Let $g'(c) < 0$. Then there exists a positive δ such that $g(x) < g(c)$ for all $x \in [a, b]$ satisfying $c < x < c + \delta$. This contradicts that $g(c)$ is the minimum value of g on $[a, b]$. Therefore $g'(c) \not< 0$.

Consequently, $g'(c) = 0$, i.e., $f'(c) = k$ and the theorem is established.

Corollary. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be differentiable on I. If $f'(a)f'(b) < 0$ then there exists a point c in (a, b) such that $f'(c) = 0$.

Note. Darboux's theorem is the intermediate-value property of the derived function f' on $[a, b]$. Although a derived function f' may not be a continuous function on $[a, b]$ (Ex.2, Page 315) the intermediate-value property holds for a derived function.

Worked Example.

1. Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = 0, x \in [-1, 0]$
$= 1, x \in (0, 1]$.

Does there exist a function g such that $g'(x) = f(x), x \in [-1, 1]$?

If possible, let there exist a function $g : [-1, 1] \to \mathbb{R}$ such that $g'(x) = f(x), x \in [-1, 1]$.

Then g is differentiable on $[-1, 1]$ and $g'(x) = 0, x \in [-1, 0]$
$= 1, x \in (0, 1]$.

Since g is differentiable on $[-1, 1]$ and $g'(-1) \neq g'(1)$, by Darboux's theorem g' must assume every real number lying between $g'(-1)$ and $g'(1)$, i.e., between 0 and 1 on $[-1, 1]$. But this is not so and therefore g does not exist.

Theorem 9.4.2. Let I be an interval and a function $f : I \to \mathbb{R}$ be differentiable on I. Then $f'(I)$ is an interval.

[A subset S of $\mathbb{R}$ is an interval if for any two points $c, d \in S$ with $c < d$, the closed interval $[c, d] \subset S$].

Proof. Let $p, q \in f'(I)$ and $p < q$. There exists points $c, d \in I$ such that $f'(c) = p, f'(d) = q$. Let $r \in (p, q)$. Then $p < r < q$.

By Darboux's theorem, there exists a point x_0 in (c, d) [or (d, c)] such that $f'(x_0) = r$. Therefore $r \in f'(I)$ and that implies $(p, q) \subset f'(I)$. Also $p \in f'(I)$ and $q \in f'(I)$. Hence $[p, q] \subset f'(I)$.

Therefore $f'(I)$ is an interval.

Theorem 9.4.3. If $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ then the derived function f' cannot have a jump discontinuity on $[a, b]$.

Proof. If for some $c \in (a, b]$ $\lim\limits_{x \to c-} f'(x)$ does not exist, or for some $c \in [a, b)$ $\lim\limits_{x \to c+} f'(x)$ does not exist then clearly c cannot be a point of jump discontinuity of f'.

Therefore let us assume that for each $c \in (a, b]$, $\lim_{x \to c-} f'(x)$ exists and also for each $c \in [a, b)$, $\lim_{x \to c+} f'(x)$ exists. Now it is sufficient to prove

(i) $\lim_{x \to c-} f'(x) = f'(c)$ for all $c \in (a, b]$, and

(ii) $\lim_{x \to c+} f'(x) = f'(c)$ for all $c \in [a, b)$.

Let $c \in (a, b]$ and let $\lim_{x \to c-} f'(x) = l$. We prove that $f'(c) = l$.

If not, let $l < f'(c)$. Let us choose $\epsilon > 0$ such that $l + \epsilon < f'(c)$.

Since $\lim_{x \to c-} f'(x) = l$, there exists a positive δ such that

$l - \epsilon < f'(x) < l + \epsilon$ for all $x \in (c - \delta, c) \cap (a, b]$.

Let $d \in (c - \delta, c) \cap (a, b]$. Then $l - \epsilon < f'(d) < l + \epsilon < f'(c)$.

By Darboux's theorem on $[d, c] \subset [a, b]$, there exists a point ξ in (d, c) such that $f'(\xi) = l + \epsilon$.

But $\xi \in (d, c) \Rightarrow \xi \in (c - \delta, c) \cap (a, b]$ and this implies $f'(\xi) < l + \epsilon$, a contradiction. Therefore $l \not< f'(c)$.

Next let $l > f'(c)$. Let us choose $\epsilon_1 > 0$ such that $l - \epsilon_1 > f'(c)$.

Since $\lim_{x \to c-} f'(x) = l$, there exists a positive δ_1 such that

$l - \epsilon_1 < f'(x) < l + \epsilon_1$ for all $x \in (c - \delta_1, c) \cap (a, b]$.

Let $d_1 \in (c - \delta_1, c) \cap (a, b]$. Then $f'(c) < l - \epsilon_1 < f'(d_1)$.

By Darboux's theorem on $[d_1, c] \subset [a, b]$, there exists a point ξ_1 in (d_1, c) such that $f'(\xi_1) = l - \epsilon_1$.

But $\xi_1 \in (d_1, c) \Rightarrow \xi_1 \in (c - \delta_1, c) \cap (a, b]$ and this implies $f'(\xi_1) > l - \epsilon_1$, a contradiction. Therefore $l \not> f'(c)$.

Thus $\lim_{x \to c-} f'(x) = f'(c)$ for all $c \in (a, b]$.

In a similar manner it can be proved that $\lim_{x \to c+} f'(x) = f'(c)$ for all $c \in [a, b)$. This completes the proof.

Note 1. A derived function on an interval can have a discontinuity of second kind. (Ex.3.)

2. If f' exists in some deleted neighbourhood of c and $\lim_{x \to c-} f'(x) \neq \lim_{x \to c+} f'(x)$ then f cannot be differentiable at c. (Ex.4.)

3. If a function f be differentiable on an interval I and f' is monotonic on I then f' is continuous on I.

It follows from the property that a monotone function can have only jump discontinuities in its domain.

Worked Examples (continued).

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 \sin \frac{1}{x}, x \neq 0$
$= 0, \quad x = 0.$

Show that f is differentiable on $\mathbb{R}$ but f' is not continuous on $\mathbb{R}$.

For $x \neq 0, f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$.

$\lim\limits_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x \to 0} x \sin \frac{1}{x} = 0$, by Theorem 7.1.7.

Therefore $f'(0) = 0$. Hence f is differentiable on $\mathbb{R}$ and the derived function f' is defined by $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}, x \neq 0$
$= 0, x = 0.$

Now $\lim\limits_{x \to 0} f'(x)$ does not exist, since $\lim\limits_{x \to 0} \cos \frac{1}{x}$ does not exist. This proves that f' is not continuous at 0.

Note. f' is bounded on any neighbourhood $(-\delta, \delta)$ of 0. 0 is a point of oscillatory discontinuity of f'.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 \sin \frac{1}{x^2}, x \neq 0$
$= 0, \quad x = 0.$

Show that f is differentiable on $\mathbb{R}$ but f' is not continuous on $\mathbb{R}$.

For $x \neq 0, f'(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}$.

$\lim\limits_{x \to 0} \frac{f(x)-f(c)}{x-0} = \lim\limits_{x \to 0} \dfrac{x^2 \sin \frac{1}{x^2}}{x} = \lim\limits_{x \to 0} x \sin \frac{1}{x^2} = 0.$

Therefore $f'(0) = 0$. Hence f is differentiable on $\mathbb{R}$ and the derived function f' is defined by

$$\begin{aligned} f'(x) &= 2x \sin \tfrac{1}{x^2} - \tfrac{2}{x} \cos \tfrac{1}{x^2}, x \neq 0 \\ &= 0, x = 0. \end{aligned}$$

Now $\lim\limits_{x \to 0} f'(x)$ does not exist, since $\lim\limits_{x \to 0} \frac{2}{x} \cos \frac{1}{x^2}$ does not exist. This proves that f' is not continuous at 0.

f' is unbounded on any neighbourhood of 0. 0 is a point of infinite discontinuity of f'.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|, x \in \mathbb{R}$.

$$\begin{aligned} \text{Then } f(x) &= x, x > 0 \\ &= 0, x = 0 \\ &= -x, x < 0. \end{aligned}$$

For $x > 0, f'(x) = 1$ and when $x < 0, f'(x) = -1$. $\lim\limits_{x \to 0-} f'(x) \neq \lim\limits_{x \to 0+} f'(x)$. This does not contradict the Theorem 9.4.3, because f is not differentiable at 0.

9.5. Rolle's theorem and Mean value theorems.

Theorem 9.5.1. (Rolle's theorem)

Let a function $f : [a, b] \to \mathbb{R}$ be such that

(i) f is continuous on $[a, b]$,

(ii) f is differentiable at every point of (a, b), and

(iii) $f(a) = f(b)$.

Then there exists at least one point ξ in (a, b) such that $f'(\xi) = 0$.

Proof. Since f is continuous on $[a, b]$, f is bounded on $[a, b]$.

Let $\sup\limits_{x\in[a,b]} f(x) = M$, $\inf\limits_{x\in[a,b]} f(x) = m$.

By the property of continuity there exists a point c in $[a, b]$ such that $f(c) = M$ and there exists point d in $[a, b]$ such that $f(d) = m$.

Two cases arise.

Case 1. $M = m$.

In this case $f(x) = M$ for all $x \in [a, b]$. Therefore $f'(x) = 0$ for all $x \in [a, b]$. The theorem holds trivially in this case.

Case 2. $M \neq m$.

In this case at least one of M and m, if not both, must be unequal to $f(a)$ (and $f(b)$).

Let $M \neq f(a)$. Then $c \neq a, c \neq b$. $\therefore$ $a < c < b$.

By condition (ii), $f'(c)$ exists.

If possible, let $f'(c) > 0$. Then $\lim\limits_{x\to c} \dfrac{f(x) - f(c)}{x - c} > 0$.

Hence there exists a positive δ such that $\frac{f(x)-f(c)}{x-c} > 0$ for all $x \in [a, b]$ satisfying $0 <| x - c |< \delta$.

This implies $f(x) - f(c) > 0$ for all $x \in [a, b]$ satisfying $c < x < c + \delta$
i.e., $f(x) > M$ for all $x \in [a, b]$ satisfying $c < x < c + \delta$.

This contradicts that M is the supremum of f on $[a, b]$.
Consequently, $f'(c) \not> 0$ ... ... (i)

If possible, let $f'(c) < 0$. Then $\lim\limits_{x\to c} \dfrac{f(x) - f(c)}{x - c} < 0$.

Hence there exists a positive δ such that $\frac{f(x)-f(c)}{x-c} < 0$ for all $x \in [a, b]$ satisfying $0 <| x - c |< \delta$.

This implies $f(x) > f(c)$ for all $x \in [a, b]$ satisfying $c - \delta < x < c$
i.e., $f(x) > M$ for all $x \in [a, b]$ satisfying $c - \delta < x < c$.
This contradicts that M is the supremum of f on $[a, b]$.
Consequently, $f'(c) \not< 0$ ... ... (ii)

From (i) and (ii) it follows that $f'(c) = 0$. Therefore $\xi = c$.

If however, $M = f(a) = f(b)$, then $m \neq f(a)$ and therefore $d \neq a, d \neq b$ and $a < d < b$.

Proceeding with similar arguments we can prove $f'(d) = 0$. Therefore $\xi = d$.

This completes the proof of the theorem.

Note. The set of conditions in Rolle's theorem is a set of sufficient conditions in the sense that $f'(x)$ may be zero at some point ξ in (a, b) even if the conditions of the theorem do not hold together. To establish this, let us consider the function

$f(x) = |x| + |x - 1|, x \in [-1, 2]$.

$$\begin{aligned} f(x) &= 2x - 1, 1 < x \leq 2 \\ &= 1, \quad 0 \leq x \leq 1 \\ &= 1 - 2x, -1 \leq x < 0. \end{aligned}$$

f is continuous on $[-1, 2]$; f is not differentiable at 0 and 1; $f(-1) = f(2) = 3$.

Therefore f does not satisfy the second condition of Rolle's theorem.

But $f'(x) = 0$ for all $x \in (0, 1)$.

Although f does not satisfy all the conditions of Rolle's theorem together on $[-1, 2], f'(x) = 0$ at some points in $(-1, 2)$.

Corollary. Rolle's theorem for polynomials.

If a polynomial function f has at least two real roots, then between any two real roots there exists at least one real root of the derived polynomial function f'.

Let α, β be two real roots of the polynomial function $f, \alpha < \beta$.

Then (i) f is continuous on $[\alpha, \beta]$, (ii) f is differentiable on (α, β) and (iii) $f(\alpha) = f(\beta)$.

Therefore by Rolle's theorem, there exists at least one real number ξ in (α, β) such that $f'(\xi) = 0$. That is, ξ is a real root of the derived polynomial function f'.

Geometrical Interpretation.

If a function f has a graph which is a continuous curve on the interval $[a, b]$; and the curve has a tangent at every point on it with abscissa between a and b; and the ordinates $f(a), f(b)$ are equal, then there exists at least one point ξ in (a, b) such that the tangent to the curve at $(\xi, f(\xi))$ is parallel to the x-axis.

Worked Examples.

1. If $p(x)$ is a polynomial of degree > 1 and $k \in \mathbb{R}$, prove that between any two real roots of $p(x) = 0$ there is a real root of $p'(x) + kp(x) = 0$.

Let $f(x) = e^{kx}p(x), x \in \mathbb{R}$.
Then $f'(x) = e^{kx}[kp(x) + p'(x)], x \in \mathbb{R}$.
Let α, β be two real roots of $p(x)$ and $\alpha < \beta$. Then $p(\alpha) = 0, p(\beta) = 0$.
Therefore $f(\alpha) = e^{k\alpha}p(\alpha) = 0$. $f(\beta) = e^{k\beta}p(\beta) = 0$.
f is continuous on $[\alpha, \beta]$; $f'(x)$ exists for all $x \in (\alpha, \beta)$; and $f(\alpha) = f(\beta)$.
By Rolle's theorem, $f'(\gamma) = 0$ for some γ in (α, β).
or, $e^{k\gamma}[kp(\gamma) + p'(\gamma)] = 0$.
This implies $kp(\gamma) + p'(\gamma) = 0$, since $e^{k\gamma} \neq 0$.
That is, γ is a root of $kp(x) + p'(x) = 0$, where $\alpha < \gamma < \beta$.

2. The functions u, v, u' and v' are all continuous on $\mathbb{R}$ and $uv' - u'v \neq 0$ in $\mathbb{R}$. Prove that between any two consecutive real roots of $u = 0$ lies one real root of $v = 0$ and between any two consecutive real roots of $v = 0$ lies one real root of $u = 0$.

Let α, β be any two consecutive real roots of $u = 0, \alpha < \beta$. We prove that there exists a real root of $v = 0$ in (α, β)

Since $u(\alpha) = 0$ and $u(\beta) = 0, v(\alpha) \neq 0$ and $v(\beta) \neq 0$ by the given condition.

If possible, let $v = 0$ has no real root in (α, β). Then $v \neq 0$ in $[\alpha, \beta]$. Let $f = \frac{u}{v}$ in $[\alpha, \beta]$.

Then the function f is continuous on $[\alpha, \beta]$; $f' = \frac{vu' - v'u}{v^2}$ exists in (α, β); and $f(\alpha) = 0 = f(\beta)$.

By Rolle's theorem, there exists a point ξ in (α, β) such that $f'(\xi) = 0$.

This implies $uv' - u'v = 0$ at ξ. This is a contradiction to the hypothesis. Therefore $v = 0$ has a real root lying in (α, β).

Let γ, δ be two consecutive real roots of $v = 0, \gamma < \delta$. Similar arguments will establish that $u = 0$ has a real root lying in (γ, δ).

Observation. Taking $u = \sin x, v = \cos x$ it follows that between any two consecutive zeroes of $\sin x$ there is a zero of $\cos x$, and conversely.

3. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $f''(x)$ exists for all $x \in (a, b)$. Let $a < c < b$. Prove that there exists a point ξ in (a, b) such that

$$f(c) = \frac{b-c}{b-a}f(a) + \frac{c-a}{c-b}f(b) + \frac{1}{2}(c-a)(c-b)f''(\xi).$$

Let $\phi : [a, b] \to \mathbb{R}$ be defined by

$$\phi(x) = f(x) - \frac{(x-b)(x-c)}{(a-b)(a-c)}f(a) - \frac{(x-c)(x-a)}{(b-c)(b-a)}f(b) - \frac{(x-a)(x-b)}{(c-a)(c-b)}f(c), \ x \in [a, b].$$

ϕ is continuous on $[a,b]$, since f is continuous on $[a,b]$.

Since $f''(x)$ exists for all $x \in (a,b)$, f' is continuous on (a,b).

Therefore $\phi''(x)$ exists for all $x \in (a,b)$ and ϕ' is continuous on (a,b) and hence ϕ is differentiable on (a,b).

$\phi(a) = 0, \phi(b) = 0, \phi(c) = 0.$

Applying Rolle's theorem to the function ϕ on $[a,c]$ and $[c,b]$,

$\phi'(\xi_1) = 0$ for some $\xi_1 \in (a,c)$ and $\phi'(\xi_2) = 0$ for some $\xi_2 \in (c,b)$.

Applying Rolle's theorem to the function ϕ' on $[\xi_1, \xi_2]$,

$\phi''(\xi) = 0$ for some $\xi \in (\xi_1, \xi_2)$. That is, $\phi''(\xi) = 0$ for some $\xi \in (a,b)$.

But $\phi''(\xi) = f''(\xi) - \frac{2f(a)}{(a-b)(a-c)} - \frac{2f(b)}{(b-c)(b-a)} - \frac{2f(c)}{(c-a)(c-b)}$.

Hence $f(c) = \frac{b-c}{b-a} f(a) + \frac{c-a}{b-a} f(b) + \frac{1}{2}(c-a)(c-b)f''(\xi), a < \xi < b$.

Theorem 9.5.2. Mean value theorem (Lagrange)

Let a function $f : [a,b] \to \mathbb{R}$ be such that

(i) f is continuous on $[a,b]$, and

(ii) f is differentiable at every point of (a,b).

Then there exists at least a point ξ in (a,b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(\xi).$$

Proof. Let us define $\phi : [a,b] \to \mathbb{R}$ by

$\phi(x) = f(x) + Ax, x \in [a,b]$ where A is a constant.

Clearly, ϕ is continuous on $[a,b]$, since f is continuous on $[a,b]$; and ϕ is differentiable at every point of (a,b), since f is differentiable at every point of (a,b).

Let us choose A such that $\phi(a) = \phi(b)$.

Then $f(a) + Aa = f(b) + Ab$ and this determines $A = \frac{f(b)-f(a)}{a-b}$.

For this choice of A, ϕ satisfies all conditions of Rolle's theorem on $[a,b]$. Therefore there exists at least a point ξ in (a,b) such that $\phi'(\xi) = 0$.

But $\phi'(\xi) = f'(\xi) + A$ and therefore $0 = f'(\xi) + \frac{f(b)-f(a)}{a-b}$

or, $f'(\xi) = \frac{f(b)-f(a)}{b-a}$.

Another form.

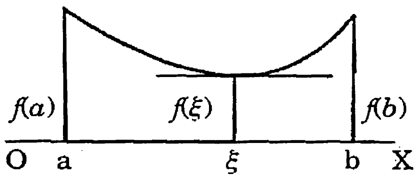
If $b = a + h$ then $b - a = h$ and $\xi = a + \theta h$ for some real number θ satisfying $0 < \theta < 1$. The theorem takes the following form.

Let $f : [a, a+h] \to \mathbb{R}$ be such that (i) f is continuous on $[a, a+h]$, and (ii) f is differentiable at every point of $(a, a+h)$.

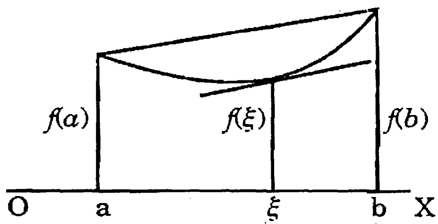
Then there exists a real number θ satisfying $0 < \theta < 1$ such that $f(a+h) = f(a) + hf'(a + \theta h)$.

Geometrical interpretation.

If a function f has a graph which is a continuous curve on the interval $[a, b]$ and the curve has a tangent at every point on it with abscissa between a and b then there exists a point ξ in (a, b) such that the tangent to the curve at $(\xi, f(\xi))$ is parallel to the line segment joining the points $(a, f(a))$ and $(b, f(b))$.



Rolle's theorem



Mean value theorem

Remark.

Rolle's theorem is a particular case of Mean value theorem. If $f(a) = f(b)$ holds in addition to the two conditions of Mean value theorem, then $f(b) - f(a) = 0$ and consequently $f'(\xi) = 0$.

In the particular case, the geometrical interpretation is that there is a point $(\xi, f(\xi))$ on the curve, the tangent at which is parallel to the x-axis.

Theorem 9.5.3. Let a function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ for all $x \in (a, b)$ then f is a constant on $[a, b]$.

Proof. Let $x_1, x_2 \in [a, b]$ and $a \leq x_1 < x_2 \leq b$.

Then f is continuous on $[x_1, x_2]$, and f is differentiable on (x_1, x_2).

By the Mean value theorem there exists a point ξ in (x_1, x_2) such that $f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

But $f'(\xi) = 0$, by hypothesis. Therefore $f(x_2) = f(x_1)$.

Since x_1 and x_2 are arbitrary points in $[a, b]$, f is a constant on $[a, b]$.

Corollary. Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be both continuous on $[a, b]$ and they are both differentiable on (a, b). If $f'(x) = g'(x)$ for all $x \in (a, b)$ then $f = g + c$, where $c (\in \mathbb{R})$ is a constant.

Theorem 9.5.4. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $f'(x) \geq 0$ for all $x \in (a, b)$. Then f is a monotone increasing function on $[a, b]$.

Proof. Let $x_1, x_2 \in [a,b]$ and $a \leq x_1 < x_2 \leq b$.

Then f is continuous on $[x_1, x_2]$, and f is differentiable on (x_1, x_2).

By the Mean value theorem there exists a point ξ in (x_1, x_2) such that $f'(\xi) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. But $f'(\xi) \geq 0$.

Therefore $f(x_2) \geq f(x_1)$, since $x_2 - x_1 > 0$.

Since x_1 and x_2 are arbitrary points in $[a,b]$, f is a monotone increasing function on $[a,b]$.

Note. If $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and $f'(x) > 0$ for all $x \in (a,b)$ then f is a strictly increasing function on $[a,b]$.

Theorem 9.5.5. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and $f'(x) \leq 0$ for all $x \in (a,b)$. Then f is a monotone decreasing function on $[a,b]$.

The proof is similar and left to the reader.

Note. If $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and $f'(x) < 0$ for all $x \in (a,b)$ then f is a strictly decreasing function on $[a,b]$.

Worked Examples.

1. Prove that $\frac{2x}{\pi} < \sin x$ for $0 < x < \frac{\pi}{2}$.

Let $f(x) = \frac{\sin x}{x}$, $0 < x < \frac{\pi}{2}$.

f is continuous on $[\delta, \frac{\pi}{2}]$ for some $\delta > 0$. $f'(x) = \frac{x \sin x - \cos x}{x}$ on $[\delta, \frac{\pi}{2}]$.

Because $x < \tan x$ in $0 < x < \frac{\pi}{2}$, $f'(x) < 0$ in $\delta < x < \frac{\pi}{2}$.

Therefore f is a strictly decreasing function on $(0, \frac{\pi}{2}]$.

Because $f(\frac{\pi}{2}) = \frac{2}{\pi}$, it follows that $f(x) > \frac{2}{\pi}$ for $0 < x < \frac{\pi}{2}$, i.e., $\frac{2x}{\pi} < \sin x$ for $0 < x < \frac{\pi}{2}$.

2. Prove that $\frac{x}{1+x} < \log(1+x) < x$ for all $x > 0$.

Let $f(x) = \log(1+x) - \frac{x}{1+x}, x \geq 0$.

f is continuous on $[0, \infty)$. $f'(x) = \frac{x}{(1+x)^2} > 0$ for all $x > 0$.

Therefore f is a strictly increasing function on $[0, \infty)$.

So $f(x) > f(0)$ for all $x > 0$.

Consequently, $\log(1+x) > \frac{x}{1+x}$ for all $x > 0$ ... ... (i)

Let $g(x) = x - \log(1+x), x \geq 0$.

g is continuous on $[0, \infty)$. $g'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0$ for all $x > 0$.

Therefore g is a strictly increasing function on $[0, \infty)$.

So $g(x) > g(0)$ for all $x > 0$.

Consequently, $x > \log(1+x)$ for all $x > 0$ ... ... (ii)

From (i) and (ii), $\frac{x}{1+x} < \log(1+x) < x$ for all $x > 0$.

3. Prove that $(1+\frac{1}{x})^x > (1+\frac{1}{y})^y$ if $x, y \in \mathbb{R}$ and $x > y > 0$.

Let $f(x) = (1+\frac{1}{x})^x, x > 0$.

$$\begin{aligned}\text{Then } f'(x) &= (1+\tfrac{1}{x})^x[\log(1+\tfrac{1}{x}) + x.\tfrac{1}{1+\frac{1}{x}}.(-\tfrac{1}{x^2})] \\ &= (1+\tfrac{1}{x})^x[\log(1+\tfrac{1}{x}) - \tfrac{\frac{1}{x}}{1+\frac{1}{x}}].\end{aligned}$$

Let $\phi(x) = \log(1+x) - \frac{x}{1+x}, x \geq 0$.

Then ϕ is continuous on $[0,\infty)$ and $\phi'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{1+x^2} > 0$ for $x > 0$.

Hence ϕ is a strictly increasing function on $[0,\infty)$.

Since $\phi(0) = 0, \phi(x) > 0$ for $x > 0$. That is, $\log(1+x) > \frac{x}{1+x}$ for $x > 0$.

It follows that $\log(1+\frac{1}{x}) > \frac{\frac{1}{x}}{1+\frac{1}{x}}$ for $x > 0$.

We also have $(1+\frac{1}{x})^x > 0$ for $x > 0$. Therefore $f'(x) > 0$ for $x > 0$, showing that f is a strictly increasing function for $x > 0$.

Hence $x > y > 0 \Rightarrow (1+\frac{1}{x})^x > (1+\frac{1}{y})^y$.

4. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $| f(x) - f(y) | \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function on $\mathbb{R}$.

By the condition,

$-(x-y)^2 \leq f(x) - f(y) \leq (x-y)^2$ for all $x, y \in \mathbb{R}$.

Let $c \in \mathbb{R}$. Then $-h^2 \leq f(c+h) - f(c) \leq h^2$ for all $h \in \mathbb{R}$.

That is, $-h \leq \frac{f(c+h)-f(c)}{h} \leq h$ if $h > 0$ and $h \leq \frac{f(c+h)-f(c)}{h} \leq -h$ if $h < 0$.

By Sandwich theorem, it follows from the first that $\lim\limits_{h\to 0+} \frac{f(c+h)-f(c)}{h} = 0$; and it follows from the second that $\lim\limits_{h\to 0-} \frac{f(c+h)-f(c)}{h} = 0$.

Consequently, $\lim\limits_{h\to 0} \dfrac{f(c+h)-f(c)}{h} = 0$. That is, $f'(c) = 0$ for all $c \in \mathbb{R}$.

This proves that f is a constant function on $\mathbb{R}$.

5. Use Mean value theorem to prove $0 < \frac{1}{x}\log\frac{e^x-1}{x} < 1$, for $x > 0$.

Let $x_0 > 0$. Let $f(x) = e^x, x \in [0, x_0]$.

Then f is continuous on $[0, x_0]$ and f is differentiable on $(0, x_0)$.

By Mean value theorem, there exists a real number θ satisfying $0 < \theta < 1$ such that $\frac{f(x_0)-f(0)}{x_0} = f'(\theta x_0)$

or, $e^{x_0} - 1 = x_0 e^{\theta x_0}$
or, $\log \frac{e^{x_0}-1}{x_0} = \theta x_0$ or, $\frac{1}{x_0} \log \frac{e^{x_0}-1}{x_0} = \theta$.

Since $0 < \theta < 1$, we have $0 < \frac{1}{x_0} \log \frac{e^{x_0}-1}{x_0} < 1$.

Since $x_0 (> 0)$ is arbitrary, it follows that $0 < \frac{1}{x} \log \frac{e^x - 1}{x} < 1$ for all $x > 0$.

6. A function f is twice differentiable on $[a, b]$ and $f(a) = f(b) = 0$ and $f(c) < 0$ for some c in (a, b). Prove that there is at least one point ξ in (a, b) for which $f''(\xi) > 0$.

Since f'' exists on $[a, b]$, f and f' are both continuous on $[a, b]$.

Applying Mean value theorem on $[a, c]$ and $[c, b]$, we have
$\frac{f(c)-f(a)}{c-a} = f'(\xi_1), a < \xi_1 < c$ and $\frac{f(b)-f(c)}{b-c} = f'(\xi_2), c < \xi_2 < b$.

But $f(a) = f(b) = 0$. Therefore $f'(\xi_1) = \frac{f(c)}{c-a}, f'(\xi_2) = \frac{f(c)}{c-b}$.

Applying Mean value theorem to the function f' on $[\xi_1, \xi_2]$,
$\frac{f'(\xi_2)-f'(\xi_1)}{\xi_2-\xi_1} = f''(\xi)$ for some ξ in (ξ_1, ξ_2).

But $f'(\xi_2) - f'(\xi_1) = \frac{f(c)(b-a)}{(c-b)(c-a)} > 0$, since $f(c) < 0$.
Therefore $f''(\xi) > 0$, since $\xi_2 - \xi_1 > 0$.

Since $\xi_1 < \xi < \xi_2$, ξ lies between a and b and $f''(\xi) > 0$.

7. Let $a, b \in \mathbb{R}$ and $a < b$. If a function f has finite derivative at each point in (a, b), and for $c \in (a, b)$ if $\lim\limits_{x \to c+} f'(x)$ is finite $(= l)$, prove that $f'(c) = l$.

[That is, if $f'(c+0)$ and $f'(c)$ both exist finitely, then $f'(c+0)$ cannot be different from $f'(c)$.]

Let us choose $\delta > 0$ such that $(c, c + \delta) \subset (a, b)$.
Since f is differentiable on (a, b), f is differentiable on $[c, c + \delta]$.

Let $\{h_n\}$ be a sequence of points such that $h_n > 0$ for all $n \in \mathbb{N}$ and $\lim h_n = 0$. Then there exists a natural number k such that $0 < h_n < \delta$ for all $n \geq k$.

By Mean value theorem applied to f on $[c, c + h_n]$ for $n \geq k$,
$\frac{f(c+h_n)-f(c)}{h_n} = f'(c + \theta_n h_n)$ where $0 < \theta_n < 1$.

As $\theta_n > 0, \theta_n h_n > 0$ for all $n \in \mathbb{N}$. As $0 < \theta_n < 1$ and $h_n \to 0$, $\lim \theta_n h_n = 0$.
Since $\lim\limits_{x \to c+} f'(x) = l$, we have $\lim\limits_{\theta_n h_n \to 0+} f'(c + \theta_n h_n) = l$ by sequential criterion.

It follows that $\lim\limits_{h_n \to 0+} \frac{f(c+h_n)-f(c)}{h_n} = l$.

Since f has a finite derivative at c, $\lim\limits_{h \to 0+} \frac{f(c+h)-f(c)}{h} = Rf'(c)$.

By sequential criterion, $\lim\limits_{h_n \to 0+} \frac{f(c+h_n)-f(c)}{h_n} = Rf'(c)$.

Therefore $Rf'(c) = l$.

Since $f'(c)$ exists, $f'(c) = Rf'(c) = l$.

Note. If $\lim\limits_{x \to c-} f'(x)$ and $f'(c)$ be both finite, then $f'(c-0) = f'(c)$.

8. Let $a, b \in \mathbb{R}$ and $a < b$. If a function f be continuous on (a, b) and f has finite derivative at each point in (a, b) except possibly at c. If $\lim\limits_{x \to c} f'(x)$ is finite $(= l)$ then prove that $f'(c) = l$.

[That is, if $f'(c+0)$ and $f'(c-0)$ both exist finitely and be equal, then f has a derivative at c and $f'(c) = f'(c+0) = f'(c-0)$.]

Let us choose $\delta > 0$ such that $(c, c+\delta) \subset (a, b)$.

f is continuous on $[c, c+\delta]$ and f is differentiable on $(c, c+\delta)$.

Let $\{h_n\}$ be a sequence of points such that $h_n > 0$ for all $n \in \mathbb{N}$ and $\lim h_n = 0$. Then there exists a natural number k such that $0 < h_n < \delta$ for all $n \geq k$.

By Mean value theorem applied to f on $[c, c+h_n]$ for $n \geq k$,

$\frac{f(c+h_n)-f(c)}{h_n} = f'(c+\theta_n h_n), 0 < \theta_n < 1$.

As $\theta_n > 0, \theta_n h_n > 0$ for all $n \in \mathbb{N}$. As $0 < \theta_n < 1$ and $h_n \to 0, \lim \theta_n h_n \to 0$.

Since $\lim\limits_{x \to c} f'(x) = l$, $\lim\limits_{x \to c+} f'(x) = l$.

By sequential criterion, $\lim\limits_{\theta_n h_n \to 0+} f'(c+\theta_n h_n) = l$.

It follows that $\lim\limits_{h_n \to 0+} \frac{f(c+h_n)-f(c)}{h_n} = l$.

Since $\{h_n\}$ is an arbitrary sequence converging to 0,

$\lim\limits_{h \to 0+} \frac{f(c+h)-f(c)}{h} = l$. That is, $Rf'(c) = l$.

Also since $\lim\limits_{x \to c-} f'(x)$ exists and equals l, proceeding with similar arguments we can establish that $Lf'(c) = l$. Hence $f'(c) = l$.

Note. We have shown that under the given conditions one-sided derivative $Rf'(c)$ exists if $f'(c+0)$ exists, and $Lf'(c)$ exists if $f'(c-0)$ exists.

The next example shows that if the right hand limit (one-sided limit) of the derivative f' exists at a and f be right continuous at a, then the right hand derivative at a exists.

9. Let $a \in \mathbb{R}$ and a function f be differentiable for $x > a$, right continuous at a and $f'(a+0)$ exists. Show that $Rf'(a)$ exists and $Rf'(a) = f'(a+)$.

Let us consider the interval $[a, a+\delta]$ for some $\delta > 0$.

As f is differentiable on $(a, a+\delta]$, f is continuous on $(a, a+\delta]$. Since f is right continuous at a, f is continuous on $[a, a+\delta]$.

Let us consider a sequence of points $\{h_n\}$ such that $h_n > 0$ for all $n \in \mathbb{N}$ and $\lim h_n = 0$. Then there exists a natural number k such that $h_n < \delta$ for all $n \geq k$. Then $[a, a+h_n] \subset [a, a+\delta]$ for all $n \geq k$.

By Lagrange's Mean value theorem, for $n \geq k$,
$f(a+h_n) - f(a) = h_n f'(a+\theta_n h_n)$ for some θ_n satisfying $0 < \theta_n < 1$.

Therefore $\frac{f(a+h_n)-f(a)}{h_n} = f'(a+\theta_n h_n)$.

Since $h_n \to 0+$ and $0 < \theta_n < 1, \theta_n h_n \to 0+$.

Since $f'(a+)$ exists, $\lim\limits_{\theta_n h_n \to 0+} f'(a+\theta_n h_n) = \lim\limits_{x \to a+} f'(x) = f'(a+0)$.

Therefore $\lim\limits_{h_n \to 0+} \dfrac{f(a+h_n)-f(a)}{h_n}$ exists and equals $f'(a+0)$.

Since $\{h_n\}$ is an arbitrary sequence converging to 0 (from the right), by sequential criterion for limits it follows that
$\lim\limits_{h \to 0+} \frac{f(a+h)-f(a)}{h}$ exists and equals $f'(a+0)$.

But $\lim\limits_{h \to 0+} \frac{f(a+h)-f(a)}{h} = Rf'(a)$, since the limit exists.

Hence $Rf'(a) = f'(a+0)$.

Note. If the left hand limit of the derivative f' exists at b and f be left continuous at b, then the left hand derivative $Lf'(b)$ exists and $Lf'(b) = f'(b-0)$.

However these conditions are sufficient conditions for the existence of the one-sided derivatives. For example, let us consider the function f defined on $\mathbb{R}$ by $f(x) = x^2 \sin\frac{1}{x}, x \neq 0$
$= 0, x = 0$.

Then $f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}, x \neq 0$
$= 0, x = 0$.

Here $f'_+(0) = f'_-(0) = 0$, but neither $f'(0+)$ nor $f'(0-)$ exists.

10. Let $I = [a, b]$ be a closed and bounded interval and $f : I \to \mathbb{R}$ be differentiable on I. Let $J = f(I)$ and $g : J \to \mathbb{R}$ be differentiable on J. Use Mean value theorem to prove that the composite function $g \circ f$ is differentiable on I and $(g \circ f)'(x) = g'(f(x))f'(x)$ for all $x \in I$, assuming that f' and g' are continuous on I and J respectively.

Let c be an interior point of I.

$g \circ f$ is differentiable at c if $\lim_{h \to 0} \dfrac{g \circ f(c+h) - g \circ f(c)}{h}$ exists.

f satisfies all conditions of Mean value theorem on $[c, c+h]$ or on $[c+h, c]$. By Mean value theorem, $f(c+h) = f(c) + hf'(c+\theta h)$ for some real number θ satisfying $0 < \theta < 1$.

Let $f(c) = d \in J, f(c+h) = d+k \in J$.

g satisfies all conditions of Mean value theorem on $[d, d+k]$ or on $[d+k, d]$. By Mean value theorem, $g(d+k) = g(d) + kg'(d+\theta' k)$ for some real number θ' satisfying $0 < \theta' < 1$.

i.e., $g \circ f(c+h) = g \circ f(c) + hf'(c+\theta h)g'[f(c) + \theta' hf'(c+\theta h)]$.

Therefore $\lim_{h \to 0} \dfrac{g \circ f(c+h) - g \circ f(c)}{h}$

$= \lim_{h \to 0} f'(c+\theta h).g'[f(c) + \theta' hf'(c+\theta h)]$

$= f'(c).g'(f(c))$, since f' and g' are both continuous at c.

Therefore $(g \circ f)'(c) = f'(c).g'(f(c))$.

Similar proof for $c = a$ and $c = b$.

Since c is arbitrary, $(g \circ f)'(x) = g'(f(x)).f'(x)$ for all $x \in [a, b]$.

