Title

Wenhao

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1 Lagrange method

1.1 Equation of motion

We define the Lagrangian[1] from the kinetic energy T and potential energy V as:

$$\mathcal{L}(x, \dot{x}, t) = T - V \tag{1}$$

and action $\mathcal{S}$ with the unit of energy $\times$ time:

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t) dt \tag{2}$$

with the initial condition $x(t_1) = x_0, \dot{x}(t_0) = \dot{x}_0$. The **equation of motion** can be determined by the **Principle of least action** which states that $\mathcal{S}$ will be a extrema (stationary point) for the dynamic of system from time t_1 to time t_2.

The equation of motion is obtained as:

$$\begin{aligned}
\delta\mathcal{S} &= \int_{t_1}^{t_2} \mathcal{L}(x + \delta x, \dot{x} + \delta\dot{x}, t) dt - \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t) dt \\
&= \int_{t_1}^{t_2} \left(\mathcal{L} + \frac{\delta\mathcal{L}}{\delta x}\delta x + \frac{\delta\mathcal{L}}{\delta\dot{x}}\delta\dot{x} \right) dt - \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t) dt \\
&= \int_{t_1}^{t_2} \left(\frac{\delta\mathcal{L}}{\delta x}\delta x + \frac{\delta\mathcal{L}}{\delta\dot{x}}\delta\dot{x} \right) dt \\
&= \int_{t_1}^{t_2} \left(\frac{\delta\mathcal{L}}{\delta x}\delta x + \frac{\delta\mathcal{L}}{\delta\dot{x}}\delta\dot{x} \right) dt \\
&= \frac{\delta\mathcal{L}}{\delta\dot{x}}\delta x\bigg|_1^2 + \int_{t_1}^{t_2} \left(\frac{\delta\mathcal{L}}{\delta x}\delta x - \frac{d}{dt}\left(\frac{\delta\mathcal{L}}{\delta\dot{x}}\right)\delta x \right) dt && (3) \\
& && (4)
\end{aligned}$$

where we used integral by parts to obtain the final equation. the boundary condition for choosing δx is that they are 0 at the initial coordinate and final coordinate, therefore, $(\delta\mathcal{L}/\delta\dot{x})\delta x|_1^2 = 0$. δx at other time can be choosen arbitrary. The requirement that $\mathcal{S} = 0$ thus lead to:

$$\frac{\delta\mathcal{L}}{\delta x} - \frac{d}{dt}\left(\frac{\delta\mathcal{L}}{\delta\dot{x}}\right) = 0 \tag{5}$$

In terms of multiply coordinates, we have:

$$\frac{\partial\mathcal{L}}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}_i}\right) = 0 \tag{6}$$

since the deviation of $\mathcal{L}$ is to first order in each coordiante.

[1] From David Morin's chapter in *Introduction to Classical Mechanics*

1.2 Change of coordinates

We consider changing coordinate x_i to q_i as:

$$q_i = \mathbf{q_i}(x_1, x_2, \cdots, x_N, t) \tag{7}$$

which does not depend on $\dot{x}$. Using the relationship:

$$\dot{x}_i = \sum_{i=1}^{N} \frac{\partial x_i}{\partial q_m} \dot{q}_m + \frac{\partial x_i}{\partial t} \tag{8}$$

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_m} = \frac{\partial x_i}{\partial q_m} \tag{9}$$

The equation of motion is given by:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_m}\right) &= \frac{d}{dt}\left(\sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial \dot{q}_m}\right) \\
&= \sum_{i=1}^{N} \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) \frac{\partial x_i}{\partial q_m} + \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_m}\right)\right] \\
&= \sum_{i=1}^{N} \left[\frac{\partial \mathcal{L}}{\partial x_i} \frac{\partial x_i}{\partial q_m} + \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial q_m}\right] \\
&= \sum_{i=1}^{N} \left[\frac{\partial \mathcal{L}}{\partial x_i} \frac{\partial x_i}{\partial q_m}\right] = \frac{\partial \mathcal{L}}{\partial q_m}
\end{aligned} \tag{10}$$

where in the final step, $\partial \dot{x}_i / \partial q_m = 0$ is from Eq.8. We can see that the equation of motion still hold after the change of coordinate [2].

