Recovering implied local volatility of options

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Abstract. In this paper, we studied two papers from Jiang Lishang ([4],[2]), who use well-posed algorithms to recover implied volatility under the Black-Scholes theoretical framework. Then, we did some numerical experiments which show that, with the help of the techniques developed in the field of optimal control, the local volatility function is recovered very well.

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1. Introduction

In our assumption during deriving the Black-Scholes formula, we have assumed that the volatility of the underlying asset is a constant which doesn't depend on drift. But in fact, the volatility value for the underlying asset of an option is not directly observable, and is certainly unpredictable. However, we can recovering it from options market. More precisely, if the Black-Scholes option pringcing theory is correct, then the price of options should reflect and reveal the price volatility of the underlying asset. And that is called the **implied volatility**[3].

In fact, the implied volatility σ inferred from options with different strike prices and expiration dates is a function of K,T, i.e. $\sigma=\sigma(K,T)$. (This should assume that t_0 and S_0 are equal.) Then, for a given T, the volatility σ varies with strike price K typically in one of two ways: a **volatility smile curve** or a **volatility skew curve**. Also, if we fix K, then we have a similar figure to the skew curve. Hence, we know that σ is not a constant.

A more realistic model is: σ is a function of t and asset price S. Then the stocahstic process of the asset price movement under the risk neutral measure is

$$\frac{\mathrm{d}S}{S}=(r-q)\,\mathrm{d}t+\sigma(S,t)\,\mathrm{d}W_t \tag{1}$$

Correspondingly, the Black-Scholes equation becomes

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 V}{\partial S^2}+(r-q)S\frac{\partial V}{\partial S}-rV=0 \tag{2}$$

In general, we have no explicit solution of this equation, usually, we use numerical methods to solve it. Here arise our question: how can we determine the implied volatility if we have the imformation in t_0? Here we consider an European call option, which is the problem Q as follows:

Problem 1. Let $V=V(S,t;\sigma,K,T)$ be a call option price, satisfying

$$\begin{aligned}\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 V}{\partial S^2}+(r-q)S\frac{\partial V}{\partial S}-rV &= 0, \quad (0\leqslant S<\infty, 0\leqslant t<T) && (3)\\ V(S,T) &= (S-K)^+ \quad (0\leqslant S<\infty) && (4)\end{aligned}$$

Suppose at $t=t^\star, (0\leqslant t^\star<T_1), S=S^\star$,

$$V(S^\star,t^\star;\sigma,K,T)=F(K,T) \quad (0<K<\infty, T_1\leqslant T\leqslant T_2)$$

is given, find $\sigma=\sigma(S,t)$, $(0\leqslant S<\infty, T_1\leqslant t\leqslant T_2)$

2. Dupire Method

2.1. Dupire's formula.

The inverse problem (1) was first considered by Dupire in [1]. He showed that option prices given for all possible strikes and maturities completely determine the local volatility function.

Let $V=V(S,t;K,T)$ be a european call option price, and define

$$\frac{\partial^2 V}{\partial K^2}=G(S,t;K,T) \tag{5}$$

From the equations (3, 4), G satisfies

$$\begin{cases}\frac{\partial G}{\partial t}+\frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 G}{\partial S^2}+(r-q)S\frac{\partial G}{\partial S}-rG=0\\ G(S,t)=\delta(K-S)=\delta(S-K)\end{cases} \tag{6}$$

This is the definition of fundamental solution of Black-Scholes eqation (6) at $t=T$. Then by the theorem given by Ladyzenskaya etc., it is also the fundamental solotion of the adjoint equation of the problem given by (6), i.e.

$$\begin{cases}-\frac{\partial G}{\partial T}+\frac{1}{2}\frac{\partial^2}{\partial K^2}(\sigma^2(K,T)K^2G)+(r-q)\frac{\partial}{\partial K}(KG)-rG=0 & (0\leqslant K<\infty, t\leqslant T)\\ G(S,t;K,t)=\delta(K-S)\end{cases} \tag{7}$$

After substituting (5) into equation (7), and simplifying some terms in (7), we have a Cauchy problem as follows:

$$\begin{cases} -\frac{\partial V}{\partial T}+\frac{1}{2}K^2\sigma^2(K,T)\frac{\partial^2 V}{\partial K^2}+(r-q)K\frac{\partial V}{\partial K}-qV=0 \\ V|_{T=t}=(S-K)^+ \end{cases} \quad (0\leqslant K<\infty, t\leqslant T) \tag{8}$$

From equation (8), we get

$$\sigma(K,T)=\sqrt{\frac{\frac{\partial V}{\partial T}+(r-q)K\frac{\partial V}{\partial K}+qV}{\frac{1}{2}K^2\frac{\partial^2 V}{\partial K^2}}} \tag{9}$$

However, this solution is ill-posed since a small error in F can result in big changes in its derivatives, especially the second derivatives. Dupire's formula (9) is found to be difficult to apply in practice. It therefore needs to be improved.