Note. This is an alternative proof for differentiability of the composite function $g \circ f$ under wider conditions.

Theorem 9.5.6. Let I be an interval. If a function $f : I \to \mathbb{R}$ be such that f' exists and is bounded on I then f is uniformly continuous on I.

Proof. Let $x_1, x_2 \in I$ with $x_1 < x_2$.

Since f' is bounded on I, there exists a positive real number k such that $| f'(x) | \leq k$ for all $x \in I$.

f satisfies both the conditions of Mean value theorem on $[x_1, x_2]$ and therefore there exists a point ξ in (x_1, x_2) such that $f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

Therefore $|\frac{f(x_2) - f(x_1)}{x_2 - x_1}| \leq k$. That is, $|f(x_2) - f(x_1)| \leq k|x_2 - x_1|$.

It follows that $| f(x_2) - f(x_1) | \leq k | x_2 - x_1 |$ for all $x_1, x_2 \in I$.

Let us choose $\epsilon > 0$. Then there exists a positive $\delta (= \frac{\epsilon}{k})$ such that $|f(x_2) - f(x_1)| < \epsilon$ for all x_1, x_2 in I satisfying $|x_2 - x_1| < \delta$.

Hence f is uniformly continuous on I.

Note. Under the stated conditions f satisfies a Lipschitz's condition on I.

Worked Example.

1. Prove that the function f defined on $\mathbb{R}$ by $f(x) = \frac{1}{x^2+1}, x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

$f'(x) = -\frac{2x}{(x^2+1)^2}, x \in \mathbb{R}$. Therefore $| f'(x) |< 2$ for all $x \in \mathbb{R}$.

Let x_1, x_2 be any two points in $\mathbb{R}$ such that $x_1 < x_2$.

f is continuous on $[x_1, x_2]$ and f is differentiable on (x_1, x_2). By the Mean value theorem, there exists a point ξ in (x_1, x_2) such that $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(\xi)$.

Since $|f'(x)| < 2$ for all $x \in \mathbb{R}$, $|f(x_1) - f(x_2)| < 2|x_2 - x_1|$.

Let us choose $\epsilon > 0$. There exists a positive $\delta(= \frac{\epsilon}{2})$ such that $| f(x_1) - f(x_2) |< \epsilon$ for all x_1, x_2 in $\mathbb{R}$ satisfying $| x_2 - x_1 |< \delta$.

This proves that f is uniformly continuous on $\mathbb{R}$.

Theorem 9.5.7. (Generalised Mean value theorem)

Let the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be such that

(i) f and g are both continuous on $[a, b]$, and

(ii) f and g are both differentiable on (a, b).

Then there exists a point ξ in (a, b) such that $[g(b) - g(a)]f'(\xi) = [f(b) - f(a)]g'(\xi)$.

Proof. Let us define $\phi : [a, b] \to \mathbb{R}$ by $\phi(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)], x \in [a, b]$.

ϕ is continuous on $[a, b]$, since f and g are continuous on $[a, b]$.

ϕ is differentiable on (a, b), since f and g are differentiable on (a, b). Also $\phi(a) = f(a)g(b) - g(a)f(b) = \phi(b)$.

By Rolle's theorem, $\phi'(\xi) = 0$ for some ξ in (a, b).
But $\phi'(x) = f'(x)[g(b) - g(a)] - g'(x)[f(b) - f(a)]$.
$\phi'(\xi) = 0$ gives $f'(\xi)[g(b) - g(a)] = g'(\xi)[f(b) - f(a)]$.

This completes the proof.

Note. If $g'(x) \neq 0$ for all $x \in (a, b)$, then $g(b) \neq g(a)$; because the condition $g(b) = g(a)$ together with the conditions satisfied by g in the theorem would imply $g'(c) = 0$ for some $c \in (a, b)$, by Rolle's theorem. Therefore if $g'(x) \neq 0$ for all $x \in (a, b)$, the conclusion of the theorem can be expressed as $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$ for some ξ in (a, b).

Theorem 9.5.8. Mean value theorem (Cauchy)

Let the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be such that

(i) f and g are both continuous on $[a, b]$,

(ii) f and g are both differentiable on (a, b), and

(iii) $g'(x) \neq 0$ for all $x \in (a, b)$.

Then there exists a point ξ in (a, b) such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$.

Proof. $g(a) \neq g(b)$. Because, if $g(a) = g(b)$ then g would satisfy all conditions of Rolle's theorem on $[a, b]$ and consequently $g'(x)$ would be equal to 0 for some point c in (a, b) and this would contradict the condition (iii) of the theorem.

Let us define $\phi : [a, b] \to \mathbb{R}$ by $\phi(x) = f(x) + Ag(x), x \in [a, b]$, where A is a constant.

ϕ is continuous on $[a, b]$, since f and g are both continuous on $[a, b]$; ϕ is differentiable on (a, b), since f and g are both differentiable on (a, b).

Let us choose A such that $\phi(a) = \phi(b)$. Then $f(a)+Ag(a) = f(b)+Ag(b)$.

or, $A[g(a) - g(b)] = f(b) - f(a)$. Since $g(a) \neq g(b), A = \frac{f(b)-f(a)}{g(a)-g(b)}$.

For this choice of A, ϕ satisfies all conditions of Rolle's theorem on $[a, b]$. Therefore there exists a point ξ in (a, b) such that $\phi'(\xi) = 0$.

But $\phi'(\xi) = f'(\xi) - \frac{f(b)-f(a)}{g(a)-g(b)} g'(\xi)$.

As $a < \xi < b, g'(\xi) \neq 0$ and therefore $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$.

This completes the proof.

Note 1. Lagrange's Mean value theorem can be deduced from Cauchy's Mean value theorem by taking $g(x) = x, x \in [a, b]$.

Note 2. Both f and g satisfy the conditions of Lagrange's Mean value theorem. Consequently, there exist points c and d in (a, b) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$ and $\frac{g(b)-g(a)}{b-a} = g'(d)$.

c and d are different points in (a, b) in general, and therefore a single point ξ in (a, b) may not be found to satisfy the conclusion of Cauchy's Mean value theorem, unless the third condition "$g'(x) \neq 0$ for all $x \in (a, b)$" is imposed on the function g.

Exercises 15

1. Show that there does not exist a function ϕ such that $\phi'(x) = f(x)$ where

(i) $f(x) = [x], x \in [0, 2]$, (ii) $f(x) = x - [x], x \in [0, 2]$.

2. Let I be an interval and a function $f : I \to \mathbb{R}$ is differentiable on I. If f' be monotonic on I prove that f' is continuous on I.

3. Verify Rolle's theorem for the following funcfions on the indicated intervals.

(i) $f(x) = x^2 - 5x + 10$ on $[2, 3]$, (ii) $f(x) = (x-a)^3(x-b)^4$ on $[a, b]$,

(iii) $f(x) = x^2 + \cos x$ on $[-\frac{\pi}{4}, \frac{\pi}{4}]$, (iv) $f(x) = \sin(\frac{1}{x})$ on $[\frac{1}{3\pi}, \frac{1}{2\pi}]$.

4. Show that the following functions do not satisfy the conditions of Rolle's theorem on the indicated intervals.

(i) $f(x) = 1 - \mid x - 1 \mid$ on $[0, 2]$, (ii) $f(x) = 1 - (x-1)^{2/3}$ on $[0, 2]$.

5. Verify the hypothesis and the conclusion of Mean value theorem for the following functions on the indicated intervals.

(i) $f(x) = x(x-1)(x-2)$ on $[0, \frac{1}{2}]$, (ii) $f(x) = \frac{x}{x-1}$ on $[2, 4]$,

(iii) $f(x) = x^3 - 3x + 1$ on $[1, 3]$, (iv) $f(x) = 4 - (6-x)^{2/3}$ on $[5, 7]$,

(v) $f(x) = \cos(\frac{1}{x}), x \neq 0$
$= 0,\ x = 0$ on $[-1, 1]$.

6. Calculate ξ in Cauchy's mean value theorem for each of the following pairs of functions.

(i) $f(x) = \sin x,\ g(x) = \cos x$ on $[\frac{\pi}{4}, \frac{3\pi}{4}]$,

(ii) $f(x) = \sqrt{x},\ g(x) = \frac{1}{\sqrt{x}}$ on $[1, 3]$,

(iii) $f(x) = \log x,\ g(x) = \frac{1}{x}$ on $[1, e]$,

(iv) $f(x) = (1+x)^{3/2},\ g(x) = \sqrt{1+x}$ on $[0, \frac{1}{2}]$.

7. A function f is differentiable on $[0, 2]$ and $f(0) = 0, f(1) = 2, f(2) = 1$. Prove that $f'(c) = 0$ for some c in $(0, 2)$.

[**Hint.** Apply Lagrange's Mean value theorem to f on $[0, 1]$ and on $[1, 2]$. Then $f'(c_1) > 0$ and $f'(c_2) < 0$ for some $c_1 \in (0, 1)$ and some $c_2 \in (1, 2)$. Apply Darboux's theorem to f' on $[c_1, c_2]$.]

8. Prove that the equation $(x-1)^3 + (x-2)^3 + (x-3)^3 + (x-4)^3 = 0$ has only one real root.

9. If $c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \cdots + \frac{c_n}{n+1} = 0$ where $c_0, c_1, c_2, \cdots, c_n$ are real, show that the equation $c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0$ has at least one real root between 0 and 1.

10. Prove that between any two real roots of the equation $e^x \cos x + 1 = 0$ there is at least one real root of the equation $e^x \sin x + 1 = 0$.

[**Hint.** Let $f(x) = e^{-x} + \cos x, x \in \mathbb{R}$. Apply Rolle's theorem on $[c, d]$ where c and d $(c < d)$ are any two real roots of the equation $f(x) = 0$.]

11. Prove that between any two real roots of the equation $e^x \sin x + 1 = 0$ there is at least one real root of the equation $\tan x + 1 = 0$.

[**Hint.** Let $f(x) = e^x \sin x + 1, x \in \mathbb{R}$. Proceed as in Ex.10]

12. If f is differentiable on $[0,1]$ show by Cauchy's Mean value theorem that the equation $f(1) - f(0) = \frac{f'(x)}{2x}$ has at least one solution in $(0,1)$.

13. A function $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and $f''(x)$ exists for all $x \in (a,b)$. If $a < c < b$ and $f(a) = f(b) = 0$, prove that there exists a point ξ in (a,b) such that $f(c) = \frac{1}{2}(c-a)(c-b)f''(\xi)$.

14. A function f is twice differentiable on $[a,b]$ and $f(a) = f(b) = 0$. If $f(c) > 0$ for some $c \in (a,b)$, prove that there exists a point ξ in (a,b) such that $f''(\xi) < 0$.

15. A function $f : [a,b] \to \mathbb{R}$ is said to satisfy a *Lipschitz condition* of order α on $[a,b]$ if there exists a real number $M > 0$ such that

$| f(x) - f(y) | < M \mid x - y \mid^{\alpha}$ for all x, y in $[a,b]$.

If f satisfies a Lipschitz condition of order $\alpha > 1$ on $[a,b]$, prove that f is a constant on $[a,b]$.

16. A function f is thrice differentiable on $[a,b]$ and $f(a) = f(b) = 0, f'(a) = f'(b) = 0$. Prove that $f'''(c) = 0$ for some $c \in (a,b)$.

17. If $f''(x) \geq 0$ on $[a,b]$ prove that $f(\frac{x_1+x_2}{2}) \leq \frac{1}{2}[f(x_1) + f(x_2)]$ for any two points x_1, x_2 in $[a,b]$.

Use the principle of induction to prove that

$f(\frac{x_1+x_2+\cdots+x_n}{n}) \leq \frac{1}{n}[f(x_1) + f(x_2) + \cdots + f(x_n)]$ for any n points $x_1, x_2, \ldots, x_n$ in $[a,b]$.

[**Hint.** Let $x_1 < x_2$. Apply Mean value theorem on $[x_1, \frac{x_1+x_2}{2}]$ and on $[\frac{x_1+x_2}{2}, x_2]$.]

18. A function f is continuous on $[a,b]$ and $f''(x)$ is finite for every $x \in (a,b)$. If the line segment joining the points $A(a, f(a))$ and $B(b, f(b))$ intersects the graph of f at some point P different from A and B, prove that $f''(\xi) = 0$ for some ξ in (a,b).

[**Hint.** Let $P = (c, f(c))$. Then $\frac{f(c)-f(a)}{c-a} = \frac{f(b)-f(c)}{b-c}$.]

19. If $\phi(x) = f(x) + f(1-x), x \in [0,1]$ and $f''(x) < 0$ for all $x \in [0,1]$, show that ϕ is an increasing function on $[0, \frac{1}{2}]$ and a decreasing function on $[\frac{1}{2}, 1]$.

[**Hint.** f' is a strictly decreasing function on $[0,1]$.

$\phi'(0) = f'(0) - f'(1) > 0, \phi'(\frac{1}{2}) = 0, \phi'(1) = f'(1) - f'(0) < 0.$

$0 < x < \frac{1}{2} \Rightarrow 0 < x < 1 - x < 1 \Rightarrow f'(x) > f'(1-x) \Rightarrow \phi'(x) > 0.$

$\frac{1}{2} < x < 1 \Rightarrow 0 < 1 - x < x < 1 \Rightarrow f'(1-x) > f'(x) \Rightarrow \phi'(x) < 0.$]

20. Let $f(x) = \frac{\sin x}{x}, x \in (0, \frac{\pi}{2})$, $\phi(x) = \frac{\tan x}{x}, x \in (0, \frac{\pi}{2})$. Show that f is a strictly decreasing function on $(0, \frac{\pi}{2})$ and ϕ is a strictly increasing function on $(0, \frac{\pi}{2})$.

21. Prove that

(i) $\frac{2x}{\pi} < \sin x < x$ for $0 < x < \frac{\pi}{2}$, (ii) $x < \tan x < \frac{4x}{\pi}$ for $0 < x < \frac{\pi}{4}$,

(iii) $\frac{x}{\sin x} < \frac{\tan x}{x}$ for $x \in (0, \frac{\pi}{2})$, (iv) $x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}}$ for $0 < x < 1$.

22. Use Mean value theorem to prove $0 < \frac{1}{\log(1+x)} - \frac{1}{x} < 1$ for $x > 0$.

23. Use Mean value theorem to prove that the function f defined by

(i) $f(x) = \sin x$, $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$,

(ii) $f(x) = \tan^{-1} x$, $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

24. Prove that $\frac{x}{1+x} < \log(1 + x) < x$ for all $x > 0$. Deduce that

$\log \frac{2n+1}{n+1} < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \log 2$, n being a positive integer.

9.6. The nth order derivatives.

1. Let $f(x) = x^\alpha, \alpha \in \mathbb{R}$.

$$\begin{aligned}
\text{Then } f'(x) &= \alpha x^{\alpha-1} \\
f''(x) &= \alpha(\alpha-1)x^{\alpha-2} \\
f'''(x) &= \alpha(\alpha-1)(\alpha-2)x^{\alpha-3} \\
\cdots & \quad \cdots \\
f^n(x) &= \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)x^{\alpha-n}.
\end{aligned}$$

Note 1. Here the result has been obtained by inference. But it can be proved by the principle of induction.

2. The domain D of the function f is $(0, \infty)$ when α is irrational; $D = [0, \infty)$ when $\alpha = \frac{m}{n}, m \in \mathbb{N}, n(> 1) \in \mathbb{N}$; $D = (0, \infty)$ when $\alpha = -\frac{m}{n}, m \in \mathbb{N}, n(> 1) \in \mathbb{N}$; $D = \mathbb{R}$ when α is an integer ≥ 0 and $D = \mathbb{R} - \{0\}$ when α is an integer < 0.

The domain D of the function f^n is $(0, \infty)$ when α is not an integer; $D = \mathbb{R}$ when α is an integer ≥ 0 and $D = \mathbb{R} - \{0\}$ when α is an integer < 0.

Corollary. Let $f(x) = x^m$ where m is a positive integer.

$$\begin{aligned}
\text{Then } f'(x) &= mx^{m-1} \\
f''(x) &= m(m-1)x^{m-2} \\
f'''(x) &= m(m-1)(m-2)x^{m-3} \\
\cdots & \quad \cdots \\
f^m(x) &= m! \text{ and} \\
f^{m+r}(x) &= 0 \text{ for } r = 1, 2, 3, \ldots
\end{aligned}$$

In the following discussion the functions and their deriatives of different orders are assumed to be defined on their respective domains.

In each of the following cases, the nth derivative is obtained by inference. But in every case it can be proved by the principle of induction.

2. Let $f(x) = \frac{1}{ax+b}$, $a \in \mathbb{R}, b \in \mathbb{R}, ax + b \neq 0$.

Then
$$\begin{aligned} f'(x) &= -1(ax+b)^{-2}.a \\ f''(x) &= (-1)^2 2!(ax+b)^{-3}.a^2 \\ f'''(x) &= (-1)^3 3!(ax+b)^{-4}.a^3 \\ \dots & \quad \dots \\ f^n(x) &= (-1)^n n!(ax+b)^{-n-1}.a^n. \end{aligned}$$

3. Let $f(x) = e^{ax+b}$, $a \in \mathbb{R}, b \in \mathbb{R}$.

Then
$$\begin{aligned} f'(x) &= e^{ax+b}.a \\ f''(x) &= e^{ax+b}.a^2 \\ \dots & \quad \dots \\ f^n(x) &= e^{ax+b}.a^n \end{aligned}$$

4. Let $f(x) = \log(ax+b)$, $a \in \mathbb{R}, b \in \mathbb{R}, ax + b > 0$.

Then
$$\begin{aligned} f'(x) &= (ax+b)^{-1}.a \\ f''(x) &= -1.(ax+b)^{-2}.a^2 \\ f'''(x) &= (-1)^2 2!.(ax+b)^{-3}.a^3 \\ \dots & \quad \dots \\ f^n(x) &= (-1)^{n-1}(n-1)!(ax+b)^{-n}.a^n. \end{aligned}$$

5. Let $f(x) = \sin ax$, $a \in \mathbb{R}$.

Then
$$\begin{aligned} f'(x) &= \cos ax.a = \sin(\pi/2 + ax).a \\ f''(x) &= -\sin ax.a^2 = \sin(2\pi/2 + ax).a^2 \\ f'''(x) &= -\cos ax.a^3 = \sin(3\pi/2 + ax).a^3 \\ \dots & \quad \dots \\ f^n(x) &= \sin(n\pi/2 + ax).a^n. \end{aligned}$$

6. Let $f(x) = \cos ax$, $a \in \mathbb{R}$. Then $f^n(x) = \cos(n\pi/2 + ax).a^n$.

7. Let $f(x) = e^{ax} \sin bx$, $a \in \mathbb{R}, b \in \mathbb{R}$.
Then $f'(x) = e^{ax}(a \sin bx + b \cos bx)$.
Let $a = r\cos\theta, b = r\sin\theta; -\pi < \theta \leq \pi$. Then $r^2 = a^2 + b^2$.
$f'(x) = re^{ax}(\sin bx \cos\theta + \cos bx \sin\theta) = re^{ax}\sin(bx+\theta)$
$f''(x) = r^2 e^{ax} \sin(bx + 2\theta)$
$\dots \quad \dots$
$f^n(x) = r^n e^{ax}\sin(bx + n\theta)$, where $r\cos\theta = a$, $r\sin\theta = b$.

8. Let $f(x) = e^{ax}\cos bx$, $a \in \mathbb{R}, b \in \mathbb{R}$.
Then $f'(x) = e^{ax}(a\cos bx - b\sin bx)$.
Let $a = r\cos\theta, b = r\sin\theta; -\pi < \theta \leq \pi$. Then $r^2 = a^2 + b^2$.
$$\begin{aligned} f'(x) &= re^{ax}(\cos bx\cos\theta - \sin bx \sin\theta) = re^{ax}\cos(bx+\theta) \\ \dots & \quad \dots \\ f^n(x) &= r^n e^{ax}\cos(bx + n\theta), \text{ where } r\cos\theta = a, r\sin\theta = b. \end{aligned}$$

Worked Examples.

1. If $f(x) = \frac{x^3}{x^2-1}$, prove that for $n > 1$, $f^n(0) = 0$ if n be even $= -n!$ if n be odd.

$f(x) = x + \frac{x}{x^2-1} = x + \frac{1}{2}[\frac{1}{x+1} + \frac{1}{x-1}]$.
$f'(x) = 1 + \frac{1}{2}[(-1)(x+1)^{-2} + (-1)(x-1)^{-2}]$.
$f''(x) = \frac{1}{2}[(-1)^2 2!(x+1)^{-3} + (-1)^2 2!(x-1)^{-3}]$.
... ...
Therefore $f^n(x) = \frac{1}{2}[\frac{(-1)^n n!}{(x+1)^{n+1}} + \frac{(-1)^n n!}{(x-1)^{n+1}}]$ for $n > 1$.
That is , for $n > 1, f^n(0) = \frac{1}{2}(-1)^n n![1 + (-1)^{n+1}]$.
Therefore for $n > 1, f^n(0) = 0$ if n be even $= -n!$ if n be odd.

2. If $y = \frac{1}{x^2+a^2}$, prove that $y_n = \frac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta$, where $\cot\theta = \frac{x}{a}$.

Let $x = a\cot\theta$. Then $\frac{dx}{d\theta} = -a \operatorname{cosec}^2\theta$, $y = \frac{1}{a^2}\sin^2\theta$.

Therefore $y_1 = \frac{1}{a^2}\sin 2\theta.\frac{d\theta}{dx} = \frac{1}{a^2}\sin 2\theta(-\frac{\sin^2\theta}{a}) = -\frac{1}{a^3}\sin^2\theta\sin 2\theta$.

$y_2 = -\frac{1}{a^3}[2\sin\theta\cos\theta\sin 2\theta + 2\sin^2\theta\cos 2\theta].\frac{d\theta}{dx}$
$= \frac{-2}{a^3}\sin\theta[\sin 2\theta\cos\theta + \cos 2\theta\sin\theta].(-\frac{\sin^2\theta}{a}) = \frac{(-1)^2 2!}{a^4}\sin^3\theta\sin 3\theta$.

$y_3 = \frac{2!}{a^4}[3\sin^2\theta\cos\theta\sin 3\theta + 3\sin^3\theta\cos 3\theta].(-\frac{\sin^2\theta}{a})$
$= \frac{(-1)^3 3!}{a^5}\sin^4\theta(\cos\theta\sin 3\theta + \sin\theta\cos 3\theta) = \frac{(-1)^3 3!}{a^5}\sin^4\theta\sin 4\theta$.
... ...

$y_n = \frac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta$, where $\cot\theta = \frac{x}{a}$.

3. If $y = \tan^{-1}x$, prove that $y_n = (-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$ where $\cot\theta = x$.

$y_1 = \frac{1}{1+x^2}$. Let $y_1 = z$. Then by Example 2,

$z_n = (-1)^n n!\sin^{n+1}\theta\sin(n+1)\theta$, where $\cot\theta = x$.
Hence $y_n = z_{n-1} = (-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$, where $\cot\theta = x$.

4. If $y = \cot^{-1}x$, prove that $y_n = (-1)^n(n-1)!\sin^n y\sin ny$.

$y_1 = -\frac{1}{1+x^2} = -\frac{1}{cosec^2 y} = -\sin^2 y = -\sin y.\sin y$.
$y_2 = -2\sin y\cos y\frac{dy}{dx} = -\sin 2y(-\sin^2 y) = (-1)^2\sin^2 y\sin 2y$.

$y_3 = (-1)^2[2\sin y\cos y\sin 2y + \sin^2 y.2\cos 2y].\frac{dy}{dx}$
$= (-1)^2 2\sin y[\sin 2y\cos y + \sin y\cos 2y](-\sin^2 y)$
$= (-1)^3.2!\sin^3 y\sin 3y$.
... ...
$y_n = (-1)^n(n-1)!\sin^n y\sin ny$.

Theorem 9.6.1. (Leibnitz)

If f and g be two functions each differentiable n times at a, then the nth derivative of the product function fg at a is given by

$(fg)^n(a) = f^n(a)g(a) + n_{C_1} f^{n-1}(a)g'(a) + n_{C_2} f^{n-2}(a)g''(a) + \cdots + n_{C_r} f^{n-r}(a)g^r(a) + \cdots + f(a)g^n(a)$

$= \sum\limits_{r=0}^{n} n_{C_r} D^{n-r} f(a) D^r g(a)$, where $D^r f(a) = f^r(a)$, $r \geq 1$; $D^0 f(a) = f(a)$.

Proof. $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.

This shows that the theorem is true for $n = 1$.

Let us assume the theorem to be true for $n = m$, where m is a natural number.

Then $(fg)^m(a) = f^m(a)g(a) + m_{C_1} f^{m-1}(a)g'(a) + + m_{C_r} f^{m-r}(a)g^r(a) + \cdots + f(a)g^m(a)$.

Differentiating the function $(fg)^m$ at a in L.H.S., and the function $f^m g + m_{C_1} f^{m-1} g' + \cdots + m_{C_r} f^{m-r} g^r + \cdots + f g^m$ at a in R.H.S., we have

$(fg)^{m+1}(a) = [f^{m+1}(a)g(a) + f^m(a)g'(a)] + m_{C_1}[f^m(a)g'(a) + f^{m-1}(a)g''(a)] + \cdots + m_{C_r}[f^{m-r+1}(a)g^r(a) + f^{m-r}(a)g^{r+1}(a)] + \cdots + [f'(a)g^m(a) + f(a)g^{m+1}(a)]$

$= f^{m+1}(a)g(a) + (1 + m_{C_1})[f^m(a)g'(a)] + (m_{c_1} + m_{c_2})[f^{m-1}(a)g''(a)] + \cdots + (m_{c_{r-1}} + m_{C_r})[f^{m+1-r}(a)g^r(a)] + \cdots + f(a)g^{m+1}(a)$

$= f^{m+1}(a)g(a) + (m+1)_{C_1} f^m(a)g'(a) + (m+1)_{c_2} f^{m-1}(a)g''(a) + \cdots + (m+1)_{C_r} f^{m+1-r}(a)g^r(a) + \cdots + f(a)g^{m+1}(a)$.

This shows that the theorem is true for $n = m + 1$, if it is true for $n = m$. And the theorem is true for $n = 1$.

By the principle of induction, the theorem is true for all natural numbers n.

Worked Examples.

1. If $x + y = 1$ prove that the nth derivative of $x^n y^n$ is

$n!\{y^n - (n_{C_1})^2 y^{n-1} x + (n_{C_2})^2 y^{n-2} x^2 - (n_{C_3})^2 y^{n-3} x^3 + \cdots + (-1)^n x^n\}$.

By Leibnitz's theorem,

$D^n(x^n y^n) = D^n(x^n) y^n + n_{C_1} D^{n-1}(x^n) D(y^n) + n_{C_2} D^{n-2}(x^n) D^2(y^n) + n_{C_3} D^{n-3}(x^n) D^3(y^n) + \cdots + x^n D(y^n)$.

If $r \leq n$, $D^r(x^n) = n(n-1)\cdots(n-r+1)x^{n-r} = \frac{n!}{(n-r)!} x^{n-r}$.

If $r \leq n$, $D^r(y^n) = D^r(1-x)^n = \frac{n!}{(n-r)!} y^{n-r}.(-1)^r$.

Therefore $D^n(x^ny^n) = n!y^n - n_{C_1}\frac{n!}{1!}x.\frac{n!}{(n-1)!}y^{n-1} + n_{C_2}\frac{n!}{2!}x^2.\frac{n!}{(n-2)!}y^{n-2} - n_{C_3}\frac{n!}{3!}x^3.\frac{n!}{(n-3)!}y^{n-3} + \cdots + x^n.n!(-1)^n$

$= n![y^n - (n_{C_1})^2xy^{n-1} + (n_{C_2})^2x^2y^{n-2} - (n_{C_3})^2x^3y^{n-3} + \cdots + (-1)^nx^n]$.

2. If $y = \frac{x^n}{1+x^2}$ prove that

$y_n = n!\sin\theta[\sin\theta - n_{C_1}\cos\theta\sin 2\theta + n_{C_2}\cos^2\theta\sin 3\theta - \cdots + (-1)^n\cos^n\theta.\sin(n+1)\theta]$, where $x = \cot\theta$.

Let $u = x^n, v = \frac{1}{1+x^2}$.

By Leibnitz's theorem, $y_n = u_nv + n_{C_1}u_{n-1}v_1 + n_{C_2}u_{n-2}v_2 + \cdots + uv_n$.

If $r \le n, u_r = n(n-1)\cdots(n-r+1)x^{n-r} = \frac{r!}{(n-r)!}x^{n-r}$.

If $r \le n, v_r = (-1)^r r!\sin^{r+1}\theta\sin(r+1)\theta$, where $x = \cot\theta$.

$y_n = n!\sin^2\theta + n_{C_1}.\frac{n!}{1!}x(-\sin^2\theta\sin 2\theta) + n_{C_2}.\frac{n!}{2!}x^2.(2!\sin^3\theta\sin 3\theta) + n_{C_3}\frac{n!}{3!}x^3(-3!\sin^4\theta\sin 4\theta) + \cdots + x^n.(-1)^n n!\sin^{n+1}\theta\sin(n+1)\theta$

$= n![\sin^2\theta - n_{C_1}\cot\theta\sin^2\theta\sin 2\theta + n_{C_2}\cot^2\theta\sin^3\theta\sin 3\theta - \cdots + (-1)^n\cot^n\theta\sin^{n+1}\theta\sin(n+1)\theta]$

$= n!\sin\theta[\sin\theta - n_{C_1}\cos\theta\sin 2\theta + n_{C_2}\cos^2\theta\sin 3\theta - \cdots + (-1)^n\cos^n\theta\sin(n+1)\theta]$.

3. If $y = (x+\sqrt{1+x^2})^m$ find the value of $y_n(0)$.

$y_1 = m(x+\sqrt{1+x^2})^{m-1}(1+\frac{x}{\sqrt{1+x^2}}) = \frac{my}{\sqrt{1+x^2}}$.

$y_2 = \frac{my_1}{\sqrt{1+x^2}} - \frac{mxy}{(1+x^2)^{3/2}} = \frac{m^2y}{1+x^2} - \frac{xy_1}{1+x^2}$

or, $(1+x^2)y_2 + xy_1 - m^2y = 0$.

Differentiating n times by Leibnitz's theorem, we have

$[(1+x^2)y_{n+2} + n_{C_1}.2xy_{n+1} + n_{C_2}.2y_n] + [xy_{n+1} + ny_n] - m^2y_n = 0$

or, $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$.

At $x = 0, y_{n+2}(0) = (m^2-n^2)y_n(0)$ ... ... (i)

But $y(0) = 1, y_1(0) = m, y_2(0) = m^2$. Using (i) we have

$y_3(0) = (m^2-1^2)y_1(0) = m(m^2-1^2)$.

$y_4(0) = (m^2-2^2)y_2(0) = m^2(m^2-2^2)$.

... ...

Therefore

$y_n(0) = m(m^2-1^2)(m^2-3^2)\cdots[m^2-(n-2)^2]$, if n be odd.

$y_n(0) = m^2(m^2-2^2)(m^2-4^2)\cdots[m^2-(n-2)^2]$, if n be even.

Exercises 16

1. If $y = \tan^{-1} x$, prove that $y_n = (-1)^{n-1}(n-1)! \sin^n \theta \sin n\theta$, where $\cot\theta = x$.

2. If $y = \tan^{-1} \frac{1+x}{1-x}$, prove that $y_n = (-1)^{n-1}(n-1)! \sin^n \theta \sin n\theta$, where $\cot\theta = x$.

3. If $y = \frac{1}{1+x+x^2+x^3}$, prove that
$y_n = \frac{1}{2}(-1)^n n! \sin^{n+1}\theta[\sin(n+1)\theta - \cos(n+1)\theta + (\sin\theta + \cos\theta)^{-n-1}]$,
where $\cot\theta = x$.

4. If $y = \frac{\log x}{x}$, prove that $y_n = \frac{(-1)^n n!}{x^{n+1}}[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots \cdots - \frac{1}{n}]$.

5. If $y = x \log \frac{x-1}{x+1}$, prove that $y_n = (-1)^n(n-2)![\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}]$.

6. If $y = x^{n-1}\log x$, prove that $y_n = \frac{(n-1)!}{x}$.

7. If $y = x^n \log x$, prove that $y_n = n![\log x + 1 + \frac{1}{2} + \cdots + \frac{1}{n}]$.

8. If $y = x^{n-1}e^{1/x}$, prove that $y_n = \{(-1)^n e^{1/x}\}/x^{n+1}$.

9. If $y = \tan^{-1} x$, prove that

(i) $(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$;

(ii) $(y_n)_0 = 0$, if n be even
$= (-1)^{\frac{1}{2}(n-1)}(n-1)!$, if n odd.

10. If $y = \cos(m \sin^{-1} x)$, prove that

(i) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$;

(ii) $(y_n)_0 = 0$, if n be odd
$= -m^2(2^2 - m^2)(4^2 - m^2)\cdots\{(n-2)^2 - m^2\}$, if n be even.

11. If $y = e^{a \sin^{-1} x}$, prove that

(i) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + a^2)y_n = 0$;

(ii) $(y_n)_0 = \{(n-2)^2 + a^2\}\{(n-4)^2 + a^2\}\cdots(2^2 + a^2)a^2$, if n be even
$= \{(n-2)^2 + a^2\}\{(n-4)^2 + a^2\}\cdots(1^2 + a^2)a$, if n be odd.

12. If $y^{1/m} + y^{-1/m} = 2x$, prove that
$(x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$.

13. If $y = a\cos(\log x) + b\sin(\log x)$, prove that
$x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.

14. If $y = \log(x + \sqrt{1+x^2})$, prove that
$y_{2n}(0) = 0$; $y_{2n+1}(0) = (-1)^n.1^2.3^2.5^2\cdots(2n-1)^2$.

15. If $y = (\sinh^{-1} x)^2$, prove that
$y_{2n+1}(0) = 0$; $y_{2n}(0) = (-1)^{n-1}2.2^2.4^2.6^2\cdots(2n-2)^2$.

16. If $y = (1+x^2)^{m/2}\sin(m \tan^{-1} x)$, show that

(i) $(1+x^2)y_{n+2}+2(n-m+1)xy_{n+1}+(n-m)(n+1-m)y_n=0$;

(ii) $y_{2n}(0)=0$; $y_{2n+1}(0)=(-1)^n m(m-1)\cdots(m-2n)$.

17. If $y=e^{-x}x^n$, prove that $xy_{n+2}+(x+1)y_{n+1}+(n+1)y_n=0$.

Deduce that $xL_n''(x)+(1-x)L_n'(x)+nL_n(x)=0$, where $L_n(x)=\frac{1}{n!}e^x D^n(e^{-x}x^n)$.

9.7. Taylor's theorem and expansion of functions.

Theorem 9.7.1. (Taylor's theorem)

Let a function $f:[a,a+h]\to\mathbb{R}$ be such that

(i) f^{n-1} is continuous on $[a,a+h]$, and

(ii) f^{n-1} is differentiable on $(a,a+h)$.

Then there exists a real number θ satisfying $0<\theta<1$ such that $f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h)$, where p is a positive integer $\leq n$ ... ... (I)

Proof. Since f^{n-1} is continuous on $[a,a+h]$, $f,f',f'',\ldots,f^{n-1}$ are all continuous on $[a,a+h]$.

Since f^{n-1} is differentiable on $(a,a+h)$, $f,f',f'',\ldots,f^{n-1}$ are all differentiable on $(a,a+h)$.

Let us consider the function $\phi:[a,a+h]\to\mathbb{R}$ defined by $\phi(x)=f(x)+(a+h-x)f'(x)+\frac{(a+h-x)^2}{2!}f''(x)+\cdots+\frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x)+A(a+h-x)^p$, for $x\in[a,a+h]$ where A is a constant to be determined under the condition $\phi(a)=\phi(a+h)$.

$\phi(a+h)=\phi(a)$ gives

$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+Ah^p$.

or, $A=\frac{1}{h^p}[f(a+h)-f(a)-hf'(a)-\cdots-\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)]$ ... ... (i)

ϕ is continuous on $[a,a+h]$, since $f,f',\ldots,f^{n-1}$ are continuous on $[a,a+h]$.

ϕ is differentiable on $(a,a+h)$, since $f,f',\ldots,f^{n-1}$ are differentiable on $(a,a+h)$. Also $\phi(a)=\phi(a+h)$.

By Rolle's theorem, there exists a real number θ satisfying $0<\theta<1$ such that $\phi'(a+\theta h)=0$.

But $\phi'(x)=f'(x)+[-f'(x)+(a+h-x)f''(x)]+[-(a+h-x)f''(x)+\frac{(a+h-x)^2}{2!}f'''(x)]+\cdots+[-\frac{(a+h-x)^{n-2}}{(n-2)!}f^{n-1}(x)+\frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x)]-pA(a+h-x)^{p-1}$

$=\frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x)-pA(a+h-x)^{p-1}$.

$\phi'(a+\theta h)=0$ gives $\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h)=pAh^{p-1}(1-\theta)^{p-1}$

or, $A = \frac{h^{n-p}(1-\theta)^{n-p}}{p(n-1)!} \cdot f^n(a+\theta h)$.

Therefore from (i) $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h), 0 < \theta < 1$.

The last term $\frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h)$ is called the *remainder after n terms* and it is denoted by R_n. R_n in this form is called **Schlomilch-Roche's form of remainder.**

If $p = 1, R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h)$. R_n in this form is called **Cauchy's form of remainder.**

If $p = n, R_n = \frac{h^n}{n!}f^n(a+\theta h)$. R_n in this form is called **Lagrange's form of remainder.**

Note 1. Taylor's theorem with any particular form of remainder can be proved independently by suitably defining the function ϕ. The last term of $\phi(x)$ [i.e., $A(x+h-x)^p$] is to be chosen as $A(a+h-x)$, or as $A(a+h-x)^n$ according as R_n is desired in Cauchy's form or in Lagrange's form.

Note 2. Taylor's theorem is the nth Mean value theorem. Lagrange's mean value theorem is a particular case, corresponding to $n = 1$.

Note 3. The theorem also holds if $h < 0$. In this case the interval $[a, a+h]$ is to be replaced by $[a+h, a]$.

