{L}}t\right)$. We may find that the period of the pendulum does not depend on the angle θ_0.

As we demonstrated above the period of the linearized pendulum is a constant independent of the initial displacement (and mass). This should give us some idea of why pendula are such accurate timekeepers. If the pendulum is released at a certain angle to start and friction causes the amplitude of the swings to diminish, it still swings back and forth in the same amount of time. In a clock, friction is overcome by giving the pendulum a push when it begins to slow down. The push can come from weights descending, springs uncoiling, or a battery.

The comments about the invariance of period apply to the linearized model of the pendulum. That model is not accurate for large initial displacements.

Comparison of Periods for Explicit Solutions

The period of the linearized pendulum is a constant $2\pi\sqrt{\frac{L}{g}}$.

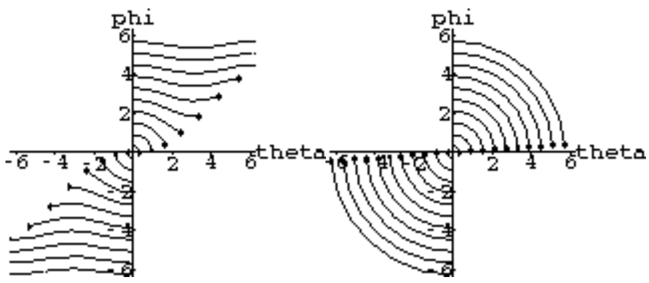
- Flow of the Linearized Pendulum

  Use the computer program **Mathematica** to solve the equations

$$\frac{d\theta}{dt}=\phi$$

$$\frac{d\phi}{dt}=-\frac{g}{L}\theta$$

Then we can draw the Flow picture as follow



Nonlinear and Linear Pendula

Comparing the flow of the linearized model to the flow of the "real" (rigid rod frictionless) pendulum, we find that the dots of current state remain in line for the linear equation and not remain in line for the nonlinear flow. It's because in fact as the theta0 is big enough phi can not be 0 for it does circular motion.What's more, some close curves are not circle,for it's no longer a simple pendulum.

The basis for our linear approximation to the equation of motion for the pendulum is the approximation $\sin\theta \approx \theta$.

The Sine Approximation

Recall the increment equation defining the derivative

$$f[x+\delta x] = f[x] + f'[\mathrm{x}] \bullet \delta x + \varepsilon \cdot \delta x$$

where $\varepsilon \approx 0$ if $\delta x \approx 0$. Use this equation with *x=0*, $\delta x = \theta$ and $f[x] = \sin x$ to give a first approximation to the meaning of "$\sin\theta \approx \theta$." In the Mathematical Background there is a higher order increment equation, Taylor's second-order "small oh" formula,

$$f[x+\delta x] = f[x] + f'[\mathrm{x}] \bullet \delta x + \frac{1}{2} f''[x] \bullet \delta x^2 + \varepsilon \bullet \delta x^2$$

Substitute into this equation with *x=0*, $\delta x = \theta$, and $f[x] = \sin x$ to give us a better approximation to the meaning of "$\sin\theta \approx \theta$". The Taylor formula can be used to show that

$$\left| f[x+\delta x] - f[x] + f'[\mathrm{x}] \bullet \delta x \right| \le \frac{1}{2} Max[| f''[\xi] | : x - \delta x \le \xi \le x + \delta x] \bullet \delta x^2$$

Then take $f[x] = \sin x$ & *x=0*, $\delta x = \theta$,we get

$$|\sin\theta - \theta| \le \frac{\theta^2}{2} Max[| -\sin\xi | : -\theta \le \xi \le \theta] = \frac{\theta^2}{2}\left(-\sin(-\theta)\right) = \frac{\theta^2}{2}\sin\theta \le \frac{\theta^2}{2}$$

If a pendulum swings no more than 10°, the error between sin θ and θ is less than

$$\frac{\left(\frac{\pi}{36}\right)^2}{2}\sin\frac{\pi}{36} \approx 0.000331864$$

