STOCHASTIC PROCESSES LECTURE#6

6

6.1 NON-DECREASING FAMILIES OF σ-ALGEBRAS IN Ω

With any stochastic process $(X_t, t \in T)$ we can associate a family of σ-algebras in the sample space Ω
(6–1)

$$\mathcal{F}_t = \sigma(X_u, u \le t) = \sigma\left\{\{\omega \in \Omega : X_u(\omega) \in C\}, u \le t, C \in \mathcal{S}\right\} \qquad t \in T$$

The family (6–1) is non-decreasing: $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$ for $t_1 < t_2$; it is often referred to as a filtration.

6.2 THE WIENER PROCESS

A Wiener process is defined as a stochastic process W_t, where the time parameter t changes in some interval, satisfying the following conditions:

1) For $t_0 < t_1 < \ldots < t_n$ the increments

$$W(t_1) - W(t_0) \quad W(t_2) - W(t_1) \quad \ldots \quad W(t_n) - W(t_{n-1})$$

are independent random variables.

2) For two time moments $s < t$, the increment $W_t - W_s$ has the normal distribution with parameters $(0, t-s)$ (i.e., with the expectation equal to 0 and the variance equal to $t - s$).

3) The trajectories $W_\bullet(\omega)$ are continuous functions of t for every $\omega \in \Omega$.

Question: is the Wiener process determined uniquely by the requirements 1) and 2) in the sense of its finite-dimensional distributions? The answer is NO because if W_t is a Wiener process, so is $W_t + C$ where C is an arbitrary constant. But if we impose on the Wiener process the condition that it is equal to some constant x_0 at some time moment t_0 then its finite-dimensional distributions are determined in a unique way by the requirements 1–2).

THEOREM 6.1

Let W_t be the Wiener process defined for $t \in [t_0, +\infty[$ and such that

$$W(t_0) = x_0$$

The n-dimensional distribution $\mu_{t_1,\ldots,t_n}$ for n time moments $t_1 < \ldots < t_n$ strictly greater than t_0 is a continuous distribution characterized by the following probability density function:

$$p_{t_1,\ldots,t_n}(x_1,\ldots,x_n) =$$

$$\frac{1}{\sqrt{2\pi(t_1-t_0)}} e^{-\frac{(x_1-x_0)^2}{2(t_1-t_0)}} \times \frac{1}{\sqrt{2\pi(t_2-t_1)}} e^{-\frac{(x_2-x_1)^2}{2(t_2-t_1)}} \times \ldots \times \frac{1}{\sqrt{2\pi(t_n-t_{n-1})}} e^{-\frac{(x_n-x_{n-1})^2}{2(t_n-t_{n-1})}}$$

Now the question that remains is: does such a stochastic process exist? Here the Kolmogorov consistency condition comes into play (see Lecture #2).

THE CONSISTENCY CONDITION: It is enough to verify that integrating with respect to x_i

$$\int_{-\infty}^{+\infty} \left[\frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} e^{-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}} \times \frac{1}{\sqrt{2\pi(t_{i+1} - t_i)}} e^{-\frac{(x_{i+1} - x_i)^2}{2(t_{i+1} - t_i)}} \right] dx_i \tag{6–2}$$

results in

$$\frac{1}{\sqrt{2\pi(t_{i+1} - t_{i-1})}} e^{-\frac{(x_{i+1} - x_{i-1})^2}{2(t_{i+1} - t_{i-1})}}$$

The rest of the factors in the densities

$$p_{t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_n}(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$$

and

$$p_{t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

are the same.
The integral in Eq.(6–2) is the formula for the distribution density of the sum of two independent random variables

$$p_{U+V}(v) = \int_{-\infty}^{+\infty} p_U(u) p_V(v - u)\, du \qquad (u = x_i\,,\, v = x_{i+1})$$

where U has the normal distribution with parameters $(x_{i-1}, [t_i - t_{i-1}])$ and V has the normal distribution with parameters $(0, [t_{i+1} - t_i])$. $U + V$ being the sum of two independent normal random variables, it also has the normal distribution with expectation $x_{i-1} + 0$ and variance $(t_i - t_{i-1}) + (t_{i+1} - t_i) = t_{i+1} - t_{i-1}$. □

Back to the Wiener process definition: Let $\{\mathcal{F}_t\}_{t \geq t_0}$ be the non-decreasing family of σ-algebras (or filtration) generated by the Wiener process $(W_t, t \geq t_0)$ starting from x_0. The condition 1) can be reformulated as

1′) For $s < t$ the increments $W_t - W_s$ is independent of the σ-algebra $\mathcal{F}_s = \sigma(W_u, u \leq s)$.