Non-Affine Transforms in 2D?

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A recent query on the MSDN Forum for WPF asked if it's possible to apply a non-affine transform to two-dimensional graphics. The simple answer is *No*. The 3×3 *Matrix* structure in the *System.Windows.Media* namespace does not allow setting the third column of the matrix required for non-affine transforms.

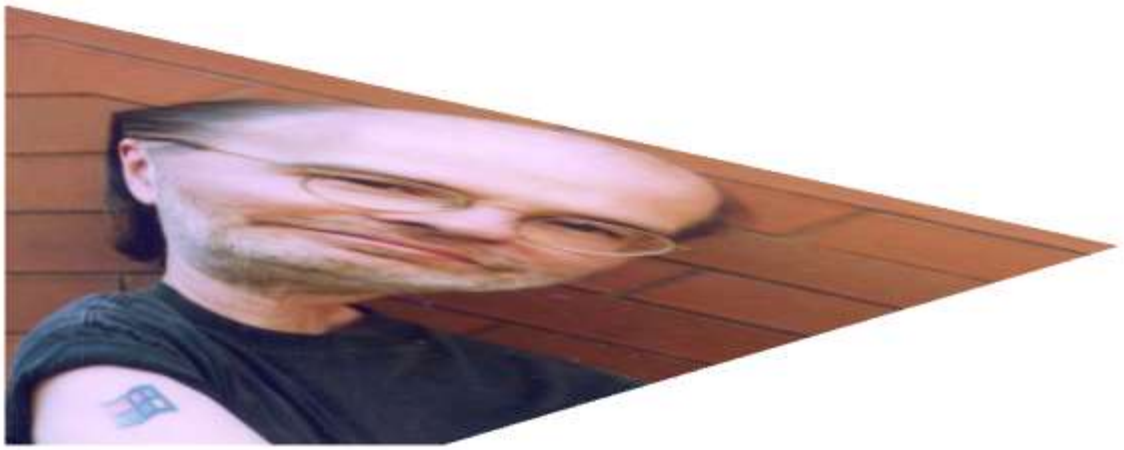
However, non-affine transforms *are* allowed in WPF 3D and, indeed, you might be able to get the effect you want without getting involved with transforms at all.

The two programs presented here both display a photo of myself in a square. Using the mouse, you can grab any one of the corners and drag it. *You must click within the image!* The nearest corner will jump to the mouse position and then you can drag the corner somewhere else. As you drag a corner, the other corners remain fixed. Here's the first version:

NonAffineImageTransform1.xbap

This is the simple approach: It uses WPF 3D to display a square on the XY plane. The 3D coordinates defined in the *Positions* collection of the *MeshGeometry3D* are (0, 0, 0), (0, 1, 0), (1, 0, 0), and (1, 1, 0). Whenever the image is clicked or dragged, the program gets the two-dimensional mouse point, converts it to 3D coordinates (with a simple method that only works with *OrthographicCamera* pointed straight back along the Z axis), and sets the proper item in the *Positions* collection. Here's the source code.

The problem with this technique is that the image is divided into two triangles, one on the lower left and the other on the upper right, and if you drag the bottom-left or top-right corner, only half the image is stretched, like this:



You can practically see the diagonal from the upper-left corner to the lower-right. Nothing below that diagonal is distorted.

Perhaps a better approach is to apply an actual non-affine transform to the *GeometryModel3D*. This is done in the following program:

NonAffineImageTransform2.xbap

Yes, you can easily drag a corner to a place where the transform breaks down and the image flips over in strange ways. But, as you can see, whenever any corner is dragged, the entire image is affected:

Here's the source code for this second version. For the theoretical analysis that follows, I'll be assuming that we're just working with two dimensions. Converting to a flat surface in three dimensions where Z equals 0 is trivial.

