Optimal Control Simulation of Inverted Pendulum System Using LQR: Visualization and Analysis

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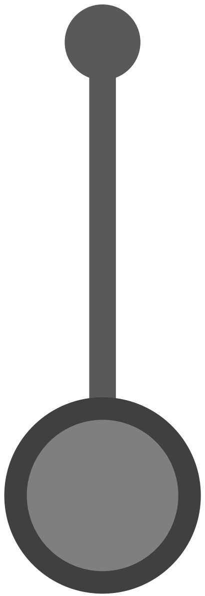
20154563

Segway

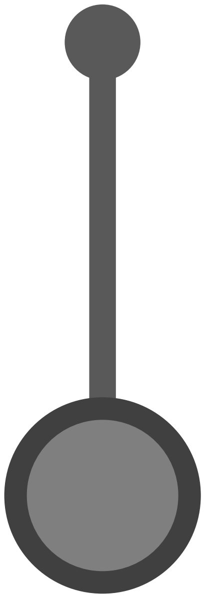
Segway

Segway



instead

LQR

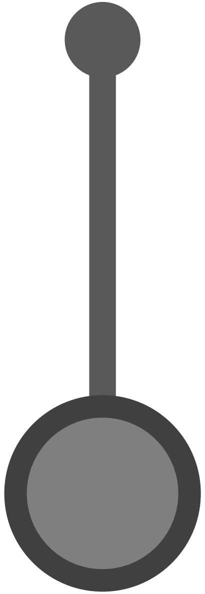


Dynamics: $X = [\theta \quad \dot{\theta} \quad x \quad \dot{x}]^T$

Constraint: $f(X, U, t) = \dot{X} = AX + BU$

Cost: $J = \frac{1}{2} X_f^T N X_f + \frac{1}{2} \int_{t_0}^{t_f} (X^T Q X + U^T R U) dt$

LQR



✓ Dynamics: $X = [\theta \quad \dot{\theta} \quad x \quad \dot{x}]^T$

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Transitional Dynamics

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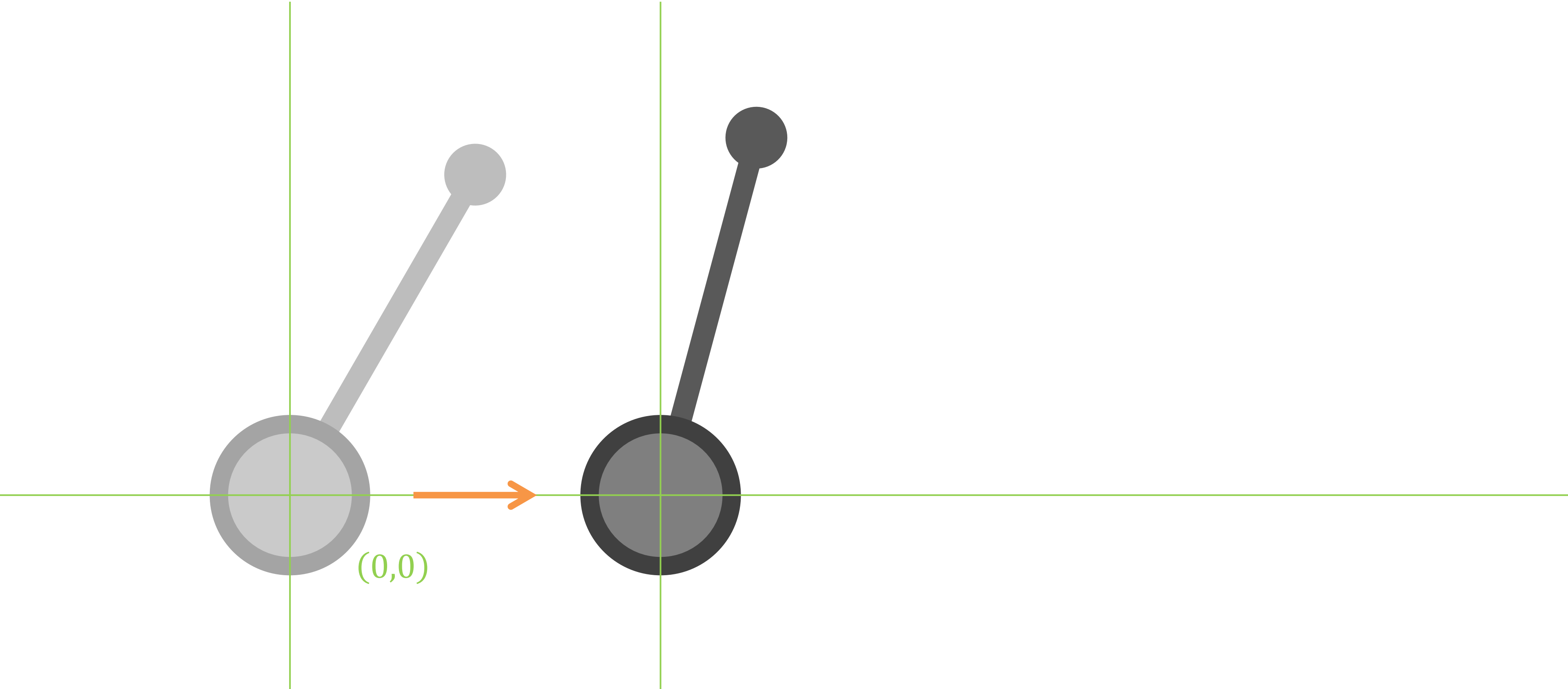
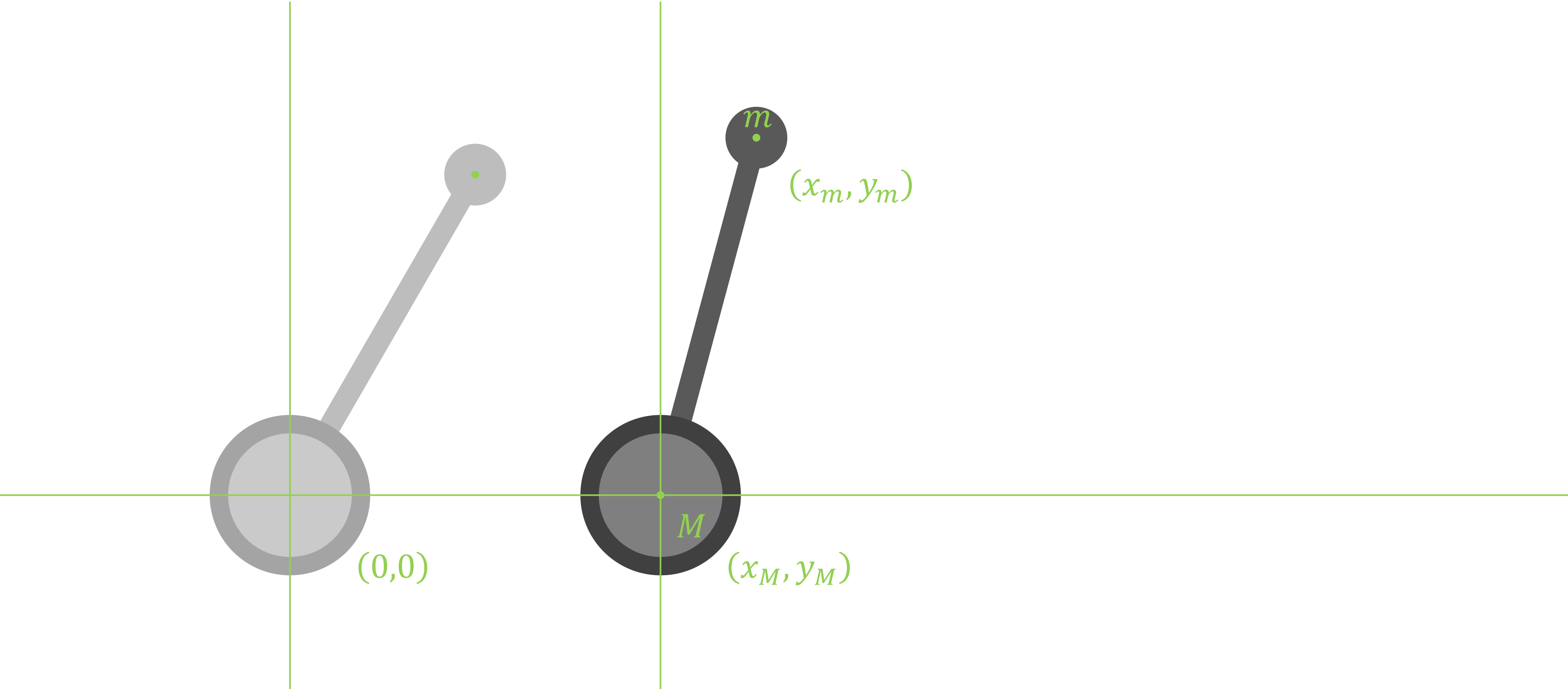
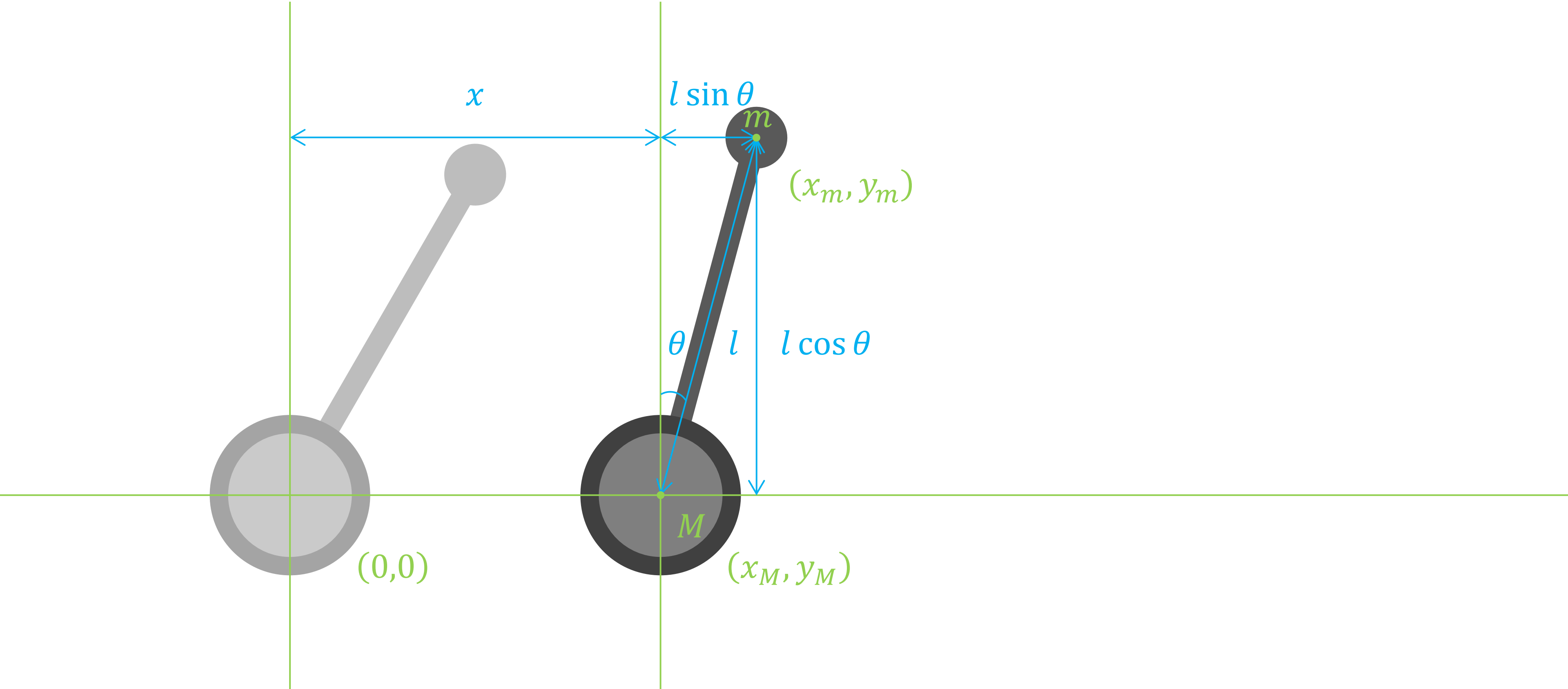


