2.A

(a) $D_1 = S$, $D_2 = C$, $D_3 = C$, $D_4 = R$

$P = 1 \times 0.2 \times 0.4 \times 0.2$

$= \underline{0.016}$ #

2.B

Code:

```
import numpy as np
trans = [
    [0.8, 0.4, 0.2],
    [0.2, 0.4, 0.6],
    [0, 0.2, 0.2]
]
today = [[0], [0], [1]]
states = [["s"], ["c"], ["r"]]
num = int(input("How many days: "))
if today[0][0] == 1:
    print("day 1 is sunny")
elif today[1][0] == 1:
    print("day 1 is cloud")
else:
    print("day 1 is rainy")
for i in range(2, num+1):
    tormorrow_p = np.dot(trans, today)
    tomorrow_d = np.random.choice(np.reshape(
        states, 3), replace=True, p=np.reshape(tormorrow_p, 3))
    if tomorrow_d == "s":
        print("day {} probable is sunny".format(i))
        today = [[1], [0], [0]]
    elif tomorrow_d == "c":
        print("day {} probable is cloudy".format(i))
        today = [[0], [1], [0]]
    else:
        print("day {} probable is rainy".format(i))
        today = [[0], [0], [1]]
```

輸出

第一天是 sunny:

```
PS D:\OneDrive - 國立陽明交通大學\NCTU\課程\自駕車\HW2> python 2_b.py
How many days: 9
day 1 is sunny
day 2 probable is sunny
day 3 probable is sunny
day 4 probable is cloudy
day 5 probable is rainy
day 6 probable is sunny
day 7 probable is sunny
day 8 probable is cloudy
day 9 probable is sunny
```

第一天是 cloudy:

```
PS D:\OneDrive - 國立陽明交通大學\NCTU\課程\自駕車\HW2> python 2_b.py
How many days: 9
day 1 is cloud
day 2 probable is rainy
day 3 probable is rainy
day 4 probable is cloudy
day 5 probable is cloudy
day 6 probable is sunny
day 7 probable is cloudy
day 8 probable is cloudy
day 9 probable is cloudy
```

第一天是 rainy:

```
PS D:\OneDrive - 國立陽明交通大學\NCTU\課程\自駕車\HW2> python 2_b.py
How many days: 9
day 1 is rainy
day 2 probable is rainy
day 3 probable is cloudy
day 4 probable is sunny
day 5 probable is sunny
day 6 probable is cloudy
day 7 probable is rainy
day 8 probable is sunny
day 9 probable is sunny
```

2.C

Code:

```python
import numpy as np
trans = [
    [0.8, 0.4, 0.2],
    [0.2, 0.4, 0.6],
    [0, 0.2, 0.2]
]
```

```python
states = [["s"], ["c"], ["r"]]
s_count, c_count, r_count = (0, 0, 0)


def sim(days):
    today = [[1], [0], [0]]
    num = days
    for i in range(2, num+1):
        tomorrow_p = np.dot(trans, today)
        tomorrow_d = np.random.choice(np.reshape(
            states, 3), replace=True, p=np.reshape(tomorrow_p, 3))
        if tomorrow_d == "s":
            today = np.array([[1], [0], [0]])
        elif tomorrow_d == "c":
            today = np.array([[0], [1], [0]])
        else:
            today = np.array([[0], [0], [1]])
    return tomorrow_d


for i in range(10000):
    wheather = sim(49)
    if wheather == "s":
        s_count = s_count+1
    elif wheather == "c":
        c_count = c_count+1
    else:
        r_count = r_count+1
stationary_distrubution = [s_count/10000, c_count/10000, r_count/10000]
print("Stationary Distriburion is" + str(stationary_distrubution))
```

輸出:

```
PS D:\OneDrive - 國立陽明交通大學\NCTU\課程\自駕車\HW2> python 2_c.py
Stationary Distriburion is[0.6443, 0.2818, 0.0739]
```