The transform we want produces the following mappings (and I hope you're seeing arrows between the pairs of points):

$(0, 0) \rightarrow (x_0, y_0)$
$(0, 1) \rightarrow (x_1, y_1)$
$(1, 0) \rightarrow (x_2, y_2)$
$(1, 1) \rightarrow (x_3, y_3)$

The coordinates on the left of each line are the original coordinates of the corners of the image; the coordinates on the right are the four points we want for those corners. In general, this is a non-affine transform between it maps a square to an arbitrary quadrilateral. Affine transforms always map

squares to parallelograms. The transform we desire will be much easier to derive if we break it down into two transforms:

$(0, 0) \rightarrow (0, 0) \rightarrow (x_0, y_0)$
$(0, 1) \rightarrow (0, 1) \rightarrow (x_1, y_1)$
$(1, 0) \rightarrow (1, 0) \rightarrow (x_2, y_2)$
$(1, 1) \rightarrow (a, b) \rightarrow (x_3, y_3)$

The first transform is obviously a non-affine transform that I'll call **B**. The second transform is something that I'll force to be an affine transform called **A** (for "affine"). The composite transform is **B**×**A**. The task here is to derive the two transforms plus the point (a, b). Let's derive the affine transform first.

An affine transform always maps a square to a parallelogram, so it is completely determined by the mappings of three points. I'll use the first three in the list:

$(0, 0) \rightarrow (x_0, y_0)$
$(0, 1) \rightarrow (x_1, y_1)$
$(1, 0) \rightarrow (x_2, y_2)$

A 3×3 affine matrix can be represented like this (using the property names of the *Matrix* structure):

M11	M12	0
M21	M22	0
OffsetX	OffsetY	1

The transform formulas are:

$x' = M11 \cdot x + M21 \cdot y + OffsetX$
$y' = M12 \cdot x + M22 \cdot y + OffsetY$

It is easy to apply the transform to the points (0, 0), (0, 1), and (1, 0), and solve for the elements of the matrix:

$M11 = x_2 - x_0$
$M12 = y_2 - y_0$
$M21 = x_1 - x_0$
$M22 = y_1 - y_0$
$OffsetX = x_0$
$OffsetY = y_0$

It's not necessary to know this, but the fourth point of the square, which is (1, 1), is mapped to (M11 + M21 + OffsetX, M12 + M22 + OffsetY), which is the fourth point of the parallelogram. But we're not actually concerned with this point in this exercise. Instead, we want this affine transform to map a point (a, b) to the point (x_3, y_3). What is this point (a, b)? If we apply the affine transform to (a, b) and solve for a and b, we get:

$a = (M22 \cdot x_3 - M21 \cdot y_3 + M21 \cdot OffsetY - M22 \cdot OffsetX) / (M11 \cdot M22 - M12 \cdot M21)$
$b = (M11 \cdot y_3 - M12 \cdot x_3 + M12 \cdot OffsetX - M11 \cdot OffsetY) / (M11 \cdot M22 - M12 \cdot M21)$

Now let's take a shot at the non-affine transform, which needs to yield the following mappings:

(0, 0) → (0, 0)
(0, 1) → (0, 1)
(1, 0) → (1, 0)
(1, 1) → (a, b)

The generalized non-affine transform (using property names that are *not* defined in the *Matrix* structure) is:

M11	M12	M13
M21	M22	M23
OffsetX	OffsetY	M33

And the transform formulas are:

x' = (M11•x + M21•y + OffsetX) / (M13•x + M23•y + M33)
y' = (M12•x + M22•y + OffsetY) / (M13•x + M23•y + M33)

The point (0, 0) is mapped to (0, 0), which tells us that OffsetX and OffsetY are zero, and M33 is non-zero. Let's go out on a limb and say that M33 is 1.

The point (0, 1) is mapped to (0, 1), which tells us that M21 is zero and M23 = M22 – 1.

The point (1, 0) is mapped to (1, 0), which tells us that M12 is zero and M13 = M11 – 1.

The point (1, 1) is mapped to (a, b), which requires a bit of algebra to derive the following:

M11 = a / (a + b – 1)
M22 = b / (a + b – 1)

The a and b values have already been calculated in connection with the affine transform.

The derivations of the affine matrix **A** and the non-affine matrix **B** are implemented in the *CalculateNonAffineTransform* method in the NonAffineImageTransform2.cs file. Of course, the method actually returns a *Matrix3D* object that is applied to the *GeometryModel3D* containing the image.

Using 3D graphics to implement a two-dimensional non-affine transform may sound a bit extravagant, but keep in mind that *Viewport3D* is a WPF element much like any other. You can easily mix it in with panels, *TextBlock* elements, controls, et cetera. In particular, it's very easy to determine the required size of the *Viewport3D* based on the size of figures viewed with *OrthographicCamera*, and to convert between the two coordinate systems.