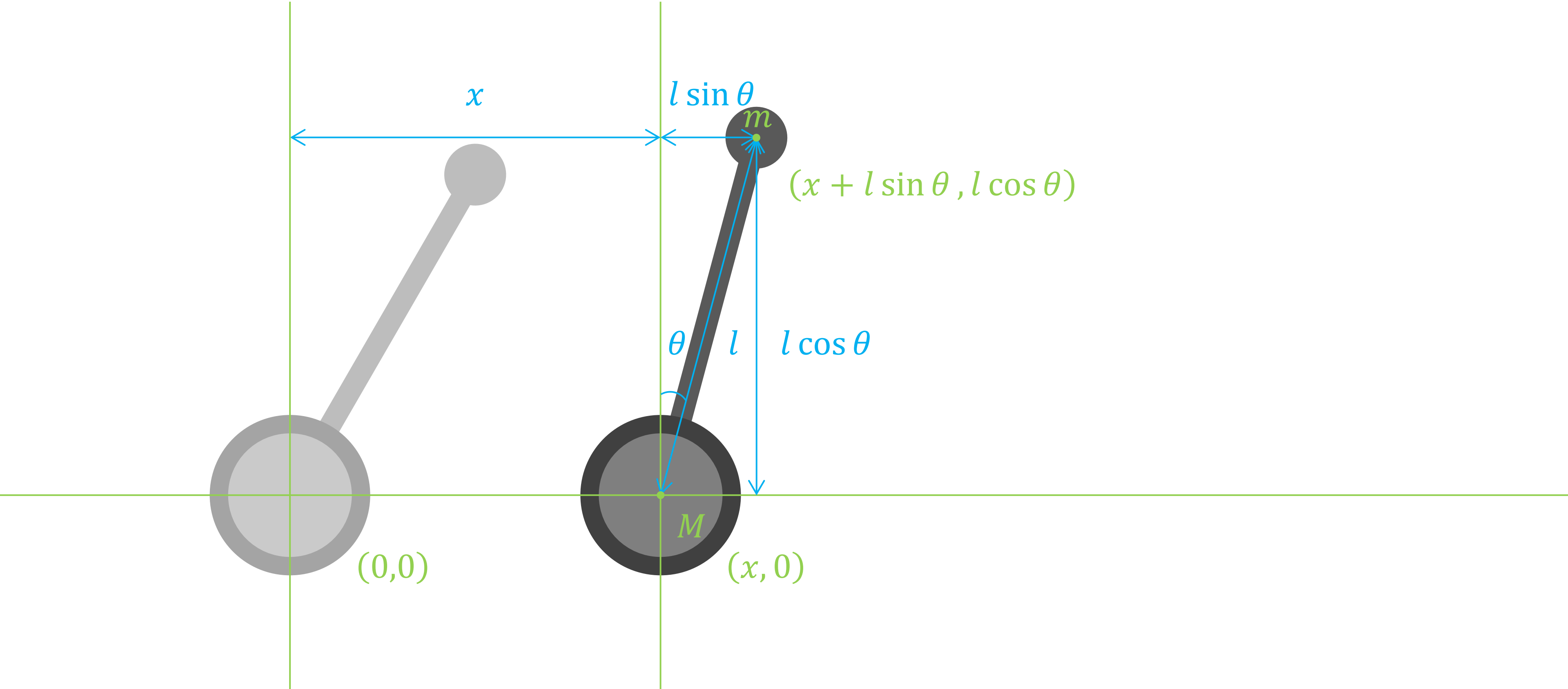
Transitional Dynamics



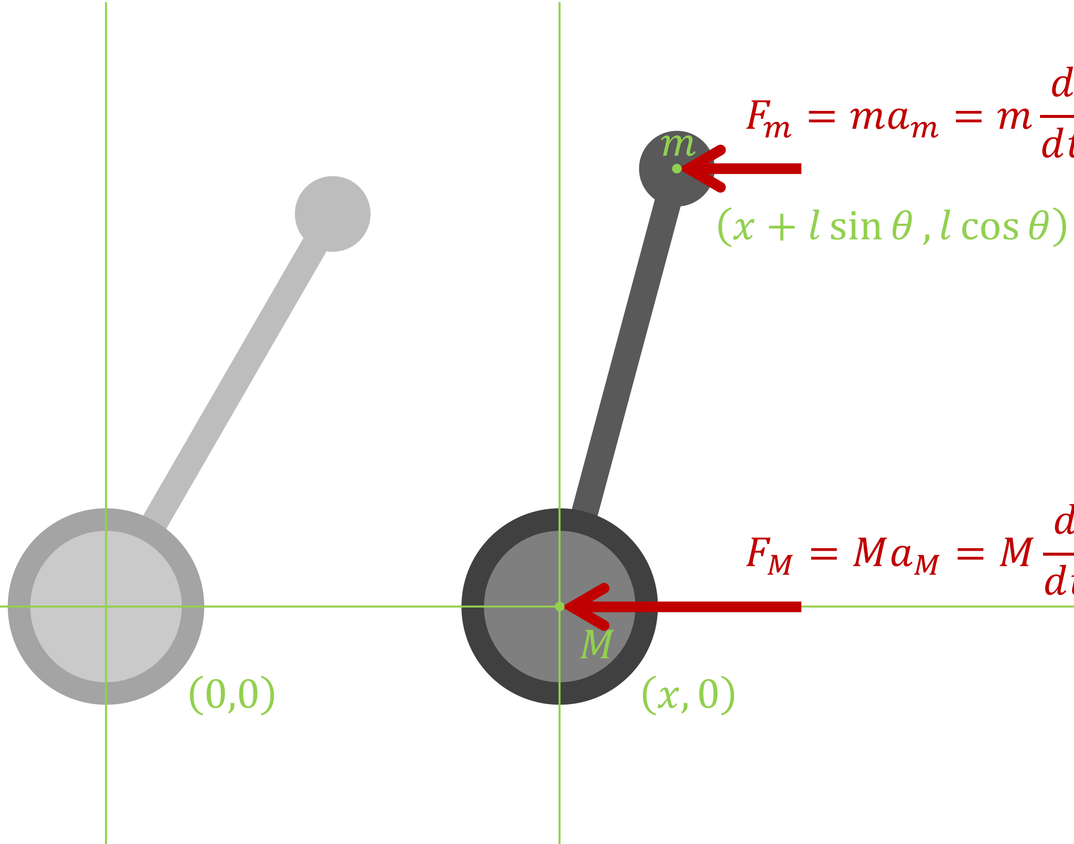
Transitional Dynamics



Transitional Dynamics



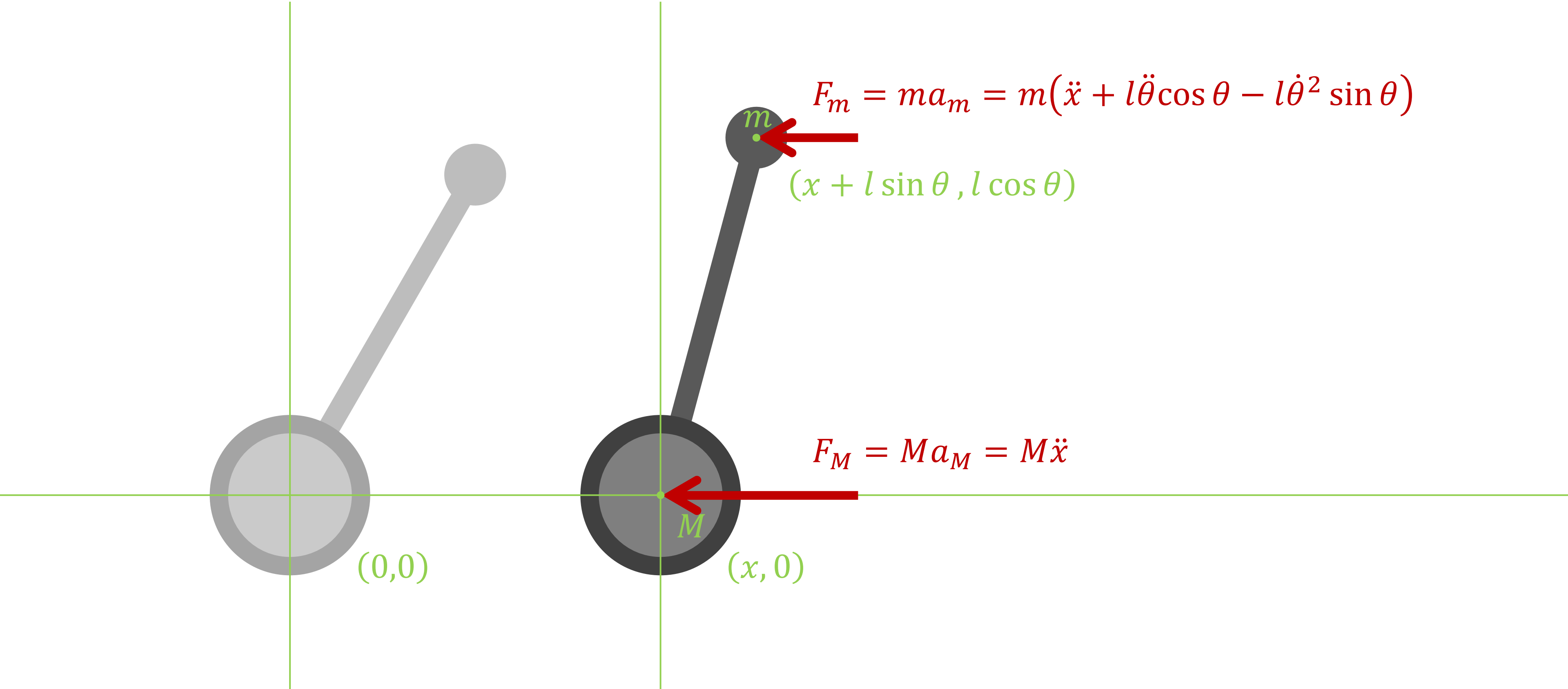
Transitional Dynamics



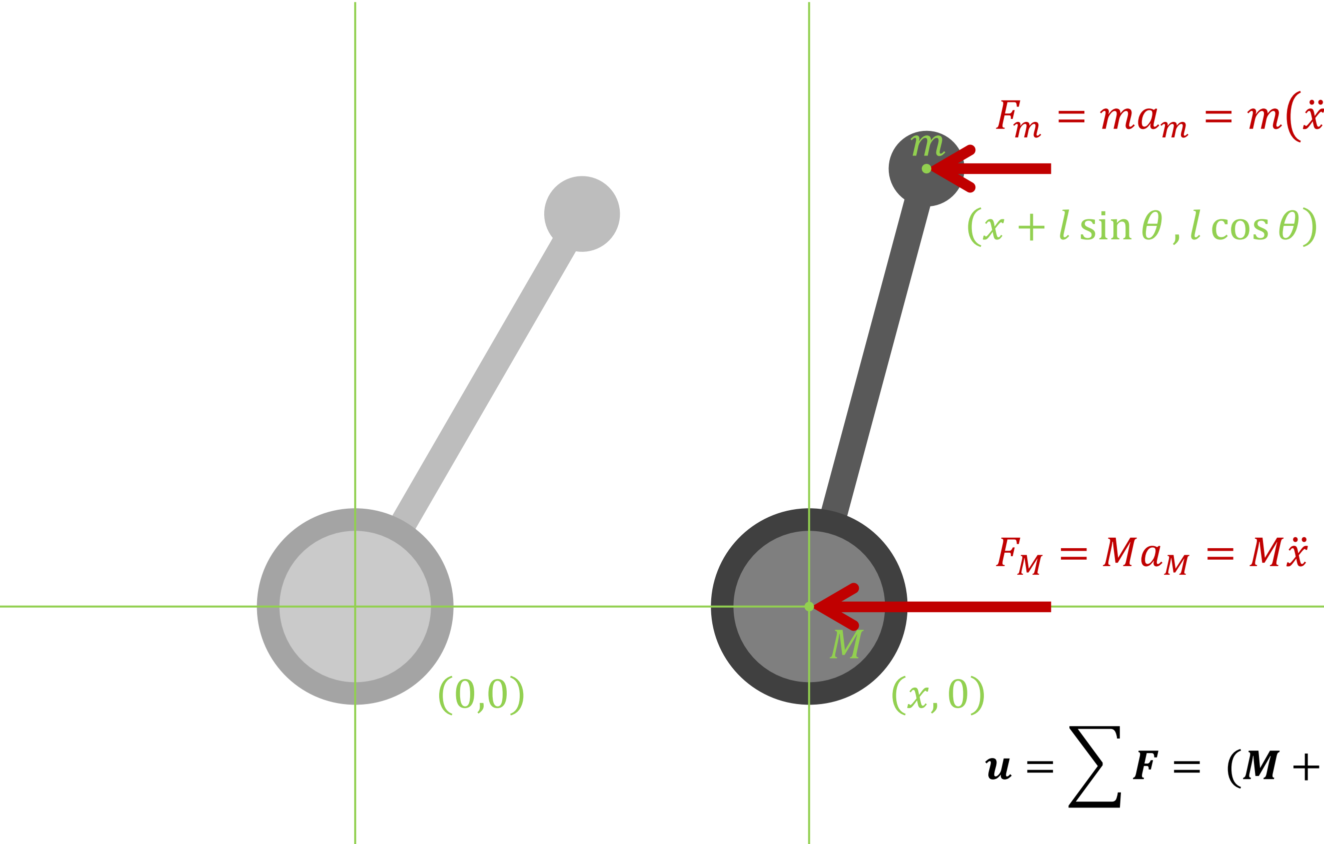
$$F_m = ma_m = m\frac{d^2}{dt^2}(x + l\sin\theta)$$

$$F_M = Ma_M = M\frac{d^2}{dt^2}x$$

Transitional Dynamics



Transitional Dynamics



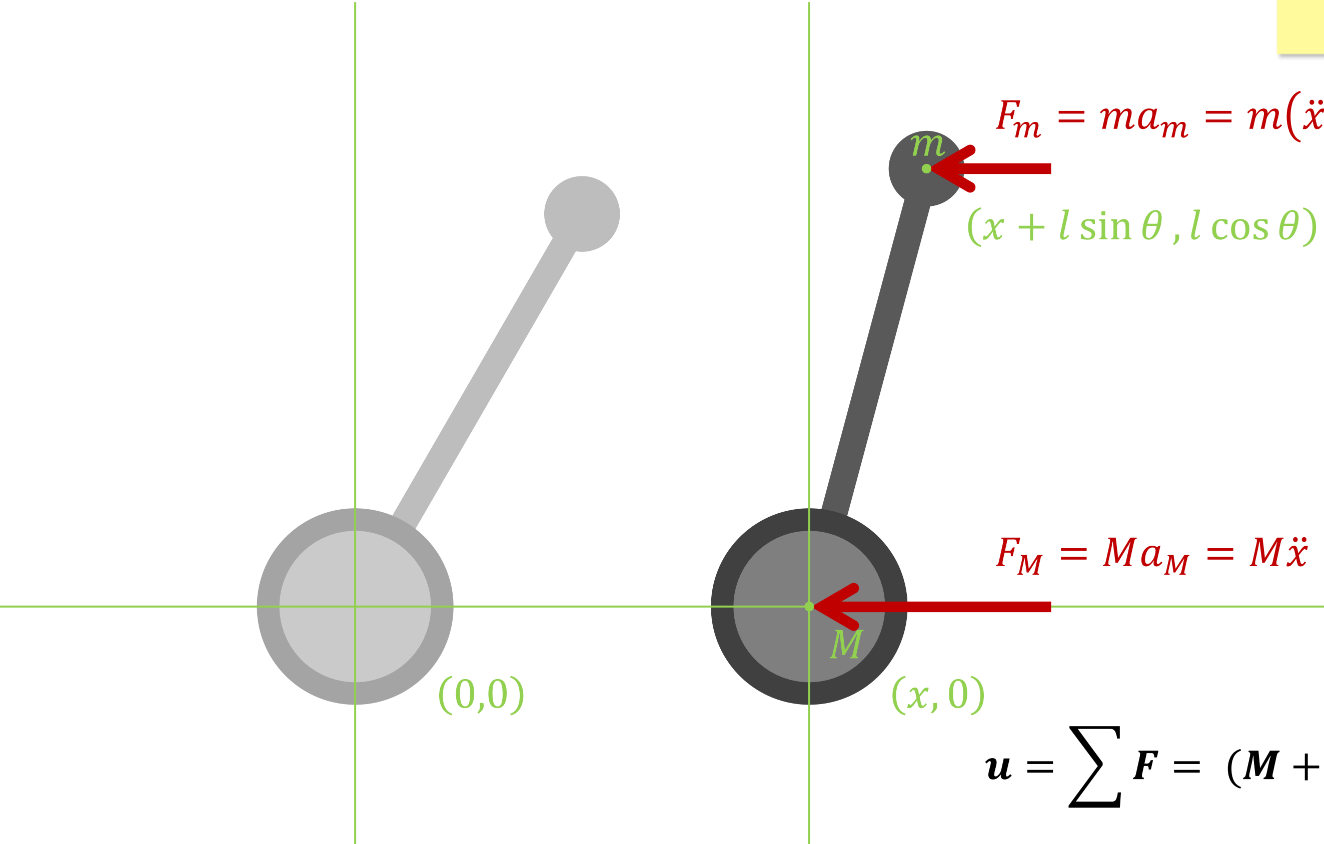