2.DE

(d) $x = \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.6 \\ 0 & 0.2 & 0.2 \end{bmatrix} x_0 = Ax = 0$

$\Rightarrow A - I = \begin{bmatrix} -0.2 & 0.4 & 0.2 \\ 0.2 & -0.6 & 0.6 \\ 0 & 0.2 & -0.8 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -1 & 2 & 1 \\ 1 & -3 & 3 \\ 0 & 1 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 2 & 1 \\ 0 & -1 & 4 \\ 0 & 1 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 2 & 1 \\ 0 & -1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow x = \begin{bmatrix} 9t \\ 4t \\ t \end{bmatrix}$

$\Rightarrow$ stationary distribution $= \begin{bmatrix} \frac{9t}{14} \\ \frac{4t}{14} \\ \frac{t}{14} \end{bmatrix} \cong \underline{\begin{bmatrix} 0.642857 \\ 0.285714 \\ 0.071428 \end{bmatrix}}$ #

(e) entrophy $= -\sum_{x} p(x) \log_2 p(x)$

$= -\left(\frac{9}{14} \log_2 \frac{9}{14} + \frac{4}{14} \log_2 \frac{4}{14} + \frac{1}{14} \log_2 \frac{1}{14} \right)$

$\cong -(-0.409725 - 0.51638711 - 0.271954)$

$= \underline{1.19811611}$ #

2.FG

(f) let yesterday $= X_{k-1}$, today $= X_k$

$P(X_{k-1} \mid X_k) = \frac{P(X_k \mid X_{k-1}) P(X_{k-1})}{P(X_k)} = \frac{P(X_k \mid X_{k-1}) P(X_{k-1})}{\sum_{X_{k-1}} P(X_k \mid X_{k-1}) P(X_{k-1})}$。 其中 $P(X_{k-1})$ 為 $\begin{bmatrix} \frac{9}{14} \\ \frac{4}{14} \\ \frac{1}{14} \end{bmatrix}$

today \ yesterday	sunny	Cloudy	Rainy
Sunny	$\frac{0.8 \times \frac{9}{14}}{0.8 \times \frac{9}{14} + 0.4 \times \frac{4}{14} + 0.2 \times \frac{1}{14}}$	$\frac{0.4 \times \frac{4}{14}}{0.8 \times \frac{9}{14} + 0.4 \times \frac{4}{14} + 0.2 \times \frac{1}{14}}$	$\frac{0.2 \times \frac{1}{14}}{0.8 \times \frac{9}{14} + 0.4 \times \frac{4}{14} + 0.2 \times \frac{1}{14}}$
Cloudy	$\frac{0.2 \times \frac{9}{14}}{0.2 \times \frac{9}{14} + 0.4 \times \frac{4}{14} + 0.6 \times \frac{1}{14}}$	$\frac{0.4 \times \frac{4}{14}}{0.2 \times \frac{9}{14} + 0.4 \times \frac{4}{14} + 0.6 \times \frac{1}{14}}$	$\frac{0.6 \times \frac{4}{14}}{0.2 \times \frac{9}{14} + 0.4 \times \frac{4}{14} + 0.6 \times \frac{1}{14}}$
Rainy	0	$\frac{0.2 \times \frac{1}{14}}{0 + 0.2 \times \frac{4}{14} + 0.2 \times \frac{1}{14}}$	$\frac{0.2 \times \frac{1}{14}}{0 + 0.2 \times \frac{4}{14} + 0.2 \times \frac{1}{14}}$

$\Rightarrow$

today \ yesterday	sunny	cloudy	Rainy
Sunny	0.8	0.1778	0.0222
cloudy	0.45	0.4	0.15
Rainy	0	0.8	0.2

