Optimal Control Simulation of Wheeled Inverted Pendulum System Using LQR: Visualization and Analysis

GyuHyeon Choi
291, Daehak-ro, Yuseong-gu,
Daejeon,
Korea
wiany11@kaist.ac.kr

Abstract

Linear Quadratic Regulator (LQR) is one of the optimal control techniques making optimal control decisions using the current state of dynamical system. As a way of understanding LQR, wheeled inverted pendulum is used in modeling and simulation. Modeling process and simulation algorithm are explained. Several optimal control profiles resulting from different situations are provided. Lastly, impact of cost function is analyzed.

1 Introduction

The inverted pendulum (IP) is a classic example of optimal control problem. It is a highly unstable system with nonlinear dynamics. Controlling inverted pendulum has been an important research in the field of control engineering due to its potential usage especially in systems which need to be constantly balanced. For example, IP is representative of a class of attitude control problems whose goal is to maintain the desired vertically oriented position at all times [1]. Recently, due to development of technology, vehicles characterized as wheeled inverted pendulums have received attention in the robotics area. Segway is the most typical example of vehicles with wheeled inverted pendulum system.

The control problem requires a dynamic model of a system to determine control strategies to achieve the desired state of the system. Many methods have been introduced into optimal control to operate a dynamic system at minimum cost, often defined as a sum of the deviations of key measurements of the dynamics of the system. LQR is one of the main solutions in the optimal control theory.

The aim of this case stud is to understand LQR for optimal control through computer simulation of a wheeled inverted pendulum.

2 Experiment

The wheeled inverted pendulum is designed in two-dimensional space and driven by two motors, one is for transitional movement and the other one is for rotational movement. The motor for transition moves a wheel then it decides the acceleration. The other motor rotates a rod then it decides the angular acceleration. By moving the wheel and rotating the rod, the wheeled inverted pendulum will be in stable state, the rod in the upright position, with the minimum cost.

2.1 Transitional dynamics of wheeled inverted pendulum system

The free body diagram of the wheeled inverted pendulum is shown in Fig. 1. This figure describes transitional and rotational movement of the wheeled inverted pendulum within the short time range.

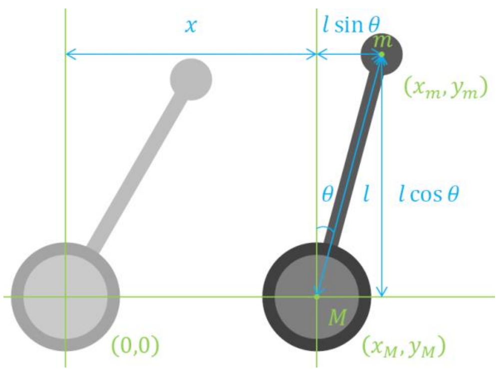

Figure 1 Two motors driven wheeled inverted pendulum

The centroid of the wheel and the rod are denoted as M and m. The position of the wheeled inverted pendulum is M and it is initially at the origin, (0,0). From the stable state of the wheeled inverted pendulum at (0,0), when it becomes unstable it moves its body and rotates the rod to stabilize the system. By defining the distance the wheeled inverted pendulum moved as x, the length between M and m as l and the angle between the vertical line and the rod as θ, coordinates of M and m can be written as

$$(x_M, y_M) = (x, 0) \tag{1}$$

and

$$(x_m, y_m) = (x + l\sin\theta, l\cos\theta). \tag{2}$$

Since transition is made only in the x direction, the system can be described using horizontal acceleration. Acceleration in the x direction of M is:

$$a_M = \frac{d}{dt}x_M = \ddot{x}. \tag{3}$$

In the same way, acceleration in the x direction of m is:

$$a_m = \frac{d}{dt}x_m = \ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta. \tag{4}$$

According to Newton's laws of motion, $F = ma$. Vector forces affecting on M and m are $F_M = Ma_M$ and $F_m = ma_m$.

$$F_M = M\ddot{x} \tag{5}$$

$$F_m = m\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta \tag{6}$$

Acceleration decreases during transition because the system should be stabilized and stop. Force has the same direction with acceleration which is opposite to velocity, direction of movement. The free body of the wheeled inverted pendulum with force affecting on each centroid is shown in the figure 2.

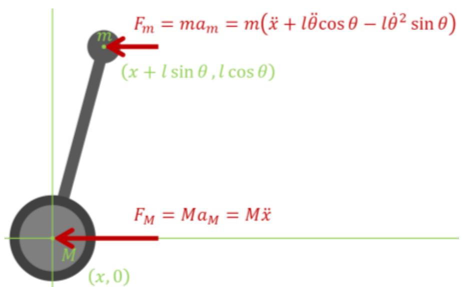

Figure 2 Forces affecting wheeled inverted pendulum

Input U generated by two motors is supposed to be sum of vector forces affecting on the system.

$$U = (M + m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta \tag{7}$$

2.2 Rotational dynamics of wheeled inverted pendulum system

Figure 3 describes rotational dynamics of wheeled inverted pendulum.