Note 4. For a given function f and the given interval $[a, a+h]$, θ appearing in the expansion of $f(a+h)$ depends on n. This dependence can be properly indicated by writing $R_n = \frac{h^n(1-\theta_n)^{n-p}}{p(n-1)!}f^n(a+\theta_n h)$, $0 < \theta_n < 1$.

Another form of Taylor's theorem.

Let a function $f : [a, a+h] \to \mathbb{R}$ be such that

(i) f^{n-1} is continuous on $[a, a+h]$, and

(ii) f^{n-1} is differentiable on $(a, a+h)$.

Then for any $x \in (a, a+h]$ there exists a real number θ satisfying $0 < \theta < 1$ such that

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)$$
$$+ \frac{(x-a)^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta(x-a)), \text{ where } p \text{ is a positive integer} \leq n \ \ldots \ \text{(II)}$$

Since f satisfies the conditions of Taylor's theorem on $[a, a+h]$ and since $x \in (a, a+h]$, f satisfies the conditions on $[a, x]$ also.

Therefore replacing $a+h$ by x in (I) we have (II).

Note 1. For an independent proof of the theorem in this form, we are to consider the function $\phi : [a, x] \to \mathbb{R}$ defined by

$$\phi(t) = f(t)+(x-t)f'(t)+\frac{(x-t)^2}{2!}f''(t)+\cdots+\frac{(x-t)^n}{(n-1)!}f^{n-1}(t)+A(x-t)^p,$$

for $t \in [a, x]$.

Note 2. Taking $a = 0$, the form (II) reduces to

$$f(x) = f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\cdots+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n(1-\theta)^{n-p}}{p(n-1)!}f^n(\theta x).$$

This is **Maclaurin's** theorem.

Theorem 9.7.2. (Maclaurin's theorem)

Let a function $f : [0, h] \to \mathbb{R}$ be such that

(i) f^{n-1} is continuous on $[0, h]$, and

(ii) f^{n-1} is differentiable on $(0, h)$.

Then for any $x \in (0, h]$ there exists a real number θ satisfying $0 < \theta < 1$ such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}(0) + \frac{x^n(1-\theta)^{n-p}}{p(n-1)!}f^n(\theta x),$$

where p is a positive integer $\leq n$.

The last term $\frac{x^n(1-\theta)^{n-p}}{p(n-1)!}f^n(\theta x)$ is called the remainder after n terms and it is denoted by R_n.

R_n in this form is called Schlomich-Roche's form of remainder.

If $p = 1$, $R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(a + \theta x)$. (Cauchy's form)

If $p = n$, $R_n = \frac{x^n}{n!}f^n(a + \theta x)$. (Lagrange's form)

For an independent proof of the theorem we are to consider the function $\phi : [0, x] \to \mathbb{R}$ defined by

$\phi(t) = f(t) + (x - t)f'(t) + \frac{(x-t)^2}{2!}f''(t) + \cdots + \frac{(x-t)^{n-1}}{(n-1)!}f^{n-1}(t) + A(x - t)^p$, for $t \in [0, x]$, where A is a constant to be determined under the condition $\phi(0) = \phi(x)$.

For the proof of the theorem with a particular form of remainder, the last term of $\phi(t)$ [i.e., $A(x-t)^p$] is to be chosen as $A(x-t)$ or as $A(x-t)^n$ according as R_n is desired in Cauchy's form or in Lagrange's form.

Theorem 9.7.3. (General form of Taylor's theorem)

Let $a \in \mathbb{R}$. Let a real function f defined on some neighbourhood $N(a)$ of a be such that f^{n-1} is differentiable on $N(a)$.

Then for any $x \in N'(a)$ [i.e., $N(a) - \{a\}$] there exists a real number θ satisfying $0 < \theta < 1$ such that

$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(x-a)^n(1-\theta)^{n-p}}{(n-1)!}f^n(a+\theta(x-a))$, where p is a positive integer $\leq n$.

Proof. Since f^{n-1} is differentiable on $N(a), f, f', f'', \ldots, f^{n-1}$ are all differentiable on $N(a)$ and therefore $f, f', f'', \ldots, f^{n-1}$ are all continuous on $N(a)$.

Let us consider the function $\phi : I \to \mathbb{R}$ defined by

$\phi(t) = f(t) + (x-t)f'(t) + \frac{(x-t)^2}{2!}f''(t) + \cdots + \frac{(x-t)^{n-1}}{(n-1)!}f^{n-1}(t) + A(x-t)^p$, for $t \in I, I$ being $[a,x] \subset N(a)$ or $[x,a] \subset N(a)$, where A is a constant to be determined under the condition $\phi(x) = \phi(a)$.

$\phi(x) = \phi(a)$ gives $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a) + A(x-a)^p$.

or, $A = \frac{1}{(x-a)^p}[f(x) - f(a) - (x-a)f'(a) - \cdots - \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)]$.

ϕ is continuous on I since $f, f', f'', \ldots, f^{n-1}$ are continuous on I. ϕ is differentiable on I since $f, f', f'', \ldots, f^{n-1}$ are differentiable on I. Also $\phi(a) = \phi(x)$.

By Rolle's theorem, there exists a real number θ satisfying $0 < \theta < 1$ such that $\phi'(a + \theta(x-a)) = 0$.

But $\phi'(t) = f'(t) + [-f'(t) + (x-t)f''(t)] + [-(x-t)f''(t) + \frac{(x-t)^2}{2!}f'''(t)] + \cdots + [-\frac{(x-t)^{n-2}}{(n-2)!}f^{n-1}(t) + \frac{(x-t)^{n-1}}{(n-1)!}f^n(t)] - Ap(x-t)^{p-1}$
$= \frac{(x-t)^{n-1}}{(n-1)!}f^n(t) - Ap(x-t)^{p-1}$.

$\phi'(a+\theta(x-a)) = 0$ gives
$\frac{(x-a)^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta(x-a)) = Ap(x-a)^{p-1}(1-\theta)^{p-1}$
or, $A = \frac{(x-a)^{n-p}(1-\theta)^{n-p}}{p(n-1)!}.f^n(a+\theta(x-a))$.

It follows that $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(x-a)^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta(x-a)), 0 < \theta < 1$.

The last term is the remainder R_n in Schlomilch-Roche's form.

If $p = 1, R_n = \frac{(x-a)^n(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta(x-a))$ (Cauchy's form).

If $p = n, R_n = \frac{(x-a)^n}{n!}f^n(a+\theta(x-a))$ (Lagrange's form).

In particular, if $a = 0$, for any $x \in N'(0)$ there exists a real number θ satisfying $0 < \theta < 1$ such that
$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n$,

where $R_n = \frac{x^n(1-\theta)^{n-p}}{p(n-1)!} f^n(\theta x)$ (Schlomilch-Roche's form)

$= \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x)$ (Cauchy's form)

$= \frac{x^n}{n!} f^n(\theta x)$ (Lagrange's form).

This is the **general form of Maclaurin's theorem.**

Worked Examples.

1. Use Taylor's theorem to prove that

$1 + \frac{x}{2} - \frac{x^3}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$, if $x > 0$.

Let $f(x) = \sqrt{1+x}$, $x \geq 0$.

Then $f'(x) = \frac{1}{2\sqrt{1+x}}$, $f''(x) = -\frac{1}{4(1+x)^{3/2}}$, $f'''(x) = \frac{3}{8(1+x)^{5/2}}$.

By Taylor's theorem with Lagrange's form of remainder (after 3 terms), for any $x > 0$,

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(c)$, for some $c \in (0, x)$.

or, $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16(1+c)^{5/2}}$.

Therefore for $x > 0$, $\sqrt{1+x} > 1 + \frac{x}{2} - \frac{x^3}{8}$, since $\frac{x^3}{16(1+c)^{5/2}} > 0$.

By Taylor's theorem with Lagrange's form of remainder (after 2 terms), for any $x > 0$,

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(d)$, for some $d \in (0, x)$.

or, $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8(1+d)^{3/2}}$.

Therefore for $x > 0$, $\sqrt{1+x} < 1 + \frac{x}{2}$, since $\frac{x^2}{8(1+d)^{3/2}} > 0$.

From (i) and (ii), $1 + \frac{x}{2} - \frac{x^3}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$, if $x > 0$.

2. Let $c \in \mathbb{R}$ and a real function f be such that f'' is continuous on some neighbourhood of c. Prove that $\lim\limits_{h \to 0} \dfrac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c)$.

Let f'' be continuous on $(c - \delta, c + \delta)$ for some $\delta > 0$.

By Taylor's theorem with Lagrange's form of remainder (after 2 terms), for any h satisfying $0 < h < \delta$

$f(c+h) = f(c) + hf'(c) + \frac{h^2}{2!} f''(c + \theta h), 0 < \theta < 1$

and $f(c-h) = f(c) - hf'(c) + \frac{h^2}{2!} f''(c - \theta' h), 0 < \theta' < 1$.

Therefore $f(c+h) + f(c-h) - 2f(c) = \frac{h^2}{2}[f''(c + \theta h) + f''(c - \theta' h)]$

or, $\frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = \frac{1}{2}[f''(c + \theta h) + f''(c - \theta' h)]$.

Since f'' is continuous at c,

$$\lim_{h\to 0} f''(c+\theta h) = f''(c), \lim_{h\to 0} f''(c-\theta' h) = f''(c).$$

Hence $\displaystyle\lim_{h\to 0} \frac{f(c+h)-2f(c)+f(c-h)}{h^2} = f''(c)$.

3. Let $a \in \mathbb{R}$ and a real function f defined on some neighbourhood $N(a)$ of a be such that f'' is continuous at a and $f''(a) \neq 0$. Prove that $\lim_{h\to 0} \theta = \frac{1}{2}$, where θ is given by $f(a+h) = f(a) + hf'(a+\theta h)(0 < \theta < 1)$.

Since f'' is a continuous at a, f'' exists in some neighbourhood $(a-\delta, a+\delta) \subset N(a)$.

Case 1. Let $0 < h < \delta$.

By Taylor's theorem with Lagrange's form of remainder after 2 terms, $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a+\theta' h), 0 < \theta' < 1$.

Therefore $f'(a+\theta h) = f'(a) + \frac{h}{2}f''(a+\theta' h)$.

Applying Mean value theorem to f' on the interval $[a, a+\theta h]$, we have $f'(a+\theta h) = f'(a) + \theta h f''(a+\theta\theta'' h), 0 < \theta'' < 1$.

Therefore $\frac{1}{2}f''(a+\theta' h) = \theta f''(a+\theta\theta'' h)$.

Taking limit as $h \to 0+$, we have

$\frac{1}{2}\lim_{h\to 0+} f''(a+\theta' h) = \lim_{h\to 0+} [\theta . f''(a+\theta\theta'' h)]$.

Since f'' is continuous at a, $\lim_{h\to 0+} f''(a+\theta' h) = f''(a)$, $\lim_{h\to 0+} f''(a+\theta\theta'' h) = f''(a)$. Therefore $\lim_{h\to 0+} \theta = \frac{1}{2}$, since $f''(a) \neq 0$.

Case 2. Let $-\delta < h < 0$. Proceeding similarly, we have $\lim_{h\to 0-} \theta = \frac{1}{2}$.

Hence $\lim_{h\to 0} \theta = \frac{1}{2}$.

4. Let a function $f : [a, \infty) \to \mathbb{R}$ be twice differentiable on $[a, \infty)$ and there exist positive real numbers A and B such that $|f(x)| \leq A, |f''(x)| \leq B$ for all $x \in [a, \infty)$. Prove that $| f'(x) | \leq 2\sqrt{AB}$ for all $x \in [a, \infty)$.

Let $x \geq a$ and $h > 0$.

By Taylor's theorem with Lagrange's form of remainder after 2 terms, $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(c)$ for some $c \in (x, x+h)$.

Therefore $|hf'(x)| = |f(x+h) - f(x) - \frac{h^2}{2}f''(c)|$

$\leq |f(x+h)| + |f(x)| + |\frac{h^2}{2}f''(c)|$. Therefore $| f'(x) | \leq \frac{2A}{h} + \frac{h}{2}B$, for all $x \geq a$ and for all $h > 0$.

Let $\phi(h) = \frac{2A}{h} + \frac{h}{2}B, h > 0$. Then $\phi(h) \geq 2\sqrt{\frac{2A}{h} . \frac{h}{2}B}$, i.e., $\geq 2\sqrt{AB}$.

Consequently, $|f'(x)| \leq 2\sqrt{AB}$ for all $x \in [a, \infty)$.

5. If $f(x) = \sin x$ prove that $\lim_{h\to 0} \theta = \frac{1}{\sqrt{3}}$, where θ is given by $f(h) = f(0) + hf'(\theta h)$, $0 < \theta < 1$.

$f(h) = f(0) + hf'(\theta h)$ gives $\sin h = h\cos\theta h$, $0 < \theta < 1$ ... (i)

Also $f(h) = f(0) + hf'(0) + \frac{h^2}{2}f''(0) + \frac{h^3}{6}f'''(\theta' h)$, $0 < \theta' < 1$

and $f'(\theta h) = f'(0) + \theta h f''(0) + \frac{\theta^2 h^2}{2}f'''(\theta''\theta h)$, $0 < \theta'' < 1$.

Therefore $\sin h = h - \frac{h^3}{6}\cos(\theta' h)$, $0 < \theta' < 1$ (ii)

and $\cos\theta h = 1 - \frac{\theta^2 h^2}{2}\cos(\theta''\theta h)$, $0 < \theta'' < 1$ (iii)

From (i) and (ii) $\cos\theta h = 1 - \frac{h^2}{6}\cos(\theta' h)$ ... (iv)

From (iii) and (iv) $3\theta^2\cos(\theta''\theta h) = \cos(\theta' h)$.

$\lim_{h\to 0}\theta = \frac{1}{\sqrt{3}}$, since $\lim_{h\to 0}\cos(\theta' h) = 1$ and $\lim_{h\to 0}\cos(\theta''\theta h) = 1$.

Note. This does not contradict worked Ex.3, since here $f''(0) = 0$.

6. Let $f : [a, b] \to \mathbb{R}$ be such that $f''(x)$ exists in $[a, b]$ and $f'(a) = f'(b)$. Prove that $f(\frac{a+b}{2}) = \frac{1}{2}[f(a) + f(b)] + \frac{1}{8}(b-a)^2 f''(c)$ for some $c \in (a, b)$.

By Mean value theorem applied to f on $[a, \frac{a+b}{2}]$ and on $[\frac{a+b}{2}, b]$,

$f(\frac{a+b}{2}) = f(a) + \frac{b-a}{2}f'(a) + \frac{(b-a)^2}{8}f''(\xi_1)$ for some $\xi_1 \in (a, \frac{a+b}{2})$ ... (i)

$f(\frac{a+b}{2}) = f(b) - \frac{b-a}{2}f'(b) + \frac{(b-a)^2}{8}f''(\xi_2)$ for some $\xi_2 \in (\frac{a+b}{2}, b)$ ... (ii)

From (i) and (ii) $f(\frac{a+b}{2}) = \frac{1}{2}[f(a) + f(b)] + \frac{1}{8}(b-a)^2[\frac{f''(\xi_1)+f''(\xi_2)}{2}]$.

If $f''(\xi_1) \neq f''(\xi_2)$, by intermediate value property of the derived function f'', $\frac{f''(\xi_1)+f''(\xi^2)}{2} = f''(c)$ for some $c \in (\xi_1, \xi_2)$.

Therefore $f(\frac{a+b}{2}) = \frac{1}{2}[f(a) + f(b)] + \frac{1}{8}(b-a)^2 f''(c)$ for some $c \in (a, b)$.

If however, $f''(\xi_1) = f''(\xi_2)$, then $c = \xi_1$ and the result holds.

7. Let $a \in \mathbb{R}$ and a real function f be such that $f''(x)$ exists in $[a-h, a+h]$ for some $h > 0$. Prove that $\frac{f(a+h)-2f(a)+f(a-h)}{h^2} = f''(c)$ for some $c \in (a-h, a+h)$.

By Mean value theorem applied to f on $[a, a+h]$ and on $[a-h, a]$,

$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2}f''(\xi_1)$ for some $\xi_1 \in (a, a+h)$ ... (i)

$f(a-h) = f(a) - hf'(a) + \frac{h^2}{2}f''(\xi_2)$ for some $\xi_2 \in (a-h, a)$ ... (ii)

From (i) and (ii) $f(a+h) + f(a-h) - 2f(a) = h^2[\frac{f''(\xi_1)+f''(\xi_2)}{2}]$.

If $f''(\xi_1) \neq f''(\xi_2)$, by intermediate value property of the derived function f'', $\frac{f(a+h)-2f(a)+f(a-h)}{h^2} = f''(c)$ for some $c \in (a-h, a+h)$.

If however, $f''(\xi_1) = f''(\xi_2)$, then $c = \xi_1$ and the result holds.

9.7.4. Taylor's infinite series.

Let $a \in \mathbb{R}$. Let a real function f defined on some neighbourhood $N(a)$ of a be such that f^{n-1} is differentiable on $N(a)$.

Then for any $x \in N'(a)$ $[= N(a) - \{a\}]$, $f(x) = P_n(x) + R_n(x)$, where $R_n(x)$ is the remainder after n terms and

$$P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a).$$

$P_n(x)$ is a polynomial of degree $n-1$. $P_n(x)$ is such that

$$P_n(a) = f(a), P_n'(a) = f'(a), P_n''(a) = f''(a), \ldots, P_n^{n-1}(a) = f^{n-1}(a).$$

$P_n(x)$ is called the n-th **Taylor polynomial** of f about the point a.

If f be a function such that for all $n \in \mathbb{N}$, f^n exists on $N(a)$, then the polynomial $P_n(x)$ for any $x \in N'(a)$ takes the form of an infinite series

$$f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots \qquad \cdots \text{(i)}$$

The questions now arise:

1. Does the infinite series (i) converge?

2. If it be convergent, does it converge to $f(x)$?

The infinite series (i) will be convergent if and only if the sequence of partial sums [i.e., the sequence $\{P_n(x)\}$] be convergent.

In order that the sum of the series (i) may be $f(x)$, the limit of the sequence $\{P_n(x)\}$ must be $f(x)$.

Since $f(x) = P_n(x) + R_n(x)$ for any n, the series (i) will converge to $f(x)$ if and only if $\lim\limits_{n\to\infty} R_n(x) = 0$.

Therefore if $f : N(a) \to \mathbb{R}$ be such that $f^n(x)$ exists on $N(a)$ for all $n \in \mathbb{N}$, then for any x belonging to some subset $A \subset N'(a)$ for which $\lim\limits_{n\to\infty} R_n(x) = 0$, $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots \qquad \cdots$ (ii)

The equality (ii) also holds for $x = a$ trivially.

The infinite series in the right hand side of (ii) is called **Taylor's infinite series** for the function f about the point a, the region of convergence of the series being $A \cup \{a\}$.

In particular, if $a = 0$, then Taylor's infinite series for the function f about 0 takes the form $f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots\cdots$

This is called **Maclaurin's infinite series** for f.

9.7.5. Expansion of some functions.

1. Let $f(x) = e^x, x \in \mathbb{R}$.

For all $n \in \mathbb{N}$, $f^n(x) = e^x$ for all $x \in \mathbb{R}$.

By Taylor's theorem with Lagrange's form of remainder after n terms, for a non-zero $x \in \mathbb{R}$,
$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n(x)$, where $R_n(x) = \frac{x^n}{n!}f^n(\theta x)$ for some real number θ satisfying $0 < \theta < 1$ ... ... (i)

Since for all $n \in \mathbb{N}$, $f^n(x)$ exists for all real x, the right hand polynomial in (i) takes the form of an infinite series as $n \to \infty$. The infinite series will converge to $f(x)$ for those non-zero real x for which $\lim\limits_{n\to\infty} R_n = 0$.

$R_n = \frac{x^n}{n!}e^{\theta x}$.

Let $u_n = \frac{|x|^n}{n!}, x \neq 0$. Then $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{|x|}{n+1} = 0$. This proves $\lim u_n = 0$. [Theorem 5.8.1.]

For all real $x, e^{\theta x}$ is bounded. Hence $\lim\limits_{n\to\infty} | R_n |= 0$ for all real $x \neq 0$ and this implies $\lim R_n = 0$ for all real $x \neq 0$. [Theorem 5.6.1.]

Consequently, the infinite series $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ converges to e^x for all real $x \neq 0$.

At $x = 0$, the convergence holds trivially.

So $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ for all $x \in \mathbb{R}$.

2. Let $f(x) = \sin x, x \in \mathbb{R}$.

For all $n \in \mathbb{N}, f^n(x) = \sin(\frac{n\pi}{2} + x)$ exists for all $x \in \mathbb{R}$.

By Taylor's theorem with Lagrange's form of remainder after n terms, for any non-zero $x \in \mathbb{R}$,
$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n(x)$, where $R_n(x) = \frac{x^n}{n!}f^n(\theta x)$ for some real number θ satisfying $0 < \theta < 1$ ... ... (i)

$$\begin{aligned} f(0) = 0, f^n(0) = \sin\tfrac{n\pi}{2} &= 0 \text{ if } n \text{ be even} \\ &= 1 \text{ if } n = 4k+1, k \text{ being an integer} \\ &= -1 \text{ if } n = 4k+3, k \text{ being an integer.} \end{aligned}$$

Therefore $f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{x^n}{n!}\sin(\frac{n\pi}{2} + \theta x)$.

Since for all $n \in \mathbb{N}$, $f^n(x)$ exists for all real x, the right hand polynomial in (i) takes the form of an infinite series as $n \to \infty$. The infinite series will converge to $f(x)$ for those non-zero real x for which $\lim\limits_{n\to\infty} R_n = 0$.

Now $| R_n |=| \frac{x^n}{n!}\sin(\frac{n\pi}{2} + \theta x) | \leq \frac{|x|^n}{n!}$.

Let $u_n = \frac{|x|^n}{n!}, x \neq 0$. Then $\lim \frac{u_{n+1}}{u_n} = \lim \frac{|x|}{n+1} = 0$. This proves $\lim u_n = 0$. [Theorem 5.8.1.]

Hence $\lim | R_n |= 0$ for all real $x \neq 0$ and this implies $\lim R_n =$ (all real $x \neq 0$. [Theorem 5.6.1.]

Consequently, the infinite series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ converges to s for all real $x \neq 0$.

At $x = 0$, the convergence holds trivially.

Therefore $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ for all real x.

3. Let $f(x) = (1+x)^m, x \in \mathbb{R}$.

Case 1. Let m be a positive integer.

$$\begin{aligned} \text{Then } f^n(x) &= m(m-1)\cdots(m-n+1)(1+x)^{m-n} \text{ if } 1 \leq n \cdot \\ &= m! \text{ if } n = m \\ &= 0 \text{ if } n > m. \end{aligned}$$

By Taylor's theorem with remainder after $m+1$ terms, for any n zero $x \in \mathbb{R}$,

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^m}{m!}f^m(0)$.

But $f(0) = 1, f^n(0) = m_{c_n}$ for $1 \leq n \leq m$. Therefore

$(1+x)^m = 1 + m_{c_1}x + m_{c_2}x^2 + \cdots + m_{c_m}x^m$ for all non-zero $x \in$

At $x = 0$, the equality holds trivially.

Therefore $(1+x)^m = 1 + m_{c_1}x + m_{c_2}x^2 + \cdots + m_{c_m}x^m$ for all $x \in$

Thus we obtain a *finite series* expansion in this case.

Case 2. Let m be not a positive integer.

In this case f is defined for all $x \neq -1$, if m be a negative integer a f is defined for all $x > -1$, if m be not an integer.

Considering all cases, f is defined for all $x > -1$; and for all n $\mathbb{N}, f^n(x) = m(m-1)\cdots(m-n+1)(1+x)^{m-n}$ for all $x > -1$.

By Taylor's theorem with Cauchy's form of remainder after n tern for any real non-zero $x > -1$,

$f(x) = f(0) + xf(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n(x)$, whe $R_n(x) = \frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x)$ for some real θ satisfying $0 < \theta < 1$ ... ... (

But $f(0) = 1, f^n(0) = m(m-1)\cdots(m-n+1), f^n(\theta x) = m(m$ $1)\cdots(m-n+1)(1+\theta x)^{m-n}$.

Therefore $f(x) = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1)\cdots(m-n+1)}{(n-1)!}x^{n-1}$ $R_n(x)$, where $R_n(x) = \frac{x^n(1-\theta)^{n-1}}{(n-1)!}m(m-1)\cdots(m-n+1)(1+\theta x)^{m-r}$

Since for all $n \in \mathbb{N}$, $f^n(x)$ exists for all $x > -1$, the right hand sid

$$R_n = (-1)^{n-1}.(1-\theta)^{n-1}.\frac{x^n}{(1+\theta x)^n} \text{ (Cauchy's form).}$$

Since for all $n \in \mathbb{N}$, $f^n(x)$ exists for all $x > -1$, the right hand polynomial takes the form of an infinite series as $n \to \infty$.

The infinite series will converge to $f(x)$ for those non-zero $x > -1$ for which $\lim\limits_{n\to\infty} R_n = 0$.

Case 1. Let $0 < x \leq 1$. We take R_n in Lagrange's form.

$R_n = \frac{x^n}{n!}\frac{(-1)^{n-1}.(n-1)!}{(1+\theta x)^n} = \frac{(-1)^{n-1}}{n}(\frac{x}{1+\theta x})^n$.

In $0 < x < 1$, $1 + \theta x > x > 0$. Therefore $0 < \frac{x}{1+\theta x} < 1$.

When $x = 1$, $\frac{x}{1+\theta x} = \frac{1}{1+\theta} < 1$.
So in $0 < x \leq 1$, $0 < \frac{x}{1+\theta x} < 1$ and hence $0 < (\frac{x}{1+\theta x})^n < 1$.

Therefore $\lim |R_n| = \lim \frac{1}{n}|\frac{x}{1+\theta x}|^n = 0$, since $\lim \frac{1}{n} = 0$ and $|\frac{x}{1+\theta x}|^n$ is bounded. Hence $\lim\limits_{n\to\infty} R_n = 0$. [Theorem 5.6.1.]

Case 2. Let $-1 < x < 0$. We take R_n in Cauchy's form.

Here $R_n = \frac{(-1)^{n-1}x^n(1-\theta)^{n-1}}{(1+\theta x)^n}$. $|R_n| = |x|^n|\frac{1-\theta}{1+\theta x}|^{n-1}.\frac{1}{|1+\theta x|}$.

In $-1 < x < 0$, $0 < 1 - \theta < 1 + \theta x < 1$.
Therefore $0 < \frac{1-\theta}{1+\theta x} < 1$ and hence $0 < |\frac{1-\theta}{1+\theta x}|^{n-1} < 1$.

In $-1 < x < 0$, $\lim |x|^n = 0$.

For all real x, $-|x| \leq x \leq |x|$. Hence $-|x| < -\theta|x| \leq \theta x \leq \theta|x| < |x|$, since $0 < \theta < 1$.

In $-1 < x < 0$, $0 < 1 - |x| < 1 + \theta x$. Therefore $\frac{1}{|1+\theta x|} < \frac{1}{1-|x|}$.
Hence $\lim\limits_{n\to\infty} |R_n| = 0$ and this implies $\lim\limits_{n\to\infty} R_n = 0$. [Theorem 5.6.1.]

Thus the infinite series $x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ converges to $\log(1 + x)$ for all non-zero $x \in (-1, 1]$.

At $x = 0$, the convergence holds trivially.

So $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ for all $x \in (-1, 1]$.

Worked Example.

1. Use Taylor's theorem to the function $f(x) = e^x, x \in \mathbb{R}$ to prove that e is irrational.

f satisfies all conditions of Taylor's theorem on $\mathbb{R}$.

By Taylor's theorem with Lagrange's form of remainder, for any real $x \neq 0$ there exists a real number θ satisfying $0 < \theta < 1$ such that

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{n!}f^n(\theta x)$
or, $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}e^{\theta x}$.

Therefore $e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!} + \frac{e^\theta}{n!}$.
Since $e^\theta > 0, e > 2$.

If possible, let e be rational and $e = \frac{p}{q}$ where p, q are positive integers prime to each other. Let us choose $n > q$ and $n > 2$.

Then $\frac{p}{q} = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!} + \frac{e^\theta}{n!}$

or, $\frac{p(n-1)!}{q} - (n-1)!\{1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!}\} = \frac{e^\theta}{n}$.

Since $n > q$, $\frac{p(n-1)!}{q}$ is an integer. And $(n-1)!(1+1+\frac{1}{2!}+\cdots+\frac{1}{(n-1)!})$ is also an integer. Therefore $\frac{e^\theta}{n}$ turns out to be an integer.

Since $0 < \theta < 1$, we have $0 < e^\theta < e < 3$ and therefore $0 < \frac{e^\theta}{n} < 1$, since $n > 2$. This shows that $\frac{e^\theta}{n}$ is a proper fraction.

Thus we arrive at a contradiction that an integer is equal to a proper fraction. Therefore e must be irrational.

Note. e lies between 2 and 3. The approximation of e upto ten places of decimal is given by $e = 2.7182818284$.

Exercises 17

1. Stating all conditions to be satisfied by f for the expansion, expand the polynomial $f(x)$ in powers of $x - 1$.

(i) $f(x) = x^4 + x^3 + x^2 + x + 1$; (ii) $f(x) = x^5 + x^3 + x$.

2. Use Taylor's theorem to prove that

(i) $\cos x \geq 1 - \frac{x^2}{2}$ for $-\pi < x < \pi$;

(ii) $x - \frac{x^3}{6} < \sin x < x$ for $0 < x < \pi$;

(iii) $x - \frac{x^2}{2} < \log(1 + x) < x$ for $x > 0$.

[**Hint.** (i) Let $x \in (0, \pi)$. By Taylor's theorem, $\cos x = 1 - \frac{x^2}{2} + \frac{x^3}{6}\sin c, 0 < c < x < \pi$. Therefore $\cos x > 1 - \frac{x^2}{2}$.

Let $x \in (-\pi, 0)$. By Taylor's theorem, $\cos x = 1 - \frac{x^2}{2} + \frac{x^3}{6}\sin c, -\pi < x < c < 0$. Therefore $\cos x > 1 - \frac{x^2}{2}$.]

3. If $x \in [0, 1]$ prove that $| \log(1 + x) - (x - \frac{x^2}{2} + \frac{x^3}{3}) | < \frac{1}{4}$.

4. If $x \in [-1, 1]$ prove that $| \sin x - (x - \frac{x^3}{3!} + \frac{x^5}{5!}) | < \frac{1}{7!}$.

5. Let $a \in \mathbb{R}$ and a real function f defined on some neighbourhood $N(a)$ of a be such that f^n is continuously differentiable on $N(a)$ and $f^{n+1}(a) \neq 0$. If for $a + h \in N(a)$, $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}f^n(a + \theta h), (0 < \theta < 1)$, prove that $\lim\limits_{h \to 0} \theta = \frac{1}{n+1}$.

6. Verify Maclaurin's infinite series expansion of the following functions on the indicated intervals.

(i) $\cos^2 x = 1 - \frac{2x^2}{2!} + \frac{2^3x^4}{4!} - \frac{2^5x^6}{6!} + \cdots$ on $\mathbb{R}$;

(ii) $\sin^2 x = \frac{2x^2}{2!} - \frac{2^3x^4}{4!} + \frac{2^5x^6}{6!} - \cdots$ on $\mathbb{R}$;

(iii) if $a > 0, a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \cdots$ on $\mathbb{R}$;

(iv) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ on $\mathbb{R}$;

(v) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$ on $\mathbb{R}$;

(vi) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ for $-1 < x < 1$;

(vii) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ for $-1 < x < 1$;

(viii) $\log(1 + 2x) = 2x - \frac{2^2x^2}{2} + \frac{2^3x^3}{3} - \cdots$ for $-\frac{1}{2} < x \leq \frac{1}{2}$;

(ix) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ for $-1 \leq x \leq 1$.

9.8. Maxima and minima.

Let I be an interval.

A function $f : I \to \mathbb{R}$ is said to have a *global maximum* (or an *absolute maximum*) on I if there exists a point $c \in I$ such that $f(c) \geq f(x)$ for all $x \in I$. c is said to be a *global maximum point* for f on I.

f is said to have a *global minimum* (or an *absolute minimum*) on I if there exists a point $c \in I$ such that $f(c) \leq f(x)$ for all $x \in I$. c is said to be a *global minimum point* for f on I.

A function $f : I \to \mathbb{R}$ is said to have a *local maximum* (or a *relative maximum*) at a point $c \in I$ if there exists a neighbourhood $N(c, \delta)$ of c such that $f(c) \geq f(x)$ for all $x \in N(c, \delta) \cap I$.

f is said to have a *local minimum* (or a *relative minimum*) at a point $c \in I$ if there exists a neighbourhood $N(c, \delta)$ of c such that $f(c) \leq f(x)$ for all $x \in N(c, \delta) \cap I$.

We say that f has a *local extremum* (or a *relative extremum*) at a point $c \in I$ if f has either a local maximum or a local minimum at c.

Note. If $f : I \to \mathbb{R}$ has a local maximum (a local minimum) at a point $c \in I$ then c is a global maximum point (a global minimum point) for f on $N(c, \delta) \cap I$ for some suitable $\delta > 0$

Theorem 9.8.1. Let $f : I \to \mathbb{R}$ be such that f has a local extremum at an interior point c of I. If $f'(c)$ exists then $f'(c) = 0$.

Proof. We prove the theorem for the case when f has a local maximum at c. The proof of the other case is similar.

Since $f'(c)$ exists, either $f'(c) > 0$, or $f'(c) < 0$, or $f'(c) = 0$.

Let $f'(c) > 0$. Then $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} > 0$.

Therefore there exists a positive δ such that $\frac{f(x)-f(c)}{x-c} > 0$ for all $x \in N'(c, \delta) \subset I$.

Let $c < x < c + \delta$. Then $x - c > 0$ and therefore $f(x) > f(c)$ for all $x \in (c, c + \delta)$. This contradicts that f has a local maximum at c.

Consequently, $f'(c) \not> 0$ ... ... (i)

Let $f'(c) < 0$. Then $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} < 0$.

Therefore there exists a positive δ such that $\frac{f(x)-f(c)}{x-c} < 0$ for all $x \in N'(c, \delta) \subset I$.

Let $c - \delta < x < c$. Then $x - c < 0$ and therefore $f(x) > f(c)$ for all $x \in (c - \delta, c)$. This contradicts that f has a local maximum at c.

Consequently, $f'(c) \not< 0$ ... ... (ii)

From (i) and (ii) we have $f'(c) = 0$.

This proves the theorem.

Corollary. Let $f : I \to \mathbb{R}$ and c be an interior point of I, where f has a local extremum. Then either $f'(c)$ does not exist, or $f'(c) = 0$.

Note 1. The theorem says that if the derivative $f'(c)$ exists at an interior point c of local extremum, $f'(c)$ must be 0. A function may, however have a local extremum at an interior point c of its domain without being differentiable at c. For example, the function defined by $f(x) = |x|, x \in \mathbb{R}$ has a local minimum at 0 but $f'(0)$ does not exist.

Note 2. The condition $f'(c) = 0$ (when $f'(c)$ exists) is only a necessary condition for an interior point c to be a point of local extremum of the function f.

For example, for the function f defined by $f(x) = x^3, x \in \mathbb{R}$, 0 is an interior point of the domain of f. $f'(0) = 0$ but 0 is neither a point of local maximum nor a point of local minimum of the function f.

Note 3. The theorem holds if c is an interior point of I.

Let a function f be defined on $[0, 1]$ by $f(x) = x, x \in [0, 1]$. Then f has a local maximum at 1 (not an interior point of I), f is differentiable at 1, but $f'(1) \neq 0$.

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Theorem 9.8.2. (First derivative test for extrema)

Let f be continuous on $I = [a, b]$ and c be an interior point of I. Let f be differentiable on (a, c) and (c, b).

1. If there exists a neighbourhood $(c-\delta, c+\delta) \subset I$ such that $f'(x) \geq 0$ for $x \in (c - \delta, c)$ and $f'(x) \leq 0$ for $x \in (c, c + \delta)$ then f has a local maximum at c.

2. If there exists a neighbourhood $(c-\delta, c+\delta) \subset I$ such that $f'(x) \leq 0$ for $x \in (c-\delta, c)$ and $f'(x) \geq 0$ for $x \in (c, c+\delta)$ then f has a local minimum at c.

3. If $f'(x)$ keeps the same sign on $(c - \delta, c)$ and $(c, c + \delta)$ then f has no extremum at c.

Proof. **1.** Let $x \in (c - \delta, c)$. Applying Mean value theorem to the function f on $[x, c]$, there exists a point ξ in (x, c) such that $f(c) - f(x) = (c - x)f'(\xi)$.

Since $f'(\xi) \geq 0$, we have $f(x) \leq f(c)$ for $x \in (c - \delta, c)$.

Let $x \in (c, c+\delta)$. Applying Mean value theorem to the function f on $[c, x]$, there exists a point η in (c, x) such that $f(x) - f(c) = (x - c)f'(\eta)$.

Since $f'(\eta) \leq 0$, we have $f(x) \leq f(c)$ for $x \in (c, c + \delta)$.

It follows that $f(c) \geq f(x)$ for all $x \in N(c, \delta) \cap I$.

Therefore f has a local maximum at c.

2. Similar proof.

3. Let $f'(x) > 0$ for $x \in (c - \delta, c)$ and for $x \in (c, c + \delta)$.

Then $f(x) < f(c)$ for $x \in (c - \delta, c)$ and $f(c) < f(x)$ for $x \in (c, c + \delta)$. Therefore f has neither a maximum nor a minimum at c.

Similar proof if $f'(x) < 0$ for $x \in (c - \delta, c)$ and for $(c, c + \delta)$.

Note. The converse of the theorem is not true.

For example, let $f(x) = 2x^2 + x^2 \sin \frac{1}{x}, x \neq 0$
$= 0, \quad x = 0.$

Then f has a local minimum at 0.

$f'(x) = 4x + 2x \sin \frac{1}{x} - \cos \frac{1}{x}, x \neq 0$
$= 0, \quad x = 0.$

f' takes both positive and negative values on both sides of 0 (in the immediate neighbourhood).

Examples.

1. Let $f(x) = |\, x \,|, x \in \mathbb{R}$.

f is continuous on $\mathbb{R}$. f is not differentiable at 0.

$f'(x) < 0$ for $x \in (-\delta, 0)$ and $f'(x) > 0$ for $x \in (0, \delta)$ for some $\delta > 0$. Therefore f has a local minimum at 0.