#

(g) It will still follow Markov property。

because state X_t is determined by X_{t-1} (Season is another state) #

3.A

3. (a) $P(x_5 \mid z_{2:5}, x_1) = \dfrac{P(z_5 \mid x_5, z_{2:4}, x_1)\, P(x_5 \mid z_{2:4}, x_1)}{P(z_5 \mid z_{2:4}, x_1)} = \eta P(z_5 \mid x_5) P(x_5 \mid z_{2:4}, x_1)$

其中 $P(x_5 \mid z_{2:4}, x_1) = \sum_{x_4} P(x_5 \mid x_4, z_{2:4}, x_1)\, P(x_4 \mid z_{2:4}, x_1)$

$= 0.2 \times 1 = 0.1 \quad = 0.2$

$\Rightarrow \eta P(z_5 \mid x_5) * 0.2 = \eta\, 0.6 \times 0.2$

$\eta = [P(z_5 \mid z_{2:4}, x_1)]^{-1}, \quad P(z_5 \mid z_{2:4}, x_1) = \sum_{x_5'} P(z_5 \mid x_5', z_{2:4}, x_1) * P(x_5' \mid z_{2:4}, x_1)$

$\Rightarrow \sum_{x_5'} P(z_5 = \text{Sunny} \mid x_5')\, p(x_5' \mid z_4)$

$= 0.6 \times 0.2 + 0.3 \times 0.6 + 0 \times 0.2$

$\Rightarrow \dfrac{0.6 \times 0.2}{0.6 \times 0.2 + 0.3 \times 0.6} = \dfrac{2}{5} = \underline{0.4}$ #

3.B

(b) Day 2 future

$P(x_2 \mid z_{2:4}, x_1) = \eta P(z_{2:4} \mid x_2, x_1) \boxed{P(x_2 \mid x_1)}$ $\begin{bmatrix} 0.8 \\ 0.2 \\ 0 \end{bmatrix}$

$\Rightarrow P(z_{2:4} \mid x_2, x_1) = P(z_2 \mid z_{3:4}, x_2, x_1)\, P(z_{3:4} \mid x_2, x_1)$

$= \boxed{P(z_2 \mid x_2)} P(z_{3:4} \mid x_2, x_1)$ $\quad \begin{bmatrix} 0.6 \\ 0.3 \\ 0 \end{bmatrix}$

$P(z_{3:4} \mid x_2, x_1) = \sum_{x_3'} P(z_{3:4} \mid x_3' x_2 x_1)\, P(x_3' \mid x_2, x_1)$

$= \sum_{x_3'} P(z_{3:4} \mid x_3', x_2) \boxed{P(x_3' \mid x_2)} \rightarrow \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.4 & 0.4 & 0.2 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}$

$P(z_3: x_3', x_2) \Rightarrow P(z_{3:4} \mid x_3') = \boxed{P(z_3 \mid x_3')} P(z_4 \mid x_3')$ $\rightarrow \begin{bmatrix} 0.6 \\ 0.3 \\ 0 \end{bmatrix}$

$P(z_4 \mid x_3') = \sum_{x_4'} \boxed{P(z_4 \mid x_4')} \boxed{P(x_4' \mid x_3')}$

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.4 & 0.4 & 0.2 \\ 0.2 & 0.6 & 0.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0.2 \\ 0.2 \end{bmatrix}$

$P(z_{3:4} \mid x_3', x_2) \Rightarrow \begin{bmatrix} 0 \\ 0.2 \\ 0.2 \end{bmatrix} \times \begin{bmatrix} 0.6 \\ 0.3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.06 \\ 0 \end{bmatrix}$

$P(z_{3:4} \mid x_2, x_1) = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.4 & 0.4 & 0.2 \\ 0.2 & 0.6 & 0.2 \end{bmatrix} \begin{bmatrix} 0 \\ 0.06 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.012 \\ 0.024 \\ 0.036 \end{bmatrix}$

$P(z_{2:4} \mid x_2, x_1) = \begin{bmatrix} 0.6 \\ 0.3 \\ 0 \end{bmatrix} \begin{bmatrix} 0.012 \\ 0.024 \\ 0.036 \end{bmatrix}$