1.3 Conservation law

Conserved quantity If $\mathcal{L}$ does not explicitly depend on coordinate q_k, then

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) = \frac{\partial \mathcal{L}}{\partial q_k} = 0 \tag{12}$$

thus, $\partial \mathcal{L} / \partial \dot{q}_k$ is a constant of motion.

Now, we define a quantity E as:

$$E = \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L} \tag{13}$$

[2] we show that $\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_m}\right) = \frac{\partial \dot{x}_i}{\partial q_m}$ is true:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_m}\right) &= \sum_{k=i}^{N} \frac{\partial}{\partial q_k}\left(\frac{\partial x_i}{\partial q_m}\right) \dot{q}_k + \frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial q_m}\right) \\
&= \sum_{k=i}^{N} \frac{\partial}{\partial q_k}\left(\frac{\partial x_i}{\partial q_m}\right) \dot{q}_k + \frac{\partial}{\partial q_m}\left(\frac{\partial x_i}{\partial t}\right) \\
&= \sum_{k=i}^{N} \frac{\partial}{\partial q_m}\left(\frac{\partial x_i}{\partial q_k}\right) \dot{q}_k + \frac{\partial}{\partial q_m}\left(\frac{\partial x_i}{\partial t}\right) \\
&= \frac{\partial}{\partial q_m}\left[\sum_{k=i}^{N}\left(\frac{\partial x_i}{\partial q_k}\right) \dot{q}_k + \left(\frac{\partial x_i}{\partial t}\right)\right] = \frac{\partial \dot{x}_i}{\partial q_m}
\end{aligned} \tag{11}$$

we can show that:

$$\begin{aligned}
\frac{dE}{dt} &= \sum_{i=1}^{N} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i\right) - \frac{d\mathcal{L}}{dt} \\
&= \sum_{i=1}^{N}\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i\right] - \left[\sum_{i=1}^{N}\left(\frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i\right) + \frac{\partial \mathcal{L}}{\partial t}\right] \\
&= -\frac{\partial \mathcal{L}}{\partial t}
\end{aligned} \tag{14}$$

so that if $\mathcal{L}$ does not explicitly depend on time t as $(\partial\mathcal{L}/\partial t = 0)$, than E is a constant of motion.

1.4 E and total energy

The quantity E is a constant of motion if $\mathcal{L}$ does not explicitly depend on time, but E is not necessarily the total energy of the system.

Theorem A necessary and sufficient condition for E to the the total energy of a system whose $\mathcal{L}$ is written in terms of a set of coordinates q_i is that these q_i are related to a cartesian set of coordinates x_i by:

$$x_i = \mathbf{x}_i(q_1, q_2, \cdots, q_N) \tag{15}$$

which does not include t or $\dot{q}$ dependence.

We use three example as argument to see this is indeed correct:

Example 1. consider a particle in a horizontal plane connected to the origin by a spring. the potential energy is $V = k(x^2+y^2)/2$ and the kinetic energy is $T = m(\dot{x}^2+\dot{y}^2)/2$.

$$\begin{aligned}
\mathcal{L} &= m(\dot{x}^2+\dot{y}^2)/2 - k(x^2+y^2)/2 \\
&= m(\dot{r}^2 + r^2\dot{\theta}^2) - kr^2/2
\end{aligned}$$

with $x = r\cos\theta,\ y = r\sin\theta$. In the coordinate (r,θ), we have $E = m(\dot{r}^2 + r^2\dot{\theta}^2) + kr^2/2$ is the total energy.

