GROUP THEORY AND REPRESENTATION

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1. Group Theory

Definition 1 (Group). *A group is a set plus an operation, that map an ordered pair of group element (g, h) of G into another element $g \cdot h \in G$, satisfying the following properties:*

(1) *operation is associative: $g \cdot (h \cdot k) = (g \cdot h) \cdot k$ for $g, h, k \in G$;*
(2) *G contain an identity element e, that satisfies $g \cdot e = e \cdot g = g$ for all $g \in G$ and*
(3) *Each element of G has an inverse, denoted by g^{-1}.*

Definition 2 (Order of the group). *Order of the group G, or the cardinality of the group, is the number of elements in the set G, denoted by $|G|$*

Definition 3 (Abelian group). *A group is called abelian if for all $g, h \in G$, $g \cdot h = h \cdot g$ (commutative)*

Permutation group. We denote a set by X. All the bijections of X to itself form a group, which we denote $\mathrm{Sym}(X)$. If $|X| = n$, then $|\mathrm{Sym}(X)| = n!$. If $|X| = |Y|$, then $\mathrm{Sym}(X) = \mathrm{Sym}(Y)$ and we denote it as $\mathrm{Sym}(n)$ or S_n For example, $S_3 = \{e, (1,2), (2,3), (1,3), (1,2,3), (3,2,1)\}$. We have $|S_3| = 6$. The permutation $(3,2,1)$ means $3 \to 2, 2 \to 1, 1 \to 3$ with 1,2,3 indicate the position in the set.

Linear transformation group. Denoting V as a vector space, we write $\mathrm{GL}(V)$ as the group of all linear transformation in V

Definition 4 (Subgroup). *Definition: H is a non empty subset of G and H is a group, then H is a subgroup of G*

Theorem 1. *The intersections of subgroups of G is also a subgroup of G.*

Proof. Suppose H and L are subgroups of G. M is the intersections between H and L, then:

(1) identity $e \in M$;
(2) if $h_1, h_2 \in H$ and $h_1 \cdot h_2 = e$, If $h_1 \in L$, then inevitably $h_2 \in L$, therefore the intersections of H and L is closed under inverse;
(3) similarly, if $h_1, h_2 \in H$ and $h_1, h_2 \in L$, then $h_1 \cdot h_2$ belong to both H and L are therefore in the intersections. M is closed under group operation.

□

Definition 5 (Generator). *For a set S, the intersections of all subgroups contain S is a subgroup. This intersections is denoted by $\langle S \rangle$ and we say that it is generated by S.*

For a group element g, we write that group that is generated by g as $\langle g \rangle$, the order of $\langle g \rangle$ is also called the order of g.

Theorem 2. *if a group G is finite, we must have $g^n = e$ for $g \in G$. Since any g^a is a number in G*

Definition 6 (Cyclic group). *If a group is generated by a single element, i.e. $G = \langle g \rangle$ for $g \in G$*

Definition 7. *A group is called normal (self-conjugate) if*

$$gBg^{-1} = B \text{ for } g \in G \qquad \text{(group automorphism)}$$

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Theorem 3 (Rearrangement theorem)**.** *for group G and a group element $g' \in G$, the set*

$$\{g'g \mid g \in G\}$$

contain each group element once and only once.

Proof. It is equivalent to say that if $g_1 \neq g_2$, then $g'g_1 \neq g'g_2$. all group element in G are mapped to another distinct elements in G (rearrangement).

If $g'g_1 = g'g_2$ but $g_1 \neq g_2$, then

$$g'^{-1}g'g_1 = g'^{-1}g'g_2$$

which apperant conflict with the assumption □

Multiplication. For S and T, both subset of group G, we define their produce:

$$ST = \{st \mid s \in S, t \in T\} \tag{1}$$

and $sT \equiv \{s\}T$ and $Ts \equiv T\{s\}$ for $s \in S$.

Definition 8 (Left cosets)**.** *For H a subgroup of G and $g \in G$, gH is called a left coset. Hg is called a right coset. The set $\{gH \mid g \in G, H$ is subgroup of $G\}$ is written as $G \setminus H$*

For example, for $S_3 = \{e, (12), (23), (13), (123), (132)\}$ and $H = \{e, (123), (132)\}$, We can work out the following relationship:

$$\begin{aligned} (123)(123) &= (132) & (132)(123) &= e \\ (132)(132) &= (123) & (123)(132) &= e \end{aligned}$$

i.e., H is a subset of S_4. Applying element $g \in \{(12), (23), (13)\}$ on H give the set $\{(12), (23), (13)\}$. Therefore, the left cosets of H is:

$$\{\, \{e, (123), (132)\}, \{(12), (23), (13)\} \,\}$$

Theorem 4. *The left cosets of the subgroup H of G partition G*

Proof. This is equivalent to say that gH is either H itself, or share no comment elements with H. if $g \in H$, then $gH = H$. On the other hand, if $g \notin H$ but $gh \in H$ for an element $h \in H$, then, by the requirement of group $h^{-1} \in H$. $ghh^{-1} = g \in H$ which conflict with the assumption. Therefore, we gH cannot share element with H: $|gH| = |H|$, so that left cosets of a subgroup partition the group. □

As a result, the whole group can be written as:

$$G = H + g_1H + g_2H + \cdots + g_nH$$

Theorem 5 (Lagrange's theorem)**.** *For a finite group G and H is a subgroup of G, $|H|$ can divide G.*

Definition 9 (Index of H in G)**.** *The number of left cosets of a subgroup H is called the index of H in G, denoted as $[G : H]$.*

If G is a finite group and $g \in G$. Then the order of $\langle g \rangle$ divide $|G|$. This is because $\langle g \rangle$ is a subgroup of G.

2. Representation Theory

3. Crystal Structure

Appendix A