The relative error compared to the maximum angle is

$$\frac{0.000331864}{\frac{\pi}{36}} \approx 0.00380289$$

5. The Period of the "Real" Pendulum

Use the computer program ***Mathematica*** *to solve the equations*

$$\frac{d\theta}{dt} = \phi$$
$$\frac{d\phi}{dt} = -\frac{g}{L}\sin\theta$$

More over we can find T by program:

By the Binomial Theorem:

$$\left(1 - k^2\sin^2(\varphi)\right)^{-1/2} = 1 + \frac{1}{2}k^2\sin^2\varphi + \left(\frac{1\cdot 3}{2\cdot 4}\right)k^4\sin^4\varphi + \cdots + \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}k^{2n}\sin^{2n}\varphi\cdots,$$

and

$$\int_0^{\pi/2}\sin^{2n}\varphi\, d\varphi = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\frac{\pi}{2},$$

Thus we can get

$$T = 2\pi\sqrt{\frac{\ell}{g}}\left(1 + \left(\frac{1}{2}\right)^2\sin^2\left(\frac{\theta_0}{2}\right) + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\sin^4\left(\frac{\theta_0}{2}\right) + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\sin^6\left(\frac{\theta_0}{2}\right) + \cdots\right)$$
$$= 2\pi\sqrt{\frac{\ell}{g}}\cdot\sum_{n=0}^{\infty}\left[\left(\frac{(2n)!}{(2^n\cdot n!)^2}\right)^2\cdot\sin^{2n}\left(\frac{\theta_0}{2}\right)\right].$$

$$T = 2\pi\sqrt{\frac{\ell}{g}}\left(1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \frac{173}{737280}\theta_0^6 + \frac{22931}{1321205760}\theta_0^8 + \frac{1319183}{951268147200}\theta_0^{10} + \frac{233526463}{2009078326886400}\theta_0^{12} + \ldots\right)$$

6. Friction in the Pendulum (Optional)

A common observation is that the periodic motion of a pendulum gradually diminishes and that, eventually, a pendulum will come to rest. In fact all large systems that are not continually energized lose energy . There is no such thing as perpetual motion. In a mechanical system, the causes of energy loss are multitudinous, and include friction between solid surfaces and drag on the system from fluids and gases. Objects that travel through the atmosphere or other "fluids" at relatively high speeds experience a drag force that is proportional to the square of the velocity.

Furthermore, other frictional effects for real pendulums seem also to be accounted for with the same velocity dependence. For now, we approximate the aggregate effect of drag or friction for typical pendulum as

$$F_D = -2\gamma \frac{d\theta}{dt}$$

where γ is a constant. The equation of motion for the linearized pendulum now has an added term and becomes

$$\frac{d^2\theta}{dt^2} + 2\gamma \frac{d\theta}{dt} + \frac{g}{L}\theta = 0$$

The general solution of this second-order differential equation has the form

$$\theta[t] = Ae^{r_1 t} + Be^{r_2 t}$$

where A and B are constants to be determined from initial conditions, and r1 and r2 are the two roots of the quadratic equation created by substitution of a trial solution, e^rt , into Eq.

$$r^2 + (2\gamma)r + \frac{g}{L} = 0$$

$$r_1 = -\gamma + \sqrt{\gamma^2 - \frac{g}{L}}$$

$$r_2 = -\gamma - \sqrt{\gamma^2 - \frac{g}{L}}$$

The angular frequency of undamped oscillations is $\omega_0 = \sqrt{\frac{g}{L}}$. The strength of the damping now determines the time-dependent pendulum behaviour. Three distinct regimes exist like the situation in damped motion.

7. The Spring Pendulum (Optional)

Now we consider a mass suspended from a spring instead of a rod. During the motion of the pendulum we will assume that the spring remains straight. The force of gravity still acts on the mass and can be resolved into tangential and radial components. There is also a force now due to extension of the spring. If the spring remains straight it will act in the radial direction, but its effect is dependent on whether the spring is stretched or compressed. Figure 46.4 illustrates the forces acting on the mass and their resolution into radial and tangential components.