$$F_m = ma_m = m\left(\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta\right)$$

$$F_M = Ma_M = M\ddot{x}$$

$$\boldsymbol{u = \sum F = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta}$$

$$u = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$$

Transitional Dynamics



$$F_m = ma_m = m\left(\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta\right)$$

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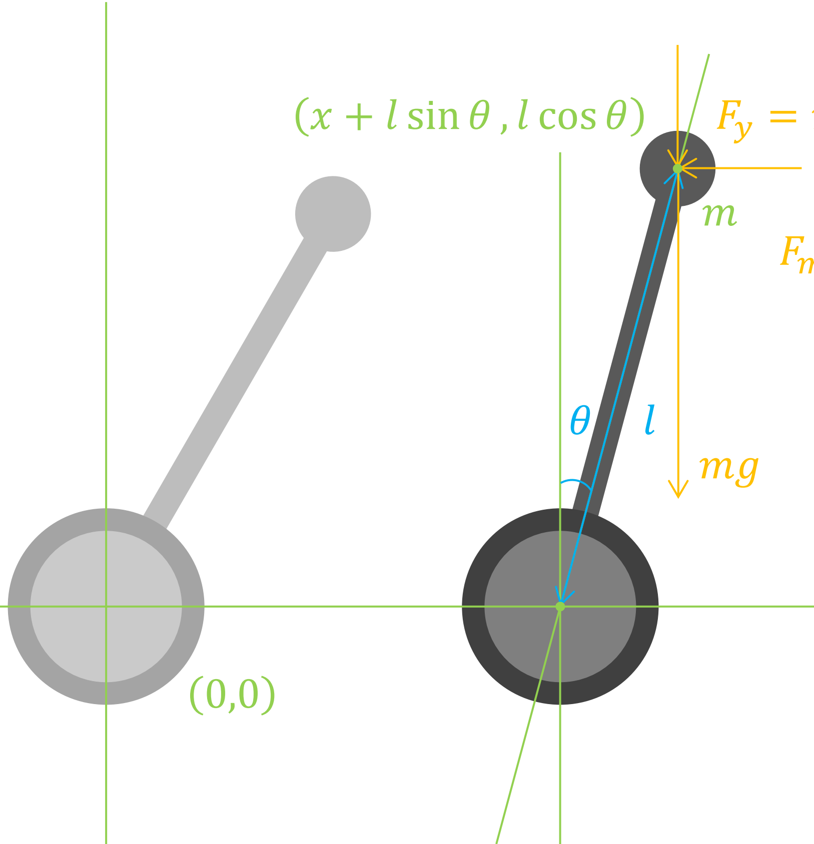
Rotational Dynamics