2.2. Duality Problem.

Using the duality skill, we have reduced problem 1. to a typical terminal state observation problem.

Problem 2. Let $V=V(S,\ t;\ \sigma,\ K,\ T)$ be a call option price, and suppose at t, $S=S_0$, $V(K,T;S_0,0)=F(K,T)$ $\quad(0<K<\infty, T_1\leqslant T\leqslant T_2)$. Find $\hat{\sigma}(K, T)=\begin{cases} \sigma_0(K) & 0\leqslant T\leqslant T_1 \\ \sigma(K,T) & T_1\leqslant T\leqslant T_2\end{cases}$, $(0<K<\infty)$, s.t. the Dupire equation (where $\tau=T-t$):

$$\begin{cases} -\frac{\partial V}{\partial \tau}+\frac{1}{2}K^2\hat{\sigma}^2(K,T)\frac{\partial^2 V}{\partial K^2}+(r-q)K\frac{\partial V}{\partial K}-qV=0 \\ V|_{T=0}=(S_0-K)^+ \end{cases} \quad (0\leqslant K<\infty, t\leqslant T) \tag{10}$$

has the solution $V(S,K;S_0,0)=F(K,T)$.

By transformation $y=\ln\frac{K}{S_0}$, $v=\frac{1}{S_0}e^{q\tau}V$, $\tau=T-t$, $v(y,\tau)=\frac{1}{S_0}V(S_0e^y,\tau)$, $v'(y,\tau)=\frac{1}{S_0}F(S_0e^y, \tau)$, $a(y)=\frac{1}{2}\sigma_0^2(S_0e^y)$, $a(y,\ \tau)=\frac{1}{2}\sigma^2(S_0e^y,\ \tau)$ and simplify the Problem 2, finally, we can get a problem decomposition as following two problems.

Problem 2.1. Find $a(y)\in A=\{y\in\mathbb{R}, 0<t\leqslant T_1\}$, such that

$$\begin{cases} \frac{\partial v}{\partial \tau}=a(y)\left(\frac{\partial^2 v}{\partial y^2}-\frac{\partial v}{\partial y}\right)-(r-q)\frac{\partial v}{\partial y} \\ v(y,0)=(1-e^y)^+ \end{cases} \tag{11}$$

has the solution $v(y,T_1;a)=v'(y,T_1)$ when $t=T_1$.

Problem 2.2. Find $a(y,\ \tau)\in B=\{y\in\mathbb{R},\ T_1\leqslant t\leqslant T_2\}$, with initial condition $a(y,T_1)=a(y)$, and such that

$$\begin{cases} \frac{\partial v}{\partial \tau}=a(y,\tau)\left(\frac{\partial^2 v}{\partial y^2}-\frac{\partial v}{\partial y}\right)-(r-q)\frac{\partial v}{\partial y} \\ v(y,T_1)=u(y,T;a_0) \end{cases} \tag{12}$$

has the solution $v(y,\tau;a)=v'(y,\tau)$.

3. REGULARIZATION METHOD

3.1. Regularization idea.

To solve the ill-posed problem, we take the regularization method which is proposed by A. N. Tikhonov in 1950s. Its main idea is as follows.

Let U,F be given metric spaces, A is an operator deffined on F, that is $A\colon F\to U$. The original problem: Given $V_0\in U$, find $\sigma_0\in F$, s.t. $A(\sigma_0)=V_0$.

There some two possibilities:

1. We know that $AF\subset U$, but $AF\neq U$, hence above equation may not have solution in F.

2. According to the given V_0, the above equation may have solution σ_0, but it is unstable. That's, σ_0 has no continous depedence on V_0. We can say that the small change in V_0 in U may lead to a big change of σ_0 in F.

Then main idea of the regularization method is to use a cluster of well-posed problems to take place of the original problem. Although it is just an approximation solution to the original problem, the process of greeting the approximation solution is quite stable. We can achieve the approximation solution in computing and use this solution to instead of the original true solution. We call the well-pose problems as the "regularization problem". It is often from the operator A and takes the parameter N to implement it.