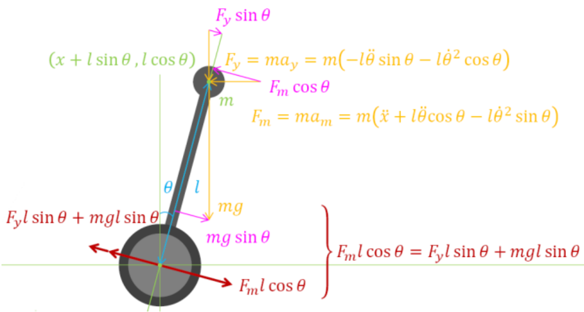

Figure 3 Rotational dynamics of wheeled inverted pendulum

Three forces are affecting on m while the system is being stabilized during transition. Gravity always affects downwards. F_m is already obtained in the equation (6). Force in the y direction F_y can be driven in the same way and it has downward direction which is opposite to upward vertical velocity.

$$\mathrm{F_y} = -ml\ddot{\theta}\sin\theta - ml\dot{\theta}^2\cos\theta \tag{7}$$

Each factor of the three forces that is perpendicular to l, generates torque on the wheel. Sum of torques on one side should be the same with a torque on the other side.

$$\mathrm{F_y}l\sin\theta + mgl\sin\theta = F_m l\cos\theta \tag{8}$$

Substituting equation (8) using equations (6) and (7), equation (8) becomes

$$\mathrm{m}\ddot{x}\cos\theta + ml\ddot{\theta} = mg\sin\theta \tag{9}$$

2.3 Dynamics of the system

Replacement of angular acceleration $\ddot{\theta}$ from equation (7) using equation (9) provides an equation about transitional acceleration $\ddot{x}$.

$$\ddot{x} = \frac{U + ml\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta}{M + m - m\cos^2\theta} \tag{10}$$

Similarly, replacement of $\ddot{x}$ from equation (9) using equation (7) gives an equation about $\ddot{\theta}$.

$$\ddot{\theta} = \frac{U\cos\theta + ml\dot{\theta}^2 \sin\theta\cos\theta - (M+m)\,g\sin\theta}{ml\cos^2\theta - (M+m)l} \tag{11}$$

Dynamics of the system is designed as:

$$\mathrm{X} = [\theta \quad \dot{\theta} \quad x \quad \dot{x}]^T \tag{12}$$

Constraint function $\mathrm{f}(X, U, t)$ is the time derivative of dynamics, $\dot{X} = [\dot{\theta} \quad \ddot{\theta} \quad \dot{x} \quad \ddot{x}]^T$.

$$\mathrm{f}(X,U,t) = \begin{bmatrix} \dot{\theta} \\ \dfrac{U\cos\theta + ml\dot{\theta}^2\sin\theta\cos\theta - (M+m)\,g\sin\theta}{ml\cos^2\theta - (M+m)l} \\ \dot{x} \\ \ddot{x} = \dfrac{U + ml\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta}{M + m - m\cos^2\theta} \end{bmatrix} \tag{13}$$

For optimal control using LQR, constraint must be in the form of $AX + BU$. Equation (13), however, cannot be in the form because θ cannot be separated from trigonometric functions.

2.4 Model Linearization

The dynamics of the system is updated after the specified time Δt. When Δt is close to 0, θ cannot be changed much then it is supposed to be close to 0. When θ is close to 0, $\dot{\theta}^2 \approx 0$, $\sin\theta \approx 0$ and $\cos\theta \approx 1$.

$$\mathrm{f}(X,U,t) = \begin{bmatrix} \dot{\theta} \\ \dfrac{U - (M+m)g\theta}{-Ml} \\ \dot{x} \\ \ddot{x} = \dfrac{U - mg\theta}{M} \end{bmatrix} \tag{14}$$

In the form of $AX + BU$, the constraint is

$$\mathrm{f}(X,U,t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{(M+m)g}{Ml} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{mg}{M} & 0 & 0 & 0 \end{bmatrix} \mathrm{X} + \begin{bmatrix} 0 \\ -\dfrac{1}{Ml} \\ 0 \\ \dfrac{1}{M} \end{bmatrix} U \tag{15}$$

where $\mathrm{X} = [\theta \quad \dot{\theta} \quad x \quad \dot{x}]^T$ when Δt is close to 0.

2.5 Optimal control simulation using LQR

Cost function of current state vector X with arbitrary 4 by 4 positive semidefinite matrix Q and a positive definite matrix R is

$$J = \frac{1}{2}X_f^T N X_f + \frac{1}{2}\int_{t_0}^{t_f} X^T Q X + U^T R U \, dt \tag{17}$$

where N is zero matrix. Cost function ignores the final state of the system because the final state is the same with the initial state, zero vector. It will be explained in the pseudo code.