$$u = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$$

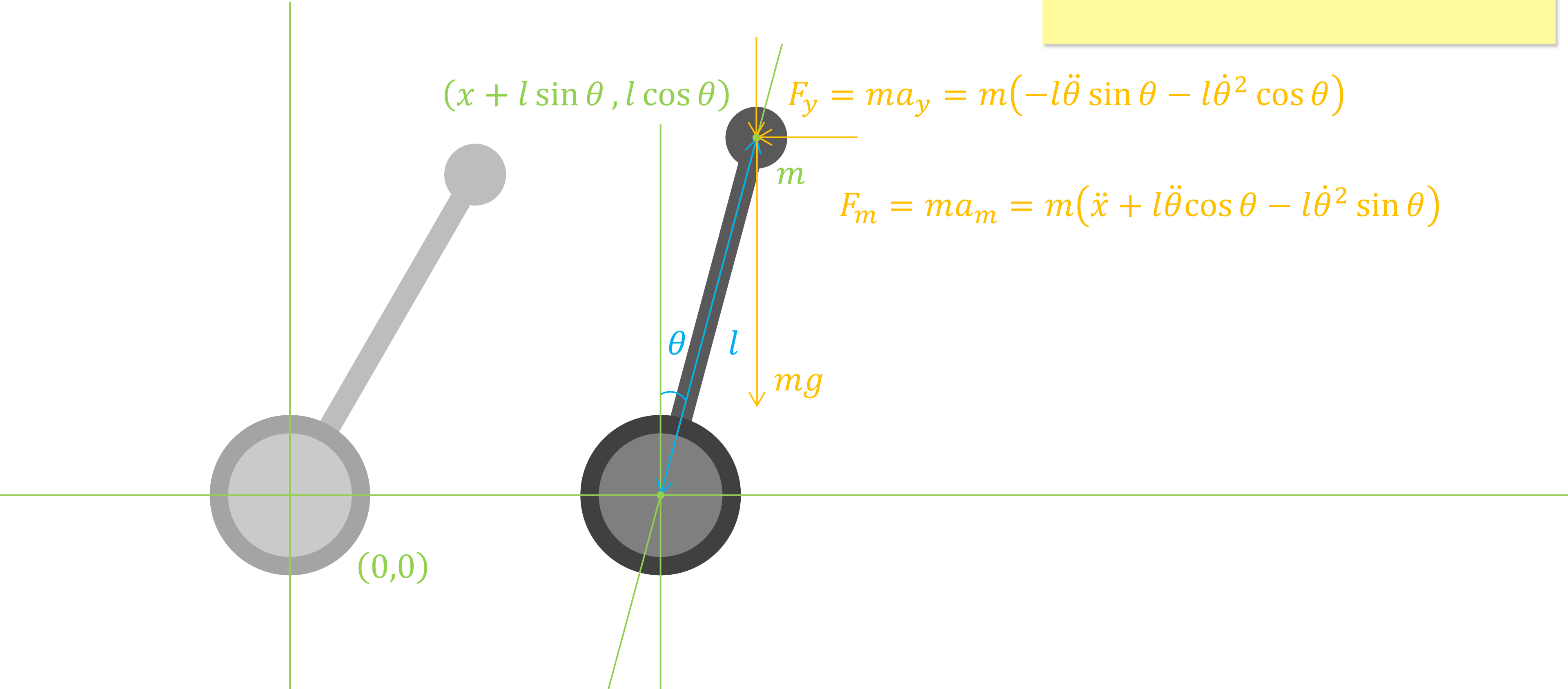


$$F_y = ma_y = m\frac{d^2}{dt^2}l\cos\theta$$

$$F_m = ma_m = m\frac{d^2}{dt^2}(x + l\sin\theta)$$

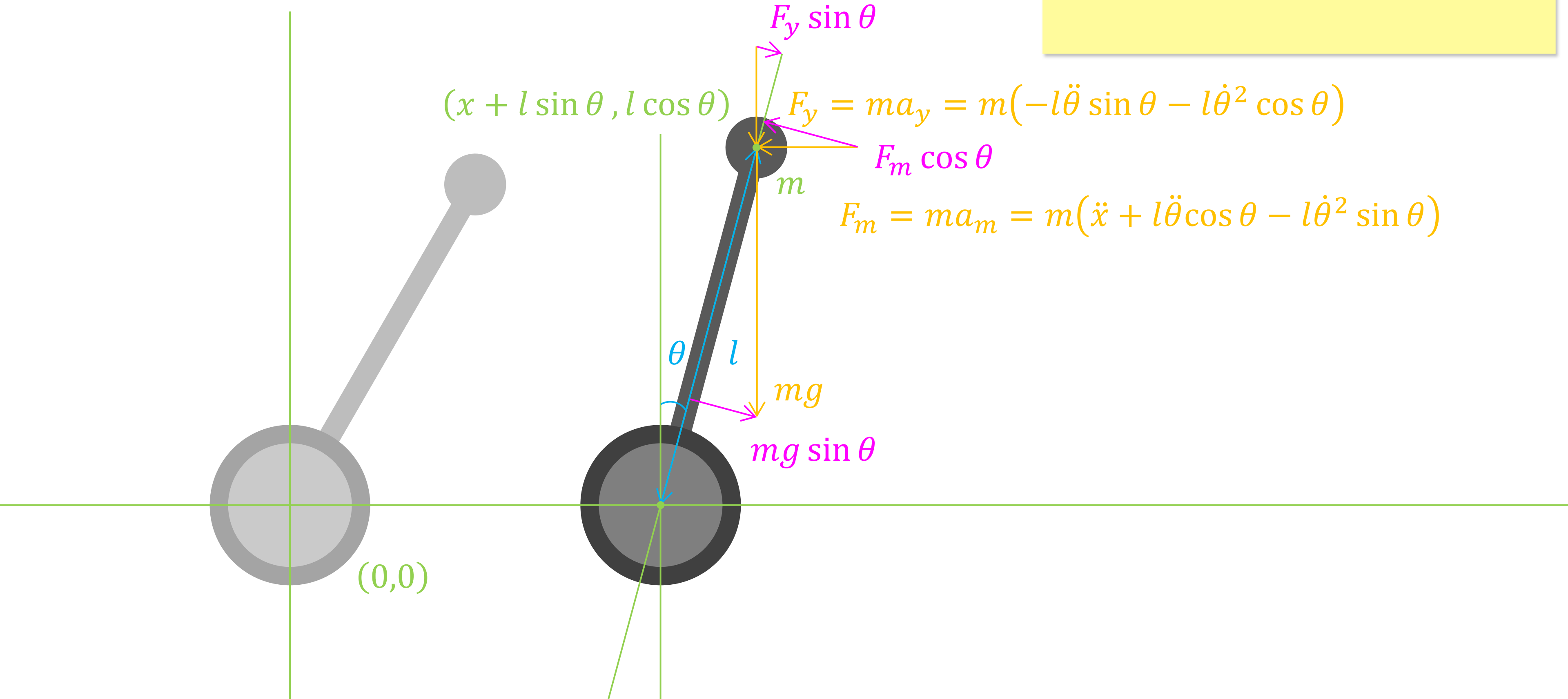
$u = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$