Regularization Problem: Given $V_0 \in U$, find $\sigma_\alpha \in F(N>0)$, s.t.

$$J_\alpha(\sigma_\alpha) = \min_{\sigma \in F} J_\alpha(\sigma) \tag{13}$$

where $J_\alpha(\sigma) = \rho^U(A\rho, V_0) + N\mathcal{R}(\sigma)$, ρ^U is the distance in space U.

We call $N\mathcal{R}(\sigma)$ the regularizatoin operator, and α the rehularization factor. The regularization method is that solving the well-posed problem (13) by using the solution σ_α instead of σ_0. The α can be any number. When $\alpha = 0$, $\sigma_\alpha = \sigma_0$ we can get the true solution, but the algorithm is unstable. When α becomes bigger, σ_α will get more far from the true solution σ_0. So in practice, we try our best to take the α smaller then we can make the process more stable.

3.2. Regularization version of our problem.

Now we go back to the problems 2.1 and 2.2. Firstly we take the regularization method into the problem 2.1 and we call it problem Q_0 as follows:

Problem $\boldsymbol{Q_0}$: Find $\bar{a}\,(y) \in \mathcal{A}$, such that $J_\alpha(\bar{a}\,) = \min_{a \in \mathcal{A}} J_\alpha(a)$, where

$$J_\alpha(a) = \frac{1}{2}\int_{\mathbb{R}} |v(y,\tau^\star;a) - v^\star(y)|^2 \mathrm{d}y + \frac{\alpha}{2}\int_{\mathbb{R}} \left|\frac{\mathrm{d}a}{\mathrm{d}y}\right|^2 \mathrm{d}y \tag{14}$$

$$v^\star(y) = \frac{1}{S^\star} e^{q\tau^\star} F(S^\star e^y)$$

, $v(y,\tau;a)$ is the solution to the Cauchy problem 2.1, and

$$\mathcal{A} = \left\{ a(y) | 0 < a_0 \leqslant a(y) \leqslant a_1, \int_{\mathbb{R}} \left|\frac{\mathrm{d}a}{\mathrm{d}y}\right|^2 \mathrm{d}y < \infty \right\} \tag{15}$$

Here, a_0, a_1 are constants, $J_\alpha(a)$ is called the cost functional, $a = a(y)$ is called the control variable, and $\bar{a}\,(y)$ is called the optimal control or minimizer. The variational problem Q_0 is called the **optimal control problem**.

In Jiang's paper [4], he proved the following theorem:

Theorem 1. *The variational problem Q_0 has at least one minimizer $\bar{a}\,(y) \in \mathcal{A}$ of $J_\alpha(a)$, i.e. $J_\alpha(\bar{a}\,) = \min_{\alpha \in \mathcal{A}} J_\alpha(a)$.*

In order to get the regurization version of problem 2.2, we can make the time range $[T_1, T_2]$ discrete. Set $T_1 = \tau_0 < \tau_1 < \ldots < \tau_N = T_2$, $h = \frac{1}{N}(T_2 - T_1)$, $\tau_n = T_1 + nh$, $n = 0, 1, 2\ldots, N$. From the problem Q_0, when $\tau = \tau_0$ we can get the $\bar{a}\,(y)$ and then we use the induction method, then we can get $\bar{a}\,(y) = a(y, \tau_n)$, when $\tau = \tau_n$.

Problem $\boldsymbol{Q_n}$: Suppose we know $\bar{a}_0(y) = a(y)$,$\bar{a}_1(y)$, ..., $\bar{a}_{n-1}(y)$ and $v\big(t, \tau; \bar{a}_k\big)$, $(k = 0, 1, 2, \ldots, n-1)$which is the solution of the equations:

$$\frac{\partial v}{\partial \tau} = \bar{a}_k(y)\left(\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y}\right) - r\frac{\partial v}{\partial y}, \quad (y \in \mathbb{R}, \tau_{k-1} \leqslant \tau \leqslant \tau_k) \tag{16}$$

$$v(y, \tau_{k-1}) = v(y, \tau_{k-1}; \bar{a}_{k-1}), \quad (y \in \mathbb{R}) \tag{17}$$

Find $\bar{a}_n(y) \in \mathcal{A}$, such that $J_\alpha^n\left(\bar{a}_n\right) = \min_{a_n \in \mathcal{A}} J_\alpha^n(a_n)$, where

$$\begin{aligned} J_\alpha^n(a_n) &= \frac{1}{2h}\int_{\mathbb{R}} |v(y,\tau_n;a_n) - v^\star(y,\tau_n)|^2 \mathrm{d}y + \\ &\quad \frac{\alpha}{2}\left\{\frac{1}{h}\int_{\mathbb{R}} |a_n(y) - \bar{a}_{n-1}|^2 \mathrm{d}y + \int_{\mathbb{R}} \left|\frac{\mathrm{d}a_n}{\mathrm{d}y}\right|^2 \mathrm{d}y\right\} \end{aligned} \tag{18}$$

$$v^\star(y) = \frac{1}{S^\star} e^{q\tau^\star} F(S^\star e^y)$$

where $v(y,\tau_n;a_n)$ is the solution to equation (16,17).