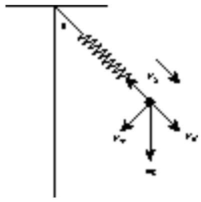


Figure 46.4: The Spring Pendulum

As with the simple pendulum, let $X[\theta[t]]$ represent the position vector of the mass. The length of the pendulum now varies with time. LetL(t) represent the length of the pendulum at time t. Express the vector $X[\theta[t]]$ in terms of *L(t)* and the unit vector $\boldsymbol{U_R}$ defined above. The force from the spring obey's Hooke's law. If the unstretched length of the pendulum is L_0, then the magnitude of the spring force is $k(L(t)-L_0)$, where k is the spring constant. Express the spring force, $\boldsymbol{F_S}$, in terms of the unit vector $\boldsymbol{U_R}$ defined above. Resolve the force of gravity into radial and tangetial components, $\boldsymbol{F_R}$ and $\boldsymbol{F_T}$ as for the simple pendulum.

The second derivative of the position vector with respect tot is again the acceleration vector. Differentiate the expression for $X[\theta[t]]$ twice with respect to t to obtain the acceleration vector. Express the acceleration vector in terms of the unit vectors $\mathbf{U}_R$ and $\mathbf{U}_T$. Be very careful. In this case both the length of the pendulum and the displacement angle are functions of t.

We again want to use Newton's law to determine the equation of motion of the pendulum. The total force acting on the mass is$\mathbf{F}_T+\mathbf{F}_R+\mathbf{F}_S$. In the exercise above, you have expressed this in terms of the unit vectors $\boldsymbol{U_R}$ and $\boldsymbol{U_T}$. You have also expressed the acceleration vector in terms of $\boldsymbol{U_R}$ and $\boldsymbol{U_T}$. Equate $\boldsymbol{F}$ and $m\boldsymbol{A}$ to obtain the equations of motion for the spring pendulum:

$$\frac{d^2L}{dt^2} - L\left(\frac{d\theta}{dt}\right)^2 - g\cos\theta + \frac{k}{m}(L - L_0) = 0$$

$$L\frac{d^2\theta}{dt^2} + 2\frac{dL}{dt}\frac{d\theta}{dt} + gL\sin\theta = 0$$

In order to solve the system of second-order differential equations numerically, we must again introduce phase variables to reduce it to a system of first-order differential equations. If we let $y_1=L$, $y_2 = \frac{dL}{dt}, y_3 = \theta, \& y_4 = \frac{d\theta}{dt}$, then the system becomes:

$$y_1' = y_2$$

$$y_2' = y_1 y_4^2 + g\cos y_3 - \frac{k}{m}(y_1 - L_0)$$

$$y_3' = y_4$$

$$y_4' = -2y_2 y_4 / y_1 - \frac{g\sin y_3}{y_1}$$

Associated with the spring pendulum are two "natural frequencies," $\omega_1 = \sqrt{\frac{g}{L_0}}$ & $\omega_2 = \sqrt{\frac{k}{m}}$. If the ratio of these frequencies is a rational number like 1, 2, or $\frac{1}{2}$, then the vibrations of the pendulum can exchange energies.

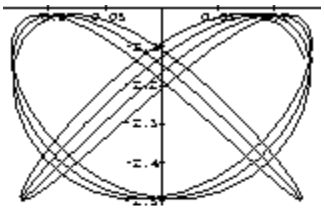


Figure 46.5: A Lissajous

To get a better idea of how the vibrations change energy, we can plot the path that the bob takes as it moves on the end of the spring. The resulting graph will be what is sometimes called a lissajous. A lissajous is generated by plotting a superposition of two periodic motions. In this instance the two periodic motions are the vibration of the spring and the swinging of the pendulum. Figure 46.5 illustrates a sample path of the pendulum bob.

Preferences:

[1]GREGORY L. BAKER,The Pendulum:A Case Study in Physics,Oxford University Press 2005

[2]wolfram scienceworld , http://scienceworld.wolfram.com/physics/Pendulum.html , *1996-2007 Eric W. Weisstein*

[3]wikipedia , http://en.wikipedia.org/wiki/Pendulum_%28mathematics%29 , last modified on 9 June 2011 at 05:12