Rotational Dynamics



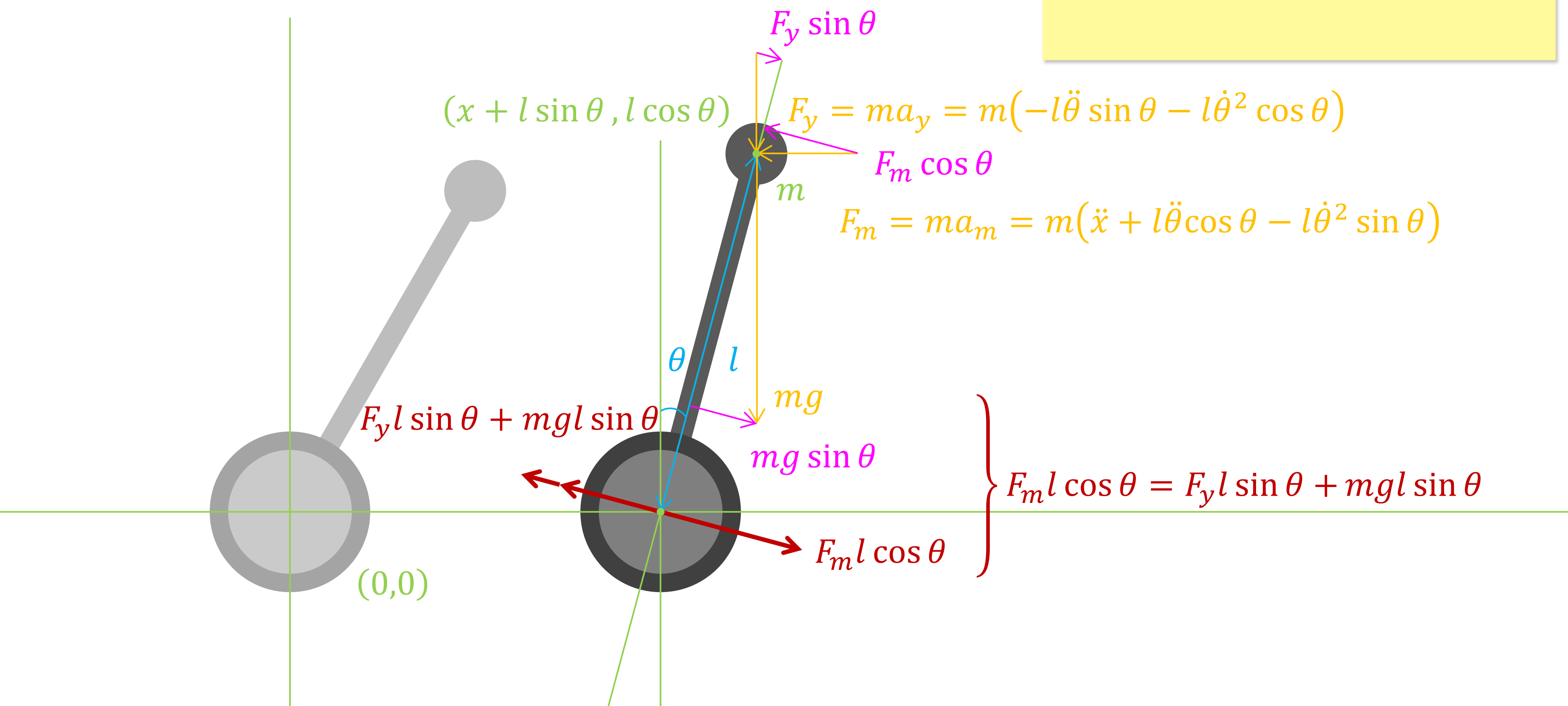
Rotational Dynamics

$$u = (M + m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$$



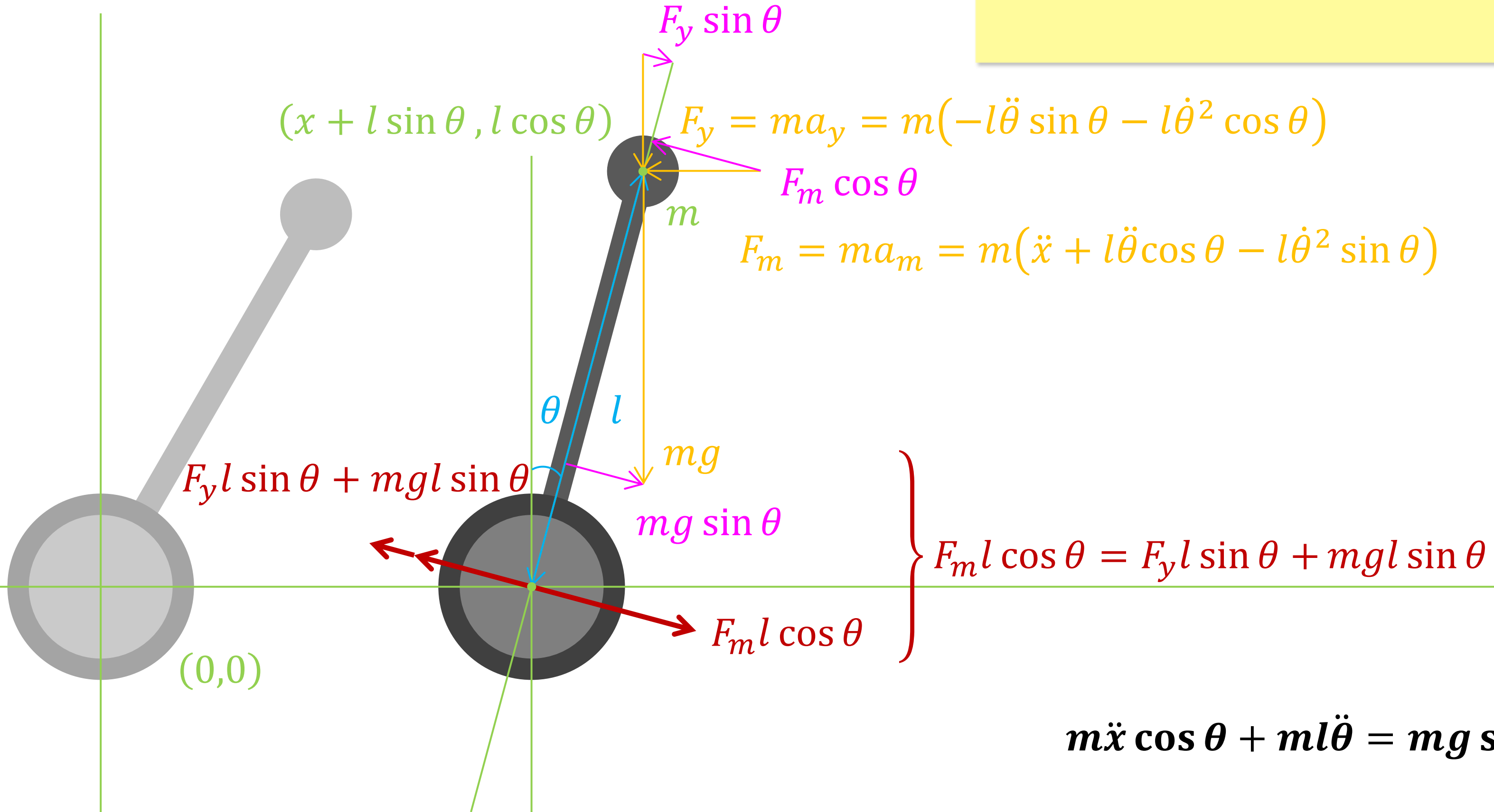
Rotational Dynamics

$u = (M + m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$



Rotational Dynamics

$$u = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$$

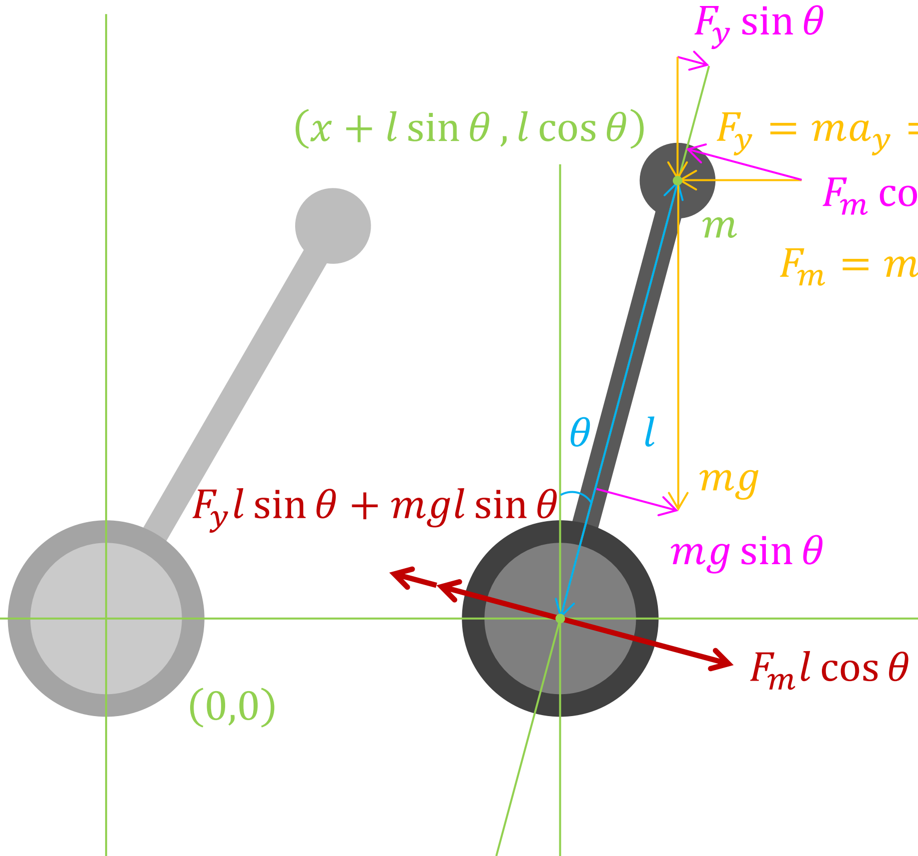


$$\boldsymbol{m\ddot{x}\cos\theta + ml\ddot{\theta} = mg\sin\theta}$$

$u = (M + m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2 \sin\theta$

$m\ddot{x}\cos\theta + ml\ddot{\theta} = mg\sin\theta$

Rotational Dynamics



$$F_m l\cos\theta = F_y l\sin\theta + mgl\sin\theta$$

$$\boldsymbol{m\ddot{x}\cos\theta + ml\ddot{\theta} = mg\sin\theta}$$

$$u = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$$

$$m\ddot{x}\cos\theta + ml\ddot{\theta} = mg\sin\theta$$

Dynamics

$$\begin{cases} u = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta \\ m\ddot{x}\cos\theta + ml\ddot{\theta} = mg\sin\theta \end{cases}$$

Dynamics: Transition

$$\begin{cases} u = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta \\ m\ddot{x}\cos\theta + ml\ddot{\theta} = mg\sin\theta \ \rightarrow ml\ddot{\theta} = mg\sin\theta - m\ddot{x}\cos\theta \end{cases}$$

$$u = (M+m)\ddot{x} + (mg\sin\theta - m\ddot{x}\cos\theta)\cos\theta - ml\dot{\theta}^2\sin\theta$$

$$u = (M+m)\ddot{x} + mg\sin\theta\cos\theta - m\ddot{x}\cos^2\theta - ml\dot{\theta}^2\sin\theta$$

$$u + ml\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta = (M+m-m\cos^2\theta)\ddot{x}$$

$$\ddot{x} = \frac{u + ml\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta}{M+m-m\cos^2\theta}$$

$$\ddot{x} = \frac{u + ml\dot{\theta}^2 \sin\theta - mg\sin\theta\cos\theta}{M + m - m\cos^2\theta}$$

Dynamics: Transition

$$\begin{cases} u = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta \\ m\ddot{x}\cos\theta + ml\ddot{\theta} = mg\sin\theta \ \rightarrow ml\ddot{\theta} = mg\sin\theta - m\ddot{x}\cos\theta \end{cases}$$

$$u = (M+m)\ddot{x} + (mg\sin\theta - m\ddot{x}\cos\theta)\cos\theta - ml\dot{\theta}^2\sin\theta$$

$$u = (M+m)\ddot{x} + mg\sin\theta\cos\theta - m\ddot{x}\cos^2\theta - ml\dot{\theta}^2\sin\theta$$

$$u + ml\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta = (M + m - m\cos^2\theta)\ddot{x}$$

$$\ddot{x} = \frac{u + ml\dot{\theta}^2 \sin\theta - mg\sin\theta\cos\theta}{M + m - m\cos^2\theta}$$

$$\ddot{x} = \frac{u + ml\dot{\theta}^2 \sin\theta - mg\sin\theta\cos\theta}{M + m - m\cos^2\theta}$$

Dynamics: Rotation

$$\begin{cases} u = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta \rightarrow \ddot{x} = \dfrac{u - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta}{M+m} \\ m\ddot{x}\cos\theta + ml\ddot{\theta} = mg\sin\theta \end{cases}$$

$$\frac{m}{M+m}\cos\theta\left(u - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta\right) + ml\ddot{\theta} = mg\sin\theta$$

$$\cos\theta\left(u - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta\right) + (M+m)l\ddot{\theta} = (M+m)g\sin\theta$$

$$u\cos\theta - ml\ddot{\theta}\cos^2\theta + ml\dot{\theta}^2\sin\theta\cos\theta + (M+m)l\ddot{\theta} = (M+m)g\sin\theta$$

$$u\cos\theta + ml\dot{\theta}^2\sin\theta\cos\theta - (M+m)g\sin\theta = ml\ddot{\theta}\cos^2\theta - (M+m)l\ddot{\theta}$$

$$\ddot{\theta} = \frac{u\cos\theta + ml\dot{\theta}^2\sin\theta\cos\theta - (M+m)g\sin\theta}{ml\cos^2\theta - (M+m)l}$$

$$\ddot{x} = \frac{u + ml\dot{\theta}^2 \sin\theta - mg\sin\theta\cos\theta}{M + m - m\cos^2\theta}$$

$$\ddot{\theta} = \frac{u\cos\theta + ml\dot{\theta}^2\sin\theta\cos\theta - (M+m)g\sin\theta}{ml\cos^2\theta - (M+m)l}$$

Dynamics: Rotation

$$\begin{cases} u = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta \rightarrow \ddot{x} = \dfrac{u - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta}{M+m} \\ m\ddot{x}\cos\theta + ml\ddot{\theta} = mg\sin\theta \end{cases}$$

$$\frac{m}{M+m}\cos\theta\left(u - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta\right) + ml\ddot{\theta} = mg\sin\theta$$

$$\cos\theta\left(u - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta\right) + (M+m)l\ddot{\theta} = (M+m)g\sin\theta$$

$$u\cos\theta - ml\ddot{\theta}\cos^2\theta + ml\dot{\theta}^2\sin\theta\cos\theta + (M+m)l\ddot{\theta} = (M+m)g\sin\theta$$

$$u\cos\theta + ml\dot{\theta}^2\sin\theta\cos\theta - (M+m)g\sin\theta = ml\ddot{\theta}\cos^2\theta - (M+m)l\ddot{\theta}$$

$$\ddot{\theta} = \frac{u\cos\theta + ml\dot{\theta}^2\sin\theta\cos\theta - (M+m)g\sin\theta}{ml\cos^2\theta - (M+m)l}$$