Similarly, We also can prove that there exists a minimizer in variation problem Q_n. i.e. we have following theorem:

Theorem 2. *The variational problem Q_n has at least one minimizer $\bar{a}_n(y) \in \mathcal{A}$ of $J_\alpha^n(a)$, i.e. $J_\alpha^n(\bar{a}_n) = \min_{a_n \in \mathcal{A}} J_\alpha^n(a_n)$.*

Now we consider when $h \to 0$, how the $\{\bar{a}_n(y)\}$ will go. We fixx h, and define the function $a^h(y,\tau)$ and $v^h(y,\tau)$ as follows:

$$\begin{aligned} a^h(y,\tau) &= \begin{cases} \bar{a}_n(y) & \tau = \tau_n \\ \frac{\tau - \tau_{n-1}}{h}\bar{a}_n(y) + \frac{\tau_n - \tau}{h}\bar{a}_{n-1}(y) & \tau_{n-1} \leqslant \tau \leqslant \tau_n \end{cases} \\ v^h(y,\tau) &= v(y,\tau;\bar{a}_n), \quad \tau_{n-1} \leqslant \tau \leqslant \tau_n, n = 1,2,...N \end{aligned}$$

We can get a result as follows:

Theorem 3. *When $h \to 0$, $\{a^h(y,\ \tau),\ v^h(y,\ \tau)\}$ uniformly convergences to the limit function $\{a(y,\tau), v(y,\tau)\}$.*

Esitimation of $a^k(y,\tau)$ are constructed which are uniformly bounded and it is independent of the mesh parameter h, so we can take the limit for the approximate sequence $a(y,\tau)$ as $h \to 0$.

3.3. Necessary optimality condition.

We now turn to necessary optimality conditions which have to be satisfied by each optimal control $\bar{a}$.

Theorem 4. *If $\bar{a}$ be the solution of the optimal control problem Q_0, then there exists a triple of functions $\{\bar{a}\,(y), \bar{v}\,(y,\tau), \bar{\varphi}\,(y,\tau)\}$ satisfying the following system in the rigion $\{y \in \mathbb{R}, 0 \leqslant \tau \leqslant T_1\}$:*

$$\begin{aligned} \frac{\partial \bar{v}}{\partial \tau} &= \bar{a}\,(y)\left(\frac{\partial^2 \bar{v}}{\partial y^2} - \frac{\partial \bar{v}}{\partial y}\right) - (r-q)\frac{\partial \bar{v}}{\partial y} \\ \bar{v}\,|_{\tau=0} &= (1 - e^y)^+ \\ -\frac{\partial \bar{\varphi}}{\partial \tau} &= \frac{\partial^2(\bar{a}\,(y)\bar{\varphi}\,)}{\partial y^2} - \frac{\partial(\bar{a}\,(y)\bar{\varphi}\,)}{\partial y} - (r-q)\frac{\partial \bar{\varphi}}{\partial y} \\ \bar{\varphi}\,|_{\tau=T_1} &= \bar{v}\,(y,T_1) - v^\star(y,T_1) \end{aligned}$$

and

$$\int_0^{T_1}\int_{\mathbb{R}} \varphi(h-a)(\bar{v}_{yy} - \bar{v}_y)\mathrm{d}y\mathrm{d}\tau + \alpha\int_{\mathbb{R}} \nabla a \nabla(h-a)\mathrm{d}y \geqslant 0$$

for any $h \in A$.

We also have a similar theorem for Q_n problem, which we omit here.

3.4. A uniqueness result.

This is a series of forward-backward parabolic equations coupled with an elliptic variation inequality. Here we will prove the implied volatility $\bar{a}\,(y)$ uniquness.

Theorem 5. *Suppose $a_1(y)$, $a_2(y)$ be two minimizers of the modified optimal control problem Q1. If there exists a point y_0 such that $a_1(y_0) = a_2(y_0)$ and $\rho(y) \geqslant \rho_0 > 0$, $\int_{\mathbb{R}} \frac{\mathrm{d}y}{\rho(y)} < \infty$, then for small $\tau^\star$ we have*

$$a_1(y) \equiv a_2(y) \quad \textit{for any } y \in R$$

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