2. Let $f(x) = |x - 1| + |x - 2|, x \in [0, 3]$.

$$\text{Then } f(x) = 3 - 2x, \text{ if } 0 \le x < 1$$
$$= 1, \text{ if } 1 \le x \le 2$$
$$= 2x - 3, \text{ if } 2 < x \le 3.$$

f is continuous on $[0, 3]$. f is not differentiable at 1 and 2.

$f'(x) < 0$ for $x \in (1 - \delta, 1)$, $f'(x) = 0$ for $x \in (1, 1 + \delta)$ for some δ satisfying $0 < \delta < 1$. Therefore f has a local minimum at 1.

$f'(x) = 0$ for $x \in (2 - \delta, 2)$, $f'(x) > 0$ for $x \in (2, 2 + \delta)$ for some δ satisfying $0 < \delta < 1$. Therefore f has a local minimum at 2.

3. $f(x) = (x - 1)^2(x - 3)^3, x \in \mathbb{R}$.

$$f'(x) = 2(x - 1)(x - 3)^3 + 3(x - 1)^2(x - 3)^2$$
$$= (x - 1)(x - 3)^2(5x - 9), x \in \mathbb{R}.$$

f is continuous on $\mathbb{R}$. $f'(x) = 0$ at the points $1, 3, \frac{9}{5}$.

$f'(x) > 0$ for $x \in (1 - \delta, 1)$ and $f'(x) < 0$ for $x \in (1, 1 + \delta)$ for some $\delta > 0$. Therefore f has a local maximum at 1.

$f'(x) > 0$ for $x \in (3 - \delta, 3)$ and $f'(x) > 0$ for $x \in (3, 3 + \delta)$, for some $\delta > 0$. Therefore f has neither a maximum nor a minimum at 3.

$f'(x) < 0$ for $x \in (\frac{9}{5} - \delta, \frac{9}{5})$ and $f'(x) > 0$ for $x \in (\frac{9}{5}, \frac{9}{5} + \delta)$ for some $\delta > 0$. Therefore f has a local minimum at $\frac{9}{5}$.

Theorem 9.8.3. (Higher order derivative test for extrema)

Let $f : I \to \mathbb{R}$ and c be an interior point of I. $n \geq 2$

If $f'(c) = f''(c) = \cdots = f^{n-1}(c) = 0$ and $f^n(c) \neq 0$, then f has

(i) no extremum at c if n be odd, and

(ii) a local extremum at c if n be even:

a local maximum if $f^n(c) < 0$, a local minimum if $f^n(c) > 0$.

Proof. Since $f^n(c) \neq 0$, $f^n(c)$ is either positive or negative.

If $f^n(c) > 0$, then f^{n-1} is increasing at c.

Therefore there exists a positive δ such that $f^{n-1}(x) < f^{n-1}(c)$ for $x \in (c - \delta, c)$ and $f^{n-1}(c) < f^{n-1}(x)$ for $x \in (c, c + \delta)$.

That is, $f^{n-1}(x) < 0$ for $x \in (c - \delta, c)$ and $f^{n-1}(x) > 0$ for $x \in (c, c + \delta)$ ... (i)

If $f^n(c) < 0$, then f^{n-1} is decreasing at c.

By similar arguments there exists a positive δ such that $f^{n-1}(x) > 0$ for $x \in (c - \delta, c)$ and $f^{n-1}(x) < 0$ for $x \in (c, c + \delta)$ ... ... (ii)

Since $f^n(c)$ exists, then $f', f'', \cdots, f^{n-1}$ all exist in some δ-neighbourhood of c.

Let $x \in (c-\delta, c)$. By Taylor's theorem with Lagrange's form of remainder after $n-1$ terms, there exists a point ξ such that $f(x) = f(c) + (x-c)f'(c) + \cdots + \frac{(x-c)^{n-2}}{(n-2)!}f^{n-2}(c) + \frac{(x-c)^{n-1}}{(n-1)!}f^{n-1}(\xi), x < \xi < c$. This gives $f(x) - f(c) = \frac{(x-c)^{n-1}}{(n-1)!}f^{n-1}(\xi)$, $x < \xi < c$ ... (iii)

Let $x \in (c, c+\delta)$. By Taylor's theorem with Lagrange's form of remainder after $n-1$ terms, there exists a point η such that $f(x) = f(c) + (x-c)f'(c) + \cdots + \frac{(x-c)^{n-2}}{(n-2)!}f^{n-2}(c) + \frac{(x-c)^{n-1}}{(n-1)!}f^{n-1}(\eta), c < \eta < x$. This gives $f(x) - f(c) = \frac{(x-c)^{n-1}}{(n-1)!}f^{n-1}(\eta)$, $c < \eta < x$ ... (iv)

Case 1. Let n be odd.

Then $\frac{(x-c)^{n-1}}{(n-1)!} > 0$ for all $x \in (c-\delta, c)$ and for all $x \in (c, c+\delta)$.

Subcase (*i*). If $f^n(c) > 0$, then $f^{n-1}(\xi) < 0$ and $f^{n-1}(\eta) > 0$, by (i).

Using (iii) and (iv) we have $f(x) < f(c)$ for all $x \in (c-\delta, c)$ and $f(x) > f(c)$ for all $x \in (c, c+\delta)$.

Therefore f has neither a maximum nor a minimum at c.

Subcase (*ii*). If $f^n(c) < 0$, then $f^{n-1}(\xi) > 0$ and $f^{n-1}(\eta) < 0$, by (ii).

Using (iii) and (iv) we have $f(x) > f(c)$ for all $x \in (c-\delta, c)$ and $f(x) < f(c)$ for all $x \in (c, c+\delta)$.

Therefore f has neither a maximum nor a minimum at c.

Case 2. Let n be even.

Then $\frac{(x-c)^{n-1}}{(n-1)!} < 0$ for all $x \in (c-\delta, c)$ and $\frac{(x-c)^{n-1}}{(n-1)!} > 0$ for all $x \in (c, c+\delta)$.

Subcase (*i*). If $f^n(c) > 0$, then $f^{n-1}(\xi) < 0$ and $f^{n-1}(\eta) > 0$, by (i).

Using (iii) and (iv) we have $f(x) > f(c)$ for all $x \in (c-\delta, c)$ and also for all $x \in (c, c+\delta)$.

Therefore f has a minimum at c.

Subcase (*ii*). If $f^n(c) < 0$, then $f^{n-1}(\xi) > 0$ and $f^{n-1}(\eta) < 0$, by (ii).

Using (iii) and (iv) we have $f(x) < f(c)$ for all $x \in (c-\delta, c)$ and also for all $x \in (c, c+\delta)$.

Therefore f has a maximum at c.

This completes the proof.

Worked Examples.

1. $f(x) = x^5 - 5x^4 + 5x^3 + 10$.

Show that f has a maximum at 1 and a minimum at 3 and f has neither a maximum nor a minimum at 0.

For an extremum $f'(x) = 0$. $f'(x) = 0$ at $x = 1, 3, 0$.
$f''(x) = 20x^3 - 60x^2 + 30x$. Therefore $f''(1) < 0$, $f''(3) > 0$, $f''(0) = 0$.

Since $f'(1) = 0$ and $f''(1) < 0, f$ has a local maximum at 1.
Since $f'(3) = 0$ and $f''(3) > 0, f$ has a local minimum at 3.

Since $f'(0)$ and $f''(0) = 0$, in order to decide the nature of f at 0, we are to examine derivatives of higher order at 0.
$f'''(x) = 60x^2 - 120x + 30$. $f'''(0) = 30 \neq 0$.

Therefore f has neither a maximum nor a minimum at 0.

2. If $f'(x) = (x-a)^{2n}(x-b)^{2m+1}$ where m, n are positive integers, show that f has neither a maximum nor a minimum at a and f has a minimum at b.

a is a multiple root of order $2n$ of the polynomial $f'(x)$.
Therefore a is a multiple root of order $2n-1$ of the polynomial $f''(x)$, a multiple root of order $2n-2$ of the polynomial $f'''(x), \cdots$, a simple root of the polynomial $f^{2n}(x)$. And a is not a root of $f^{2n+1}(x)$.
Therefore $f'(a) = f''(a) = \cdots = f^{2n}(a) = 0$ and $f^{2n+1}(a) \neq 0$.
Since $2n+1$ is odd, f has neither a maximum nor a minimum at a.
Let h be an arbitrarily small positive number.
$f'(b-h) = (b-h-a)^{2n}(-h)^{2m+1} < 0$.
$f'(b+h) = (b+h-a)^{2n}(h)^{2m+1} > 0$.

f is continuous at b. $f'(x) < 0$ for $x \in (b-\delta, b)$ and $f'(x) > 0$ for $x \in (b, b+\delta)$ for some $\delta > 0$. Hence f has a local minimum at b.

3. Find the local extremum points of the function $f(x) = \frac{x^2}{(1-x)^3}$.

$f'(x) = \frac{2(1-x)^3x + 3x^2(1-x)^2}{(1-x)^6} = \frac{x(1-x)^2(x+2)}{(1-x)^6} = \frac{x(x+2)}{(1-x)^4}$.
$f'(x) = 0$ at $x = -2, 0$.

Let h be an arbitrarily small positive number.
$f'(-2-h) > 0$, $f'(-2) = 0$, $f'(-2+h) < 0$.
$f'(0-h) < 0$, $f'(0) = 0$, $f'(0+h) > 0$.

f is continuous at -2. $f'(x) > 0$ for $x \in (-2-\delta, -2)$ and $f'(x) < 0$ for $x \in (-2, -2+\delta)$ for some $\delta > 0$.
f is continuous at 0. $f'(x) < 0$ for $x \in (-\delta, 0)$ and $f'(x) > 0$ for $x \in (0, \delta)$ for some $\delta > 0$.

Hence f has a local maximum at -2 and a local minimum at 0.

4. Find the global maximum and the global minimum of the function f on $\mathbb{R}$, where $f(x) = \frac{x^2-2x+4}{x^2+2x+4}$, $x \in \mathbb{R}$.

$f'(x) = \frac{4(x^2-4)}{(x^2+2x+4)^2}$.
$f'(x) = 0$ at $x = \pm 2$. $f'(x) < 0$ for $|x| < 2$ and $f'(x) > 0$ for $|x| > 2$.

f is continuous at 2. $f'(2+h) > 0$ and $f'(2-h) < 0$ for sufficiently small $h > 0$. Therefore f has a local minimum at 2 and $f(2) = \frac{1}{3}$.

f is continuous at -2. $f'(-2+h) < 0$ and $f'(-2-h) > 0$ for suffiociently small $h > 0$. Therefore f has a local maximum at -2 and $f(-2) = 3$.

As $f'(x) > 0$ for $x > 2$ and f is continuous at 2, f is an increasing function on $[2, \infty)$ and $\lim\limits_{x\to\infty} f(x) = 1$.

Therefore $\sup\limits_{x\in[2,\infty)} f(x) = 1$ and $\inf\limits_{x\in[2,\infty)} f(x) = f(2) = \frac{1}{3}$.

As $f'(x) > 0$ for $x < -2$ and f is continuous at -2, f is an increasing function on $(-\infty, -2]$ and $\lim\limits_{x\to-\infty} f(x) = 1$.

Therefore $\sup\limits_{x\in(-\infty,-2]} f(x) = f(-2) = 3$ and $\inf\limits_{x\in(-\infty,-2]} = 1$.

$\sup\limits_{x\in[-2,2]} f(x) = 3$ and $\inf\limits_{x\in[-2,2]} f(x) = \frac{1}{3}$.
Therefore $\sup\limits_{x\in\mathbb{R}} f(x) = f(-2) = 3$ and $\inf\limits_{x\in\mathbb{R}} f(x) = f(2) = \frac{1}{3}$.

Exercises 18

1. Examine if f has a local maximum or a local minimum at 0.

(i) $f(x) = 2x+3, x > 0$
$\quad = -3x+1, x \le 0$

(ii) $f(x) = 2x+3, x \ge 0$
$\quad = -3x+1, x < 0$

(iii) $f(x) = 2x+3, x < 0$
$\quad = -3x+1, x \ge 0$

(iv) $f(x) = 2x+3, x \le 0$
$\quad = -3x+1, x > 0$

(v) $f(x) = 2x+3, x > 0$
$\quad = -3x+3, x \le 0$

(vi) $f(x) = 2x+3, x \le 0$
$\quad = -3x+3, x > 0$

(vii) $f(x) = x - [x]$,

(viii) $f(x) = |x| + |x-1|$.

2. Find the points of local maximum and local minimum of the function f.

(i) $f(x) = 12x^5 - 45x^4 + 40x^3 + 1, x \in \mathbb{R}$

(ii) $f(x) = \frac{x^2-3x+4}{x^2+3x+4}, x \in \mathbb{R}$

(iii) $f(x) = 8x^5 - 10x^3 + 5x^2 + 1, x \in \mathbb{R}$

(iv) $f(x) = \frac{x^2+x+1}{x^2-x+1}, x \in \mathbb{R}$

(v) $f(x) = 4x + 2 - 5\log(1 + x^2), x \in \mathbb{R}$ (vi) $f(x) = \frac{x}{(1+x^2)^2}, x \in \mathbb{R}$

(vii) $f(x) = (x - 1)^4(x - 2)^2, x \in \mathbb{R}$ (viii) $f(x) = (x - 1)^5(x - 2)^4, x \in \mathbb{R}$

(ix) $f(x) = \sin^{-1} 2x\sqrt{1 - x^2}, x \in (-1, 1)$ (x) $f(x) = \sin^{-1}(3x - 4x^3), x \in (-1, 1)$.

Find the global maximum and the global minimum of the function f in (ii), (iv) and (vi) on $\mathbb{R}$.

3. Find the maximum and the minimum values of

(i) $\sin x(1 + \cos x)$ in $[0, 2\pi]$ (ii) $\cos x + \cos 2x$ in $[-\frac{\pi}{4}, \frac{5\pi}{4}]$

(iii) $\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x$ in $[0, \pi]$

(iv) $\cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 2x$ in $[0, \pi]$.

4. Find the extreme values of the function f in its domain.

(i) $f(x) = x^x$, (ii) $f(x) = x^{\frac{1}{x}}$,

(iii) $f(x) = \frac{\log x}{x}$, (iv) $f(x) = 2^x - x$.

5. If $ax^2 + 2hxy + by^2 = 1$, show that the maximum and the minimum values of $x^2 + y^2$ are given by the roots of the quadratic equation $(t - a)(t - b) = h^2$.

[**Hint.** Let $x = r\cos\theta, y = r\sin\theta$.]

6. (i) Divide the number 10 into two parts such that the sum of their cubes is the least possible.

(ii) Decompose the number 36 into two factors such that the sum of their squares is the least possible.

7. (i) The perimeter of an isosceles triangle is $2s$. What must its sides be so that the volume of the solid generated by revolving the triangle about the base is the greatest possible?

(ii) The perimeter of an isosceles triangle is $2s$. What must its sides be so that the volume of the solid generated by revolving the triangle about the altitude upon the base is the greatest possible?

8. (i) Determine the altitude of a right circular cylinder of greatest possible volume that can be inscribed in a sphere of radius r.

(ii) Determine the altitude of a right circular cone of greatest possible volume that can be inscribed in a sphere of radius r.

9. (i) Show that the semi-vertical angle of a right circular cone of maximum possible volume and of the given curved surface is $\sin^{-1}(\frac{1}{\sqrt{3}})$.

(ii) Show that the semi-vertical angle of a right circular cone of minimum possible curved surface and of the given volume is $\sin^{-1}(\frac{1}{\sqrt{3}})$.

10. (i) Show that the semi-vertical angle of a right circular cone of maximum

possible volume and of the given surface is $\sin^{-1}(\frac{1}{3})$.

(ii) Show that the semi-vertical angle of a right circular cone of minimum possible surface and of the given volume is $\sin^{-1}(\frac{1}{3})$.

11. (i) One corner of a rectangular sheet of paper is folded over so as to reach the opposite edge (lengthwise) of the sheet. If the area of the folded part be minimum , show that the crease divides the width in the ratio 2 : 3.

(ii) One corner of a long rectangular sheet of paper of width b is folded over so as to reach the opposite edge (lengthwise) of the sheet. Show that the minimum length of the crease is $\frac{3\sqrt{3}b}{4}$.

12. A line is drawn through a fixed point $(a, b)[a > 0, b > 0]$ to meet the positive direction of the co-ordinate axes at P and Q respectively. Show that

(i) the minimum value of PQ is $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$;

(ii) the minimum value of $OP + OQ$ is $(\sqrt{a} + \sqrt{b})^2$, O being the origin.

13. p is the length of perpendicular from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the normal at a variable point on the ellipse. Show that the greatest value of p is $a - b$.

14. A tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the major axis and the minor axis at P, Q respectively. Show that the least value of PQ is $a + b$.

9.9. Indeterminate forms.

In the chapter on limits it was shown that if $\lim_{x \to c} f(x) = l$ and $\lim_{x \to c} g(x) = m \neq 0$ then $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}$.

If however, $m = 0$ then the limit could not be evaluated. The case when $l = 0$ and $m = 0$ was not covered in earlier chapters. In this case the limit of the quotient $\frac{f}{g}$ is said to take the *indeterminate form* $\frac{0}{0}$

We will see that in this case the limit may be finite or infinite, or even the limit may not exist.

The other indeterminate forms are represented by the symbols $\frac{\infty}{\infty}$, $\infty - \infty$, $0.\infty$, 0^0, 1^∞, $1^{-\infty}$, ∞^0.

We now discuss several theorems concerning evaluation of indeterminate forms.

Theorem 9.9.1. Case $\frac{0}{0}$

Let $c \in \mathbb{R}$. Let f and g be two functions such that $f(c) = g(c) = 0, g(x) \neq 0$ in some deleted neighbourhood $N'(c, \delta)$; f and g are differentiable at c and $g'(c) \neq 0$. Then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{f'(c)}{g'(c)}$.

Proof. Let $x \in (c, c+\delta)$. Then $\frac{f(x)}{g(x)} = \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}}$.

Since f and g are differentiable at c, $\lim\limits_{x \to c+} \dfrac{f(x)-f(c)}{x-c} = Rf'(c) = f'(c)$ and $\lim\limits_{x \to c+} \dfrac{g(x)-g(c)}{x-c} = Rg'(c) = g'(c)$.

Therefore $\lim\limits_{x \to c+} \dfrac{f(x)}{g(x)} = \dfrac{f'(c)}{g'(c)}$ since $g'(c) \neq 0$ ... ... (i)

Let $x \in (c-\delta, c)$.

Since f and g are differentiable at c, $\lim\limits_{x \to c-} \dfrac{f(x)-f(c)}{x-c} = Lf'(c) = f'(c)$ and $\lim\limits_{x \to c-} \dfrac{g(x)-g(c)}{x-c} = Lg'(c) = g'(c)$.

Therefore $\lim\limits_{x \to c-} \dfrac{f(x)}{g(x)} = \dfrac{f'(c)}{g'(c)}$ since $g'(c) \neq 0$ ... ... (ii)

From (i) and (ii) we have $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{f'(c)}{g'(c)}$.

Corollary. Let f and g be functions on $[a, b]$ such that $f(a) = g(a) = 0, g(x) \neq 0$ on (a, b); f and g are differentiable at a and $g'(a) \neq 0$.

Then $\lim\limits_{x \to a+} \dfrac{f(x)}{g(x)} = \dfrac{f'(a)}{g'(a)}$.

Examples.

1. Let $f(x) = x^2 \sin \frac{1}{x}, x \neq 0$
   $\quad = 0,\ x = 0$;
   and $g(x) = \sin x, x \in \mathbb{R}$.
   Then $f(0) = g(0) = 0$. $g(x) \neq 0$ in some deleted neighbourhood of 0.
   $f'(0)$ and $g'(0)$ both exist and $g'(0) = 1 \neq 0$.
   Therefore $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to 0} \dfrac{\frac{f(x)-f(0)}{x-0}}{\frac{g(x)-g(0)}{x-0}} = \frac{f'(0)}{g'(0)} = 0$.

2. Let $f(x) = \sin x, x \in \mathbb{R}, g(x) = \sqrt{x}, x \in [0, \infty]$.
   The theorem can not be applied here, since $g'(0)$ does not exist.

We now come to the limit theorem known as L'Hospital's Rule where differentiability of the functions f and g at the point c are not assumed. The theorem asserts that the limiting behaviour of $\frac{f}{g}$ at c is same as that of $\frac{f'}{g'}$ under certain conditions.

Theorem 9.9.2. L'Hospital's rule. Case $\frac{0}{0}$

Let $c \in \mathbb{R}$. Let the functions f and g be continuous on some neighbourhood $N(c, \delta)$ and f, g are differentiable on the deleted neighbourhood $N'(c, \delta)$.

Let $f(c) = g(c) = 0$ and $g(x) \neq 0, g'(x) \neq 0$ on $N'(c, \delta)$. Then

(a) if $\lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)} = l (l \in \mathbb{R})$ then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = l$.

(b) if $\lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)} = \infty$ (or, $-\infty$) then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \infty$ (or, $-\infty$).

Proof. (a) Let us choose $\epsilon > 0$. Then there exists a positive $\delta_1 < \delta$ such that $|\frac{f'(x)}{g'(x)} - l| < \epsilon$ for all $x \in N'(c, \delta_1)$... ... (i)

Let $x \in (c, c + \delta_1)$. Applying Cauchy's Mean value theorem to f and g on $[c, x]$ we have $\frac{f(x)-f(c)}{g(x)-g(c)} = \frac{f'(\xi)}{g'(\xi)}$ for some ξ in (c, x).

That is, $\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}$.

Therefore for all $x \in (c, c + \delta_1), | \frac{f(x)}{g(x)} - l | = | \frac{f'(\xi)}{g'(\xi)} - l |$.

Since $\xi \in (c, x), | \frac{f'(\xi)}{g'(\xi)} - l | < \epsilon$ from (i).

Therefore for all $x \in (c, c + \delta_1), | \frac{f(x)}{g(x)} - l | < \epsilon$.

This proves that $\lim\limits_{x \to c+} \dfrac{f(x)}{g(x)} = l$ ... ... (ii)

Let $x \in (c - \delta_1, c)$. Applying Cauchy's Mean value theorem to f and g on $[x, c]$ we have $\frac{f(c)-f(x)}{g(c)-g(x)} = \frac{f'(\xi)}{g'(\xi)}$ for some $\xi \in (x, c)$.

That is, $\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}$.

Therefore for all $x \in (c - \delta_1, c), | \frac{f(x)}{g(x)} - l | = | \frac{f'(\xi)}{g'(\xi)} - l |$.

Since $\xi \in (x, c), | \frac{f'(\xi)}{g'(\xi)} - l | < \epsilon$, from (i).

Therefore for all $x \in (c - \delta_1, c), | \frac{f(x)}{g(x)} - l | < \epsilon$.

This proves that $\lim\limits_{x \to c-} \dfrac{f(x)}{g(x)} = l$ ... . (iii)

From (ii) and (iii) it follows that $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = l$.

(b) Let us choose $G > 0$. Then there exists a positive $\delta_1 < \delta$ such that $\frac{f'(x)}{g'(x)} > G$ for all $x \in N'(c, \delta_1)$.

Let $x \in (c, c+\delta_1)$. Applying Cauchy's Mean value theorem to f and g on $[c, x]$, we have

$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}$ for some $\xi \in (c, x)$.

Therefore for all $x \in (c, c+\delta_1)$, $\frac{f(x)}{g(x)} > G$, since $\xi \in (c, x)$.

This proves that $\lim\limits_{x \to c+} \dfrac{f(x)}{g(x)} = \infty$ ... ... (i)

Let $x \in (c-\delta_1, c)$. Applying Cauchy's Mean value theorem to f and g on $[x, c]$, we have

$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}$ for some $\xi \in (x, c)$.

Therefore for all $x \in (c-\delta_1, c)$, $\frac{f(x)}{g(x)} > G$. since $\xi \in (x, c)$.

This proves that $\lim\limits_{x \to c-} \dfrac{f(x)}{g(x)} = \infty$ ... ...(ii)

From (i) and (ii) it follows that $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \infty$.

Similar proof for the case when $\lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)} = -\infty$.

Corollary. If f and g be continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = g(a) = 0; g(x) \neq 0$ and $g'(x) \neq 0$ on (a, b) then

(a) if $\lim\limits_{x \to a+} \dfrac{f'(x)}{g'(x)} = l (l \in \mathbb{R})$ then $\lim\limits_{x \to a+} \dfrac{f(x)}{g(x)} = l$.

(b) if $\lim\limits_{x \to a+} \dfrac{f'(x)}{g'(x)} = \infty$ (or $-\infty$) then $\lim\limits_{x \to a+} \dfrac{f(x)}{g(x)} = \infty$ (or $-\infty$).

We now extend the results to the case of limits at infinity. We consider the case when $x \to \infty$. The case when $x \to -\infty$ is similar.

Theorem 9.9.3. Let f and g be continuous and differentiable on $[c, \infty)$ for some positive c. Let $\lim\limits_{x \to \infty} f(x) = 0$, $\lim\limits_{x \to \infty} g(x) = 0$ and $g(x) \neq 0, g'(x) \neq 0$ on (c, ∞). Then if

$\lim\limits_{x \to \infty} \dfrac{f'(x)}{g'(x)}$ exists in $\mathbb{R}^*$ then $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to \infty} \dfrac{f'(x)}{g'(x)}$.

Proof. Let us define functions F and G on $[0, \frac{1}{c}]$ by

$$\begin{aligned} F(t) &= f(\tfrac{1}{t}), 0 < t \le \tfrac{1}{c} \qquad & G(t) &= g(\tfrac{1}{t}), 0 < t \le \tfrac{1}{c} \\ &= 0, t = 0; & &= 0, t = 0. \end{aligned}$$

We have $\lim\limits_{t \to 0+} F(t) = \lim\limits_{x \to \infty} f(x)$ and $\lim\limits_{t \to 0+} G(t) = \lim\limits_{x \to \infty} g(x)$.

F and G are continuous on $[0, \frac{1}{c}]$, differentiable on $(0, \frac{1}{c})$ and $F(0) = 0, G(0) = 0$.

$F'(t) = -(\frac{1}{t^2})f'(\frac{1}{t}), G'(t) = -(\frac{1}{t^2})g'(\frac{1}{t})$ for $0 < t < \frac{1}{c}$.

Therefore $G(t) \neq 0, G'(t) \neq 0$ on $0 < t < \frac{1}{c}$.

By the corollary of the theorem 9.9.2,

if $\lim\limits_{t\to 0+} \dfrac{F'(t)}{G'(t)}$ exists in $\mathbb{R}^*$ then $\lim\limits_{t\to 0+} \dfrac{F(t)}{G(t)} = \lim\limits_{t\to 0+} \dfrac{F'(t)}{G'(t)}$.

But $\frac{F'(t)}{G'(t)} = \frac{f'(\frac{1}{t})}{g'(\frac{1}{t})}$ for $0 < t < \frac{1}{c}$ and $\frac{F(t)}{G(t)} = \frac{f(\frac{1}{t})}{g(\frac{1}{t})}$ for $0 < t < \frac{1}{c}$.

Therefore $\lim\limits_{t\to 0+} \dfrac{F'(t)}{G'(t)} = \lim\limits_{t\to 0+} \dfrac{f'(\frac{1}{t})}{g'(\frac{1}{t})} = \lim\limits_{x\to\infty} \dfrac{f'(x)}{g'(x)}$

$\lim\limits_{t\to 0+} \dfrac{F(t)}{G(t)} = \lim\limits_{t\to 0+} \dfrac{f(\frac{1}{t})}{g(\frac{1}{t})} = \lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)}$.

Therefore if $\lim\limits_{x\to\infty} \dfrac{f'(x)}{g'(x)}$ exists in $\mathbb{R}^*$, then $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \dfrac{f'(x)}{g'(x)}$.

Theorem 9.9.4. Another form of the rule. Case $\frac{0}{0}$

Let $c \in \mathbb{R}$. Let the functions f and g be differentiable on some deleted neighbourhood $N'(c, \delta)$ and $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} g(x) = 0, g'(x) \neq 0$ on $N'(c, \delta)$. Then

(a) if $\lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)} = l (l \in \mathbb{R})$ then $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = l$.

(b) if $\lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)} = \infty$ (or $-\infty$) then $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \infty$ (or $-\infty$).

Proof. (a) Let us choose $\epsilon > 0$. Then there exists a positive $\delta_1 < \delta$ such that

$| \frac{f'(x)}{g'(x)} - l | < \epsilon$ for all $x \in N'(c, \delta_1)$ ... ... (i)

Let us define functions F and G by

$F(x) = f(x)$ for $x \in N'(c, \delta_1)$ $\quad$ $G(x) = g(x)$ for $x \in N'(c, \delta_1)$
$\quad = 0$ for $x = c$; $\quad\quad$ $= 0$ for $x = c$.

F and G are differentiable on $N'(c, \delta_1)$.

Since $\lim\limits_{x\to c} F(x) = \lim\limits_{x\to c} f(x) = 0 = F(c), F$ is continuous at c.

Similarly G is continuous at c.

Let $x \in (c, c + \delta_1)$. Applying Cauchy's Mean value theorem to F and G on $[c, x]$, we have $\frac{F(x) - F(c)}{G(x) - G(c)} = \frac{F'(\xi)}{G'(\xi)}$ for some ξ in (c, x).

That is, $\frac{F(x)}{G(x)} = \frac{F'(\xi)}{G'(\xi)}$, for some ξ in (c, x).

Hence $\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}$ for $x \in (c, c+\delta_1)$.

Therefore for all $x \in (c, c+\delta_1)$, $\left| \frac{f(x)}{g(x)} - l \right| = \left| \frac{f'(\xi)}{g'(\xi)} - l \right|$.

Since $\xi \in (c, x)$, $\left| \frac{f'(x)}{g'(x)} - l \right| < \epsilon$ from (i).

Therefore for all $x \in (c, c+\delta_1)$, $\left| \frac{f(x)}{g(x)} - l \right| < \epsilon$.

This proves that $\lim\limits_{x \to c+} \dfrac{f(x)}{g(x)} = l$ ... ... (ii)

Let $x \in (c - \delta_1, c)$. Applying Cauchy's Mean value theorem to F and G on $[x, c]$ we can prove in a similar manner

$\lim\limits_{x \to c-} \dfrac{f(x)}{g(x)} = l$ ... ... (iii)

From (ii) and (iii) it follows that $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = l$.

(b) Let us choose $B > 0$. Then there exists a positive $\delta_1 < \delta$ such that

$\left| \frac{f'(x)}{g'(x)} \right| > B$ for all $x \in N'(c, \delta_1)$ ... ... (i)

Let us define functions F and G as in **(a)**.

Let $x \in (c, c+\delta_1)$. Applying Cauchy's Mean value theorem to F and G on $[c, x]$, we have $\frac{F(x)}{G(x)} = \frac{F'(\xi)}{G'(\xi)}$ for some ξ in (c, x).

Therefore $\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}$ for $x \in (c, c+\delta_1)$.

Since $\xi \in (c, x)$, $\frac{f'(\xi)}{g'(\xi)} > B$ from (i).

Therefore $\frac{f(x)}{g(x)} > B$ for all $x \in (c, c+\delta_1)$.

This proves that $\lim\limits_{x \to c+} \dfrac{f(x)}{g(x)} = \infty$ ... ... (ii)

Let $x \in (c - \delta_1, c)$. Applying Cauchy's Mean value theorem to F and G on $[x, c]$ we can prove in a similar manner

$\lim\limits_{x \to c-} \dfrac{f(x)}{g(x)} = \infty$ ... ... (iii)

From (ii) and (iii) it follows that $\lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)} = \infty$.

Similar proof for the case when $\lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)} = -\infty$.

Corollary. If f and g be continuous on $[a, b]$ and differentiable on (a, b) and $\lim\limits_{x \to a+} f(x) = 0$, $\lim\limits_{x \to a+} g(x) = 0$; $g(x) \neq 0, g'(x) \neq 0$ on (a, b) then

(a) if $\lim\limits_{x \to a+} \dfrac{f'(x)}{g'(x)} = l (l \in \mathbb{R})$ then $\lim\limits_{x \to a+} \dfrac{f(x)}{g(x)} = l$;

(b) if $\lim\limits_{x \to a+} \dfrac{f'(x)}{g'(x)} = \infty$ (or $-\infty$) then $\lim\limits_{x \to a+} \dfrac{f(x)}{g(x)} = \infty$ (or $-\infty$).

Note. Under the conditions, stated in the theorem, satisfied by f and g, if $\lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)}$ exists then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)}$ exists and equals $\lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)}$.

However, $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)}$ may exist even if $\lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)}$ does not exist.

For example, let $f(x) = x^2 \sin(\frac{1}{x}), x \neq 0$
$= 0, x = 0$

and $g(x) = x, x \in \mathbb{R}$.

Then $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)} = 0$ but $\lim\limits_{x \to 0} \dfrac{f'(x)}{g'(x)}$ does not exist.

9.9.5. Generalised L'Hospital's rule. Case $\frac{0}{0}$

Let $c \in \mathbb{R}$. Let f and g be such that $f^n(x), g^n(x)$ exist on some deleted neighbourhood $N'(c,\delta)$, $g^n(x) \neq 0$ on $N'(c,\delta)$ and

$$\lim_{x \to c} f(x) = \lim_{x \to c} f'(x) = \cdots = \lim_{x \to c} f^{n-1}(x) = 0,$$
$$\lim_{x \to c} g(x) = \lim_{x \to c} g'(x) = \cdots = \lim_{x \to c} g^{n-1}(x) = 0.$$

Then if $\lim\limits_{x \to c} \dfrac{f^n(x)}{g^n(x)}$ exists in $\mathbb{R}^*$, then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to c} \dfrac{f^n(x)}{g^n(x)}$.

Examples.

1. Let $f(x) = \sin x, x \in \mathbb{R}$, $g(x) = \sqrt{x}, x \in [0, \infty)$.

Then
$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} \frac{f'(x)}{g'(x)} \quad \text{[by L'Hospital's rule, } \tfrac{0}{0}]$$
$$= \lim_{x \to 0+} 2\sqrt{x} \cos x$$
$$= 0.$$

2. Evaluate the limit
$$\lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}.$$

Here the limit takes the indeterminate form $\frac{0}{0}$. We have to apply L'Hospital's rule successively. The evaluation of the limit can be exhibited as follows-

$$\lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x} \quad (= \tfrac{0}{0})$$
$$= \lim_{x \to 0} \frac{e^x + e^{-x} - \frac{2}{1+x}}{x \cos x + \sin x} \quad (= \tfrac{0}{0})$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} + \frac{2}{(1+x)^2}}{-x \sin x + 2\cos x} = 1.$$

Form $\frac{\infty}{\infty}$.

Theorem 9.9.6. Let $c \in \mathbb{R}$. Let f and g be differentiable in some deleted neighbourhood $N'(c, \delta)$ of c. Let $\lim_{x \to c} f(x) = \infty$, $\lim_{x \to c} g(x) = \infty$ and $g(x) \neq 0, g'(x) \neq 0$ on $N'(c, \delta)$. Then

(a) if $\lim_{x \to c} \frac{f'(x)}{g'(x)} = l (l \in \mathbb{R})$ then $\lim_{x \to c} \frac{f(x)}{g(x)} = l$;

(b) if $\lim_{x \to c} \frac{f'(x)}{g'(x)} = \infty$ (or $-\infty$) then $\lim_{x \to c} \frac{f(x)}{g(x)} = \infty$ (or $-\infty$).

Proof. (a) Let us choose ϵ such that $0 < \epsilon < \frac{1}{2}$. Then there exists a positive $\delta_1 < \delta$ such that $\mid \frac{f'(x)}{g'(x)} - l \mid < \epsilon$ for all $x \in N'(c, \delta_1)$.

Let us choose $c_1 \in (c, c + \delta_1)$. Then $c < c_1 < c + \delta_1$

Since $\lim_{x \to c+} f(x) = \infty$, we can choose c_2 in (c, c_1) such that $f(x) \neq f(c_1)$ for all $x \in (c, c_2)$.

Since $g'(x) \neq 0$ on $N'(c, \delta), g(x) \neq g(c_1)$ for all $x \in (c, c_2)$.

Let us define a function ϕ on $[c, c_2]$ by $\phi(x) = \frac{1 - \frac{f(c_1)}{f(x)}}{1 - \frac{g(c_1)}{g(x)}}$.

Since $\lim_{x \to c+} f(x) = \infty and \lim_{x \to c+} g(x) = \infty$, $\lim_{x \to c+} \phi(x) = 1$.

Therefore there exists a c_3 in (c, c_2) such that $1 - \epsilon < \phi(x) < 1 + \epsilon$ for all $x \in (c, c_3)$ or, $\frac{1}{2} < \phi(x) < \frac{3}{2}$ for all $x \in (c, c_3)$.

Now $\frac{f(x)}{g(x)} = \frac{f(x) - f(c_1)}{g(x) - g(c_1)} \cdot \frac{1}{\phi(x)}$ for all $x \in (c, c_2)$.

Applying Cauchy's Mean value theorem to f and g on $[c, c_1]$, $\frac{f(x) - f(c_1)}{g(x) - g(c_1)} = \frac{f'(\xi)}{g'(\xi)}$ for some ξ in (c, c_1).

Therefore $\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)} \cdot \frac{1}{\phi(x)}$ for all $x \in (c, c_2)$.

$$\begin{aligned} \text{Now } \left| \frac{f(x)}{g(x)} - l \right| &= \left| \frac{f'(\xi)}{g'(\xi)} \cdot \frac{1}{\phi(x)} - l \right| \\ &= \left| \frac{f'(\xi)}{g'(\xi)} - l\phi(x) \right| \cdot \frac{1}{|\phi(x)|} \\ &\leq \left\{ \left| \frac{f'(\xi)}{g'(\xi)} - l \right| + |\phi(x) - 1| \cdot |l| \right\} \cdot \frac{1}{|\phi(x)|}. \end{aligned}$$

Therefore for all $x \in (c, c_3)$, $\mid \frac{f(x)}{g(x)} - l \mid < 2\epsilon(1 + |l|)$.

Since ϵ is arbitrary, $\lim_{x \to c+} \frac{f(x)}{g(x)} = l$ ... ... (i)

Let us choose c_4 in $(c - \delta_1, c)$. Then $c - \delta_1 < c_4 < c$.