Example 2. consider similar to the above case, but now we have the coordinate transformation $x = r\cos(\omega t),\ y = r\sin(\omega t)$ depend on time t. The quantity E is still conserved but is no longer the total energy:

$$\begin{aligned}
\mathcal{L} &= m(\dot{r}^2 + r^2\omega^2) - kr^2/2 \\
E &= m(\dot{r}^2 - r^2\omega^2) + kr^2/2
\end{aligned}$$

Example 3. A particle is fixed on a rod accelerating in y direction: $y = at^2/2$. The Lagrangian and E is:

$$\mathcal{L} = m(\dot{x}^2 + (at)^2) - (mg)at^2/2E = m(\dot{x}^2 - (at)^2) + (mg)at^2/2$$

Still, E is conserved but E is not total energy since coordinate y depend on time t: $(x = x;\ y = at)$

We note that the for the later two examples, we have an accelerating frame of reference (rotating, accelerating) and E is not the total energy of the system.

1.5 Noether's theorem

Noether's theorem states that if $\mathcal{L}$ is invariant under transformation $q_i \to q_i + \varepsilon K_i(q)$ ($\mathcal{L}$ does not explicitly depend on ε), then some quantity will be conserved.

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \varepsilon} &= \sum_{i=1}^{N}\left[\frac{\partial \mathcal{L}}{\partial q_i}\frac{\partial q_i}{\partial \varepsilon} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \varepsilon}\right] \\
&= \sum_{i=1}^{N}\left[\frac{\partial \mathcal{L}}{\partial q_i}K_i(q) + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{K}_i(q)\right] \\
&= \sum_{i=1}^{N}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{K}_i(q) + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)K_i(q)\right] \\
&= \sum_{i=1}^{N}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{K}_i(q) + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}K_i(q)\right) - \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{K}_i(q)\right] \\
&= \frac{d}{dt}\left(\sum_{i=1}^{N}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}K_i(q)\right) \qquad (16)
\end{aligned}
$$

So that if $\partial \mathcal{L}/\partial \varepsilon = 0$, the quantity $\sum_{i=1}^{N}(\partial \mathcal{L}/\partial \dot{q}_i)K_i(q)$ is constant of motion.

As an example, suppose for $\mathcal{L}$ depend on coordinates x that is invariant under translation $x \to x + \varepsilon$, then the quantity $\partial \mathcal{L}/\partial \dot{x}$ is conserved, which is momentum if $\mathcal{L} = m\dot{x}^2/2 - V(x)$

2 Hamiltanion method

2.1 Hamiltonian

The quantity E is given:

$$
E(q, \dot{q}) = \sum_{i=1}^{N}\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L}(q, \dot{q}) \qquad (17)
$$

Define the conjugate momentum and Hamiltonian:

$$
p_i = \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_i} \qquad (18)
$$

$$
\mathcal{H} = \sum_{i=1}^{N} p_i \dot{q}_i - \mathcal{L}(q, \dot{q}) \qquad (19)
$$

where $\dot{q}$ are implicit function of (q, p) we can find:

$$
\begin{cases}
\frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i - \frac{\partial \mathcal{L}}{\partial p_i} = \dot{q}_i \\
\frac{\partial \mathcal{H}}{\partial q_i} = 0 - \frac{\partial \mathcal{L}}{\partial q_i} = -\dot{p}_i
\end{cases}
$$

and thus the Hamiltanion equation:

$$
\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad (20)
$$

$$
\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \qquad (21)
$$

The cyclic coordinates are still the ones which $\mathcal{H}$ does not explicitly depend on.

2.2 Legendre transformation

Legendre transformation states that if $Z(x) = Y(X) - xX$, then we should have the relationship:

$$
\frac{\partial Z}{\partial x} = -X \qquad (22)
$$

Using the relationship for Lagrangian and Hamiltanion, we can identify, for one specific coordinate i:

$$\begin{gathered}\mathcal{H}(p_i) = p_i\dot{q}_i - \mathcal{L}(\dot{q}_i) \\ Z(x) = -H(p_i) \\ Y(X) = L(\dot{q}_i) \\ \text{leading to } \frac{\partial H(p_i)}{\partial p_i} = \dot{q}_i \end{gathered} \tag{23}$$