Project 39: Logistic Growth with Hunting

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Abstract:
A field of logistically growing mice is visited by Voodoo the barn cat. This project begins with a review of some of the basic ideas we have learnt. We discover the logistic model by using the example of mice and its hunt.

Keywords:
Logistic Growth, Basic Fertility, Voodoo, hunting

1. Basic Fertility

Each spring, mice reproduce prolifically in the field behind my house. One Monday before work, I counted about 1000 mouse pairs per acre. When I came home the mouse census had increased to 1100 per acre.

VARIABLES

t=time in days
x=mouse couples per acre

PARAMETERS

We begin with the single parameter *r*, the per capita birth rate of mice.

- The differential equation

$$\frac{dx}{dt} = rx$$

The instantaneous per capita birth rate of mice is *r*. So the increasing rate is r , the increment is rx & we have the above equation.

- the solution to the initial value problem

$$x[0] = 1000$$
$$\frac{dx}{dt} = rx$$

is $x[t]=1000e^{rt}$.

Proof:

$$\frac{dx}{dt} = rx$$
$$\frac{dx}{x} = rdt$$
$$\ln x = rt + C$$
$$x = e^{rt+C} = Ce^{rt}$$
$$substitute\_x[0]$$
$$we\_get\_\_x[t] = 1000e^{rt}$$

- Use the information x=1100 in 8 hours,we can find that

$$1000e^{r/3} = 1100$$

then r≈0.2859305394

When the mouse population is small compared with the food and shelter available to them in my small 7-acre field, the birth rate can continue at the per capita rate r≈0.2859305394. However, this cannot continue for even 1 month.

2. Logistic Growth

ANOTHER PARAMETER: *c*

We introduce a parameter *c* called the carrying capacity of the ecosystem. Our introduction is mathematical, and your job is to explain the biological significance of this parameter.

The former owner of my field said he noticed that the birth rate of the mice dropped off sharply each spring as the population density reached about *5000* mouse couples per acre. When the population is small, the basic fertility of mice is the constant *r*,

$$\frac{1}{x}\frac{dx}{dt} = r$$

- but as *x* grows toward *5000*, food and shelter become difficult for the mice and the per capita growth declines toward zero.

- The Logistic Growth Law

the differential equation

$$\frac{1}{x}\frac{dx}{dt} = r\left(1 - \frac{x}{c}\right)$$

says that the per capita rate of growth of mice is a decreasing function with a basic fertility rate of *r* but a limiting population of *c* mouse couples per acre. We can show that the limit of *x* as *t* tends to infinity is *c by phase line*,

$$\lim_{t\to\infty} x[t] = c$$

39.3 Voodoo Discovers the Mice

Voodoo the barn cat came with our property. I believe he feels that HE owns the property and has to tolerate a new tenant. He lets us feed and pet him and seems to like the new insulated cat house I built. But hunting is Voodoo's life. We prefer that he leave the song birds alone, but when the mice start coming in the house, we're grateful for his diligence.

Voodoo has noticed all the mouse activity in the back field and has decided to concentrate his efforts there. Being well-fed and preferring rabbits anyway, he often brings us some of his extra catch, so we have some idea of his mousing success.

ANOTHER PARAMETER:

We notice that Voodoo's catch increases with increasing density of mice, so we want to explore some descriptions of his impact on the mouse population. We introduce another parameter h - Voodoo's hunting success rate. Our first try will be hunting success that increases linearly with population.

- Linear Hunting

Suppose Voodoo's hunting success rate is a linear function of mouse density,

$$\frac{dx}{dt} = rx\left(1 - x / c\right) - hx$$

The biological meaning of the term hx is the hunted amount of the mice,and h is the hunted rate.
We can show that

$$\lim_{t \to \infty} x[t] = c\left(1 - h / r\right)$$

If x[t] has a limit,then dx/dt→0， let dx/dt=0,then we get x=c(1-h/r)

Discuss the importance of the ratio *h/r*.
If h/r>=1,then the specie may die out;
If h/r<1,then the specie may not die out.

The previous hint assumes that Voodoo will be proportionately as successful at low densities as he is at high densities. Here is another possible model of hunting success:

- Nonlinear Hunting

Suppose the mouse population is affected by Voodoo as follows,

$$\frac{dx}{dt} = rx\left(1 - x / c\right) - hx^2$$

We can also show that

$$\lim_{t \to \infty} x[t] = \frac{c}{1 + h / r}$$

For

$$\frac{dx}{dt} = rx\left(1 - x / c\right) - hx^2$$ equals to 0

Then,we get

$$x = \frac{c}{1 + h / r}$$

Notice that the linear hunting model predicts that Voodoo can hunt the mice to extinction, whereas the nonlinear one does not. Because, when x get smaller enough, the x^2 get much smaller than x. X^2 is the less step infinitesimal.