Since $\lim_{x\to c-} f(x) = \infty$, we can choose c_5 in (c_4, c) such that $f(x) \neq f(c_4)$ for all $x \in (c_5, c)$.

Since $g'(x) \neq 0$ on $N'(c,\delta), g(x) \neq g(c_4)$ for all $x \in (c_5, c)$.

Defining $\phi(x) = \frac{1-\frac{f(c_4)}{f(x)}}{1-\frac{g(c_4)}{g(x)}}$ for $x \in (c_5, c)$ and proceeding similarly as above we can prove $\lim_{x\to c-} \frac{f(x)}{g(x)} = l$ ... ... (ii)

From (i) and (ii) $\lim_{x\to c} \frac{f(x)}{g(x)} = l$.

(b) Let us choose $G > 0$. Then there exists a positive $\delta_1 < \delta$ such that $\frac{f'(x)}{g'(x)} > G$ for all $x \in N'(c, \delta_1)$.

Let us choose $c_1 \in (c, c+\delta_1)$. Then $c < c_1 < c + \delta_1$.

Since $\lim_{x\to c+} f(x) = \infty$, we can choose c_2 in (c, c_1) such that $f(x) \neq f(c_1)$ for all $x \in c, c_2)$.

Since $g'(x) \neq 0$ on $N'(c,\delta), g(x) \neq g(c_1)$ for all $x \in (c, c_2)$.

Let us define ϕ on $[c, c_2]$ by $\phi(x) = \frac{1-\frac{f(c_1)}{f(x)}}{1-\frac{g(c_1)}{g(x)}}$.

Since $\lim_{x\to c+} f(x) = \infty and \lim_{x\to c+} g(x) = \infty$, we have $\lim_{x\to c+} \phi(x) = 1$.

Let us choose $\epsilon = \frac{1}{2}$. Then there exists a c_3 in (c, c_2) such that $| \phi(x) - 1 | < \frac{1}{2}$ for all $x \in (c, c_3)$. Therefore $\frac{1}{2} < \phi(x) < \frac{3}{2}$.

Now $\frac{f(x)}{g(x)} = \frac{f(x)-f(c_1)}{g(x)-g(c_1)} \cdot \frac{1}{\phi(x)}$ for all $x \in (c, c_2)$.

Applying Cauchy's Mean value theorem to f and g on $[c, c_1]$,

$\frac{f(x)-f(c_1)}{g(x)-g(c_1)} = \frac{f'(\xi)}{g'(\xi)}$ for some ξ in (c, c_1).

Therefore $\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)} \cdot \frac{1}{\phi(x)}$ for all $x \in (c, c_2)$.

Consequently, $\frac{f(x)}{g(x)} > \frac{2}{3}G$ for all $x \in (c, c_3)$.

This proves that $\lim_{x\to c+} \frac{f(x)}{g(x)} = \infty$ ... ... (i)

In a similar manner we can prove that $\lim_{x\to c-} \frac{f(x)}{g(x)} = \infty$ ... ... (ii)

From (i) and (ii) $\lim_{x\to c} \frac{f(x)}{g(x)} = \infty$.

The case when $\lim_{x\to c} \frac{f(x)}{g(x)} = -\infty$ can be similarly dealt with.

Corollary. Let f and g be continuous on $[a,b]$ and differentiable on (a,b). Let $\lim_{x\to a+} f(x) = \infty$, $\lim_{x\to a+} g(x) = \infty$ and $g(x) \neq 0, g'(x) \neq 0$ on (a,b). Then

(a) if $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = l(l \in \mathbb{R})$ then $\lim_{x\to a+} \frac{f(x)}{g(x)} = l$;

(b) if $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = \infty$ (or $-\infty$) then $\lim_{x\to a+} \frac{f(x)}{g(x)} = \infty$ (or $-\infty$).

Other indeterminate forms.

Indeterminate forms such as $\infty - \infty$, $0.\infty$, 1^∞, 0^0, ∞^0 can be reduced to either of the forms $\frac{0}{0}$, $\frac{\infty}{\infty}$ by algebraic manipulations and use of logarithmic and exponential functions.

Worked Examples.

1. Evaluate $\lim_{x\to 0+} (\frac{1}{x} - \frac{1}{\sin x})$.

Let $f(x) = \frac{1}{x}, x \in (0,1), g(x) = \frac{1}{\sin x}, x \in (0,1)$.

$\lim_{x\to 0+} [f(x) - g(x)]$ takes the indeterminate form $\infty - \infty$.

We have
$$\lim_{x\to 0+} (\tfrac{1}{x} - \tfrac{1}{\sin x}) = \lim_{x\to 0+} \frac{\sin x - x}{x\sin x} \quad (= \tfrac{0}{0})$$
$$= \lim_{x\to 0+} \frac{\cos x - 1}{\sin x + x\cos x} \quad (= \tfrac{0}{0})$$
$$= \lim_{x\to 0+} \frac{-\sin x}{2\cos x - x\sin x} = 0.$$

2. Evaluate $\lim_{x\to 0+} x\log x$.

Let $f(x) = x, x \in (0,\infty), g(x) = \log x, x \in (0,\infty)$.
Then $\lim_{x\to 0+} x = 0$, $\lim_{x\to 0+} \log x = \infty$.
$\lim_{x\to 0+} x\log x$ takes the indeterminate form $0.\infty$.

We have
$$\lim_{x\to 0+} x\log x = \lim_{x\to 0+} \frac{\log x}{\frac{1}{x}} \quad (= \tfrac{\infty}{\infty})$$
$$= \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0+} (-x) = 0.$$

3. Evaluate $\lim_{x\to 0} (\frac{\sin x}{x})^{\frac{1}{x}}$.

Let $f(x) = \frac{\sin x}{x}$, $x \neq 0$, $g(x) = \frac{1}{x}$, $x \neq 0$.

Then $\lim_{x\to 0+} f(x) = 1$, $\lim_{x\to 0+} g(x) = \infty$.

$\lim_{x\to 0+} f(x)^{g(x)}$ takes the indeterminate form 1^∞.

$$\lim_{x\to 0+} \log(\tfrac{\sin x}{x})^{\frac{1}{x}} = \lim_{x\to 0+} \frac{\log\frac{\sin x}{x}}{x} \quad (= \tfrac{0}{0})$$

$$= \lim_{x\to 0+} \frac{\frac{x}{\sin x}\cdot\frac{x\cos x - \sin x}{x^2}}{1}$$

$$= \lim_{x\to 0+} \frac{x\cos x - \sin x}{x\sin x} \quad (= \tfrac{0}{0})$$

$$= \lim_{x\to 0+} \frac{-x\sin x}{x\cos x + \sin x} \ (= \tfrac{0}{0})$$

$$= \lim_{x\to 0+} \frac{-x\cos x - \sin x}{-x\sin x + 2\cos x} = 0.$$

Therefore $\lim_{x\to 0+} (\frac{\sin x}{x})^{\frac{1}{x}} = e^0 = 1$.

Also we have $\lim_{x\to 0-} f(x) = 1$ and $\lim_{x\to 0-} g(x) = -\infty$.

$\lim_{x\to 0-} f(x)^{g(x)}$ takes the indeterminate form $1^{-\infty}$.

Proceeding similarly, we have $\lim_{x\to 0-} (\frac{\sin x}{x})^{\frac{1}{x}} = 1$.

Consequently, $\lim_{x\to 0} (\frac{\sin x}{x})^{\frac{1}{x}} = 1$.

4. Evaluate $\lim_{x\to 0+} x^x$.

Let $f(x) = x, x > 0$; $g(x) = x, x > 0$. Then $\lim_{x\to 0+} f(x) = 0$, $\lim_{x\to 0+} g(x) = 0$.

$\lim_{x\to 0+} [f(x)]^{g(x)}$ takes the indeterminate form 0^0.

$$\lim_{x\to 0+} \log x^x = \lim_{x\to 0+} \frac{\log x}{\frac{1}{x}} \quad (= \tfrac{\infty}{\infty}) = \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0+} -x = 0.$$

Therefore $\lim_{x\to 0+} x^x = 1$.

5. Evaluate $\lim_{x\to\infty} (\frac{3x}{3x+1})^{3x+1}$.

Let $f(x) = (\frac{3x}{3x+1})^{3x+1}$, $x \in [0, \infty)$. Then $f(x) = e^{(3x+1)\log\frac{3x}{3x+1}}$.

$$\lim_{x\to\infty} (3x+1)\log\frac{3x}{3x+1} = \lim_{x\to\infty} \frac{\log\frac{3x}{3x+1}}{\frac{1}{3x+1}} \quad (= \tfrac{0}{0})$$

$$= \lim_{x\to\infty} \frac{\frac{3x+1}{3x}\cdot\frac{3}{(3x+1)^2}}{-\frac{1}{(3x+1)^2}} = \lim_{x\to\infty} -\frac{3x+1}{x} = -3.$$

Since the exponential function is continuous,

$\lim_{x\to\infty} e^{(3x+1)\log\frac{3x}{3x+1}} = e^{-3}$, i.e., $\lim_{x\to\infty} (\frac{3x}{3x+1})^{3x+1} = e^{-3}$.

6. Evaluate $\lim\limits_{n\to\infty}(1-\frac{1}{2n})^{n+1}$.

Let $f(x)=(1-\frac{1}{2x})^{x+1}, x>1$. Then $\log f(x)=(x+1)\log(1-\frac{1}{2x})$.

$$\lim_{x\to\infty}\log f(x)=\lim_{x\to\infty}\frac{\log(1-\frac{1}{2x})}{\frac{1}{x+1}}\quad(=\tfrac{0}{0})$$

$$=\lim_{x\to\infty}\frac{\frac{1}{1-\frac{1}{2x}}\cdot\frac{1}{2x^2}}{-(\frac{1}{x+1})^2}$$

$$=\lim_{x\to\infty}\frac{2x(x+1)^2}{-(2x-1).2x^2}=\lim_{x\to\infty}\frac{x^2+2x+1}{-(2x^2-x)}=-\tfrac{1}{2}.$$

Hence $\lim\limits_{x\to\infty}f(x)=e^{-\frac{1}{2}}$.

Let us consider the sequence $\{n\}$ that diverges to ∞.
By sequential criterion for limits, $\lim\limits_{n\to\infty}f(n)=e^{-\frac{1}{2}}$.
That is, $\lim\limits_{n\to\infty}(1-\frac{1}{2n})^{n+1}=e^{-\frac{1}{2}}$.

7. Evaluate $\lim\limits_{n\to\infty}(1+\frac{3}{n})^{2n}$.

Let $f(x)=(1+\frac{3}{x})^{2x},\ x\in(0,\infty)$.
$\lim\limits_{x\to\infty}f(x)$ takes the indeterminate form 0^{∞}.

$$\lim_{x\to\infty}\log f(x)=\lim_{x\to\infty}\frac{\log(1+\frac{3}{x})}{\frac{1}{2x}}\quad(=\tfrac{\infty}{\infty})$$

$$=\lim_{x\to\infty}\frac{\frac{1}{1+\frac{3}{x}}\cdot\frac{-3}{x^2}}{-\frac{1}{2x^2}}=\lim_{x\to\infty}\frac{6}{1+\frac{3}{x}}=6.$$

Hence $\lim\limits_{x\to\infty}f(x)=e^6$, i.e., $\lim\limits_{x\to\infty}(1+\frac{3}{x})^{2x}=e^6$.

Let us consider the sequence $\{n\}$ that diverges to ∞.
By sequential criterion for limits, $\lim\limits_{n\to\infty}f(n)=e^6$.
That is, $\lim\limits_{n\to\infty}(1+\frac{3}{n})^{2n}=e^6$.

Exercises 19

1. Prove that

(i) $\lim\limits_{x\to0}\dfrac{xe^x-\log(1+x)}{x^2}=\dfrac{3}{2}$, (ii) $\lim\limits_{x\to0}\dfrac{x-\log(1+x)}{1-\cos x}=1$,

(iii) $\lim\limits_{x\to0}\dfrac{\tan x-x}{x^2\tan x}=1/3$ (iv) $\lim\limits_{x\to0}\dfrac{\cosh x-\cos x}{x\sin x}=\cancel{\dfrac{1}{3}}$, 1,

(v) $\lim\limits_{x\to0}\dfrac{e^x-e^{\sin x}}{x-\sin x}=1$, (vi) $\lim\limits_{x\to0}\dfrac{e^x-e^{-x}-2x}{x^2\sin x}=\dfrac{1}{3}$.

10. FUNCTIONS OF BOUNDED VARIATION

10.1. Introduction.

Let $[a, b]$ be a closed and bounded interval. A *partition* P of $[a, b]$ is a finite ordered set $(x_0, x_1, \ldots, x_n)$ of points of $[a, b]$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

The family of all partitions of $[a, b]$ is denoted by $\mathcal{P}[a, b]$ and the partition $P = (x_0, x_1, \ldots, x_n)$ is a member of $\mathcal{P}[a, b]$.

For example, $P = (0, \frac{1}{2}, 1)$ is a partition of $[0, 1]$, $Q = (0, \frac{1}{8}, \frac{1}{2}, \frac{7}{8}, 1)$ is another partition of $[0, 1]$.

The partition $P = (x_0, x_1, \ldots, x_n)$ of $[a, b]$ divides the interval $[a, b]$ into non-overlapping subintervals $[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]$.

Definition. Let $[a, b]$ be a closed and bounded interval and $f : [a, b] \to \mathbb{R}$ be a function. Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$. Let us consider the sum

$$V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})|.$$

For different partitions $P \in \mathcal{P}[a, b]$, $V(P, f)$ gives a set of non-negative real numbers. If the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$ be bounded above, then f is said to be a *function of bounded variation* (or a *BV-function*) on $[a, b]$.

The supremum of the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$ is said to be the *total variation* of f on $[a, b]$ and is denoted by $V_f[a, b]$ (or by V_f, if there is no confusion regarding the interval).

Note. Since each sum $V(P, f) \geq 0$, it follows that $V_f[a, b] = 0$ if and only if f is a constant function on $[a, b]$.

Examples.

1. Let $k \in \mathbb{R}$ and $f(x) = k$, $x \in [a, b]$.

Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$. Then

$$V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})| = 0.$$

For each partition P of $[a, b]$, $V(P, f) = 0$. Therefore the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$ is bounded above and the supremum of the set is 0, a finite real number.

Consequently, f is a function of bounded variation on $[a, b]$ and the total variation $V_f[a, b]$ is 0.

2. Let $f(x) = x, x \in [a, b]$. P: ordered set

Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$. Then

$$\begin{aligned} V(P, f) &= |x_1 - x_0| + |x_2 - x_1| + \cdots + |x_n - x_{n-1}| \\ &= (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) \\ &= b - a. \end{aligned}$$

For each partition P of $[a, b]$, $V(P, f) = b - a$. Therefore the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$ is bounded above and the supremum of the set is $b - a$, a finite real number.

Therefore f is a function of bounded variation on $[a, b]$ and the total variation $V_f[a, b]$ is $b - a$.

3. Let $f(x) = \sin x, x \in [a, b]$.

Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$. Then

$V(P, f) = |\sin x_1 - \sin x_0| + |\sin x_2 - \sin x_1| + \cdots + |\sin x_n - \sin x_{n-1}|$.

By Mean value theorem, $|f(x_r) - f(x_{r-1})| = |x_r - x_{r-1}||\cos \xi_r|$ for some ξ_r satisfying $x_{r-1} < \xi_r < x_r$. This holds for $r = 1, 2, \ldots, n$.

Therefore $|f(x_r) - f(x_{r-1})| \le |x_r - x_{r-1}|$, since $|\cos \xi_r| \le 1$.

$$\begin{aligned} V(P, f) &\le |x_1 - x_0| + |x_2 - x_1| + \cdots + |x_n - x_{n-1}| \\ \text{i.e., } &\le (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) \\ \text{i.e., } &\le (b - a). \end{aligned}$$

This holds for every partition P of $[a, b]$. Therefore the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$ is bounded above and $V(P, f) \le b - a$.

Therefore f is a function of bounded variation on $[a, b]$.

4. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 1$, if x be rational
$= 0$, if x be irrational.

Let $P = (x_0, x_1, \ldots, x_{2n})$ be a partition of $[a, b]$ such that $x_0, x_2, \ldots, x_{2n}$ are all rational and $x_1, x_3, \ldots, x_{2n-1}$ are all irrational. Then

$V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})| = 2n$.

Clearly, the set $\{V(P, f) : P \in \mathcal{P}[0, 1]\}$ is not bounded above and therefore f is not a function of bounded variation on $[0, 1]$.

Theorem 10.1.1. Let $[a, b] \subset \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Then f is bounded on $[a, b]$.

Proof. Let P be a partition of $[a, b]$. Since f is a function of bounded variation on $[a, b]$, $\sup\{V(P, f) : P \in \mathcal{P}[a, b]\}$ is finite.

Let $\sup\{V(P, f) : P \in \mathcal{P}[a, b]\} = M$, where M is a non-negative real number.

Let $x \in (a, b)$. Let us consider the partition $P_0 = (a, x, b)$ of $[a, b]$.

Then $V(P_0, f) = |f(x) - f(a)| + |f(b) - f(x)| \leq M$. This gives $|f(x) - f(a)| \leq M$ and therefore $|f(x)| \leq |f(a)| + M$.

If however, $x = a$, then $|f(x)| = |f(a)| \leq |f(a)| + M$ and also if $x = b$, then $V(P_0, f) = |f(x) - f(a)| + |f(b) - f(b)| = |f(x) - f(a)|$ and this implies $|f(x)| \leq |f(a)| + M$.

Thus for all $x \in [a, b]$, $|f(x)| \leq |f(a)| + M$ and this proves that f is bounded on $[a, b]$.

Note 1. The converse of the theorem is not true. A function f bounded on $[a, b]$ may not be a function of bounded variation on $[a, b]$. For example, let $f : [0, 1] \to \mathbb{R}$ be defined by
$$f(x) = x\cos\frac{\pi}{2x}, x \neq 0$$
$$= 0, x = 0.$$

Then f is bounded on $[0, 1]$, since $|f(x)| \leq 1$ for all $x \in [0, 1]$.

Let $P = (0, \frac{1}{2n}, \frac{1}{2n-1}, \cdots, \frac{1}{2}, 1)$ be a partition of $[0, 1]$.

Then $f(\frac{1}{2r}) = \frac{1}{2r}\cos\frac{2r\pi}{2} = \frac{1}{2r}.(-1)^r$, for $r = 1, 2, \ldots, n$
and $f(\frac{1}{2r-1}) = \frac{1}{2r-1}\cos\frac{(2r-1)\pi}{2} = 0$, for $r = 1, 2, \ldots, n$.

Then
$$V(P, f) = |f(\tfrac{1}{2n}) - f(0)| + |f(\tfrac{1}{2n-1}) - f(\tfrac{1}{2n})| + \cdots + |f(\tfrac{1}{2}) - f(1)|$$
$$= \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n-2} + \frac{1}{2n-2} + \cdots + \frac{1}{2} + \frac{1}{2} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Since $1 + \frac{1}{2} + \cdots + \frac{1}{n}$ tends to ∞ as n tends to ∞, the set $\{V(P, f) : P \in \mathcal{P}[0, 1]\}$ is not bounded above and therefore f is not a function of bounded variation on $[0, 1]$.

Note 2. It follows from the theorem that a function f, not bounded on $[a, b]$, cannot be a function of bounded variation on $[a, b]$.

Theorem 10.1.2. Let $f : [a, b] \to \mathbb{R}$ be monotone on $[a, b]$. Then f is a function of bounded variation on $[a, b]$.

Proof. **Case 1.** Let f be monotone increasing on $[a, b]$.

Let $P = (x_0, x_1, \ldots, x_n)$ be a partition of $[a, b]$. Then
$$V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})|$$
$$= f(b) - f(a).$$

This holds for every partition P of $[a, b]$. Therefore the supremum of the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$ is $f(b) - f(a)$, a finite real number.

This proves that f is a function of bounded variation on $[a, b]$.

Case 2. Let f be monotone decreasing on $[a, b]$.

In a similar manner it can be proved that $V(P, f) = f(a) - f(b)$ for all partitions of $[a, b]$ and f is a function of bounded variation on $[a, b]$ in this case.

This completes the proof.

Note. If f be monotone increasing on $[a, b]$, then $V_f[a, b] = f(b) - f(a)$; if f be monotone decreasing on $[a, b]$, then $V_f[a, b] = f(a) - f(b)$.

L.F

Definition. A function $f : [a, b] \to \mathbb{R}$ is said to satisfy *Lipschitz condition* on $[a, b]$ if there exists a positive real number M such that $|f(x_1) - f(x_2)| \leq M|x_1 - x_2|$ for *any two points* x_1, x_2 in $[a, b]$. In this case f is also said to be a *Lipschitz function* on $[a, b]$.

Theorem 10.1.3. Let $f : [a, b] \to \mathbb{R}$ be a Lipschitz function on $[a, b]$. Then f is a function of bounded variation on $[a, b]$.

Proof. Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$. Since f is a Lipschitz function on $[a, b]$, there is a positive real number M such that

$|f(x_r) - f(x_{r-1})| \leq M|x_r - x_{r-1}|$, for $r = 1, 2, \ldots, n$.

Therefore $V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})| \leq M[|x_1 - x_0| + |x_2 - x_1| + \cdots + |x_n - x_{n-1}|] = M(b - a)$.

For each partition P of $[a, b]$, $V(P, f) \leq M(b - a)$. Therefore the supremum of the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$ is a finite real number.

Consequently, f is a function of bounded variation on $[a, b]$

Note. The converse of the theorem is not true. A function f of bounded variation on $[a, b]$ may not be a Lipschitz function on $[a, b]$. For example, let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}, x \in [0, 1]$.

Then f being a monotone increasing function on $[0, 1]$, is a function of bounded variation on $[0, 1]$. But f is not a Lipschitz function on $[0, 1]$, because if $x_1 = 0$, no positive real number M can be found to satisfy the condition "$|f(x_2) - f(x_1)| \leq M|x_2 - x_1|$ for all $x_2 \in (0, 1]$".

Theorem 10.1.4. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, f' exists and be bounded on (a, b). Then f is a function of bounded variation on $[a, b]$.

Proof. There exists a positive real number k such that $|f'(x)| \leq k$ for all $x \in (a, b)$.

Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$. Then

$V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})|$.

By Mean value theorem, we have $f(x_r) - f(x_{r-1}) = (x_r - x_{r-1})f'(\xi_r)$ for some ξ_r satisfying $x_{r-1} < \xi_r < x_r$.

Therefore $|f(x_r) - f(x_{r-1})| \leq k|x_r - x_{r-1}|$, for $r = 1, 2, \ldots, n$.

This implies $V(P, f) \leq k(b - a)$.

Therefore the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$ is bounded above and therefore the supremum of the set is a finite real number.

Consequently, f is a function of bounded variation on $[a, b]$

Note. Boundedness of f' on (a, b) is not necessary for the function f to be of bounded variation on $[a, b]$. For example, let $f(x) = \sqrt{x}, x \in [0, 1]$. Then f is a monotone increasing function on $[0, 1]$ and therefore it is a function of bounded variation on $[0, 1]$. But f' is not bounded on $(0, 1)$.

Remark. A function f continuous on a closed and bounded interval $[a, b]$ may not be a function of bounded variation on $[a, b]$. For example, let $f(x) = x \cos \frac{\pi}{2x}$, if $x \in (0, 1]$

$= 0$, if $x = 0$.

Then f continuous on $[0, 1]$. But f is not a function of bounded variation on $[0, 1]$. [Worked out in Note 1 of Theorem 10.1.1.]

Worked Example.

1. A function $f : [0, 1] \to \mathbb{R}$ is defined by $f(x) = x^2 \cos \frac{1}{x}$, if $x \neq 0$

$= 0$, if $x = 0$.

Show that f is a function of bounded variation on $[0, 1]$.

f is continuous on $[0, 1]$. $f'(x) = 2x \cos \frac{1}{x} + \sin \frac{1}{x}, x \in (0, 1]$.

f' is bounded on $(0, 1)$, since $|f'(x)| < 3$ for all $x \in (0, 1)$.

Therefore f is a function of bounded variation on $[0, 1]$.

Theorem 10.1.5. Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be functions of bounded variation on $[a, b]$. Then

(i) $f+g$ is a function of bounded variation on $[a, b]$ and $V_{f+g} \leq V_f + V_g$;

(ii) $f - g$ is a function of bounded variation on $[a, b]$ and $V_{f-g} \leq V_f + V_g$;

(iii) cf $(c \in \mathbb{R})$ is a function of bounded variation on $[a, b]$.

Proof. (i) Let $h(x) = f(x) + g(x), x \in [a, b]$.

Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$. Then

$$V(P,f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})|;$$
$$V(P,g) = |g(x_1) - g(x_0)| + |g(x_2) - g(x_1)| + \cdots + |g(x_n) - g(x_{n-1})|;$$
$$V(P,h) = |h(x_1) - h(x_0)| + |h(x_2) - h(x_1)| + \cdots + |h(x_n) - h(x_{n-1})|.$$

Now $|h(x_r) - h(x_{r-1})| = |f(x_r) + g(x_r) - f(x_{r-1}) - g(x_{r-1})|$
$\leq |f(x_r) - f(x_{r-1})| + |g(x_r) - g(x_{r-1})|.$

Therefore $V(P,h) \leq V(P,f) + V(P,g)$.

Since f and g are functions of bounded variation on $[a,b]$, $V(P,f) \leq V_f[a,b]$, $V(P,g) \leq V_g[a,b]$ for all partitions P of $[a,b]$.

Therfore $V(P,h) \leq V_f[a,b] + V_g[a,b]$ for all partitions P of $[a,b]$.

This shows that the set $\{V(P,h) : P \in \mathcal{P}[a,b]\}$ is bounded above and $\sup\{V(P,h) : P \in \mathcal{P}[a,b]\} \leq V_f[a,b] + V_g[a,b]$, a finite number.

Hence h (i.e.,$f+g$) is a function of bounded variation on $[a,b]$ and $V_{f+g} \leq V_f + V_g$.

Note. Strict inequality in the above relation holds for some functions f and g. For example, let $f(x) = x$, $x \in [1,2]$, $g(x) = 1 - x$, $x \in [1,2]$. Then $(f+g)(x) = 1$, $x \in [1,2]$.

Since f is a monotone increasing function on $[1,2]$, $V_f[1,2] = f(2) - f(1) = 1$. Since g is a monotone decreasing function on $[1,2]$, $V_g[1,2] = g(1) - g(2) = 1$. Since $f+g$ is a constant function on $[1,2]$, $V_{f+g}[1,2] = 0$.

Clearly, $V_{f+g}[1,2] < V_f[1,2] + V_g[1,2]$.

(ii) Similar proof.

(iii) Similar proof.

Note. The class S of all BV-functions on $[a,b]$ form a *real vector space*, since $f \in S, g \in S \Rightarrow f+g \in S$ and $c \in \mathbb{R}, f \in S \Rightarrow cf \in S$.

Theorem 10.1.6. Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be functions of bounded variation on $[a,b]$. Then fg is a function of bounded variation on $[a,b]$ and $V_{fg} \leq AV_g + BV_f$, where $A = \sup\{|f(x)| : x \in [a,b]\}$, $B = \sup\{|g(x)| : x \in [a,b]\}$.

Proof. Let $h(x) = f(x).g(x)$, $x \in [a,b]$.

Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a,b]$. Then

$$V(P,f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})|;$$
$$V(P,g) = |g(x_1) - g(x_0)| + |g(x_2) - g(x_1)| + \cdots + |g(x_n) - g(x_{n-1})|;$$
$$V(P,h) = |h(x_1) - h(x_0)| + |h(x_2) - h(x_1)| + \cdots + |h(x_n) - h(x_{n-1})|.$$
$$|h(x_r) - h(x_{r-1})| = |f(x_r).g(x_r) - f(x_{r-1}).g(x_{r-1})|$$

$= |f(x_r)[g(x_r) - g(x_{r-1})] + g(x_{r-1})[f(x_r) - f(x_{r-1})]|$
$\leq |f(x_r)||g(x_r) - g(x_{r-1})| + |g(x_{r-1})||f(x_r) - f(x_{r-1})|$.

Since f and g are functions of bounded variation on $[a, b]$, f and g are bounded on $[a, b]$. There exist positive real numbers A, B such that $|f(x)| \leq A$, $|g(x)| \leq B$, for all $x \in [a, b]$.

Then $|h(x_r) - h(x_{r-1})| \leq A|g(x_r) - g(x_{r-1})| + B|f(x_r) - f(x_{r-1})|$.

Since f and g are functions of bounded variation on $[a, b]$, $V(P, f) \leq V_f[a, b]$, $V(P, g) \leq V_g[a, b]$ for all partitions P of $[a, b]$.
Therefore $V(P, h) \leq AV_g[a, b] + BV_f[a, b]$ for all partitions P of $[a, b]$.

This shows that the set $\{V(P, h) : P \in \mathcal{P}[a, b]\}$ is bounded above and therefore $\sup\{V(P, h) : P \in \mathcal{P}[a, b]\}$ is a finite real number.

Hence h (i.e., $f.g$) is a function of bounded variation on $[a, b]$ and $V_{f.g} \leq AV_g + BV_f$.

Theorem 10.1.7. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. If there exists a positive real number k such that $0 < k \leq f(x)$ for all $x \in [a, b]$, then $1/f$ is a function of bounded variation on $[a, b]$ and $V_{\frac{1}{f}} \leq \frac{V_f}{k^2}$.

Proof. Let $h(x) = 1/f(x), x \in [a, b]$.
Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$. Then

$V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})|$;
$V(P, h) = |h(x_1) - h(x_0)| + |h(x_2) - h(x_1)| + \cdots + |h(x_n) - h(x_{n-1})|$.

Now $|h(x_r) - h(x_{r-1})| = |\frac{1}{f(x_r)} - \frac{1}{f(x_{r-1})}| = \frac{|f(x_{r-1}) - f(x_r)|}{|f(x_r)f(x_{r-1})|}$.

Since $0 < k \leq f(x)$ for all $x \in [a, b]$, $|f(x_r)f(x_{r-1})| > k^2$ for all $x \in [a, b]$.
Therefore $V(P, h) < \frac{1}{k^2}.V(P, f)$.

Since f is a function of bounded variation on $[a, b]$, $V(P, f) \leq V_f[a, b]$ for all partitions P of $[a, b]$.
Therfore $V(P, h) < \frac{1}{k^2}.V_f[a, b]$ for all partitions P of $[a, b]$.

This shows that the set $\{V(P, h) : P \in \mathcal{P}[a, b]\}$ is bounded above and therefore $\sup\{V(P, h) : P \in \mathcal{P}[a, b]\}$ is a finite real number.

Hence h (i.e.,$1/f$) is a function of bounded variation on $[a, b]$ and $V_{1/f} \leq \frac{V_f}{k^2}$.

Theorem 10.1.8. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Then $|f|$ is a function of bounded variation on $[a, b]$;

Proof. Let $h(x) = |f(x)|, x \in [a, b]$.

Let $P = (x_0, x_1, \dots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$. Then

$V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})|;$

$V(P, h) = |h(x_1) - h(x_0)| + |h(x_2) - h(x_1)| + \cdots + |h(x_n) - h(x_{n-1})|.$

Now $|h(x_r) - h(x_{r-1})| = ||f(x_r)| - |f(x_{r-1})|| \le |f(x_r) - f(x_{r-1})|$.

Therefore $V(P, h) \le V(P, f)$.

Since f is a function of bounded variation on $[a, b]$, $V(P, f) \le V_f[a, b]$ for all partitions P of $[a, b]$.

Therefore $V(P, h) \le V_f[a, b]$ for all partitions P of $[a, b]$.

This shows that the set $\{V(P, h) : P \in \mathcal{P}[a, b]\}$ is bounded above and therefore $\sup\{V(P, h) : P \in \mathcal{P}[a, b]\}$ is a finite real number.

Hence h (i.e., $|f|$) is a function of bounded variation on $[a, b]$.

Refinement of a partition.

Let $[a, b]$ be a closed and bounded interval. Let $P = (x_0, x_1, x_2, \dots, x_n)$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$.

A partition Q of $[a, b]$ is said to be a *refinement* of P if P be a proper subset of Q. That is, Q is obtained by adjoining a *finite number* of additional points to P.

For example, if $P = (0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1)$ be a partition of $[0, 1]$ and $Q = (0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, 1)$ then Q is a refinement of P.

If $R = (0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{3}{4}, 1)$ then R is a refinement of P but not a refinement of Q.

Theorem 10.1.9. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and P be a partition of $[a, b]$. If Q be a refinement of P then $V(Q, f) \ge V(P, f)$.

Proof. Let $P = (x_0, x_1, x_2, \dots, x_n)$, where $a = x_0, x_n = b$.

First we examine the effect of adjoining one additional point y to P.

Let $P_1 = (x_0, x_1, \dots, x_{k-1}, y, x_k, \dots, x_n)$.

The subinterval $[x_{k-1}, x_k]$ is divided into two smaller subintervals $[x_{k-1}, y]$ and $[y, x_k]$.

$V(P, f) = |f(x_1) - f(x_0)| + \cdots + |f(x_k) - f(x_{k-1})| + \cdots + |f(x_n) - f(x_{n-1})|.$
$V(P_1, f) = |f(x_1) - f(x_0)| + \cdots + |f(y) - f(x_{k-1})| + |f(x_k) - f(y)| + \cdots + |f(x_n) - f(x_{n-1})|.$

Since $|f(x_k) - f(x_{k-1})| = |f(x_k) - f(y) + f(y) - f(x_{k-1})| \leq |f(y) - f(x_{k-1})| + |f(x_k) - f(y)|$, it follows that $V(P_1, f) \geq V(P, f)$.

If Q be any refinement of P then Q can be obtained from P by adjoining a finite number of additional points to P, one at a time.

By repeating the argument a finite number of times, we have $V(Q, f) \geq V(P, f)$.

Theorem 10.1.10. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and $c \in (a, b)$. Then

(i) f is of bounded variation on $[a, c]$ and on $[c, b]$; and

(ii) $V_f[a, b] = V_f[a, c] + V_f[c, b]$.

Proof. (i) Let P_1 be a partition of $[a, c]$ and P_2 be a partition of $[c, b]$. Let $P = P_1 \cup P_2$. Then P is a partition of $[a, b]$

Clearly, $V(P_1, f) + V(P_2, f) = V(P, f)$.

Since f is a function of bounded variation on $[a, b]$, $V(P, f) \leq V_f[a, b]$ for all partitions P of $[a, b]$.

Since each of $V(P_1, f)$ and $V(P_2, f)$ is non-negative, it follows that $V(P_1, f) \leq V_f[a, b]$ for all partitions P_1 of $[a, c]$ and $V(P_2, f) \leq V_f[a, b]$ for all partitions P_2 of $[c, b]$.

Hence f is a function of bounded variation on $[a, c]$ and on $[c, b]$

(ii) We use here an important property of bounded sets in $\mathbb{R}$.

If S_1 and S_2 be subsets of $\mathbb{R}$ both bounded above and $T = \{x + y : x \in S_1, y \in S_2\}$, then $\sup T = \sup S_1 + \sup S_2$.

Here both the sets $S_1 = \{V(P_1, f) : P_1 \in \mathcal{P}[a, c]\}$ and $S_2 = \{V(P_2, f) : P_2 \in \mathcal{P}[c, b]\}$ are bounded above and $\sup S_1 = V_f[a, c]$, $\sup S_2 = V_f[c, b]$.

The supremum of the set $\{V(P_1, f) + V(P_2, f) : P_1 \in \mathcal{P}[a, c], P_2 \in \mathcal{P}[c, b]\} = \sup\{V(P_1, f) : P_1 \in \mathcal{P}[a, c]\} + \sup\{V(P_2, f) : P_2 \in \mathcal{P}[c, b]\}$ $= V_f[a, c] + V_f[c, b]$.

Since the set $\{V(P_1, f) + V(P_2, f) : P_1 \in \mathcal{P}[a, c], P_2 \in \mathcal{P}[c, b]\}$ is a proper subset of the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$, we have $V_f[a, c] + V_f[c, b] \leq V_f[a, b]$... ... (i)

To obtain the reverse inequality, let P be a partition of $[a, b]$ and let $P_0 = P \cup \{c\}$. Then P_0 is a refinement of P if $c \notin P$, and $P_0 = P$ if $c \in P$.

Let $P_1 = P_0 \cap [a, c]$, $P_2 = P_0 \cap [c, b]$. Then P_1 is a partition of $[a, c]$ and P_2 is a partition of $[c, b]$ and $V(P_0, f) = V(P_1, f) + V(P_2, f)$.

Since P_0 is a refinement of the partition P, $V(P_0, f) \geq V(P, f)$.

We have $V(P,f) \le V(P_0,f) = V(P_1,f)+V(P_2,f) \le V_f[a,c]+V_f[c.b]$ and this holds for all partitions P of $[a,b]$.

This shows that $V_f[a,c] + V_f[c,b]$ is an upper bound of the set $\{V(P,f) : P \in \mathcal{P}[a,b]\}$. Therefore $V_f[a,b] \le V_f[a,c] + V_f[c.b]$ ... (ii)

From (i) and (ii) we have $V_f[a,b] = V_f[a,c] + V_f[c,b]$.

This completes the proof.

Theorem 10.1.11. Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,c]$ and on $[c,b]$ where $c \in (a,b)$. Then

(i) f is of bounded variation on $[a,b]$, and

(ii) $V_f[a,c] + V_f[c,b] = V_f[a,b]$.

Proof. (i) let P be a partition of $[a,b]$ and let $P_0 = P \cup \{c\}$. Then P_0 is a refinement of P if $c \notin P$; and $P_0 = P$ if $c \in P$.

Let $P_1 = P_0 \cap [a,c]$, $P_2 = P_0 \cap [c,b]$. Then P_1 is a partition of $[a,c]$ and P_2 is a partition of $[c,b]$ and $V(P_0,f) = V(P_1,f) + V(P_2,f)$.

We also have either $P_0 = P$ or P_0 is a refinement of P. Therefore $V(P_0,f) \ge V(P,f)$

We have $V(P,f) \le V(P_0,f) = V(P_1,f) + V(P_2,f)$.

Since f is of bounded variation on $[a,c]$ and also on $[c,b]$, $V(P_1,f) \le V_f[a,c]$ and $V(P_2,f) \le V_f[c,b]$.