Input U can be obtained by solving Riccati equation

$$A^T S + SA - SBR^{-1}B^T S + Q = 0 \tag{18}$$

where A and B are from constraint function and Q and R from cost function. Optimal input U is

$$U = -KX \tag{19}$$

where $K = R^{-1}B^T S$ is the state feedback gain.

From the current state X and optimal input U for the next state, constraint $AX + BU = [\dot{\theta} \quad \ddot{\theta} \quad \dot{x} \quad \ddot{x}]^T$ provides the information to update the state.

LQRStateFeedbackControl($\theta, \dot{\theta}, x, \dot{x}$**)**:
  $X = [\theta, \dot{\theta}, x, \dot{x}]$
  if $\theta > \pi/180$: {until θ is close to zero}
    $U \leftarrow -KX$
    $[\dot{\theta}', \ddot{\theta}', \dot{x}', \ddot{x}'] \leftarrow AX + BU$
    $\theta \leftarrow \dot{\theta}'\Delta t + \frac{1}{2}\ddot{\theta}'\Delta t^2$
    $\dot{\theta} \leftarrow \dot{\theta} + \ddot{\theta}'\Delta t$
    $x \leftarrow 0$
    $\dot{x} \leftarrow \dot{x} + \ddot{x}'\Delta t$
    LQRStateFeedbackControl(θ,$\dot{\theta}$,x,$\dot{x}$**)**
  else:
    LQRStateFeedbackControl(0,0,0,0**)**

Figure 4 Pseudo code of state feedback control simulation using LQR

The term x means distance between origin and the wheeled inverted pendulum. If magnitude of x keeps increasing, magnitude of acceleration keeps enlarging as well. Increasing acceleration does not slow down transition then the system cannot be stabilized. In this regard, distance the wheeled inverted pendulum moves at the moment should be always from the origin, (0,0).

3 Results

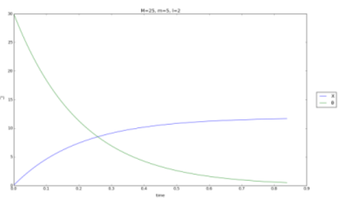

Figure 5 General profile of optimally controlled wheeled inverted pendulum with positive initial θ

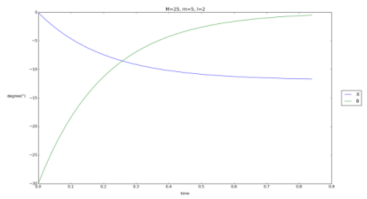

Figure 6 General profile of optimally controlled wheeled inverted pendulum with negative initial θ

Figure 5 and 6 show general transition and rotation of the optimally controlled wheeled inverted pendulum. Absolute value of θ decreases until the system becomes stable. Transition is produced in the same direction of θ.

Simulation is helpful to understand Optimal Control using LQR. However, definition of cost function still remains curiously. Q and R affect on a result of cost function so they are interesting to analyze.

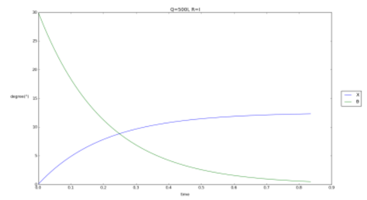

Figure 7 Optimal control profile when $Q = 500I$ and $R = I$

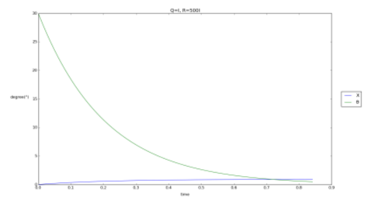

Figure 8 Optimal control profile when $Q = I$ and $R = 500I$

While rotation is not affected much from Q and R, transition is directly proportional to the size of Q and inversely proportional to the size of R.

4 Conclusion

The simulation shows that LQR can be used for optimal control of nonlinearly dynamic system with the example of wheeled inverted pendulum. The results of simulation show both transition and rotation are required to perform optimal control minimizing the cost.

Simulation of three-dimensional wheeled inverted pendulum is possible by increasing dimension of the system. An element describing transition of new axis should be introduced.

There can be infinitely many variation of Q and R so the outcome of cost function varies. In other words, one of them can make numerous outcomes with another one fixed. With fixed $R = I$, desirable profiles can be achieved by adjusting Q.

Reference

[1] Lal Bahadur Prasad, Barjeev Tyagi, Hari Om Gupta, “Optimal Control of Nonlinear Inverted Pendulum System Using PID Controller and LQR: Performance Analysis Without and With Disturbance Input,” International Journal of Automation and Computing, 661-670, December, 2014.

[2] Ashish S. Katariya, “OPTIMAL STATE-FEEDBACK AND OUTPUT-FEEDBACK CONTROLLERS FOR THE WHEELED INVERTED PENDULUM SYSTEM,” School of Electrical and Computer Engineering, Georgia Institute of Technology, 7 May, 2010.

[3] Prasanna Priyadarshi, “Optimal Controller Design for Inverted Pendulum System: An experimental Study,” Department of Electrical Engineering, National Institute of Technology, June, 2013.