Constraint

$$\ddot{x} = \frac{u + ml\dot{\theta}^2 \sin\theta - mg\sin\theta\cos\theta}{M + m - m\cos^2\theta}$$

$$\ddot{\theta} = \frac{u\cos\theta + ml\dot{\theta}^2 \sin\theta\cos\theta - (M+m)g\sin\theta}{ml\cos^2\theta - (M+m)l}$$

$$\vec{x} = \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix}$$

$$f(\vec{x}, u, t) = \dot{\vec{x}}$$

Constraint

$$\ddot{x} = \frac{u + ml\dot{\theta}^2 \sin\theta - mg\sin\theta\cos\theta}{M + m - m\cos^2\theta}$$

$$\ddot{\theta} = \frac{u\cos\theta + ml\dot{\theta}^2\sin\theta\cos\theta - (M+m)g\sin\theta}{ml\cos^2\theta - (M+m)l}$$

$$\vec{x} = \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix}$$

$$f(\vec{x}, u, t) = \dot{\vec{x}} = \frac{d}{dt}\begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix}$$

Constraint

$$\ddot{x} = \frac{u + ml\dot{\theta}^2 \sin\theta - mg\sin\theta\cos\theta}{M + m - m\cos^2\theta}$$

$$\ddot{\theta} = \frac{u\cos\theta + ml\dot{\theta}^2 \sin\theta\cos\theta - (M+m)g\sin\theta}{ml\cos^2\theta - (M+m)l}$$

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$$f(\vec{x}, u, t) = \dot{\vec{x}} = \frac{d}{dt}\begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \dfrac{u\cos\theta + ml\dot{\theta}^2 \sin\theta\cos\theta - (M+m)g\sin\theta}{ml\cos^2\theta - (M+m)l} \\ \dot{x} \\ \dfrac{u + ml\dot{\theta}^2 \sin\theta - mg\sin\theta\cos\theta}{M + m - m\cos^2\theta} \end{bmatrix}$$

Constraint

$$\ddot{x} = \frac{u + ml\dot{\theta}^2 \sin\theta - mg\sin\theta\cos\theta}{M + m - m\cos^2\theta}$$

$$\ddot{\theta} = \frac{u\cos\theta + ml\dot{\theta}^2\sin\theta\cos\theta - (M+m)g\sin\theta}{ml\cos^2\theta - (M+m)l}$$

$$\vec{x} = \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix}$$

It cannot be in the form of $A\vec{x} + Bu$.

$$f(\vec{x}, u, t) = \dot{\vec{x}} = \frac{d}{dt}\begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \boxed{\begin{bmatrix} \dot{\theta} \\ \dfrac{u\cos\theta + ml\dot{\theta}^2\sin\theta\cos\theta - (M+m)g\sin\theta}{ml\cos^2\theta - (M+m)l} \\ \dot{x} \\ \dfrac{u + ml\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta}{M + m - m\cos^2\theta} \end{bmatrix}}$$

$$\lim_{t\to 0} u = U$$

$$\lim_{t\to 0} f(\vec{x}, u, t) = f(X, U, t)$$

Constraint: Linearization

$$X = \lim_{t\to 0} \vec{x} = \lim_{t\to 0} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$f(\vec{x}, u, t) = \dot{\vec{x}} = \frac{d}{dt} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \dfrac{u\cos\theta + ml\dot{\theta}^2 \sin\theta\cos\theta - (M+m)g\sin\theta}{ml\cos^2\theta - (M+m)l} \\ \dot{x} \\ \dfrac{u + ml\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta}{M + m - m\cos^2\theta} \end{bmatrix}$$

$$\lim_{t\to 0} u = U$$

$$\lim_{t\to 0} f(\vec{x}, u, t) = f(X, U, t)$$

$$\lim_{t\to 0} \sin\theta \approx \theta$$

$$\lim_{t\to 0} \cos\theta \approx 1$$

$$\lim_{t\to 0} \dot{\theta}^2 \approx 0$$

Constraint: Linearization

$$X = \lim_{t\to 0} \vec{x} = \lim_{t\to 0} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$f(X, U, t) = \dot{X} = \lim_{t\to 0} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \lim_{t\to 0} \begin{bmatrix} \dot{\theta} \\ \dfrac{u\cos\theta + ml\dot{\theta}^2 \sin\theta\cos\theta - (M+m)g\sin\theta}{ml\cos^2\theta - (M+m)l} \\ \dot{x} \\ \dfrac{u + ml\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta}{M + m - m\cos^2\theta} \end{bmatrix}$$

$$\lim_{t\to 0} u = U \qquad \lim_{t\to 0} \sin\theta \approx \theta$$

$$\lim_{t\to 0} f(\vec{x}, u, t) = f(X, U, t) \qquad \lim_{t\to 0} \cos\theta \approx 1$$

$$\lim_{t\to 0} \dot{\theta}^2 \approx 0$$

Constraint: Linearization

$$X = \lim_{t\to 0} \vec{x} = \lim_{t\to 0} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$f(X, U, t) = \dot{X} = \lim_{t\to 0} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \lim_{t\to 0} \begin{bmatrix} \dot{\theta} \\ \dfrac{u\cos\theta + \qquad - (M+m)g\sin\theta}{ml\cos^2\theta - (M+m)l} \\ \dot{x} \\ \dfrac{u + \qquad - mg\sin\theta\cos\theta}{M + m - m\cos^2\theta} \end{bmatrix}$$

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$$f(X,U,t) = \dot{X} = \lim_{t\to 0} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \lim_{t\to 0} \begin{bmatrix} \dot{\theta} \\ \dfrac{u \quad + \quad - (M+m) g \sin\theta}{ml \quad - (M+m) l} \\ \dot{x} \\ \dfrac{u + \quad - mg \sin\theta}{M + m - m} \end{bmatrix}$$

Constraint: Linearization

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$$\lim_{t\to 0} f(\vec{x}, u, t) = f(X, U, t) \qquad \lim_{t\to 0} \cos\theta \approx 1$$

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$$X = \lim_{t\to 0} \vec{x} = \lim_{t\to 0} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$f(X, U, t) = \dot{X} = \lim_{t\to 0} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \lim_{t\to 0} \begin{bmatrix} \dot{\theta} \\ \dfrac{u \quad + \qquad - (M+m)g \quad \theta}{ml \qquad - (M+m)l} \\ \dot{x} \\ \dfrac{u + \qquad - mg \quad \theta}{M + m - m} \end{bmatrix}$$

Constraint: Linearization

$$\lim_{t\to 0} u = U \qquad \lim_{t\to 0} \sin\theta \approx \theta$$

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$$f(X, U, t) = \dot{X} = \lim_{t\to 0} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \lim_{t\to 0} \begin{bmatrix} \dot{\theta} \\ \dfrac{u - (M+m)g\theta}{-Ml} \\ \dot{x} \\ \dfrac{u - mg\theta}{M} \end{bmatrix}$$

Constraint: Linearization

$$\lim_{t\to 0} u = U \qquad \lim_{t\to 0} \sin\theta \approx \theta$$

$$\lim_{t\to 0} f(\vec{x}, u, t) = f(X, U, t) \qquad \lim_{t\to 0} \cos\theta \approx 1$$

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$$X = \lim_{t\to 0} \vec{x} = \lim_{t\to 0} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$f(X, U, t) = \dot{X} = \lim_{t\to 0} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{U - (M+m)gx_1}{-Ml} \\ x_4 \\ \dfrac{U - mgx_1}{M} \end{bmatrix}$$

Constraint: Linearization

$$\lim_{t\to 0} u = U \qquad \lim_{t\to 0} \sin\theta \approx \theta$$

$$\lim_{t\to 0} f(\vec{x}, u, t) = f(X, U, t) \qquad \lim_{t\to 0} \cos\theta \approx 1$$

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$$X = \lim_{t\to 0} \vec{x} = \lim_{t\to 0} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Now we can make it in the form of $AX + BU$.

$$f(X, U, t) = \dot{X} = \lim_{t\to 0} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \boxed{\begin{bmatrix} x_2 \\ \dfrac{U - (M+m)gx_1}{-Ml} \\ x_4 \\ \dfrac{U - mgx_1}{M} \end{bmatrix}}$$

Constraint: Linearization

$$X = \lim_{t\to 0} \vec{x} = \lim_{t\to 0} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$f(X,U,t) = \dot{X} = \lim_{t\to 0} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} x_2 \\ \left(\frac{(M+m)g}{Ml}\right) x_1 + \left(-\frac{1}{Ml}\right) U \\ x_4 \\ \left(-\frac{mg}{M}\right) x_1 + \left(\frac{1}{M}\right) U \end{bmatrix}$$

Constraint: Linearization

$$X = \lim_{t \to 0} \vec{x} = \lim_{t \to 0} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$f(X, U, t) = \dot{X} = \lim_{t \to 0} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} x_2 \\ \left(\frac{(M+m)g}{Ml}\right) x_1 \\ x_4 \\ \left(-\frac{mg}{M}\right) x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \left(-\frac{1}{Ml}\right) U \\ 0 \\ \left(\frac{1}{M}\right) U \end{bmatrix}$$

Constraint: Linearization

$$X = \lim_{t\to 0} \vec{x} = \lim_{t\to 0} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$f(X,U,t) = \dot{X} = \lim_{t\to 0} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)g}{Ml} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mg}{M} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{Ml} \\ 0 \\ \frac{1}{M} \end{bmatrix} U$$

Constraint: Linearization

$$X = \lim_{t \to 0} \vec{x} = \lim_{t \to 0} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$f(X, U, t) = \dot{X} = \lim_{t \to 0} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)g}{Ml} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mg}{M} & 0 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ -\frac{1}{Ml} \\ 0 \\ \frac{1}{M} \end{bmatrix} U$$

Constraint: Linearization

$$X = \lim_{t\to 0} \vec{x} = \lim_{t\to 0} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

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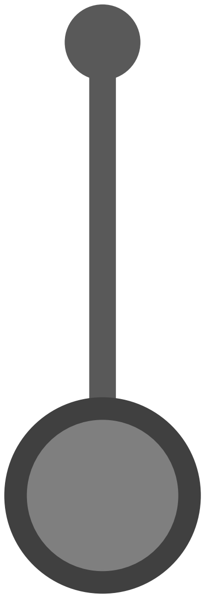
LQR



Dynamics: $X = [\theta \quad \dot{\theta} \quad x \quad \dot{x}]^T$

Constraint: $f(X, U, t) = \dot{X} = AX + BU$

Cost: $J = \frac{1}{2} X_f^T N X_f + \frac{1}{2} \int_{t_0}^{t_f} (X^T Q X + U^T R U) dt$

Cost Function

python-control

Control System Synthesis

`control.lqr(*args, **keywords)`

Linear quadratic regulator design

The lqr() function computes the optimal state feedback controller that minimizes the quadratic cost

Cost Function

Examples

```
>>> K, S, E = lqr(sys, Q, R, [N])
>>> K, S, E = lqr(A, B, Q, R, [N])
```

Returns:	K: 2-d array State feedback gains S: 2-d array Solution to Riccati equation E: 1-d array Eigenvalues of the closed loop system

Cost Function

$$K, S, E = \mathrm{lqr}(A, B, Q, R, [N])$$

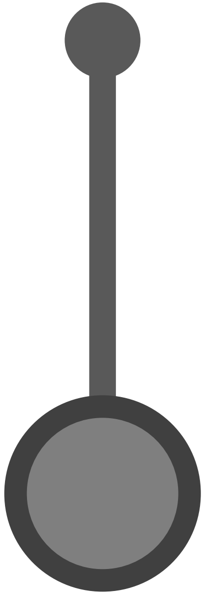
Examples

```
>>> K, S, E = lqr(sys, Q, R, [N])
>>> K, S, E = lqr(A, B, Q, R, [N])
```