Therefore the set $\{V(P,f) : P \in \mathcal{P}[a,b]\}$ is bounded above and therefore f is of bounded variation on $[a,b]$.

(ii) Since $V(P,f) \le V_f[a,c]+V_f[c,b]$, $V_f[a,c]+V_f[c,b]$ is an upper bound of the set $\{V(P,f) : P \in \mathcal{P}[a,b]\}$ and since $\sup\{V(P,f) : P \in \mathcal{P}[a,b]\}$ is $V_f[a,b]$, it follows that $V_f[a,b] \le V_f[a,c] + V_f[c,b]$ ... ... (i)

Let $\epsilon > 0$.

Since $V_f[a,c]$ is the supremum of the set $\{V(P,f) : P \in \mathcal{P}[a,c]\}$, there exists a partition Q_1 of $[a,c]$ such that $V(Q_1,f) > V_f[a,c] - \frac{\epsilon}{2}$.

Since $V_f[c,b]$ is the supremum of the set $\{V(P,f) : P \in \mathcal{P}[c,b]\}$, there exists a partition Q_2 of $[c,b]$ such that $V(Q_2,f) > V_f[c,b] - \frac{\epsilon}{2}$.

Let $Q = Q_1 \cup Q_2$. Then Q is a partition of $[a,b]$ and $V(Q,f) = V(Q_1,f) + V(Q_2,f) > V_f[a,c] + V_f[c,b] - \epsilon$.

But $V_f[a,b] \ge V(Q,f)$. Therefore $V_f[a,b] > V_f[a,c] + V_f[c,b] - \epsilon$.

Since ϵ is an arbitrarily small positive number, it follows that

$V_f[a,b] \ge V_f[a,c] + V_f[c,b]$ ... ... (ii)

Using (i) and (ii) the proof is complete.

Corollary. If $f : [a, b] \to \mathbb{R}$ and if it be possible to divide the interval $[a, b]$ into a finite number of subintervals in each of which f is monotone, then f is a BV-function on $[a, b]$.

Worked Example (continued).

2. Let $f : [0, 3] \to \mathbb{R}$ be defined by $f(x) = x^2 - 4x + 3$, $x \in [0, 3]$. Show that f is a function of bounded variation on $[0, 3]$. Calculate $V_f[0, 3]$.

f is continuous on $[0, 3]$. $f'(x) < 0$ if $x \in (0, 2)$; $f'(x) > 0$ if $x \in (2, 3)$. Therefore f is a decreasing function on $[0, 2]$ and is an increasing function on $[2, 3]$.

Hence f is a BV-function on $[0, 2]$ and on $[2, 3]$ and therefore f is a BV-function on $[0, 3]$.

$V_f[0, 2] = f(0) - f(2) = 4$, since f is a decreasing function on $[0, 2]$;
$V_f[2, 3] = f(3) - f(2) = 1$, since f is an increasing function on $[2, 3]$.
Therefore $V_f[0, 3] = V_f[0, 2] + V_f[2, 3] = 5$.

Theorem 10.1.12. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and let $\phi : [a, b] \to \mathbb{R}$ be such that ϕ is bounded on $[a, b]$ and $\phi(x) = f(x)$ except at only one point in $[a, b]$. Then ϕ is a function of bounded variation on $[a, b]$.

Proof. Let $\phi(c) \neq f(c)$, $c \in [a, b]$. Let $\phi(c) = f(c) + \mu$, $\mu \in \mathbb{R}, \mu \neq 0$.

Case 1. $c = a$.

Let us take a partition $P = (x_0, x_1, x_2, \ldots, x_n)$ of $[a, b]$. Then
$V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})|$.
$V(P, \phi) = |\phi(x_1) - \phi(x_0)| + |\phi(x_2) - \phi(x_1)| + \cdots + |\phi(x_n) - \phi(x_{n-1})|$.

$V(P, \phi) - V(P, f) = |\phi(x_1) - \phi(x_0)| - |f(x_1) - f(x_0)| = |f(x_1) + \mu - \phi(x_0)| - |f(x_1) - f(x_0)| \leq |\mu|$.

Therefore $V(P, \phi) \leq V(P, f) + |\mu|$ and this holds for every partition P of $[a, b]$.

Consequently, $\sup\{V(P, \phi) : P \in \mathcal{P}[a, b]\} \leq |\mu| + V_f[a, b]$, a finite positive number. Hence ϕ is a function of bounded variation on $[a, b]$.

Case 2. $c = b$.

Similar proof.

Case 3. $a < c < b$.

Since f is a function of bounded variation on $[a, b]$ and $a < c < b$, f is a BV-function on $[a, c]$ and on $[c, b]$.

Since ϕ is bounded on $[a, c]$ and $\phi(x) = f(x)$ for all $x \in [a, c]$ except at c, ϕ is a function of bounded variation on $[a, c]$, by Case 2.

Since ϕ is bounded on $[c, b]$ and $\phi(x) = f(x)$ for all $x \in [c, b]$ except at c, ϕ is a function of bounded variation on $[c, b]$, by Case 1.

Since ϕ is a BV-function on $[a, c]$ and on $[c, b]$, ϕ is a function of bounded variation on $[a, b]$. This completes the proof.

Note. If $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and $\phi : [a, b] \to \mathbb{R}$ be such that ϕ is bounded on $[a, b]$ and $\phi(x) = f(x)$ except at a finite number of points in $[a, b]$, then ϕ is a a function of bounded variation on $[a, b]$.

Worked Example (continued).

3. Let $f : [1, 3] \to \mathbb{R}$ be defined by $f(x) = x - [x]$, $x \in [1, 3]$. Show that f is a function of bounded variation on $[1, 3]$. Calculate $V_f[1, 3]$.

$$\begin{aligned} f(x) &= x - 1, \text{ if } 1 \leq x < 2 \\ &= x - 2, \text{ if } 2 \leq x < 3 \\ &= 0, \text{ if } x = 3. \end{aligned}$$

Let us define a function ϕ_1 on $[1, 2]$ by $\phi_1(x) = x - 1$, $x \in [1, 2]$. Then ϕ_1 is a monotone increasing function on $[1, 2]$ and therefore ϕ_1 is a BV-function on $[1, 2]$. Since f is bounded on $[1, 2]$ and $f(x) = \phi_1(x)$ for all $x \in [1, 2]$ except at 2, f is a BV-function on $[1, 2]$.

Let us define a function ϕ_2 on $[2, 3]$ by $\phi_2(x) = x - 2$, $x \in [2, 3]$. Then ϕ_2 is a monotone increasing function on $[2, 3]$ and therefore ϕ_2 is a BV-function on $[2, 3]$. Since f is bounded on $[2, 3]$ and $f(x) = \phi_2(x)$ for all $x \in [2, 3]$ except at 3, f is a BV-function on $[2, 3]$.

Since f is a BV-function on $[1, 2]$ and on $[2, 3]$, f is a BV-function on $[1, 3]$.

$$V_f[1, 2] = \sup_{x \in (1,2)} [V_f[1, x] + |f(2) - f(x)|] = \sup_{x \in (1,2)} [x - 1 + |1 - x|] = 2.$$
$$V_f[2, 3] = \sup_{x \in (2,3)} [V_f[2, x] + |f(3) - f(x)|] = \sup_{x \in (2,3)} [x - 2 + |2 - x|] = 2.$$
$$V_f[1, 3] = V_f[1, 2] + V_f[2, 3] = 4.$$

10.2. Variation function.

Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and $x \in (a, b]$. Then the total variation of f on $[a, x]$, i.e., $V_f[a, x]$ is a function of x for all $x \in (a, b]$. Let us define a function V on $[a, b]$ by

$V(a) = 0$ and $V(x) = V_f[a, x]$, if $a < x \leq b$.

V is called the *variation function* of f on $[a, b]$. The variation function of f is also denoted by V_f.

Therefore $V_f(x) = V_f[a, x]$, if $a < x \leq b$ and $V_f(a) = 0$.

Theorem 10.2.1. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Then the variation function V defined by

$$V(a) = 0 \text{ and } V(x) = V_f[a, x], \text{ if } a < x \leq b$$

is a monotone increasing function on $[a, b]$.

Proof. If $a < x < y \leq b$, then $V(y) - V(x) = V_f[a, y] - V_f[a, x] = V_f[x, y] \geq 0$.

If $a = x < y \leq b$, then $V(y) - V(x) = V_f[a, y] - V(a) = V_f[a, y] \geq 0$.

Therefore $a \leq x < y \leq b \Rightarrow V(x) \leq V(y)$ and this proves that V is a monotone increasing function on $[a, b]$.

Theorem 10.2.2. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and V be the variation function of f on $[a, b]$. Then

(i) $V + f$ is a monotone increasing function on $[a, b]$;

(ii) $V - f$ is a monotone increasing function on $[a, b]$.

Proof. (i) Let $F(x) = V(x) + f(x), x \in [a, b]$.

If $a < x < y \leq b$ then $F(y) - F(x) = V(y) - V(x) + [f(y) - f(x)] = V_f[a, y] - V_f[a, x] + [f(y) - f(x)] = V_f[x, y] - [f(x) - f(y)]$.

But $V_f(x, y) \geq |f(y) - f(x)|$, by definition.

Therefore if $a < x < y \leq b$ then $F(y) - F(x) \geq 0$.

If $a = x < y \leq b$ then $F(y) - F(x) = V(y) - V(x) + [f(y) - f(x)] = V_f[a, y] - V(a) + [f(y) - f(a)] = V_f[a, y] - [f(a) - f(y)] \geq 0$, by the foregoing argument.

Therefore $a \leq x < y \leq b \Rightarrow F(x) \leq F(y)$ and this proves that F, i.e., $V + f$ is a monotone increasing function on $[a, b]$.

(ii) Let $G(x) = V(x) - f(x), x \in [a, b]$.

If $a < x < y \leq b$ then $G(y) - G(x) = V(y) - V(x) - [f(y) - f(x)] = V_f[a, y] - V_f[a, x] - [f(y) - f(x)] = V_f[x, y] - [f(y) - f(x)]$.

But $V_f(x, y) \geq |f(y) - f(x)|$, by definition.

Therefore if $a < x < y \leq b$ then $G(y) - G(x) \geq 0$.

If $a = x < y \leq b$ then $G(y) - G(x) = V(y) - V(x) - [f(y) - f(x)] = V_f[a, y] - V(a) - [f(y) - f(a)] = V_f[a, y] - [f(y) - f(a)] \geq 0$, by the foregoing argument.

Therefore $a \leq x < y \leq b \Rightarrow V(x) \leq V(y)$ and this proves that G, i.e., $V - f$ is a monotone increasing function on $[a, b]$.

This completes the proof.

With the help of the two previous theorems we have an important characterisation of a function of bounded variation.

Theorem 10.2.3. Let $f : [a, b] \to \mathbb{R}$. Then f is a function of bounded variation on $[a, b]$ if and only if f can be expressed as the difference of two monotone increasing functions on $[a, b]$.

Proof. Let f be a function of bounded variation on $[a, b]$.

Then the variation function V is defined on $[a, b]$ by $V(x) = V_f[a, x]$, for $a < x \leq b$ and $V(a) = 0$. We prove that V and $V + f$ are both monotone increasing functions on $[a, b]$.

If $a < x < y \leq b$ then $V(y) - V(x) = V_f[a, y] - V_f[a, x] = V_f[x, y] \geq 0$.
If $a = x < y \leq b$ then $V(y) - V(x) = V_f[a, y] - V(a) = V_f[a, y] \geq 0$.

Therefore $a \leq x < y \leq b \Rightarrow V(x) \leq V(y)$ and this proves that V is a monotone increasing function on $[a, b]$.

Let $F(x) = V(x) + f(x), x \in [a, b]$.

If $a < x < y \leq b$ then $F(y) - F(x) = V(y) - V(x) + [f(y) - f(x)] = V_f[a, y] - V_f[a, x] + [f(y) - f(x)] = V_f[x, y] - [f(x) - f(y)] \geq 0$, since $V_f(x, y) \geq |f(y) - f(x)|$, by definition.

If $a = x < y \leq b$ then $F(y) - F(x) = V(y) - V(x) + [f(y) - f(x)] = V_f[a, y] - V(a) + [f(y) - f(a)] = V_f[a, y] - [f(a) - f(y)] \geq 0$, by the foregoing argument.

Therefore $a \leq x < y \leq b \Rightarrow F(x) \leq F(y)$ and this proves that F, i.e., $V + f$ is a monotone increasing function on $[a, b]$.

f can be expressed as $f = (V + f) - V$. Thus f is expressed as the difference of two monotone increasing functions $V + f$ and V.

Conversely, let f be expressed as the difference of two monotone increasing functions on $[a, b]$.

Since a monotone increasing function on $[a, b]$ is a function of bounded variation on $[a, b]$ and the difference of two BV-functions on $[a, b]$ is a BV-function on $[a, b]$, it follows that f is a function of bounded variation on $[a, b]$.

This completes the proof.

Note. f can also be expressed as $f = V - (V - f)$, where V is the variation function of f on $[a, b]$. Since V and $V - f$ are both monotone increasing functions on $[a, b]$, f is expressed as the difference of two monotone increasing functions on $[a, b]$. This shows that the representation of f as the difference of two monotone increasing functions is not unique.

Worked Examples (continued).

4. $f(x) = x^2, x \in [-1, 1]$. Show that f is a function of bounded variation on $[-1, 1]$. Find the variation function V on $[-1, 1]$. Express f as the difference of two monotone increasing functions on $[-1, 1]$.

f is continuous on $[-1, 1]$. $f'(x) < 0$ on $(-1, 0)$ and $f'(x) > 0$ on $(0, 1)$. Therefore f is a decreasing function on $[-1, 0]$ and is an increasing function on $[0, 1]$.

Hence f is a BV-function on $[-1, 0]$ and on $[0, 1]$ and therefore f is a BV-function on $[-1, 1]$.

$V(-1) = 0$.

If $-1 < x \leq 0$, then $V(x) = V_f[-1, x]$
$= f(-1) - f(x)$, since f is decreasing on $[-1, 0]$
$= 1 - x^2$.

If $0 < x \leq 1$, then $V(x) = V_f[-1, x] = V_f[-1, 0] + V_f[0, x]$
$= [f(-1) - f(0)] + [f(x) - f(0)]$, since f is a decreasing on $[-1, 0]$
$= 1 + x^2$. and increasing on $[0, 1]$

Therefore $V(x) = 1 - x^2$, if $-1 \leq x \leq 0$
$= 1 + x^2$, if $0 < x \leq 1$.

V is a monotone increasing function on $[-1, 1]$.

$(V + f)(x) = 1$, if $-1 \leq x \leq 0$
$= 1 + 2x^2$, if $0 < x \leq 1$.

$V + f$ is a monotone increasing function on $[-1, 1]$.

f can be expressed as $(V + f) - V$, the difference of two monotone increasing functions.

Note. Here $(V - f)(x) = 1 - 2x^2$, if $-1 \leq x \leq 0$
$= 1$, if $0 < x \leq 1$.

$V - f$ is a monotone increasing function on $[-1, 1]$.

f can also be expressed as $V - (V - f)$, the difference of two monotone increasing functions on $[-1, 1]$.

5. $f(x) = [x], x \in [1, 3]$. Show that f is a function of bounded variation on $[1, 3]$. Find the variation function V on $[1, 3]$. Express f as the difference of two monotone increasing functions on $[1, 3]$.

$f(x) = 1$, if $1 \leq x < 2$
$= 2$, if $2 \leq x < 3$
$= 3$, if $x = 3$.

f is a monotone increasing function on $[1, 3]$. Therefore f is a function of bounded variation on $[1, 3]$.

$V(1) = 0$.

If $1 < x < 2$ then $V(x) = V_f[-1, x]$
$$= f(x) - f(1), \text{ since } f \text{ is increasing on } [1,2]$$
$$= 0.$$

$V(2) = V_f[1,2] = f(2) - f(1) = 1$.

If $2 < x < 3$ then $V(x) = V_f[1, x] = V_f[1,2] + V_f[2, x]$
$$= [f(2) - f(1)] + [f(x) - f(2)], \text{ since } f \text{ is a increasing on } [1,3]$$
$$= (2-1) + 0 = 1.$$

$V(3) = V_f[1,3] = V_f[1,2] + V_f[2,3] = [f(2) - f(1)] - [f(3) - f(2)] = 2.$

Therefore $V(x) = 0$, if $1 \le x < 2$
$= 1$, if $2 \le x < 3$
$= 2$, if $x = 3$.

V is a monotone increasing function on $[1,3]$.

$(V+f)(x) = 1$, if $1 \le x < 2$
$= 3$, if $2 \le x < 3$
$= 5$, if $x = 3$.

$V + f$ is a monotone increasing function on $[1,3]$.

f can be expressed as $(V+f) - V$, the difference of two monotone increasing functions.

Theorem 10.2.4. Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$ then f can have only discontinuities of the first kind and the points of discontinuity of f form a countable set.

Proof. Since f is a function of bounded variation on $[a,b]$, f can be expressed as $f(x) = g(x) - h(x)$, where g and h are monotone increasing functions on $[a,b]$.

A monotone function can have only discontinuities of the first kind.

Let $c \in (a,b)$. Then each of $g(c+0), g(c-0), h(c+0), h(c-0)$ exists and therefore each of $f(c+0), f(c-0)$ exists.

By similar arguments, each of $f(a+0), f(b-0)$ exists.

It follows that f can have only discontinuities of the first kind on $[a,b]$.

Let E_1, E_2 be respectively the sets of points of discontinuity of g and h. Then $E_1 \cup E_2$ is the set of points of discontinuity of f.

By Theorem 8.6.4, E_1 and E_2 are both countable sets. Therefore the set $E_1 \cup E_2$ is countable.

This completes the proof.

Theorem 10.2.5. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and let V be the variation function on $[a, b]$. If f be continuous at a point $c \in [a, b]$ then V is continuous at c, and conversely.

Proof. Let $c \in [a, b)$ and f is right continuous at c.

Let us choose a positive ϵ. There exists a positive δ_1 such that $|f(x) - f(c)| < \frac{\epsilon}{2}$ for all $x \in [c, c + \delta_1) \cap [c, b]$.

Since $V_f[c, b]$ is the supremum of the set $\{V(P, f) : P \in \mathcal{P}[c, b]\}$,there exists a partition P_0 of $[c, b]$ such that $V_f[c, b] - \frac{\epsilon}{2} < V(P_0, f) \leq V_f[c, b]$.

Let $P_0 = (c, x_1, x_2, \ldots, x_{n-1}, b)$. Let $\delta < \min\{x_1 - c, \delta_1\}$.

Let us choose a point x_0 in $(c, c + \delta)$.

Let $P_1 = P_0 \cup \{x_0\}$. Then the partition P_1 is a refinement of P_0 and $V(P_1, f) \geq V(P_0, f)$.

Therefore for all $x_0 \in (c, c + \delta)$, we have $V_f[c, b] - \frac{\epsilon}{2} < V(P_0, f) \leq V(P_1, f) \leq V_f[c, b]$ and also we have $|f(x_0) - f(c)| < \frac{\epsilon}{2}$.

But $V(P_1, f) = |f(x_0) - f(c)| + |f(x_1) - f(x_0)| + \cdots + |f(b) - f(x_{n-1}|$
$< \frac{\epsilon}{2} + V_f[x_0, b]$.

Therefore for all $x_0 \in (c, c + \delta)$, $V(b) - V(c) - \frac{\epsilon}{2} < \frac{\epsilon}{2} + V(b) - V(x_0)$.
or, $V(x_0) - V(c) < \epsilon$ for all $x_0 \in (c, c + \delta)$.

Also we have $V(x_0) \geq V(c)$ for all $x_0 \in [c, c+\delta)$, since V is a monotone increasing function on $[a, b]$. Therefore $|V(x_0) - V(c)| < \epsilon$ for all $x_0 \in [c, c + \delta)$ and this proves that V is right continuous at c.

By similar arguments, if f be left continuous at $c \in (a, b]$ then V is left continuous at c.

It follows that V is continuous at every point $c \in [a, b]$ where f is continuous.

Conversely, let $c \in [a, b)$ and V be right continuous at c.

Let us choose a positive ϵ. Then there exists a positive δ such that $|V(x) - V(c)| < \epsilon$ for all $x \in [c, c + \delta) \cap [c, b]$.

For all $c \in [a, b)$ and for all $x \in (c, c + \delta)$, we have $|f(x) - f(c)| \leq V_f[c, x] \leq |V(x) - V(c)|$.

Therefore $|f(x) - f(c)| < \epsilon$ for all $x \in [c, c + \delta) \cap [c, b]$.

This proves that f is right continuous at c.

By similar arguments, if $c \in (a, b]$ and V be left continuous at c then f is left continuous at c.

It follows that f is continuous at every point $c \in [a, b]$ where V is continuous.

This completes the proof.

Corollary. If $f : [a, b] \to \mathbb{R}$ be continuous and be of bounded variation on $[a, b]$ then f can be expressed as the difference of two monotone increasing and continuous functions on $[a, b]$, and conversely.

10.3. Positive variation, Negative variation.

Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Let $P = (x_0, x_1, x_2, \ldots, x_n)$ be a partition of $[a, b]$. Let us consider the sum

$$\Delta f_1 + \Delta f_2 + \cdots + \Delta f_n, \text{ where } \Delta f_r = f(x_r) - f(x_{r-1}) \quad \ldots \ldots \quad \text{(i)}$$

We have $V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})| = |\Delta f_1| + |\Delta f_2| + \cdots + |\Delta f_n|$.

Let $V_+(P, f)$ denote the sum $\sum\limits_{\Delta f_i > 0} \Delta f_i$ (i.e., the sum of all positive differences in (i)) and $V_-(P, f)$ denote the sum $\sum\limits_{\Delta f_i < 0} |\Delta f_i|$ (i.e., the sum of absolute values of all negative differences in (i)).

Then $V_+(P, f) - V_-(P, f) = f(b) - f(a)$ and
$V_+(P, f) + V_-(P, f) = V(P, f)$.

Therefore $2V_+(P, f) = V(P, f) + f(b) - f(a)$ and
$2V_-(P, f) = V(P, f) - f(b) + f(a) \quad \ldots \ldots \quad$ (ii)

Since f is a function of bounded variation on $[a, b]$, the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$ is bounded above. It follows from (ii) that the both the sets $\{V_+(P, f) : P \in \mathcal{P}[a, b]\}$ and $\{V_-(P, f) : P \in \mathcal{P}[a, b]\}$ are bounded above.

The supremum of the set $\{V_+(P, f) : P \in \mathcal{P}[a, b]\}$ is said to be the *positive variation* of f on $[a, b]$ and is denoted by $(V_+)_f[a, b]$ or by $p_f[a, b]$.

The supremum of the set $\{V_-(P, f) : P \in \mathcal{P}[a, b]\}$ is said to be the *negative variation* of f on $[a, b]$ and is denoted by $(V_-)_f[a, b]$ or by $n_f[a, b]$.

Note. It follows from (ii) that $2p_f[a, b] = V_f[a, b] + f(b) - f(a)$ and
$2n_f[a, b] = V_f[a, b] - f(b) + f(a)$.

Therefore $p_f[a, b] + n_f[a, b] = V_f[a, b]$ and $p_f[a, b] - n_f[a, b] = f(b) - f(a)$.

Positive variation function, Negative variation function.

Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and let $x \in (a, b]$.

The positive variation of f on $[a, x]$ is a function of x for all $x \in (a, b]$. Let us define a function V_+ on $[a, b]$ by

$V_+(a) = 0$ and $V_+(x) = p_f[a, x]$, if $x \in (a, b]$.

V_+ is called the *positive variation function* of f on $[a, b]$. It is also denoted by p. Therefore $p(x) = p_f[a, x]$, if $x \in (a, b]$ and $p(a) = 0$.

The negative variation of f on $[a, x]$ is a function of x for all $x \in (a, b]$. Let us define a function V_- on $[a, b]$ by

$V_-(a) = 0$ and $V_-(x) = n_f[a, x]$, if $x \in (a, b]$.

V_- is called the *negative variation function* of f on $[a, b]$. It is also denoted by n. Therefore $n(x) = n_f[a, x]$, if $x \in (a, b]$ and $n(a) = 0$.

Theorem 10.3.1. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Then

(i) $p(x) + n(x) = V(x)$ for all $x \in [a, b]$,

(ii) $p(x) - n(x) = f(x) - f(a)$ for all $x \in [a, b]$,
where V is the variation function, p is the positive variation function and n is the negative variation function of f on $[a, b]$.

Proof. (i) We have $p(a) = 0$, $n(a) = 0$ and $V(a) = 0$.
Therefore $V(a) = p(a) + n(a)$ ... ... (i)

If $a < x \leq b$, then $V(x) = V_f[a, x]$, $p(x) = p_f[a, x]$, $n(x) = n_f[a, x]$.
Therefore $p(x) + n(x) = V(x)$ for all $x \in (a, b]$, since $p_f[a, x] + n_f[a, x] = V_f[a, x]$ for all $x \in (a, b]$ ... ... (ii)
From (i) and (ii) $p(x) + n(x) = V(x)$ for all $x \in [a, b]$.

(ii) Similar proof.

Theorem 10.3.2. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Then the positive variation function p is a monotone increasing function on $[a, b]$.

Proof. By definition, $2p(x) = V(x) + f(x) - f(a)$.

If $a < x < y \leq b$, then $p(y) - p(x) = \frac{1}{2}\{V(y) - V(x)\} + \frac{1}{2}[f(y) - f(x)]$
$= \frac{1}{2}\{V_f[a, y] - V_f[a, x]\} + \frac{1}{2}[f(y) - f(x)]$
$= \frac{1}{2}[V_f[x, y] - (f(x) - f(y))] \geq 0$, since $V_f[x, y] \geq |f(y) - f(x)|$.

If $a = x < y \leq b$, then $p(y) - p(x) = \frac{1}{2}\{V(y) - V(x)\} + \frac{1}{2}[f(y) - f(x)]$
$= \frac{1}{2}\{V_f[a, y]\} + \frac{1}{2}[f(y) - f(a)]$, since $V(x) = V(a) = 0$
$= \frac{1}{2}[V_f[a, y] - (f(a) - f(y))] \geq 0$, since $V_f[a, y] \geq |f(y) - f(a)|$.

Therefore $a \leq x < y \leq b \Rightarrow p(x) \leq p(y)$ and this proves that p is a monotone increasing function on $[a, b]$.

Theorem 10.3.3. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Then the negative variation function n is a monotone increasing function on $[a, b]$.

Similar proof.

Note 1. In view of the theorem 10.2.5, the functions p and n are both continuous at the point where f is continuous.

2. Since $p(x) - n(x) = f(x) - f(a)$ for all $x \in [a, b]$, we have $f(x) = [p(x) + f(a)] - n(x)$. Since p and n are both monotone increasing functions on $[a, b]$, f is expressed as the difference of two monotone increasing functions $p + f(a)$ and n.

Worked Examples (continued).

6. $f(x) = |x|, x \in [-1, 2]$. Show that f is a function of bounded variation on $[1, 3]$. Calculate the positive variation, the negative variation and the total variation of f on $[-1, 2]$. Find the functions p and n on $[-1, 2]$.

f is a monotone decreasing function on $[-1, 0]$ and a monotone increasing function on $[0, 2]$. Therefore f is a function of bounded variation on $[-1, 0]$ and on $[0, 2]$ and consequently, f is a BV-function on $[-1, 2]$.

$V_f[-1, 2] = V_f[-1, 0] + V_f[0, 2] = [f(-1) - f(0)] + [f(2) - f(0)]$. since f is monotone decreasing on $[-1, 0]$ and monotone increasing on $[0, 2]$.

Therefore $V_f[-1, 2] = 1 + 2 = 3$.

We have $2p_f[-1, 2] = V_f[-1, 2] + f(2) - f(-1) = 3 + (2 - 1) = 4$ and $2n_f[-1, 2] = V_f[-1, 2] - f(2) + f(-1) = 3 + (-2 + 1) = 2$.

or, $p_f[-1, 2] = 2$ and $n_f[-1, 2] = 1$.

$V(-1) = 0$.

If $-1 < x \leq 0$, then $V(x) = V_f[-1, x]$
$= f(-1) - f(x)$, since f is decreasing on $[-1, 0]$
$= 1 + x$.

If $0 < x \leq 2$, then $V(x) = V_f[-1, x] = V_f[-1, 0] + V_f[0, x]$
$= [f(-1) - f(0)] + [f(x) - f(0)]$, since f is decreasing on $[-1, 0]$
$= 1 + x$. and increasing on $[0, 2]$

Therefore $V(x) = 1 + x$, if $-1 \leq x \leq 0$
$= 1 + x$, if $0 < x \leq 2$.

That is, $V(x) = 1 + x, \ -1 \leq x \leq 2$.

The positive variation function p is given by $2p(x) = V(x) + f(x) - f(0)$ for all $x \in [-1, 2]$.

Therefore $p(x) = \frac{1}{2}[(1 + x) + (-x)] = \frac{1}{2}$, if $-1 \leq x < 0$
$= \frac{1}{2}[(1 + x) + (x)] = \frac{1}{2}(1 + 2x)$, if $0 \leq x \leq 2$.

The negative variation function n is given by $2n(x) = V(x) - f(x) + f(0)$ for all $x \in [-1, 2]$.

Therefore $n(x) = \frac{1}{2}[(1+x)+x] = \frac{1}{2}(1+2x)$, if $-1 \leq x < 0$
$= \frac{1}{2}[(1+x)-(x)] = \frac{1}{2}$, if $0 \leq x \leq 2$.

7. $f(x) = x - [x], x \in [1,3]$. Show that f is a function of bounded variation on $[1,3]$. Find the positive variation function, the negative variation function and express f as the difference of two monotone increasing functions on $[1,3]$.

$f(x) = x - 1$, if $1 \leq x < 2$
$= x - 2$, if $2 \leq x < 3$
$= 0$, if $x = 3$.

f is a function of bounded variation on $[1,3]$. [worked Ex. 3]

$V(1) = 0$. For $1 < x < 2$, $V(x) = V_f[1,x] = f(x) - f(1) = x - 1$.
$V(2) = V_f[1,2] = 2$. [worked Ex. 3]

For $2 < x < 3$, $V(x) = V_f[0,2] + V_f[2,x] = V(2) + f(x) - f(2) = 2 + (x-2) = x$.
$V(3) = V_f[1,3] = V_f[1,2] + V_f[2,3] = 2 + 2 = 4$. [worked Ex.3]

The variation function V is given by $V(x) = x - 1$, if $1 \leq x < 2$
$= x$, if $2 \leq x < 3$
$= 4$, if $x = 3$.

The positive variation function p is given by $2p(x) = V(x) + f(x) - f(1)$ for all $x \in [1,3]$. Therefore
$p(x) = \frac{1}{2}[(x-1)+(x-1)] = x - 1$, if $1 \leq x < 2$
$= \frac{1}{2}[x + (x-2)] = x - 1$, if $2 \leq x < 3$
$= \frac{1}{2}[4+0] = 2$, if $x = 3$.
That is, $p(x) = x - 1,\ x \in [1,3]$.

The negative variation function n is given by $2n(x) = V(x) - f(x) + f(1)$ for all $x \in [1,3]$. Therefore
$n(x) = \frac{1}{2}[(x-1)-(x-1)] = 0$, if $1 \leq x < 2$
$= \frac{1}{2}[(x)-(x-2)] = 1$, if $2 \leq x < 3$
$= \frac{1}{2}[4-0] = 2$, if $x = 3$.
Clearly, p and n are monotone increasing functions on $[1,3]$.

Since $p(x) - n(x) = f(x) - f(1),\ x \in [1,3]$ and $f(1) = 0$, f can be expressed as $f(x) = p(x) - n(x),\ x \in [1,3]$.

8. Let $x_1, x_2, \ldots, x_n, \ldots$ be an enumeration of all rational points in $[0,1]$ and let $f : [0,1] \to \mathbb{R}$ be defined by $f(x_n) = \frac{1}{n^2}$, $n = 1, 2, 3, \ldots$
$= 0$, elsewhere.
Prove that that f is a function of bounded variation on $[0,1]$.

Let us take a partition $P = (y_0, y_1, y_2, \ldots, y_n)$ of $[0,1]$.

Then $V(P, f) = |f(y_1) - f(y_0)| + |f(y_2) - f(y_1)| + \cdots + |f(y_n) - f(y_{n-1})|$
$\leq 2[|f(y_0)| + |f(y_1)| + |f(y_2)| + \cdots + |f(y_n)|]$.

Let us choose a natural number m such that the rational points in P form a proper subset of the set $\{x_1, x_2, \ldots, x_m\}$

Then $V(P, f) \leq 2[\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{m^2}]$

$\leq 2s$, where s is the sum of the convergent series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$.

Therefore $V(P, f) \leq 2s$ for all partitions P of $[a, b]$ and therefore the supremum of the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$ is finite.

Hence f is a function of bounded variation on $[0, 1]$.

Exercises 20

1. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and $[c.d] \subset [a, b]$. Prove that f is a function of bounded variation on $[c, d]$.

2. Give an example of a function f continuous on a closed interval $[a, b]$ but f is not a function of bounded variation on $[a, b]$.

3. Give an example of a function f not continuous on a closed interval $[a, b]$ but f is a function of bounded variation on $[a, b]$.

4. Show that the function f is not of bounded variation on $[0, 1]$.

(i) $f(x) = x \sin \frac{\pi}{x}$, if $x \in (0, 1]$ and $f(0) = 0$.

(ii) $f(x) = \sin \frac{\pi}{x}$, if $x \in (0, 1]$ and $f(0) = 0$.

Hint. (i) Consider the partition $(0, \frac{2}{2n+1}, \frac{2}{2n-1}, \cdots, \frac{2}{5}, \frac{2}{3}, 1)$ of $[0, 1]$.

5. Show that f is a function of bounded variation on $[0, \frac{\pi}{2}]$. Find the variation function V on $[0, \frac{\pi}{2}]$.

(i) $f(x) = \sin 2x$, (ii) $f(x) = \sin x + \cos x$, (iii) sgn cos $2x$.

6. Show that f is a function of bounded variation on $[0, 2]$. Find the variation function V on $[0, 2]$. Express f as the difference of two monotone increasing functions on $[0, 2]$.

(i) $f(x) = x^2 - 2x + 2$, (ii) $f(x) = [x] - x$, (iii) $f(x) = |x - 1|$.

7. Show that f is a function of bounded variation on $[0, 3]$. Calculate the total variation, the positive variation and the negative variation of f on $[0, 3]$.

(i) $f(x) = x^2 - 4x + 1$, (ii) $f(x) = \text{sgn}\ (x - 1)$, (iii) $f(x) = |x - 2|$.

8. Show that f is a function of bounded variation on $[0, 2]$. Find the positive variation function p and the negative variation function n on $[0, 2]$. Hence express f as the difference of two monotone increasing functions on $[0, 2]$.

(i) $f(x) = x^2 - 2x + 2$, (ii) $f(x) = \text{sgn}\ (x - 1)$, (iii) $f(x) = |x - 1|$.

11.1. Partition.

Let $[a, b]$ be a closed and bounded interval. A partition of $[a, b]$ is a finite ordered set $P = (x_0, x_1, \ldots, x_n)$ of points of $[a, b]$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

The family of all partitions of $[a, b]$ is denoted by $\mathcal{P}[a, b]$ and the partition $P = (x_0, x_1, \ldots, x_n)$ is a member of $\mathcal{P}[a, b]$.

For example, $P = (0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1)$ is a partition of $[0, 1], Q = (0, \frac{1}{4}, \frac{3}{8}, \frac{2}{4}, \frac{3}{4}, \frac{7}{8}, 1)$ is another partition of $[0, 1]$.

Let $P \in \mathcal{P}[a, b]$ where $P = (x_0, x_1, x_2, \ldots, x_n)$ such that $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. Then P divides the interval $[a, b]$ into non-overlapping subintervals $[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]$.

11.2. Riemann integrability.

Let $[a, b]$ be a closed and bounded interval. Let $f : [a, b] \to \mathbb{R}$ be a bounded function on $[a, b]$. Let us take a partition P of $[a, b]$ defined by $P = (x_0, x_1, x_2, \ldots, x_n)$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. Since f is bounded on $[a, b]$, f is bounded on $[x_{r-1}, x_r]$, for $r = 1, 2, \ldots, n$.

Let $M = \sup\limits_{x \in [a,b]} f(x), \quad m = \inf\limits_{x \in [a,b]} f(x);$

$M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x), \quad m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

Then $m \leq m_r \leq M_r \leq M$, for $r = 1, 2, \ldots, n$. ... (i)

The sum $M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1})$ is said to be the *upper Darboux sum* or the *upper sum* of f corresponding to the partition P and is denoted by $U(P, f)$;

and the sum $m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1})$ is said to be the *lower Darboux sum* or the *lower sum* of f corresponding to the partition P and is denoted by $L(P, f)$.

Each $P \in \mathcal{P}[a, b]$ determines two numbers $U(P, f)$ and $L(P, f)$.

By the inequality (i) we have

$$m(x_r - x_{r-1}) \leq m_r(x_r - x_{r-1}) \leq M_r(x_r - x_{r-1}) \leq M(x_r - x_{r-1}),$$

for $r = 1, 2, \ldots, n$. Therefore

$$\begin{aligned} m \sum_{r=1}^{n} (x_r - x_{r-1}) &\leq \sum_{r=1}^{n} m_r(x_r - x_{r-1}) \\ &\leq \sum_{r=1}^{n} M_r(x_r - x_{r-1}) \leq M \sum_{r=1}^{n} (x_r - x_{r-1}) \end{aligned}$$

or, $m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$ ... ... (ii)

Let us the consider the set $\mathcal{P}[a, b]$ of all partitions of $[a, b]$.

We have two sets of real numbers $\{U(P, f) : P \in \mathcal{P}[a, b]\}$ and $\{L(P, f) : P \in \mathcal{P}[a, b]\}$.

The inequality (ii) shows that both these sets are bounded sets. $M(b-a)$ is an upper bound and $m(b-a)$ is a lower bound in respect of both the sets.

The supremum (the least upper bound) of the set $\{L(P, f) : P \in \mathcal{P}[a, b]\}$ exists and it is called the *lower integral* of f on $[a, b]$ and is denoted by $\underline{\int}_a^b f(x)dx$, or by $\underline{\int}_a^b f dx$, or by $\underline{\int}_a^b f$.