$P(x_2 \mid z_{2:4}, x_1) = \begin{bmatrix} 0.8 \\ 0.2 \\ 0 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.3 \\ 0 \end{bmatrix} \begin{bmatrix} 0.012 \\ 0.024 \\ 0.036 \end{bmatrix} = \begin{bmatrix} 0.00576 \\ 0.00144 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0.8 \\ 0.2 \\ 0 \end{bmatrix} \Rightarrow$ Sunny 80%
cloudy 20%
Rainy 0% #

Day 3 past

$P(x_3 \mid x_1, z_{2:3}) = \eta P(z_3 \mid x_3, x_1, z_2)\, P(x_3 \mid x_1, z_2) = \eta P(z_3 \mid x_3)\, P(x_3 \mid x_1, z_2)$

$P(x_3 \mid x_1, z_2) = \sum_{x_2'} P(x_3 \mid x_2', x_1, z_2)\, P(x_2' \mid x_1, z_2) = \sum_{x_2'} P(x_3 \mid x_2') \underline{P(x_2' \mid z_2, x_1)}$ by 上

$= \sum_{x_2'} P(x_3 \mid x_2') \begin{bmatrix} 8/9 \\ 1/9 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.4 & \\ 0.2 & 0.4 & 0.6 \\ 0.0 & 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 8/9 \\ 1/9 \\ 0 \end{bmatrix} = \begin{bmatrix} 6.8 \\ 2.0 \\ 0.2 \end{bmatrix}$

$P(x_3 \mid x_1, z_{2:3}) = \eta \begin{bmatrix} 0.6 \\ 0.3 \\ 0 \end{bmatrix} \begin{bmatrix} 6.8 \\ 2.0 \\ 0.2 \end{bmatrix}$

$= \eta \begin{bmatrix} 4.08 \\ 0.6 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \frac{4.08}{4.08 + 0.6} \\ \frac{0.6}{4.08 + 0.6} \\ 0 \end{bmatrix} = \begin{bmatrix} 0.87179 \\ 0.128205 \\ 0 \end{bmatrix} \Rightarrow$ 87.18% Sunny
12.82% Cloudy
0% Rainy

<u>Day 3 future</u>

$$p(x_3 \mid z_{2:4}, x_1) = \eta\, p(z_4 \mid x_3, x_1, z_{2:3})\, p(x_3 \mid x_1, z_{2:3}) = \eta\, \underbrace{p(z_4 \mid x_3)}_{\begin{bmatrix} 0 \\ 0.2 \\ 0.2 \end{bmatrix}}\, p(x_3 \mid x_2)$$

$$p(x_3 \mid x_2, z_3) = \eta'\, p(z_3 \mid x_2, x_3)\, p(x_3 \mid x_2) = \eta' \begin{bmatrix} 0.6 \\ 0.3 \\ 0 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.4 \\ 0.2 \end{bmatrix} = \eta' \begin{bmatrix} 0.48 \\ 0.12 \\ 0 \end{bmatrix}$$

$$p(x_3 \mid z_{2:4}, x_1) = \eta\eta' \begin{bmatrix} 0.48 \\ 0.12 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.2 \\ 0.2 \end{bmatrix} = \eta\eta' \begin{bmatrix} 0 \\ 0.024 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow$
0% sunny
100% cloudy
0% rainy

<u>Day 4 past</u>

$$p(x_4 \mid z_{2:4}, x_1) = \eta\, p(z_4 \mid x_4, x_1, z_{2:3})\, p(x_4 \mid z_{2:3}, x_1)$$

$$= \eta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.2 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$
0% sunny
0% cloudy
100% rainy

3.C

(c) Day 3 must be cloudy

Day 4 must be rainy

Day 2 will 80% sunny & 20% cloudy

Thus, 80% probability sequence : sunny, cloudy, raining

20% " " : cloudy, cloudy, raining