Returns:	K: 2-d array State feedback gains S: 2-d array Solution to Riccati equation E: 1-d array Eigenvalues of the closed loop system

$K, S, E = \mathrm{lqr}(A, B, Q, R, [N])$

LQR



✓ Dynamics: $X = [\theta \quad \dot{\theta} \quad x \quad \dot{x}]^T$

✓ Constraint: $f(X, U, t) = \dot{X} = AX + BU$

✓ Cost: $J = \frac{1}{2} X_f^T N X_f + \frac{1}{2} \int_{t_0}^{t_f} (X^T Q X + U^T R U) dt$

Workflow

$K, S, E = \mathrm{lqr}(A, B, Q, R, [N])$

$X = [\theta \quad \dot{\theta} \quad x \quad \dot{x}]^T$

$\theta \leftarrow \dot{\theta}\Delta t + \frac{1}{2}\ddot{\theta}\Delta t^2$

$\dot{\theta} \leftarrow \dot{\theta} + \ddot{\theta}\Delta t$

$x \leftarrow 0$

$\dot{x} \leftarrow \dot{x} + \ddot{x}\Delta t$

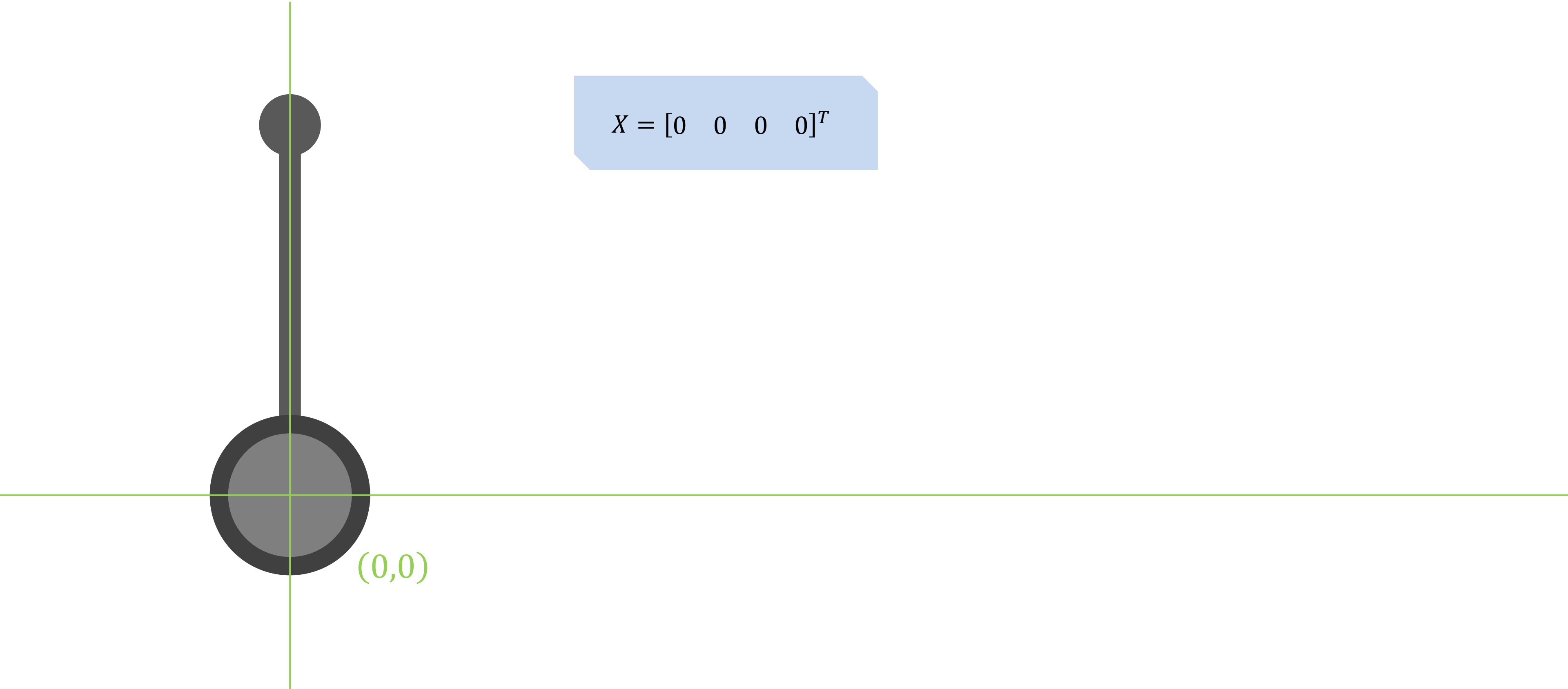
For every time slice Δt

Until $\theta \approx 0$

$U = -KX$

$f(X, U, t) = AX + BU = [\dot{\theta} \quad \ddot{\theta} \quad \dot{x} \quad \ddot{x}]^T$