The infimum (the greatest lower bound) of the set $\{U(P, f) : P \in \mathcal{P}[a, b]\}$ exists and it is called the *upper integral* of f on $[a, b]$ and is denoted by $\overline{\int}_a^b f(x)dx$, or by $\overline{\int}_a^b f dx$, or by $\overline{\int}_a^b f$.

f is said to be *Riemann integrable* on $[a, b]$ if $\underline{\int}_a^b f = \overline{\int}_a^b f$.

The common value $\underline{\int}_a^b f$ or $\overline{\int}_a^b f$ is called the *Riemann integral* of f on $[a, b]$ and it is denoted by $\int_a^b f(x)dx$, or by $\int_a^b f dx$, or by $\int_a^b f$.

In addition, we define $\int_a^a f = 0$, $\int_b^a f = -\int_a^b f$.

Note 1. It follows from the definition of the supremum and the infimum of a bounded set that

$$m(b-a) \leq \underline{\int}_a^b f \leq M(b-a), \ m(b-a) \leq \overline{\int}_a^b \leq M(b-a).$$

Note 2. The class of all Riemann integrable functions on $[a, b]$ is denoted by $\mathcal{R}[a, b]$. The class of all functions bounded on $[a, b]$ is denoted by $\mathcal{B}[a, b]$. From the definition of an integrable function it follows that $\mathcal{R}[a, b]$ is a subset of $\mathcal{B}[a, b]$.

We shall see that there are functions in $\mathcal{B}[a, b]$ which do not belong to $\mathcal{R}[a, b]$, i.e., $\mathcal{R}[a, b]$ is a proper subset of $\mathcal{B}[a, b]$.

From now on we shall often drop the adjective 'Riemann' and simply use the term 'integral' to mean Riemann integral, the term 'integrability' to mean Riemann integrability.

Worked Examples.

1. Let $[a, b]$ be a closed and bounded interval and $c \in \mathbb{R}$. A function $f : [a, b] \to \mathbb{R}$ is defined by $f(x) = c, x \in [a, b]$. Prove that $f \in \mathcal{R}[a, b]$.

f is bounded on $[a, b]$.

Let us take a partition P of $[a, b]$ defined by $P = (x_0, x_1, x_2, \ldots, x_n)$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

Let $M = \sup\limits_{x \in [a,b]} f(x), \quad m = \inf\limits_{x \in [a,b]} f(x);$

$M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x), \quad m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

Then $M = c, m = c; M_r = c, m_r = c$, for $r = 1, 2, \ldots, n$.

Now $U(P, f) = M_1(x_1 - a) + M_2(x_2 - x_1) + \cdots + M_n(b - x_{n-1})$
$= c(b - a);$

$L(P, f) = m_1(x_1 - a) + m_2(x_2 - x_1) + \cdots + m_n(b - x_{n-1})$
$= c(b - a).$

Let us consider the set $\mathcal{P}[a, b]$ of all partitions of $[a, b]$.

The set $\{L(P, f) : P \in \mathcal{P}[a, b]\}$ is the singleton set $\{c(b - a)\}$. The least upper bound of the set is $c(b - a)$, i.e., $\underline{\int_a^b} f = c(b - a)$.

Also the set $\{U(P, f) : P \in \mathcal{P}[a, b]\}$ is the singleton set $\{c(b - a)\}$. The greatest lower bound of the set is $c(b - a)$, i.e., $\overline{\int_a^b} = c(b - a)$.

As $\underline{\int_a^b} f = \overline{\int_a^b} f = c(b - a)$, f is integrable on $[a, b]$ and $\int_a^b f = c(b - a)$.

2. A function f is defined on $[0, 1]$ by

$$\begin{aligned} f(x) &= 1, \text{ if } x \text{ is rational} \\ &= 0, \text{ if } x \text{ is irrational.} \end{aligned}$$

Show that f is not integrable on $[0, 1]$.

f is bounded on $[0, 1]$.

Let us take a partition P of $[0, 1]$ defined by $P = (x_0, x_1, x_2, \ldots, x_n)$, where $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$.

Let $M = \sup\limits_{x \in [0,1]} f(x), \quad m = \inf\limits_{x \in [0,1]} f(x);$

$M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x), \quad m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

Then $M = 1, m = 0; M_r = 1, m_r = 0$, for $r = 1, 2, \ldots, n$.

$U(P, f) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1}) = 1.$
$L(P, f) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(b - x_{n-1}) = 0.$
Let us consider the set $\mathcal{P}[0, 1]$ of all partitions of $[0, 1]$.

The set $\{L(P, f) : P \in \mathcal{P}[0, 1]\}$ is the singleton set $\{0\}$. The least upper bound of the set is 0, i.e., $\underline{\int_0^1} f = 0$.

The set $\{U(P, f) : P \in \mathcal{P}[0,1]\}$ is the singleton set $\{1\}$. The greatest lower bound of the set is 1, i.e., $\overline{\int}_0^1 f = 1$.

Since $\underline{\int}_0^1 f \neq \overline{\int}_0^1 f$, f is not integrable on $[0, 1]$.

Lemma 11.2.1. Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and P be any partition of $[a, b]$. Then $L(P, f) \leq U(P, f)$.

Proof. Let $P = (x_0, x_1, x_2, \ldots, x_n)$ be a partition of $[a, b]$. Then $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $\quad m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

Then $m_r \leq M_r$, for $r = 1, 2, \ldots, n$.

Since $x_r - x_{r-1} > 0$, for $r = 1, 2, \ldots, n$ it follows that

$m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1})$

$\leq M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1})$.

That is, $L(P, f) \leq U(P, f)$. This proves the lemma.

11.3. Refinement of a partition.

Let $[a, b]$ be a closed and bounded interval. Let $P = (x_0, x_1, x_2, \ldots, x_n)$ be a partition of $[a, b]$.

A partition Q of $[a, b]$ is said to be a *refinement* of P if P be a proper subset of Q. That is, Q is obtained by adjoining a *finite number* of additional points to P.

For example, if $P = (0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1)$ be a partition of $[0, 1]$ and $Q = (0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, 1)$ then Q is a refinement of P.

If $R = (0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{3}{4}, 1)$ then R is a refinement of P but not a refinement of Q.

Lemma 11.3.1. Let a function $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and P be a partition of $[a, b]$. If Q be a refinement of P then

$$U(P, f) \geq U(Q, f) \text{ and } L(P, f) \leq L(Q, f).$$

Proof. Let $P = (x_0, x_1, x_2, \ldots, x_n)$. First we examine the effect of adjoining one additional point y to P.

Let $P_1 = (x_0, x_1, \ldots, x_{k-1}, y, x_k, \ldots, x_n)$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $\quad m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

The subinterval $[x_{k-1}, x_k]$ is divided into two smaller subintervals $[x_{k-1}, y]$ and $[y, x_k]$.

Let $M_k' = \sup\limits_{x \in [x_{k-1}, y]} f(x)$, $m_k' = \inf\limits_{x \in [x_{k-1}, y]} f(x)$,

$$M_k'' = \sup_{x\in[y,x_k]} f(x), m_k'' = \inf_{x\in[y,x_k]} f(x).$$

Then $M_k' \le M_k, M_k'' \le M_k; m_k' \ge m_k, m_k'' \ge m_k$.

$$\begin{aligned}
\text{Therefore } M_k(x_k - x_{k-1}) &= M_k[(x_k - y + y - x_{k-1})] \\
&\ge M_k''(x_k - y) + M_k'(y - x_{k-1}); \\
m_k(x_k - x_{k-1}) &= m_k[x_k - y + y - x_{k-1}] \\
&\le m_k''(x_k - y) + m_k'(y - x_{k-1}).
\end{aligned}$$

$$\begin{aligned}
U(P,f) - U(P_1,f) &= M_k(x_k - x_{k-1}) - [M_k'(y - x_{k-1}) \\
&\quad + M_k''(x_k - y)] \ge 0; \\
L(P,f) - L(P_1,f) &= m_k(x_k - x_{k-1}) - [m_k'(y - x_{k-1}) \\
&\quad + m_k''(x_k - y)] \le 0.
\end{aligned}$$

Therefore $U(P,f) \ge U(P_1,f)$ and $L(P,f) \le L(P_1,f)$.

If Q be any refinement of P then Q can be obtained from P by adjoining a finite number of additional points to P, one at a time.

By repeating the argument a finite number of times we have
$U(P,f) \ge U(Q,f)$; $L(P,f) \le L(Q,f)$.
This completes the proof.

Note. By the Lemma 11.2.1, it follows that
$L(P,f) \le L(Q,f) \le U(Q,f) \le U(P,f)$.

11.4. Norm of a partition.

Let $[a,b]$ be a closed and bounded interval and $P = (x_0, x_1, \ldots, x_n)$ be a partition of $[a,b]$.

The interval $[a,b]$ is divided into n subintervals $[a,x_1]$, $[x_1,x_2]$, $\ldots$, $[x_{n-1},x_n]$. The norm of the partition P, denoted by $\|P\|$, is defined by
$$\|P\| = \max\{(x_1 - x_0), (x_2 - x_1), \ldots, (x_n - x_{n-1})\}.$$

In other words, $\|P\|$ is the maximum length of the subintervals into which $[a,b]$ is divided by the partition P.

If Q be a refinement of P then $\|Q\| \le \|P\|$. But the converse is not true.

For example, let $P = (0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1)$, $Q = (0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1)$ be two partitions of $[0,1]$. Then $\|P\| = \frac{1}{4}, \|Q\| = \frac{1}{6}$. Here $\|Q\| < \|P\|$ but Q is not a refinment of P.

Lemma 11.4.1. Let $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and P is a partition of $[a,b]$ with $\|P\| = \delta$. If P_k be a refinement of P with k additional points of partition, then
$$\begin{aligned}
0 &\le U(P,f) - U(P_k,f) \le (M-m)k\delta, \\
0 &\le L(P_k,f) - L(P,f) \le (M-m)k\delta
\end{aligned}$$

where $M = \sup_{x\in[a,b]} f(x), \quad m = \inf_{x\in[a,b]} f(x)$.

Proof. Let $P = (x_0, x_1, \ldots, x_n)$. First we examine the effect of adjoining one additional point y to P and let $P_1 = (x_0, x_1, \ldots, x_{k-1}, y, x_k, \ldots, x_n)$.

Let $M_r = \sup_{x\in[x_{r-1},x_r]} f(x), m_r = \inf_{x\in[x_{r-1},x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

The subinterval $[x_{k-1}, x_k]$ is divided into two small subintervals $[x_{k-1}, y]$ and $[y, x_k]$.

Let $M_k' = \sup_{x\in[x_{k-1},y]} f(x), m_k' = \inf_{x\in[x_{k-1},y]} f(x)$,

$M_k'' = \sup_{x\in[y,x_k]} f(x), m_k'' = \inf_{x\in[y,x_k]} f(x)$.

Then $m \le m_k \le m_k' \le M_k' \le M_k \le M$;
$m \le m_k \le m_k'' \le M_k'' \le M_k \le M$.

$U(P, f) - U(P_1, f) = M_k(x_k - x_{k-1}) - [M_k'(y - x_{k-1}) + M_k''(x_k - y)]$
$= (M_k - M_k')(y - x_{k-1}) + (M_k - M_k'')(x_k - y)$.

Since $0 \le M_k - M_k' \le M - m$ and $0 \le M_k - M_k'' \le M - m$, it follows that $0 \le U(P, f) - U(P_1, f) \le (M - m)[y - x_{k-1}) + (x_k - y)]$
$\le (M - m)\delta$, since $(x_k - x_{k-1}) \le \delta$.

$L(P, f) - L(P_1, f) = m_k(x_k - x_{k-1}) - [m_k'(y - x_{k-1}) + m_k''(x_k - y)]$
$= (m_k - m_k')(y - x_{k-1}) + (m_k - m_k'')(x_k - y)$.

Since $0 \le m_k' - m_k \le M - m$ and $0 \le m_k'' - m_k \le M - m$, it follows that $0 \le L(P_1, f) - L(P, f) \le (M - m)[(y - x_{k-1}) + (x_k - y)]$
$\le (M - m)\delta$, since $(x_k - x_{k-1}) \le \delta$.

By introducing k additional points one by one in the partition P we obtain the partition P_k and it follows from above that

$0 \le U(P, f) - U(P_k, f) \le (M - m)k\delta$,
$0 \le L(P_k, f) - L(P, f) \le (M - m)k\delta$.

This proves the lemma.

Lemma 11.4.2. Let a function $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and P, Q be any two partitions of $[a, b]$. Then
$L(P, f) \le U(Q, f); L(Q, f) \le U(P, f)$.

Proof. Let $S = P \cup Q$. Then the partition S is a refinement of P as well as a refinement of Q.

By Lemmas 11.2.1 and 11.3.1,

$L(P, f) \le L(S, f) \le U(S, f) \le U(P, f)$
and $L(Q, f) \le L(S, f) \le U(S, f) \le U(Q, f)$.

Therefore $L(P, f) \le L(S, f) \le U(S, f) \le U(Q, f)$
and $L(Q, f) \le L(S, f) \le U(S, f) \le U(P, f)$.

Note. The Lemma states that for any two partitions in $\mathcal{P}[a,b]$, the lower sum corresponding to one does not exceed the upper sum corresponding to the other.

Theorem 11.4.3. Let a function $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$. Then $\underline{\int}_a^b f \le \overline{\int}_a^b f$.

Proof. Let P, Q be any two partitions of $[a,b]$. Then $L(P,f) \le U(Q,f)$. Keeping Q fixed, this inequality holds for every partition $P \in \mathcal{P}[a,b]$.

Therefore $U(Q,f)$ is an upper bound of the set $\{L(P,f) : P \in \mathcal{P}[a,b]\}$. But the least upper bound of the set $\{L(P,f) : P \in \mathcal{P}[a,b]\}$ is $\underline{\int}_a^b f$. Therefore $U(Q,f) \ge \underline{\int}_a^b f$.

This holds for every partition $Q \in \mathcal{P}[a,b]$. Therefore $\underline{\int}_a^b f$ is a lower bound of the set $\{U(Q,f) : Q \in \mathcal{P}[a,b]\}$. But the greatest lower bound of the set $\{U(Q,f) : Q \in \mathcal{P}[a,b]\}$ is $\overline{\int}_a^b f$.

Therefore $\underline{\int}_a^b f \le \overline{\int}_a^b f$.

Note. It follows from Note 1 of 11.2 that

$$m(b-a) \le \underline{\int}_a^b f \le \overline{\int}_a^b f \le M(b-a).$$

Worked Example.

Prove that the function f defined on $[a,b]$ by $f(x) = x, x \in [a,b]$ is integrable on $[a,b]$ by showing that $\underline{\int}_a^b f = \overline{\int}_a^b f$. Evaluate $\int_a^b f$.

f is bounded on $[a,b]$.

Let us take the partition $P_n = (a, a+h, a+2h, \ldots, a+nh)$ where $nh = b-a$. Then P_n divides the interval $[a,b]$ into n subintervals of equal length.

Let $M_r = \sup\limits_{x \in [a+(r-1)h, a+rh]} f(x), m_r = \inf\limits_{x \in [a+(r-1)h, a+rh]} f(x)$, for $r = 1, 2, \ldots, n$.

Since f is an increasing function on $[a,b]$, $M_r = a + rh, m_r = a + (r-1)h$, for $r = 1, 2, \ldots, n$.

$$\begin{aligned} U(P_n, f) &= h[(a+h) + (a+2h) + (a+nh)] \\ &= h[na + h(1+2+\cdots+n)] = nha + \tfrac{nh(nh+h)}{2} \\ &= (b-a)a + \tfrac{1}{2}(b-a)^2(1+\tfrac{1}{n}). \\ L(P_n, f) &= h[a + (a+h) + \cdots + (a + \overline{n-1}h)] \\ &= h[na + h(1+2+\cdots+\overline{n-1}] = nha + \tfrac{nh(nh-h)}{2}. \\ &= (b-a)a + \tfrac{1}{2}(b-a)^2(1-\tfrac{1}{n}). \end{aligned}$$

$$\sup\{L(P_n, f)) : n \in \mathbb{N}\} = (b-a)a + \tfrac{(b-a)^2}{2} = \tfrac{b^2-a^2}{2}$$

and $\inf\{U(P_n, f)) : n \in \mathbb{N}\} = (b-a)a + \frac{(b-a)^2}{2} = \frac{b^2-a^2}{2}$.

Proof. Let $f \in \mathcal{R}[a,b]$. Then $\underline{\int}_a^b f = \overline{\int}_a^b f$.

Let us choose $\epsilon > 0$.

Since $\underline{\int}_a^b f$ is the least upper bound of the set $\{L(P,f) : P \in \mathcal{P}[a,b]\}$, there exists a partititon P' of $[a,b]$ such that

$$\underline{\int}_a^b f - \frac{\epsilon}{2} < L(P',f) \leq \underline{\int}_a^b f.$$

Since $\overline{\int}_a^b f$ is the greatest lower bound of the set $\{U(P,f) : P \in \mathcal{P}[a,b]\}$, there exists a partititon P'' of $[a,b]$ such that

$$\overline{\int}_a^b f \leq U(P'',f) < \overline{\int}_a^b f + \frac{\epsilon}{2}.$$

Let $P = P' \cup P''$. Then P is a refinement of both P' and P''.
Therefore $L(P',f) \leq L(P,f)$ and $U(P,f) \leq U(P'',f)$.
Also we have $L(P,f) \leq U(P,f)$.
Combining, we have
$\underline{\int}_a^b f - \frac{\epsilon}{2} < L(P',f) \leq L(P,f) \leq U(P,f) \leq U(P'',f) < \overline{\int}_a^b f + \frac{\epsilon}{2}$.
Therefore $U(P,f) - L(P,f) < (\overline{\int}_a^b f + \frac{\epsilon}{2}) - (\underline{\int}_a^b f - \frac{\epsilon}{2})$

or, $U(P,f) - L(P,f) < \epsilon$, since $\overline{\int}_a^b f = \underline{\int}_a^b f$.

To prove the converse, we first observe that for any partition P of $[a,b]$, $L(P,f) \leq \underline{\int}_a^b f \leq \overline{\int}_a^b f \leq U(P,f)$.

Hence $\overline{\int}_a^b f - \underline{\int}_a^b f \leq U(P,f) - L(P,f)$.

Let us choose $\epsilon > 0$. By the condition there exists a partition P_ϵ of $[a,b]$ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.

We have $\overline{\int}_a^b f - \underline{\int}_a^b f \le U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.
Also we have $\overline{\int}_a^b f \ge \underline{\int}_a^b f$, by Theorem 11.4.3.

Therefore $0 \le \overline{\int}_a^b f - \underline{\int}_a^b f < \epsilon$. This holds for every positive ϵ.
It follows that $\overline{\int}_a^b f = \underline{\int}_a^b f$ and hence f is integrable on $[a, b]$.
This proves the theorem.

Theorem 11.4.5. (Darboux)

Let $[a, b]$ be a closed and bounded interval and a function $f[a, b] \to \mathbb{R}$ be bounded on $[a, b]$. Then

(i) to each pre-assigned positive ϵ there corresponds a positive δ such that $U(P, f) < \overline{\int}_a^b f + \epsilon$ for all partitions P of $[a, b]$ satisfying $||P|| \le \delta$; and

(ii) to each pre-assigned positive ϵ there corresponds a positive δ such that $L(P, f) > \underline{\int}_a^b f - \epsilon$ for all partitions P of $[a, b]$ satisfying $||P|| \le \delta$.

Proof. (i) Since f is bounded on $[a, b]$, there exists a positive real number B such that $| f(x) | \le B$ for all $x \in [a, b]$.

Since $\overline{\int}_a^b f$ is the infimum of the set $\{U(P, f) : P \in \mathcal{P}[a, b]\}$, for a pre-assigned positive ϵ there exists a partition Q of $[a, b]$ such that

$$U(Q, f) < \overline{\int}_a^b f + \tfrac{\epsilon}{2}.$$

Let $Q = (x_0, x_1, x_2, \ldots, x_n)$ and let $\delta = \frac{\epsilon}{4B(n-1)}$.
Let P be any partition of $[a, b]$ such that $||P|| \le \delta$.

Let $P' = P \cup Q$. Then P' is a refinement of P by adjoining $n - 1$ additional points $x_1, x_2, \ldots, x_{n-1}$ at most.

Therefore $0 \le U(P, f) - U(P', f) \le 2(n-1)B\delta$, by Lemma 11.4.1.
or, $U(P, f) \le U(P', f) + \frac{\epsilon}{2}$.
Since P' is also refinement of Q, $U(P', f) \le U(Q, f)$.
It follows that $U(P, f) \le U(Q, f) + \frac{\epsilon}{2} < \overline{\int}_a^b f + \epsilon$.

(ii) Since f is bounded on $[a, b]$ there exists a positive real number B such that $| f(x) | \le B$ for all $x \in [a, b]$.

Since $\underline{\int}_a^b f$ is the supremum of the set $\{L(P, f) : P \in \mathcal{P}[a, b]\}$, for a pre-assigned positive ϵ there exists a partition Q of $[a, b]$ such that

$$L(Q, f) > \underline{\int}_a^b f - \tfrac{\epsilon}{2}.$$

Let $Q = (x_0, x_1, x_2, \ldots, x_n)$ and let $\delta = \frac{\epsilon}{4B(n-1)}$.

Let P be any partition of $[a, b]$ such that $||P|| \leq \delta$.

Let $P' = P \cup Q$. Then P' is a refinement of P by adjoining $n-1$ additional points $x_1, x_2, \ldots, x_{n-1}$ at most.

Therefore $0 \leq L(P', f) - L(P, f) \leq 2(n-1)B\delta$, by Lemma 11.4.1.

or, $L(P, f) \geq L(P', f) - \frac{\epsilon}{2}$.

Since P' is also refinement of Q, $L(P', f) \geq L(Q, f)$.

It follows that $L(P, f) \geq L(Q, f) - \frac{\epsilon}{2} > \underline{\int}_a^b f - \epsilon$.

This completes the proof.

Theorem 11.4.6. Let a function $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$. If $\{P_n\}$ be a sequence of partitions of $[a, b]$ such that the sequence $\{||P_n||\}$ converges to 0, then

(i) $\lim_{n\to\infty} U(P_n, f) = \overline{\int}_a^b f$; and (ii) $\lim_{n\to\infty} L(P_n, f) = \underline{\int}_a^b f$.

Proof. (i) Let us choose $\epsilon > 0$. Since f is bounded on $[a, b]$, by Darboux theorem there exists a positive δ such that $\overline{\int}_a^b f \leq U(P, f) < \overline{\int}_a^b f + \epsilon$ for all partitions P satisfying $||P|| < \delta$.

Since $\lim ||P_n|| = 0$, there exists a natural number k such that $||P_n|| < \delta$ for all $n \geq k$.

Therefore $\overline{\int}_a^b f \leq U(P_n, f) < \overline{\int}_a^b f + \epsilon$ for all $n \geq k$.

Hence the inequality $| U(P_n, f) - \overline{\int}_a^b f | < \epsilon$ holds for all $n \geq k$.

This implies $\lim_{n\to\infty} U(P_n, f) = \overline{\int}_a^b f$.

(ii) Similar proof.

Worked Examples.

1. A function f is defined by $f(x) = x^2, x \in [a, b]$, where $a > 0$. Find $\overline{\int}_a^b f$ and $\underline{\int}_a^b f$. Deduce that f is integrable on $[a, b]$.

f is bounded on $[a, b]$. Let $P_n = (a, a+h, a+2h, \ldots, a+nh)$ where $h = \frac{b-a}{n}$. Then P_n is a partition of $[a, b]$ dividing $[a, b]$ into subintervals of equal length. $||P_n|| = \frac{b-a}{n}$.

Let $M_r = \sup\limits_{x\in[a+(r-1)h, a+rh]} f(x)$, $m_r = \inf\limits_{x\in[a+(r-1)h, a+rh]} f(x)$, for $r = 1, 2, \ldots, n$.

Since f is an increasing function on $[a, b]$,
$M_r = (a + rh)^2$, $m_r = (a + \overline{r-1}h)^2$, for $r = 1, 2, \ldots, n$.

$$U(P_n, f) = h[(a+h)^2 + (a+2h)^2 + \cdots + (a+nh)^2]$$
$$= h[na^2 + 2ah.\tfrac{n(n+1)}{2} + h^2.\tfrac{n(n+1)(2n+1)}{6}]$$

$= nha^2 + a.nh(nh+h) + \frac{nh(nh+h)(2nh+h)}{6}$
$= (b-a)a^2 + a(b-a)^2(1+\frac{1}{n}) + \frac{1}{6}(b-a)^3(1+\frac{1}{n})(2+\frac{1}{n}).$

$L(P_n, f) = h[a^2 + (a+h)^2 + \cdots + (a+\overline{n-1}h^2]$
$= h[na^2 + 2ah.\frac{n(n-1)}{2} + h^2.\frac{n(n-1)(2n-1)}{6}]$
$= nha^2 + a.nh(nh-h) + \frac{nh(nh-h)(2nh-h)}{6}$
$= (b-a)a^2 + a(b-a)^2(1-\frac{1}{n}) + \frac{1}{6}(b-a)^3(1-\frac{1}{n})(2-\frac{1}{n}).$

Let us consider the sequence of partitions $\{P_n\}$ of $[a,b]$.
Here $\lim\limits_{n\to\infty} ||P_n|| = \lim \frac{b-a}{n} = 0$.
Then $\overline{\int}_a^b = \lim\limits_{n\to\infty} U(P_n, f), \underline{\int}_a^b f = \lim\limits_{n\to\infty} L(P_n, f)$.
Therefore $\overline{\int}_a^b f = (b-a)a^2 + a(b-a)^2 + \frac{(b-a)^3}{3} = \frac{b^3-a^3}{3}$,
and $\underline{\int}_a^b f = (b-a)a^2 + a(b-a)^2 + \frac{(b-a)^3}{3} = \frac{b^3-a^3}{3}$.
As $\overline{\int}_a^b f = \underline{\int}_a^b f$, f is integrable on $[a,b]$ and $\int_a^b f = \frac{b^3-a^3}{3}$.

2. A function f is defined on $[a,b]$ by $f(x) = e^x$. Find $\overline{\int}_a^b f$ and $\underline{\int}_a^b f$. Deduce that f is integrable on $[a,b]$.

f is bounded on $[a,b]$.

Let $P_n = (a+h, a+2h, \ldots, a+nh)$ where $nh = b-a$. Then P_n is a partition of $[a,b]$ dividing $[a,b]$ into n subintervals of equal length. $||P_n|| = \frac{b-a}{n}$.

Let $M_r = \sup\limits_{x\in[a+(r-1)h, a+rh]} f(x)$, $m = \inf\limits_{x\in[a+(r-1)h,a+rh]} f(x)$, for $r = 1, 2, \ldots, n$.

Then $M_r = e^{a+rh}$, $m_r = e^{a+(r-1)h}$, for $r = 1, 2, \ldots, n$.

$U(P_n, f) = h[e^{a+h} + e^{a+2h} + \cdots + e^{a+nh}]$
$= h.e^{a+h}.\frac{e^{nh}-1}{e^h-1} = h.e^{a+h}.\frac{e^{b-a}-1}{e^h-1}$
$= \frac{h.e^h}{e^h-1}.(e^b - e^a);$

$L(P_n, f) = h[e^a + e^{a+h} + \cdots + e^{a+(n-1)h}]$
$= h.e^a(\frac{e^{nh}-1}{e^h-1})$
$= \frac{h}{e^h-1}(e^b - e^a).$

Let us consider the sequence of partitions $\{P_n\}$ of $[a,b]$. Here $\lim ||P_n|| = \lim \frac{b-a}{n} = 0$.

Then $\overline{\int}_a^b f = \lim\limits_{n\to\infty} U(P_n, f)$ and $\underline{\int}_a^b f = \lim\limits_{n\to\infty} L(P_n, f)$.

So $\overline{\int}_a^b f = \lim\limits_{n\to\infty} \dfrac{he^h}{e^h-1}(e^b - e^a) = e^b - e^a$

and $\underline{\int}_a^b f = \lim_{n\to\infty} \frac{e^h}{e^h - 1}(e^b - e^a) = e^b - e^a$.

As $\overline{\int}_a^b f = \underline{\int}_a^b f$, f is integrable on $[a, b]$ and $\int_a^b = e^b - e^a$.

3. A function f is defined on $[0, 1]$ by $f(x) = x, x \in [0, 1] \cap \mathbb{Q}$
$= 0, x \in [0, 1] - \mathbb{Q}$.

Find $\underline{\int}_0^1 f$ and $\overline{\int}_0^1 f$. Deduce that f is not integrable on $[0, 1]$.

f is bounded on $[0, 1]$. Let us take the partition P_n of $[0, 1]$ defined by $P_n = (0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n})$.

Let $M_r = \sup_{x\in[\frac{r-1}{n}, \frac{r}{n}]} f(x), m_r = \inf_{x\in[\frac{r-1}{n}, \frac{r}{n}]} f(x)$, for $r = 1, 2, \ldots, n$.

Then $M_r = \frac{r}{n}, m_r = 0$, for $r = 1, 2, \ldots, n$.

$$U(P_n, f) = M_1(\tfrac{1}{n} - 0) + M_2(\tfrac{2}{n} - \tfrac{1}{n}) + \cdots + M_n(\tfrac{n}{n} - \tfrac{n-1}{n})$$
$$= \tfrac{1}{n}[\tfrac{1}{n} + \tfrac{2}{n} + \cdots + \tfrac{n}{n}] = \tfrac{n+1}{2n};$$

and
$$L(P_n, f) = m_1(\tfrac{1}{n} - 0) + m_2(\tfrac{2}{n} - \tfrac{1}{n}) + \cdots + m_n(\tfrac{n}{n} - \tfrac{n-1}{n})$$
$$= 0, \text{ since each } m_r = 0.$$

Let us consider the sequence of partitions $\{P_n\}$ of $[0, 1]$. $\|P_n\| = \frac{1}{n}$ and $\lim_{n\to\infty} \|P_n\| = 0$.

Then $\lim_{n\to\infty} U(P_n, f) = \overline{\int}_0^1 f$ and $\lim_{n\to\infty} L(P_n, f) = \underline{\int}_0^1 f$.

Therefore $\overline{\int}_0^1 f = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$ and $\underline{\int}_0^1 f = 0$.

Since $\underline{\int}_0^1 f \neq \overline{\int}_0^1 f$, f is not integrable on $[0, 1]$.

4. A function f is defined on $[0, 1]$ by

$$f(x) = x \text{ if } x \text{ be rational}$$
$$= x^2 \text{ if } x \text{ be irrational} .$$

Find $\underline{\int}_0^1 f$ and $\overline{\int}_0^1 f$. Deduce that f is not integrable on $[0, 1]$.

f is bounded on $[0, 1]$. For all $x \in (0, 1), x > x^2$.

Let $I = [0, 1]$. $f/(I \cap \mathbb{Q})$ is monotone increasing on $I \cap \mathbb{Q}$.

$f/(I - \mathbb{Q})$ is monotone increasing on $I - \mathbb{Q}$.

Let P_n be the partition of $[0, 1]$ defined by $P_n = (x_0, x_1, \ldots, x_n)$ where $x_0 = 0, x_r = \frac{r}{n}, r = 1, 2, \ldots, n$.

Let $M_r = \sup_{x\in[x_{r-1}, x_r]} f(x), m_r = \inf_{x\in[x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

Since $f/(I \cap \mathbb{Q})$ is monotone increasing on $[x_{r-1}, x_r] \cap \mathbb{Q}$,

$\sup_{x\in[x_{r-1}, x_r]\cap\mathbb{Q}} f(x) = f(x_r) = \frac{r}{n}$.

Since $f/(I-\mathbb{Q})$ is monotone increasing on $[x_{r-1},x_r]-\mathbb{Q}$ and x_r is rational, $\sup\limits_{x\in[x_{r-1},x_r]-\mathbb{Q}} f(x) = \lim\limits_{n\to\infty} f(u_n)$, where $\{u_n\}$ is a sequence of irrational points in $[x_{r-1},x_r]$ converging to x_r
$$= x_r^2 = (\tfrac{r}{n})^2.$$

Since $\frac{r}{n} \geq (\frac{r}{n})^2$, $\sup\limits_{x\in[x_{r-1},x_r]} f(x) = \frac{r}{n}$. Hence $M_r = \frac{r}{n}$, for $r = 1,2,\ldots,n$.

Since $f/(I\cap\mathbb{Q})$ is monotone increasing on $[x_{r-1},x_r]\cap\mathbb{Q}$, $\inf\limits_{x\in[x_{r-1},x_r]} f(x) = f(x_{r-1}) = \frac{r-1}{n}$.

Since $f/(I-\mathbb{Q})$ is monotone increasing on $[x_{r-1},x_r]-\mathbb{Q}$ and x_{r-1} is rational, $\inf\limits_{x\in[x_{r-1},x_r]-\mathbb{Q}} f(x) = \lim\limits_{n\to\infty} f(v_n)$, where $\{v_n\}$ is a sequence of irrational points in $[x_{r-1},x_r]$ converging to x_{r-1}
$$= x_{r-1}^2 = (\tfrac{r-1}{n})^2.$$

Since $\frac{r-1}{n} \geq (\frac{r-1}{n})^2$, $\inf\limits_{x\in[x_{r-1},x_r]} f(x) = (\frac{r-1}{n})^2$. Hence $m_r = (\frac{r-1}{n})^2$, for $r = 1,2,\ldots,n$.

$$\begin{aligned} U(P_n,f) &= M_1(x_1-x_0)+M_2(x_2-x_1)+\cdots+M_n(x_n-x_{n-1}) \\ &= \tfrac{1}{n}[\tfrac{1}{n}+\tfrac{2}{n}+\cdots+\tfrac{n}{n}] = \tfrac{n+1}{2n}. \\ L(P_n,f) &= m_1(x_1-x_0)+m_2(x_2-x_1)+\cdots+m_n(x_n-x_{n-1}) \\ &= \tfrac{1}{n}[0+\tfrac{1^2}{n^2}+\tfrac{2^2}{n^2}+\cdots+\tfrac{(n-1)^2}{n^2}] = \tfrac{(n-1)(2n-1)}{6n^2}. \end{aligned}$$

Let us consider the sequence of partitions $\{P_n\}$ of $[0,1]$. $\|P_n\| = \frac{1}{n}$ and $\lim\limits_{n\to\infty}\|P_n\| = 0$.

Then $\overline{\int_0^1} f = \lim\limits_{n\to\infty} U(P_n,f)$ and $\underline{\int_0^1} f = \lim\limits_{n\to\infty} L(P_n,f)$.

So $\overline{\int_0^1} f = \lim\limits_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}$ and $\underline{\int_0^1} f = \lim\limits_{n\to\infty}\frac{(n-1)(2n-1)}{6n^2} = \frac{1}{3}$.

Since $\overline{\int_0^1} f \neq \underline{\int_0^1} f$, f is not integrable on $[0,1]$.

Theorem 11.4.7. Another condition for integrability.

Let a function $f:[a,b]\to\mathbb{R}$ be bounded on $[a,b]$. Then f is integrable on $[a,b]$ if and only if for each $\epsilon>0$ there exists a positive δ such that
$$U(P,f)-L(P,f)<\epsilon$$
for every partition P of $[a,b]$ satisfying $\|P\|\leq\delta$.

Proof. Let $f\in\mathcal{R}[a,b]$. Then $\underline{\int_a^b} f = \overline{\int_a^b} f$.

Let us choose $\epsilon>0$.

Since f is bounded on $[a,b]$, by Darboux theorem, there corresponds a positive δ_1 such that $U(P, f) < \overline{\int}_a^b f + \frac{\epsilon}{2}$ for all partitions P of $[a, b]$ satisfying $||P|| \leq \delta_1$.

Also there corresponds a positive δ_2 such that $L(P, f) > \underline{\int}_a^b f - \frac{\epsilon}{2}$ for all partitions P of $[a, b]$ satisfying $||P|| \leq \delta_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $U(P, f) < \overline{\int}_a^b f + \frac{\epsilon}{2}$ and $L(P, f) > \underline{\int}_a^b f - \frac{\epsilon}{2}$ for all partitions P of $[a, b]$ satisfying $||P|| \leq \delta$.

So $U(P, f) - L(P, f) < \epsilon$ for all partitions P of $[a, b]$ satisfying $||P|| \leq \delta$.

To prove the converse, we first observe that for any partition P of $[a, b]$, $L(P, f) \leq \underline{\int}_a^b \leq \overline{\int}_a^b f \leq U(P, f)$.

That is, $\overline{\int}_a^b f - \underline{\int}_a^b f \leq U(P, f) - L(P, f)$ for any partiton P of $[a, b]$.

Let us choose $\epsilon > 0$. By the condition, there exists a positive δ such that for all partitions P of $[a, b]$ satisfying $||P|| \leq \delta, U(P, f) - L(P, f) < \epsilon$ holds.

Therefore exists a partition, say P_ϵ of $[a, b]$ such that $||P_\epsilon|| < \delta$ and $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.

Also we have $\overline{\int}_a^b f \geq \underline{\int}_a^b f$, by Theorem 11.4.3.

Therefore $0 \leq \overline{\int}_a^b f - \underline{\int}_a^b f \leq U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.

This holds for each positive ϵ. It follows that $\overline{\int}_a^b f = \underline{\int}_a^b f$ and hence f is integrable on $[a, b]$.

This completes the proof.

11.5. Some Riemann integrable functions.

Theorem 11.5.1. Let a function $f : [a, b] \to \mathbb{R}$ be monotone on $[a, b]$. Then f is integrable on $[a, b]$.

Proof. Let f be monotone increasing on $[a, b]$. Clearly, f is bounded on $[a, b]$, $f(a)$ being a lower bound and $f(b)$ being an upper bound of f on $[a, b]$. $f(b) - f(a) \geq 0$.

Let us choose $\epsilon > 0$.

Let P be a partition of $[a, b]$ with $||P|| < \epsilon / \{f(b) - f(a) + 1\}$. Let $P = (x_0, x_1, x_2, \ldots, x_n)$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

Then $M_r = f(x_r)$ and $m_r = f(x_{r-1})$, for $r = 1, 2, \ldots, n$.

$$\begin{aligned}\text{We have } U(P,f)-L(P,f) &= \sum_{r=1}^{n}(M_r-m_r)(x_r-x_{r-1})\\ &= \sum_{r=1}^{n}\{f(x_r)-f(x_{r-1})\}(x_r-x_{r-1})\\ &\leq \|P\|\sum_{r=1}^{n}\{f(x_r)-f(x_{r-1})\}\\ &= \|P\|\{f(b)-f(a)\}<\epsilon.\end{aligned}$$

Therefore for a chosen positive ϵ, there exists a partition P of $[a,b]$ such that $U(P,f)-L(P,f)<\epsilon$.

