My initial project was to formalize L'Hôpital's theorem using only the definitions used in Assignment 5. However, after much debugging, I found that the definition of `approaches_at` provided in both Assignment 4 and 5 contains a bug. It defines $\lim_{x\to c} f(x) = L$ as

$$\forall \epsilon > 0, \exists \delta > 0 : |x - c| < \delta \implies |f(x) - L| < \epsilon.$$

This definition, however, is too strict can causes the subsequent definition of `continuous` to be content free, since it does not allow $\lim_{x\to c} f(x) = L \wedge f(x) \neq L$—a fundamental property of limits. The correct definition is

$$\forall \epsilon > 0, \exists \delta > 0 : 0 < |x - c| < \delta \implies |f(x) - L| < \epsilon,$$

which I have implemented. Debugging this fact was one of the harder parts of implementation, as well as implementing a utility lemma `approaches_at_rw`, which allows users to rewrite the inside of limits using rewrite rules that need not apply at the limit point.

L'Hôpital's theorem is used to resolve $\frac{0}{0}$ indeterminate limits using derivatives when the derivative of the denominator is not 0. The proof of L'Hôpital's that I formalized was the classical one presented in undergraduate calculus classes:

Definition: I use the limit-difference-quotient definition of the derivative

$$f'(c) = \lim x \to c \frac{f(x) - f(c)}{x - c}$$

in this proof.

Theorem: Given $f, g : \mathbb{R} \mapsto \mathbb{R} \in C^2$, $f(c) = g(c) = 0$, and $g'(c) \neq 0$,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

Proof:

$$\begin{aligned}
\lim_{x\to c} \frac{f(x)}{g(x)} &= \lim_{x\to c} \frac{f(x) - 0}{g(x) - 0} \\
&= \lim_{x\to c} \frac{f(x) - f(c)}{g(x) - g(c)} \\
&= \lim_{x\to c} \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}} \\
&= \lim_{x\to c} \frac{\lim_{x\to c} \frac{f(x)-f(c)}{x-c}}{\lim_{x\to c} \frac{g(x)-g(c)}{x-c}} \\
&= \frac{f'(c)}{g'(c)}
\end{aligned}$$

□

After proving L'Hôpital's theorem as given, I continued on to prove some dependencies such as the limit algebra laws. I have not completed the entire set needed since they are not the center of the midterm and each one is a complicated ϵ-δ proof, but I have given proof outlines of the multiplication law and reproduced the proof of scalar summation that I gave in Assignment 4.

I intentionally did not use any of `mathlib` other than `data.real`, as I wanted to give completely elementary proofs from scratch of the theorems. This aspect of my proof also was quite challenging to deal with.