Initial State



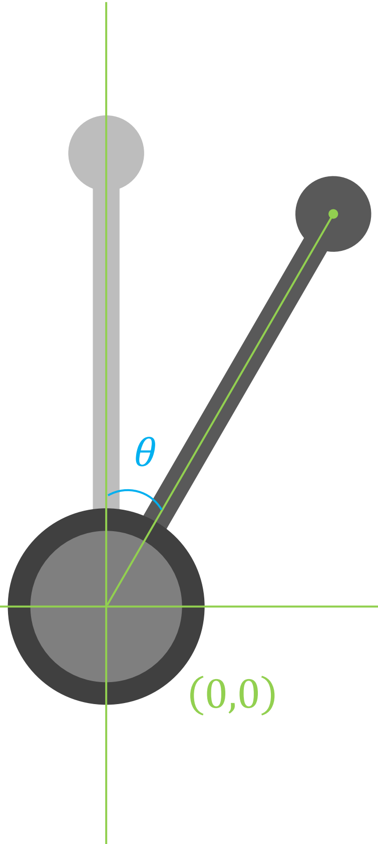
$$X = [0 \quad 0 \quad 0 \quad 0]^T$$

Initial State

Initial State

$$X = [\theta \quad \dot{\theta} \quad x \quad \dot{x}]^T$$

Initial State



$$X_0 = \left[\theta \quad \frac{\theta}{\Delta t} \quad 0 \quad 0\right]^T$$

For every time slice Δt

Until $\theta \approx 0$

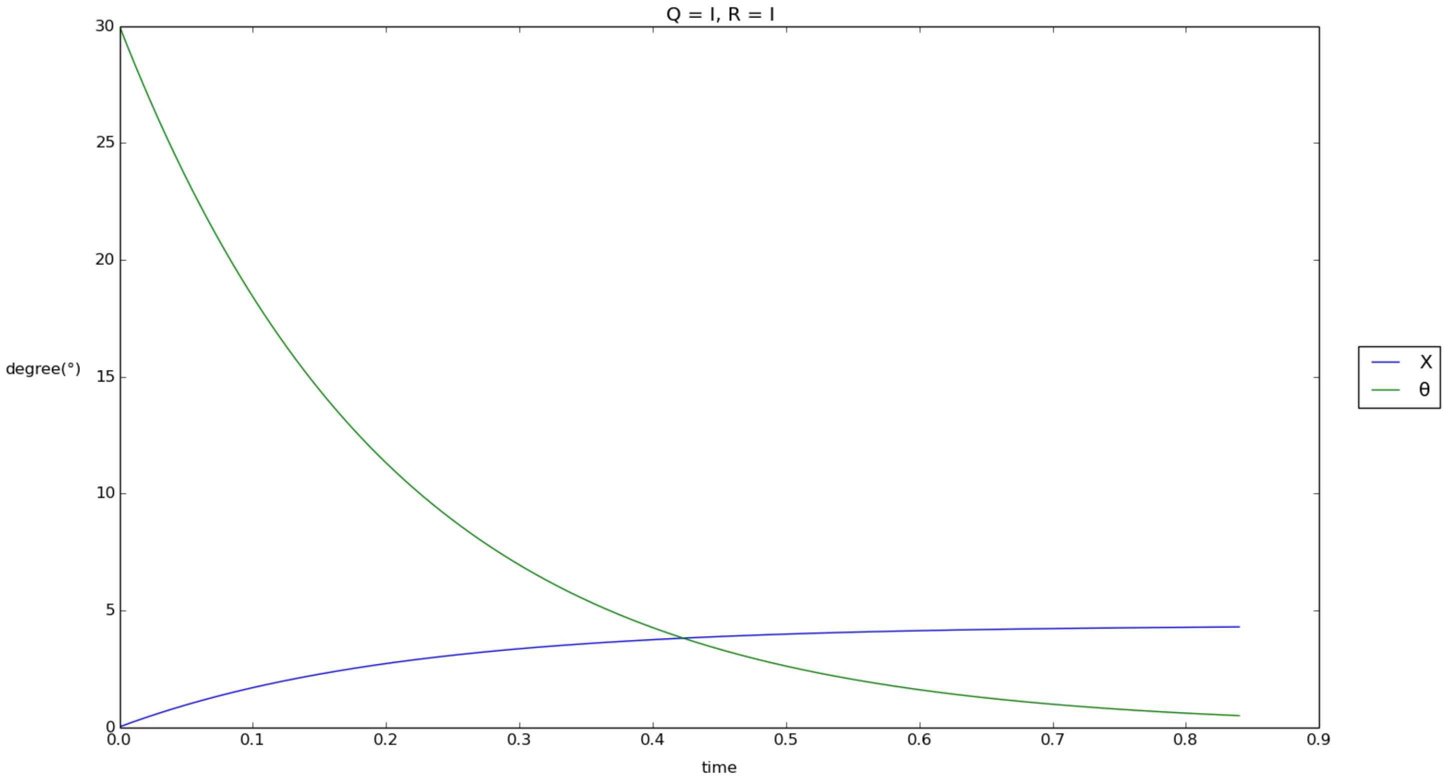
$$f(X, U, t)$$

Curiosity

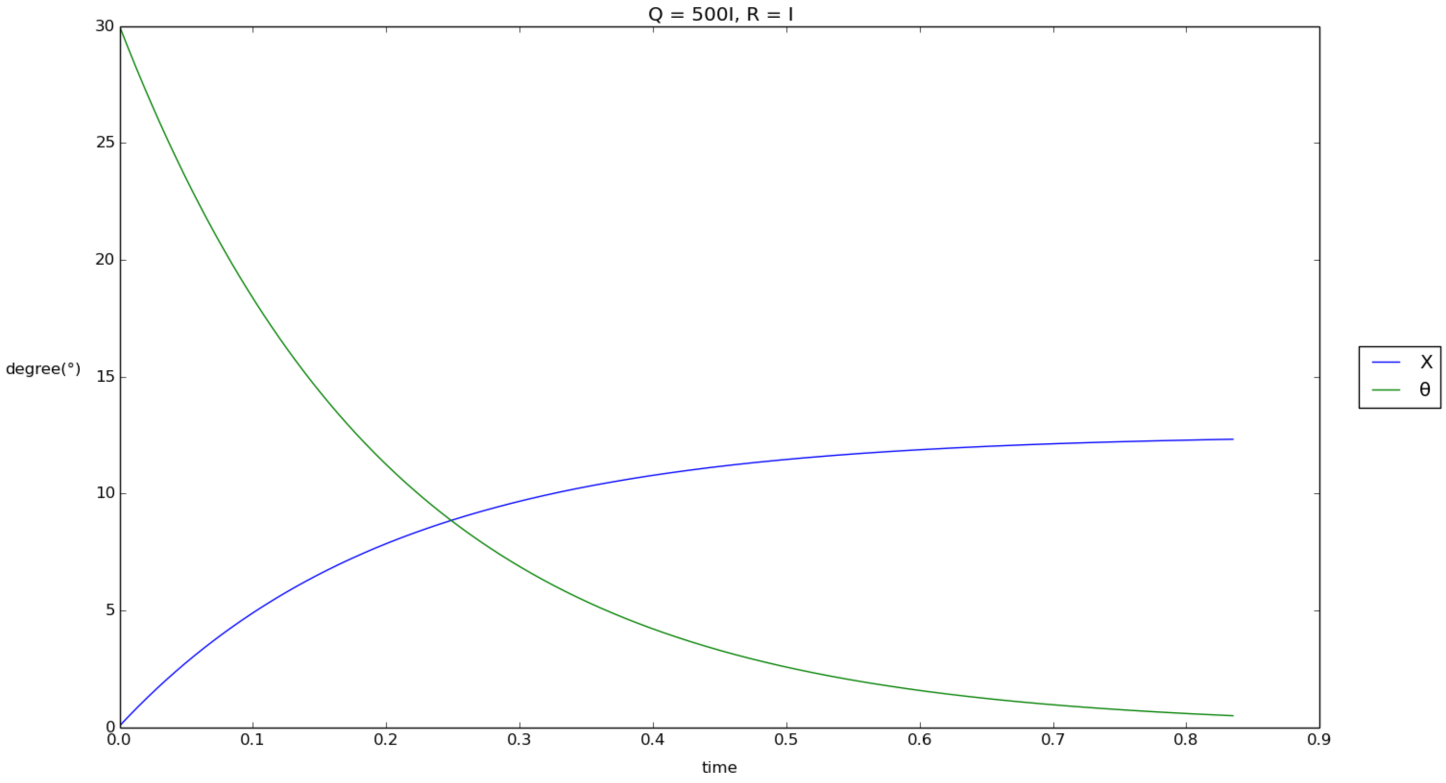
$$J = \frac{1}{2} X_f^T N X_f + \frac{1}{2} \int_{t_0}^{t_f} (X^T \boldsymbol{Q} X + U^T \boldsymbol{R} U) dt$$

How do Q and R affect on the system?

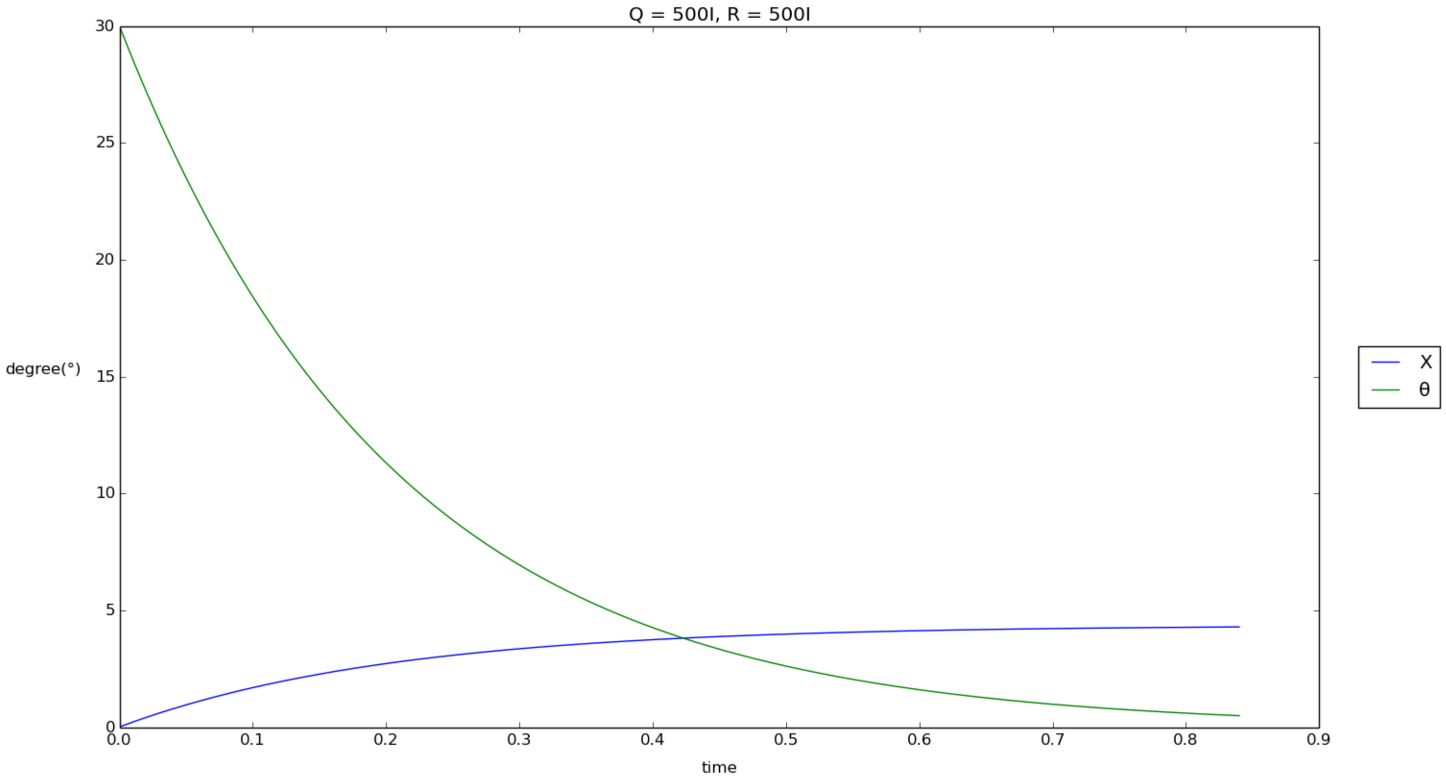
Graph when $Q = I, R = I$



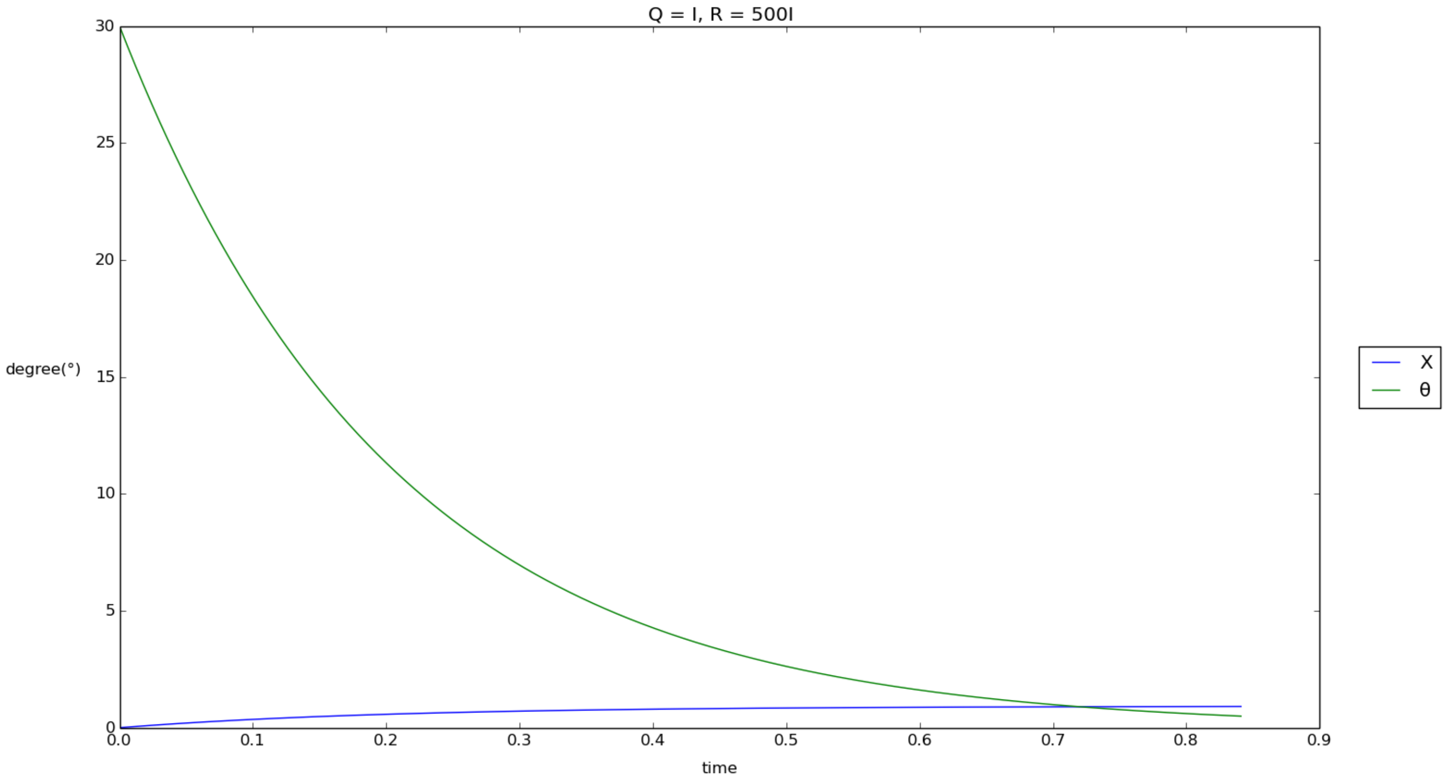
Graph when $Q = 500I, R = I$



Graph when $Q = 500I, R = 500I$



Graph when $Q = I, R = 500I$



Conclusion

Rotational dynamics is almost unaffected by Q and R.

Transitional dynamics is directly proportional to $\|Q\|$.
inversely proportional to $\|R\|$.

We may get a desirable profile by setting Q with $R = I$.

References

1. Lal Bahadur Prasad, *Barjeev Tyagi, Hari Om Gupta, Optimal Control of Nonlinear Inverted Pendulum System Using PID Controller and LQR: Performance Analysis Without and With Disturbance Input, Department f Electrical Engineering, Indian Institute of Technology Roorkee*, Uttarakhand 247667, India

2. Prasanna Priyadarshi, *Optimal Controller Design for Inverted Pendulu System: An Experimental Study*, Department of Electrical Engineering National Institute of Technology, Rourkella-769008, India, June, 2013

3. Ashish S. Ktariya, *OPTIMAL STATE-FEEDBACK AND OUTPUT-FEEDBACK CONTROLLERS FOR THE WHEELED INVERTED PENDULUM SYSTEM*, School of Electrical and Computer Engineering Georgia Institute of Technology 7 May, 2010