This being a sufficient condition for integrability, f is integrable on $[a,b]$.

Proceeding in a similar manner it can be proved that if f be monotone decreasing on $[a,b]$, then f is integrable on $[a,b]$.

This completes the proof.

Theorem 11.5.2. Let a function $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. Then f is integrable on $[a,b]$.

Proof. Since f is continuous on $[a,b]$, f is bounded on $[a,b]$. Since f is continuous on $[a,b]$, f is uniformly continuous on $[a,b]$.

Therefore for a chosen $\epsilon>0$, there exists a positive δ such that

$|f(x')-f(x'')|<\frac{\epsilon}{b-a}$ for any two ponts x',x'' in $[a,b]$ satisfying $|x'-x''|<\delta$ ... ... (i)

Let P be a partition of $[a,b]$ with $\|P\|<\delta$. Let $P=(x_0,x_1,x_2,\ldots,x_n)$, where $a=x_0<x_1<x_2<\cdots<x_n=b$.

Let $M_r=\sup\limits_{x\in[x_{r-1},x_r]} f(x)$, $m_r=\inf\limits_{x\in[x_{r-1},x_r]} f(x)$, for $r=1,2,\ldots,n$.

Since f is continuous on $[x_{r-1},x_r]$ there exist points ξ_r,η_r in $[x_{r-1},x_r]$ such that $f(\xi_r)=M_r$, $f(\eta_r)=m_r$.

From (i) $|M_r-m_r|<\frac{\epsilon}{b-a}$, for $r=1,2,\ldots,n$.

$$\begin{aligned}\text{We have } U(P,f)-L(P,f) &= \sum_{r=1}^{n}(M_r-m_r)(x_r-x_{r-1})\\ &< \frac{\epsilon}{b-a}\sum_{r=1}^{n}(x_r-x_{r-1})=\epsilon.\end{aligned}$$

Therefore for a chosen positive ϵ there exists a partition P of $[a,b]$ such that $U(P,f)-L(P,f)<\epsilon$.

This being a sufficient condition for integrability, f integrable on $[a,b]$.

This completes the proof.

Note 1. The class of all functions continuous on $[a,b]$ is denoted by $C[a,b]$. It follows from the theorem that $f\in C[a,b]\Rightarrow f\in\mathcal{R}[a,b]$.

Note 2. We shall see that $C[a,b]$ is a proper subset of $\mathcal{R}[a,b]$. There are some discontinuous functions which belong to $\mathcal{R}[a,b]$.

Theorem 11.5.3. Let a function $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and let f be continuous $[a, b]$ except for a finite number of points in $[a, b]$. Then f is integrable on $[a, b]$.

Proof. Since f is bounded on $[a, b]$, there exists a positive real number k such that $| f(x) | < k$ for all $x \in [a, b]$. Let f be discontinuous at m points $x_1, x_2, \ldots, x_m$ in $[a, b]$ such that $x_1 < x_2 < \cdots < x_m$.

Case 1. Let $a < x_1 < x_2 < \cdots < x_m < b$.

Let $\epsilon > 0$. Let us enclose m points $x_1, x_2, \ldots, x_m$ in m non-overlapping subintervals$[x_1 - \frac{\delta_1}{2}, x_1 + \frac{\delta_1}{2}], \cdots, [x_m - \frac{\delta_m}{2}, x_m + \frac{\delta_m}{2}]$ of $[a, b]$ such that $a < x_1 - \frac{\delta_1}{2}, b > x_m + \frac{\delta_m}{2}$ and $\delta_1 + \delta_2 + \cdots + \delta_m < \frac{\epsilon}{4k}$.

Let $M^{(r)} = \sup\limits_{x \in [x_r - \frac{\delta_r}{2}, x_r + \frac{\delta_r}{2}]} f(x)$, $m^{(r)} = \inf\limits_{x \in [x_r - \frac{\delta_r}{2}, x_r + \frac{\delta_r}{2}]} f(x)$, for $r = 1, 2, \ldots, m$.

Then $M^{(r)} - m^{(r)} < 2k$, for $r = 1, 2, \ldots, m$.

f is continuous on the remaining $m + 1$ subintervals $[a, x_1 - \frac{\delta_1}{2}], [x_1 + \frac{\delta_1}{2}, x_2 - \frac{\delta_2}{2}], \cdots, [x_m + \frac{\delta_m}{2}, b]$.

Then there exist partitions P_1 of $[a, x_1 - \frac{\delta_1}{2}], P_2$ of $[x_1 + \frac{\delta_1}{2}, x_2 - \frac{\delta_2}{2}], \cdots, P_{m+1}$ of $[x_m + \frac{\delta_m}{2}, b]$ such that
$U(P_k, f) - L(P_k, f) < \frac{\epsilon}{2(m+1)}$, for $k = 1, 2, \ldots, m + 1$.

The partitions $P_1, P_2, \ldots, P_{m+1}$ are disjoint.

Let $P = P_1 \cup P_2 \cup \cdots \cup P_{m+1}$. Then P is a partition of $[a, b]$.

$U(P, f) - L(P, f) = [U(P_1, f) - L(P_1, f)] + [U(P_2, f) - L(P_2, f)] + \cdots + [U(P_{m+1}, f) - L(P_{m+1}, f)] + (M^{(1)} - m^{(1)})\delta_1 + (M^{(2)} - m^{(2)})\delta_2 + \cdots + (M^{(m)} - m^{(m)})\delta_m < \frac{\epsilon}{2(m+1)} \cdot (m + 1) + 2k(\delta_1 + \delta_2 \cdots + \delta_m) < \epsilon$, since $\delta_1 + \delta_2 + \cdots + \delta_m < \frac{\epsilon}{4k}$.

Thus for a chosen positive ϵ there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$.

This being a sufficient condition for integrability, f is integrable on $[a, b]$.

Case 2. Either $a = x_1$, or $x_m = b$, or both.

If $a = x_1$, the subinterval enclosing the point x_1 can be chosen as $[a, a + \delta_1]$. If $x_m = b$, the last subinterval can be chosen as $[b - \delta_m, b]$. In any case, proceeding with similar arguments it can be proved that f is integrable on $[a, b]$.

This completes the proof.

Corollary. If $f : [a, b] \to \mathbb{R}$ be piecewise continuous on $[a, b]$ then f is integrable on $[a, b]$.

Theorem 11.5.4. Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and let f be continuous on $[a, b]$ except on a infinite subset $S \subset [a, b]$ such that the number of limit points of S is finite. Then f is integrable on $[a, b]$.

Proof. Since f is bounded on $[a, b]$, there exists a positive number k such that $| f(x) | < k$ for all $x \in [a, b]$. Let S' (the derived set of S) $= \{x_1, x_2, \ldots, x_m\}$ such that $x_1 < x_2 < \cdots < x_m$.

Case 1. Let $a < x_1 < x_2 < \cdots < x_m < b$.

Let us choose $\epsilon > 0$. Let the points $x_1, x_2, \ldots, x_m$ be enclosed by m non-overlapping subintervals $[x_1 - \frac{\delta_1}{2}, x_1 + \frac{\delta_1}{2}], [x_2 - \frac{\delta_2}{2}, x_2 + \frac{\delta_2}{2}], \cdots [x_m - \frac{\delta_m}{2}, x_m + \frac{\delta_m}{2}]$ of $[a, b]$ such that $a < x_1 - \frac{\delta_1}{2}, b > x_m + \frac{\delta_m}{2}$ and $\delta_1 + \delta_2 + \cdots + \delta_m < \frac{\epsilon}{4k}$.

Let $M^{(r)} = \sup\limits_{x \in [x_r - \frac{\delta_r}{2}, x_r + \frac{\delta_r}{2}]} f(x)$, $m^{(r)} = \inf\limits_{x \in [x_r - \frac{\delta_r}{2}, x_r + \frac{\delta_r}{2}]} f(x)$, for $r = 1, 2, \ldots, m$.

Then $M^{(r)} - m^{(r)} < 2k$, for $r = 1, 2, \ldots, m$.

On each of the remaining $m + 1$ subintervals $[a, x_1 - \frac{\delta_1}{2}], [x_1 + \frac{\delta_1}{2}, x_2 - \frac{\delta_2}{2}], \cdots, [x_m + \frac{\delta_m}{2}, b]$, f is continuous except for a finite number of points. So f is integrable on each of these intervals, by Theorem 11.5.3.

Therefore there exist partitions P_1 of $[a, x_1 - \frac{\delta_1}{2}], P_2$ of $[x_1 + \frac{\delta_1}{2}, x_2 - \frac{\delta_2}{2}], \cdots, P_{m+1}$ of $[x_m + \frac{\delta_m}{2}, b]$ such that
$U(P_k, f) - L(P_k, f) < \frac{\epsilon}{2(m+1)}$, for $k = 1, 2, \ldots, m + 1$.

The partitions $P_1, P_2, \cdots, P_{m+1}$ are disjoint.

Let $P = P_1 \cup P_2 \cup \cdots \cup P_{m+1}$. Then P is a partition of $[a, b]$.

$$U(P, f) - L(P, f) = [U(P_1, f) - L(P_1, f)] + [U(P_2, f) - L(P_2, f)] +$$
$$\cdots + [U(P_{m+1}, f) - L(P_{m+1}, f)] + (M^{(1)} - m^{(1)})\delta_1 + (M^{(2)} - m^{(2)})\delta_2 +$$
$$\cdots + (M^{(m)} - m^{(m)})\delta_m < \frac{\epsilon}{2(m+1)} . (m + 1) + 2k(\delta_1 + \delta_2 + \cdots \delta_m) < \epsilon.$$

Thus for a chosen positive ϵ there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$.

This being a sufficient condition for integrability, f is integrable on $[a, b]$.

Case 2. Either $a = x_1$, or $x_m = b$, or both.

If $a = x_1$, the subinterval enclosing the point x_1 can be chosen as $[a, a + \delta_1]$. If $x_m = b$, the last subinterval can be chosen as $[b - \delta_m, b]$. In any case, proceeding with similar arguments it can be proved that f is integrable on $[a, b]$.

This completes the proof.

Note. Since the set S is bounded and infinite, the derived set S' cannot be the null set, by Bolzano-Weierstrass theorem.

Examples.

1. Let $f(x) = \operatorname{sgn} x$, $x \in [-2, 2]$.

$$\text{Then } f(x) = -1, \ -2 \le x < 0.$$
$$= 0, \ x = 0$$
$$= 1, \ 0 < x \le 2.$$

f is bounded on $[-2, 2]$, since $|f(x)| \le 1$ for all $x \in [-2, 2]$. f is continuous on $[-2, 2]$ except at only one point, 0. Therefore f is integrable on $[-2, 2]$.

2. Let $f(x) = [x], x \in [0, 2]$.

$$f(x) = 0, \ 0 \le x < 1$$
$$= 1, \ 1 \le x < 2$$
$$= 2, \ x = 2.$$

f is bounded on $[0, 2]$, since $|f(x)| \le 2$ for all $x \in [0, 2]$. f is continuous on $[0, 2]$ except for the points 1, 2. So f is integrable on $[0, 2]$.

3. Let f be defined on $[0, 1]$ by

$f(0) = 0$ and

$f(x) = \frac{1}{2^{n-1}}, \frac{1}{2^n} < x \le \frac{1}{2^{n-1}}$, for $n = 1, 2, 3, \ldots$

f is monotone increasing and bounded on $[0, 1]$. Therefore f is integrable on $[0, 1]$.

4. Let f be defined on $[0, 1]$ by

$f(0) = 0$ and

$f(x) = (-1)^{r-1}, \ \frac{1}{r+1} < x \le \frac{1}{r}$, for $r = 1, 2, 3, \ldots$

f is continuous on $[0, 1]$ except at the points $0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ The set of points of discontinuity of f has only one limit point. Also f is bounded on $[0, 1]$. Therefore f is integrable on $[0, 1]$.

5. A function f is defined on $[0, 1]$ by $f(0) = 0$ and

$$f(x) = 0, \text{ if } x \text{ be irrational}$$
$$= \frac{1}{q}, \text{ if } x = \frac{p}{q} \text{ where } p, q \text{ are positive integers prime to each other.}$$

Show that f is integrable on $[0, 1]$ and $\int_0^1 f = 0$.

f is bounded on $[0, 1]$. Let us choose a positive ϵ such that $0 < \epsilon < 2$. Then there exists a natural number k such that $k < \frac{2}{\epsilon} \le k + 1$, by Archimedean property of $\mathbb{R}$. Let the rational numbers in $(0, 1)$ be arranged as

$$\frac{1}{1}; \ \frac{1}{2}; \ \frac{1}{3}, \frac{2}{3}; \ \frac{1}{4}, \frac{3}{4}; \ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; \ \ldots; \ \frac{1}{k}, \ldots, \frac{k-1}{k}; \ \frac{1}{k+1}, \ldots, \frac{k}{k+1}; \ \ldots$$

There are only a finite number of rational numbers of the form $\frac{p}{q}$ in $[0,1]$ with denominator $\leq k$. At every such point $f(x) \geq \frac{1}{k} > \frac{\epsilon}{2}$; and at all other rational points in $[0,1]$, $f(x) \leq \frac{\epsilon}{2}$.

Let the finite number of rational points for which $f(x) > \frac{\epsilon}{2}$ be $x_1, x_2, \cdots, x_m$ where $x_1 < x_2 < \cdots < x_m$.

Let us enclose the points by subintervals $[x_1 - \frac{\delta_1}{2}, x_1 + \frac{\delta_1}{2}]$, $[x_2 - \frac{\delta_2}{2}, x_2 + \frac{\delta_2}{2}], \cdots, [x_m - \frac{\delta_m}{2}, x_m + \frac{\delta_m}{2}]$ such that $\delta_1 + \delta_2 + \cdots + \delta_m < \frac{\epsilon}{2}$.

Since each of these subintervals contain rational as well as irrational points, the oscillation of f in each of these subintervals is less than 1.

Let $P = (0, x_1 - \frac{\delta_1}{2}, x_1 + \frac{\delta_1}{2}, x_2 - \frac{\delta_2}{2}, \cdots, x_m + \frac{\delta_m}{2}, 1)$. Then P is a partition of $[0,1]$ dividing $[0,1]$ into $2m+1$ subintervals, m of which enclose the points $x_1, x_2, \ldots, x_m$. In each of the remaining $m+1$ subintervals, the oscillation of f is less than $\frac{\epsilon}{2}$ and the sum of these $m+1$ subintervals is less than 1.

So $U(P,f) - L(P,f) < 1.\frac{\epsilon}{2} + \frac{\epsilon}{2}.1. = \epsilon$.

Therefore there exists a partition P of $[0,1]$ such that $U(P,f) - L(P,f) < \epsilon$.

This being a sufficient condition for integrability, f is integrable on $[0,1]$.

Let $P = (x_0, x_1, \ldots, x_n)$, where $0 = x_0 < x_1 < \cdots < x_n = 1$ be an arbitrary partition of $[0,1]$. Let $m_r = \inf_{x \in [x_{r-1}, x_r]} f(x)$, $r = 1, 2, \ldots, n$.

Since every subinterval $[x_{r-1}, x_r]$ contains irrational points, $m_r = 0$ for $r = 1, 2, \ldots, n$. Therefore $L(P,f) = 0$.

Consequently, $\underline{\int}_0^1 f = \sup\{L(P,f) : P \in \mathcal{P}[a,b]\} = 0$.

Since $f \in \mathcal{R}[0,1]$, $\underline{\int}_0^1 f = \int_0^1 f$ and therefore $\int_0^1 f = 0$.

Note. This function f is continuous at 0 and at every irrational point in $[0,1]$ and discontinuous at every every non-zero rational point in $[0,1]$. This example shows that a function bounded and continuous on a closed and bounded interval $[a,b]$ except for an infinite set of points $S \subset [a,b]$ having infinite number of limit points, is also $\mathcal{R}$- integrable on $[a,b]$.

Remarks.

We have seen that a bounded function f

(i) continuous on a closed interval $[a,b]$ is integrable on $[a,b]$;

(ii) continuous on $[a,b]$ except at a finite number of points of discontinuity is integrable on $[a,b]$;

(iii) continuous on $[a,b]$ with infinite number of points of discontinuity, is integrable on $[a,b]$ provided the set of points of discontinuity has only a finite number of limit points.

The reader may think that (iii) describes the most general type of bounded discontinuous functions f that are integrable on $[a, b]$. That this is not true, is established by the worked out Example 5 of 11.5.

In this respect there is a theorem of Lebesgue that a necesary and sufficient condition for a function bounded on $[a, b]$ to be Riemann integrable on $[a, b]$ is that the set of points of discontinuity of f is a set of measure zero.

Definition. A set $S \subset \mathbb{R}$ is said to be a *set of measure zero* if for each $\epsilon > 0$ there exists a countable collection of open intervals $\{I_n\}$ such that $S \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} | I_n | < \epsilon$.

Some examples of a set of measure zero.

1. A finite set $S \subset \mathbb{R}$ is a set of measure zero.

Let $S = \{x_1, x_2, \ldots, x_m\} \subset \mathbb{R}$.

Let $\epsilon > 0$. Let x_r be enclosed by the open interval $I_r = (x_r - \frac{\epsilon}{2(m+1)}, x_r + \frac{\epsilon}{2(m+1)})$, for $r = 1, 2, \ldots, m$. Then $S \subset \bigcup_{r=1}^{m} I_r$ and $|I_1| + |I_2| + \cdots + |I_m| = \frac{m\epsilon}{m+1} < \epsilon$.

Therefore S is covered by a finite collection of open intervals such that the sum of the length of the intervals is less than ϵ, proving that S is a set of measure zero.

2. An enumerable subset of $\mathbb{R}$ is a set of measure zero.

Let S be an enumerable subset of $\mathbb{R}$. The elements of S can be described as $x_1, x_2, x_3, \ldots$

Let $\epsilon > 0$. For each r, let x_r be enclosed by the open interval $I_r = (x_r - \frac{\epsilon}{2^{r+2}}, x_r + \frac{\epsilon}{2^{r+2}})$. Then $S \subset \bigcup_{r=1}^{\infty} I_r$.

$$\begin{aligned} \text{Now } | I_1 | + | I_2 | + | I_3 | + \cdots &= \frac{\epsilon}{2^2} + \frac{\epsilon}{2^3} + \frac{\epsilon}{2^4} + \cdots \\ &= \frac{\epsilon}{2^2}[1 + \tfrac{1}{2} + \tfrac{1}{2^2} + \cdots] \\ &= \frac{\epsilon}{2^2}.2 = \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Therefore S is covered by a countable number of open intervals I_n such that the sum of their lengths is less than ϵ.

This proves that S is a set of measure zero.

Corollary. Since $\mathbb{Q}$ is an ennumerable set, the set $\mathbb{Q}$ is a set of measure zero.

3. Let S be a bounded infinite subset of $\mathbb{R}$ having a finite number of limit points. Then S is a set of measure zero.

Let the limit points of S be $x_1, x_2, \ldots, x_m$.

Let the points $x_1, x_2, \ldots, x_m$ be enclosed by m disjoint open intervals $(x_1 - \frac{\delta_1}{2}, x_1 + \frac{\delta_1}{2}), (x_2 - \frac{\delta_2}{2}, x_2 + \frac{\delta_2}{2}), \cdots, (x_m - \frac{\delta_m}{2}, x_m + \frac{\delta_m}{2})$ such that $\delta_1 + \delta_2 + \cdots + \delta_m < \frac{\epsilon}{2}$.

Outside these open intervals there lie at most a finite number of points of S and those can be covered by a finite number of open intervals, the sum of whose lengths is less than $\frac{\epsilon}{2}$.

Considering the two finite families of open intervals, S is covered by a countable collection of open intervals, the sum of whose lengths is less than ϵ.

Therefore S is a set of measure zero.

11.6. Properties of Riemann integrable functions.

Theorem 11.6.1. Let $f : [a, b] \to \mathbb{R}, g : [a, b] \to \mathbb{R}$ be both integrable on $[a, b]$. Then $f + g$ is integrable on $[a, b]$ and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

Proof. Since $f \in \mathcal{R}[a, b]$ and $g \in \mathcal{R}[a, b]$, f and g are both bounded on $[a, b]$. Therefore there exist positive real numbers k_1, k_2 such that $| f(x) | < k_1$ and $| g(x) | < k_2$ for all $x \in [a, b]$.

$| f(x) + g(x) | \leq | f(x) | + | g(x) | < k_1 + k_2$ for all $x \in [a, b]$.

This shows that $f + g$ is bounded on $[a, b]$.

Let us choose $\epsilon > 0$.

Since $f \in \mathcal{R}[a, b]$, there exists a partition P_1 of $[a, b]$ such that

$$U(P_1, f) - L(P_1, f) < \frac{\epsilon}{2}.$$

Since $g \in \mathcal{R}[a, b]$, there exists a partition P_2 of $[a, b]$ such that

$$U(P_2, g) - L(P_2, g) < \frac{\epsilon}{2}.$$

Let $P_0 = P_1 \cup P_2$. Then P_0 is a refinement of P_1 as well as of P_2 and $L(P_1, f) \leq L(P_0, f) \leq U(P_0, f) \leq U(P_1, f)$;

$$L(P_2, g) \leq L(P_0, g) \leq U(P_0, g) \leq U(P_2, g).$$

So $U(P_0, f) - L(P_0, f) \leq U(P_1, f) - L(P_1, f) < \frac{\epsilon}{2}$
and $U(P_0, g) - L(P_0, g) \leq U(P_2, g) - L(P_2, g) < \frac{\epsilon}{2}$.

Let $P_0 = (x_0, x_1, \ldots, x_n)$, where $a = x_0 < x_1 < \cdots < x_n = b$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} (f + g)(x),\ m_r = \inf\limits_{x \in [x_{r-1}, x_r]} (f + g)(x)$

$M'_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x),\ m'_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$

$M''_r = \sup\limits_{x \in [x_{r-1}, x_r]} g(x),\ m''_r = \inf\limits_{x \in [x_{r-1}, x_r]} g(x)$, for $r = 1, 2, \ldots, n$.

Then $M_r \leq M'_r + M''_r,\ m_r \geq m'_r + m''_r$, for $r = 1, 2, \ldots, n$.

$$U(P_0, f + g) = M_1(x_1 - x_0) + \cdots + M_n(x_n - x_{n-1})$$
$$\leq [M'_1(x_1 - x_0) + \cdots + M'_n(x_n - x_{n-1})]$$

$$+[M_1''(x_1 - x_0) + \cdots + M_n''(x_n - x_{n-1})]$$
$$= U(P_0, f) + U(P_0, g).$$

Similarly, $L(P_0, f + g) \geq L(P_0, f) + L(P_0, g)$.

Hence $U(P_0, f+g) - L(P_0, f+g) \leq [U(P_0, f) - L(P_0, f)] + [U(P_0, g) - L(P_0, g)] < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Therefore for a chosen $\epsilon > 0$ there exists a partition P_0 of $[a, b]$ such that $U(P_0, f + g) - L(P_0, f + g) < \epsilon$.

This being a sufficient condition for integrability, $f + g$ is integrable on $[a, b]$.

Second part.

Let $\epsilon > 0$. Since $f \in \mathcal{R}[a, b]$, there exists a partition P_1 of $[a, b]$ such that $U(P_1, f) < \int_a^b f + \frac{\epsilon}{2}$.

Since $g \in \mathcal{R}[a, b]$, there exists a partition P_2 of $[a, b]$ such that $U(P_2, g) < \int_a^b g + \frac{\epsilon}{2}$.

Let $P_0 = P_1 \cup P_2$. Then P_0 is a refinement of P_1 and P_2 and $U(P_0, f) \leq U(P_1, f) < \int_a^b f + \frac{\epsilon}{2}$; $U(P_0, g) \leq U(P_2, g) < \int_a^b g + \frac{\epsilon}{2}$.

Therefore $U(P_0, f + g) \leq U(P_0, f) + U(P_0, g) < \int_a^b f + \int_a^b g + \epsilon$.

Since $f + g$ is integrable on $[a, b]$, $\int_a^b (f + g) \leq U(P_o, f + g)$.

Hence $\int_a^b (f + g) < \int_a^b f + \int_a^b g + \epsilon$. ... ...(i)

Considering the lower sums, by similar arguments we have

$\int_a^b (f + g) > \int_a^b f + \int_a^b g - \epsilon$. ... ...(ii)

From (i) and (ii) we have $\left| \int_a^b (f + g) - \int_a^b f - \int_a^b g \right| < \epsilon$.

This holds for every positive ϵ. Therefore $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

Note. For a finite number of functions $f_1, f_2, \ldots, f_n$ each integrable on $[a, b]$, $f_1 + f_2 + \cdots + f_n$ is integrable on $[a, b]$ and

$$\int_a^b (f_1 + f_2 + \cdots + f_n) = \int_a^b f_1 + \int_a^b f_2 + \cdots + \int_a^b f_n.$$

Theorem 11.6.2. Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $c \in \mathbb{R}$. Then cf is integrable on $[a, b]$ and $\int_a^b cf = c\int_a^b f$.

Proof. Since $f \in \mathcal{R}[a, b]$, f is bounded on $[a, b]$. Therefore cf is bounded on $[a, b]$.

Case 1. $c = 0$.

$cf(x) = 0$ for all $x \in [a, b]$. cf is integrable on $[a, b]$ and $\int_a^b cf = 0$ and therefore $\int_a^b cf = c \int_a^b f$.

Case 2. $c > 0$.

$\underline{\int}_a^b f =$ the supremum of the set $\{L(P,f) : P \in \mathcal{P}[a,b]\}$.

$\overline{\int}_a^b f =$ the infimum of the set $\{U(P,f) : P \in \mathcal{P}[a,b]\}$.

$\underline{\int}_a^b cf =$ the supremum of the set $\{L(P,cf) : P \in \mathcal{P}[a,b]\}$

$=$ the supremum of the set $\{cL(P,f) : P \in \mathcal{P}[a,b]\}$

$= c.$ the supremum of the set $\{L(P,f) : P \in \mathcal{P}[a,b]\}$

$= c.\underline{\int}_a^b f.$

$\overline{\int}_a^b cf =$ the infimum of the set $\{U(P,cf) : P \in \mathcal{P}[a,b]\}$

$=$ the infimum of the set $\{cU(P,f) : P \in \mathcal{P}[a,b]\}$

$= c.$ the infimum of the set $\{U(P,f) : P \in \mathcal{P}[a,b]\}$

$= c.\overline{\int}_a^b f.$

Since $f \in \mathcal{R}[a,b]$, $\underline{\int}_a^b f = \overline{\int}_a^b f = \int_a^b f$.

Hence $\underline{\int}_a^b cf = \overline{\int}_a^b cf = c\int_a^b f$.

This shows that cf is integrable on $[a,b]$ and $\int_a^b cf = c\int_a^b f$.

Case 3. $c < 0$.

Similar proof.

Note. The two theorems 11.6.1 and 11.6.2 together establish that Riemann integrals satisfy linearity property.

Theorem 11.6.3. Let a function $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Then $|f|$ is integrable on $[a,b]$.

Proof. Since $f \in \mathcal{R}[a,b]$, f is bounded on $[a,b]$. Therefore there exists a positive real number k such that $|f(x)| < k$ for all $x \in [a,b]$.

Now $|\,|f|(x)\,| = |f|(x) = |f(x)| < k$ for all $x \in [a,b]$. This shows that $|f|$ is bounded on $[a,b]$.

Let us choose $\epsilon > 0$.

Since f is integrable on $[a,b]$, there exists a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \epsilon$.

Let $P = (x_0, x_1, \ldots, x_n)$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$,

$M'_r = \sup\limits_{x \in [x_{r-1}, x_r]} |f|(x)$, $m'_r = \inf\limits_{x \in [x_{r-1}, x_r]} |f|(x)$, for $r = 1, 2, \ldots, n$.

For any two points α, β in $[x_{r-1}, x_r]$, we have

$|\,|f|(\alpha) - |f|(\beta)\,| = |\,|f(\alpha)| - |f(\beta)|\,| \le |f(\alpha) - f(\beta)| \quad \ldots \quad \ldots \text{(i)}$

We use here an important property of a bounded subset of $\mathbb{R}$.

If S be a non-empty bounded subset of $\mathbb{R}$ with $\sup S = M$ and $\inf S = m$, then the supremum of the set $\{|x-y| : x \in S, y \in S\}$ is $M - m$. [worked Ex.5, page 33]

Since f is bounded on $[x_{r-1}, x_r]$ with $\sup\limits_{x \in [x_{r-1}, x_r]} f(x) = M_r$ and $\inf\limits_{x \in [x_{r-1}, x_r]} f(x) = m_r$, the supremum of the set $\{|f(\alpha) - f(\beta)| : \alpha \in [x_{r-1}, x_r], \beta \in [x_{r-1}, x_r]\}$ is $M_r - m_r$.

Since $|f|$ is bounded on $[x_{r-1}, x_r]$ with $\sup\limits_{x \in [x_{r-1}, x_r]} |f|(x) = M'_r$ and $\inf\limits_{x \in [x_{r-1}, x_r]} |f|(x) = m'_r$, the supremum of the set $\{|\,|f|(\alpha) - |f|(\beta)\,| : \alpha \in [x_{r-1}, x_r], \beta \in [x_{r-1}, x_r]\}$ is $M'_r - m'_r$.

From the inequality (i) it follows that $M_r - m_r$ is an upper bound of the set $\{|\,|f|(\alpha) - |f|(\beta)\,| : \alpha \in [x_{r-1}, x_r], \beta \in [x_{r-1}, x_r]\}$.

Therefore $M'_r - m'_r \le M_r - m_r$. This holds for $r = 1, 2, \ldots, n$.

So $U(P, |f|) - L(P, |f|)$
$= (M'_1 - m'_1)(x_1 - x_0) + \cdots + (M'_n - m'_n)(x_n - x_{n-1})$
$\le (M_1 - m_1)(x_1 - x_0) + \cdots + (M_n - m_n)(x_n - x_{n-1})$
$= U(P, f) - L(P, f) < \epsilon$.

This being a sufficient condition for integrability, $|f|$ is integrable on $[a, b]$. This completes the proof.

Note. The converse of the theorem is not true. For example,
let $f : [a, b] \to \mathbb{R}$ be defined by $f(x) = 1, x \in [a, b] \cap \mathbb{Q}$
$= -1, x \in [a, b] - \mathbb{Q}$.

Then f is not integrable on $[a, b]$.
But $|f|(x) = 1$ for all $x \in [a, b]$ and $|f|$ is integrable on $[a, b]$.

Theorem 11.6.4. Let a function $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Then f^2 is integrable on $[a, b]$

Proof. Since $f \in \mathcal{R}[a, b]$, f is bounded on $[a, b]$. Therefore there exists a positive real number k such that $|f(x)| \le k$ for all $x \in [a, b]$. So $|f^2(x)| \le k^2$ for all $x \in [a, b]$. This shows that f^2 is bounded on $[a, b]$.

Let us choose $\epsilon > 0$.

Since f is integrable on $[a, b]$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \frac{\epsilon}{2k}$.

Let $P = (x_0, x_1, x_2, \ldots, x_n)$, where $a = x_0 < x_1 < \cdots < x_n = b$.
Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$,

$M'_r = \sup_{x\in[x_{r-1},x_r]} f^2(x)$, $m'_r = \inf_{x\in[x_{r-1},x_r]} f^2(x)$, for $r = 1, 2, \ldots, n$.

For any two points α, β in $[x_{r-1}, x_r]$, we have

$$\begin{aligned} | f^2(\alpha) - f^2(\beta) | &= | \{f(\alpha)\}^2 - \{f(\beta)\}^2 | \\ &= | f(\alpha) + f(\beta) || f(a) - f(\beta) | \\ &\leq 2k | f(\alpha) - f(\beta) | \ldots \quad \ldots \text{(i)} \end{aligned}$$

Since f is bounded on $[x_{r-1}, x_r]$ with $\sup_{x\in[x_{r-1},x_r]} f(x) = M_r$ and $\inf_{x\in[x_{r-1},x_r]} f(x) = m_r$, the supremum of the set $\{| f(\alpha) - f(\beta) |: \alpha \in [x_{r-1}, x_r], \beta \in [x_{r-1}, x_r]\}$ is $M_r - m_r$. [worked Ex.5, page 33]

Since f^2 is bounded on $[x_{r-1}, x_r]$ with $\sup_{x\in[x_{r-1},x_r]} f^2(x) = M'_r$ and $\inf_{x\in[x_{r-1},x_r]} f^2(x) = m'_r$, the supremum of the set $\{| f^2(\alpha) - f^2(\beta) |: \alpha \in [x_{r-1}, x_r], \beta \in [x_{r-1}, x_r]\} = M'_r - m'_r$. [worked Ex. 5, page 32]

It follows from the inequality (i) that $2k(M_r - m_r)$ is an upper bound of the set $\{| f^2(\alpha) - f^2(\beta) |: \alpha, \beta \in [x_{r-1}, x_r]\}$.

Therefore $M'_r - m'_r \leq 2k(M_r - m_r)$. This holds for $r = 1, 2, \ldots, n$.

$$\begin{aligned} U(P, f^2) - L(P, f^2) &= \sum_{r=1}^{n} (M'_r - m'_r)(x_r - x_{r-1}) \\ &\leq 2k \sum_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1}) \\ &= 2k[U(P, f) - L(P, f)] < \epsilon. \end{aligned}$$

This being a sufficient condition for integrability, f^2 is integrable on $[a, b]$. This completes the proof.

Theorem 11.6.5. Let the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be both integrable on $[a, b]$. Then fg is integrable on $[a, b]$.

Proof. Since $f \in \mathcal{R}[a, b]$, f is bounded on $[a, b]$. Since $g \in \mathcal{R}[a, b]$, g is bounded on $[a, b]$. Therefore fg is bounded on $[a, b]$.

Now $fg = \frac{1}{2}(f + g)^2 - \frac{1}{2}f^2 - \frac{1}{2}g^2$.

Since $f \in \mathcal{R}[a, b]$ and $g \in \mathcal{R}[a, b]$, $\frac{1}{2}(f + g)^2$, $\frac{1}{2}f^2$ and $\frac{1}{2}g^2$ are all integrable on $[a, b]$ by Theorems 11.6.1, 11.6.2 and 11.6.4.

Hence fg is integrable on $[a, b]$. This completes the proof.

Theorem 11.6.6. Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. If there exists a positive real number k such that such that $f(x) \geq k$ for all $x \in [a, b]$ then $\frac{1}{f}$ is integrable on $[a, b]$.

Proof. $|\frac{1}{f}(x)| = |\frac{1}{f(x)}| \leq \frac{1}{k}$ for all $x \in [a, b]$. This shows that $\frac{1}{f}$ is bounded on $[a, b]$.

Let us choose $\epsilon > 0$.

Since f is integrable on $[a, b]$ there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < k^2\epsilon$.

Let $P = (x_0, x_1, \ldots, x_n)$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.
Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$;
$M'_r = \sup\limits_{x \in [x_{r-1}, x_r]} \frac{1}{f}(x)$, $m'_r = \inf\limits_{x \in [x_{r-1}, x_r]} \frac{1}{f}(x)$, for $r = 1, 2, \ldots, n$.

For any two points α, β in $[x_{r-1}, x_r]$, we have
$|\frac{1}{f}(\alpha) - \frac{1}{f}(\beta)| = |\frac{1}{f(\alpha)} - \frac{1}{f(\beta)}| = \frac{1}{|f(\alpha)|\,|f(\beta)|}|f(\alpha) - f(\beta)|$
$\leq \frac{1}{k^2}|f(\alpha) - f(\beta)|$ ... ... (i)

Since f is bounded on $[x_{r-1}, x_r]$ with $\sup\limits_{x \in [x_{r-1}, x_r]} f(x) = M_r$ and $\inf\limits_{x \in [x_{r-1}, x_r]} f(x) = m_r$, the supremum of the set $\{|f(\alpha) - f(\beta)| : \alpha \in [x_{r-1}, x_r], \beta \in [x_{r-1}, x_r]\}$ is $M_r - m_r$. [worked Ex.5, page 33]

Since $\frac{1}{f}$ is bounded on $[x_{r-1}, x_r]$ with $\sup\limits_{x \in [x_{r-1}, x_r]} \dfrac{1}{f}(x) = M'_r$ and $\inf\limits_{x \in [x_{r-1}, x_r]} \dfrac{1}{f}(x) = m'_r$, the supremum of the set $\{| \frac{1}{f}(\alpha) - \frac{1}{f}(\beta) | : \alpha \in [x_{r-1}, x_r], \beta \in [x_{r-1}, x_r]\} = M'_r - m'_r$.

It follows from the inequality (i) that $\frac{1}{k^2}(M_r - m_r)$ is an upper bound of the set $\{| \frac{1}{f}(\alpha) - \frac{1}{f}(\beta) | : \alpha, \beta \in [x_{r-1}, x_r]\}$.

Therefore $M'_r - m'_r \leq \frac{1}{k^2}(M_r - m_r)$. This holds for $r = 1, 2, \ldots, n$.

$$\begin{aligned} U(P, \tfrac{1}{f}) - L(P, \tfrac{1}{f}) &= \sum_{r=1}^{n} (M'_r - m'_r)(x_r - x_{r-1}) \\ &\leq \sum_{r=1}^{n} \frac{1}{k^2}(M_r - m_r)(x_r - x_{r-1}) \\ &= \tfrac{1}{k^2}[U(P, f) - L(P, f)] < \epsilon. \end{aligned}$$

Therefore for a chosen $\epsilon > 0$ there exists a partition P of $[a, b]$ such that $U(P, \frac{1}{f}) - L(P, \frac{1}{f}) < \epsilon$.

This being a sufficient condition for integrability, $\frac{1}{f}$ is integrable on $[a, b]$. This completes the proof.

Note 1. If $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $f(x) > 0$ for all $x \in [a, b]$, then $\frac{1}{f}$ may not be integrable on $[a, b]$. For example, let

$$f : [0, 1] \to \mathbb{R} \text{ be defined by } f(x) \begin{aligned}[t] &= x, 0 < x \leq 1 \\ &= 1, x = 0. \end{aligned}$$

Then f is bounded on $[0, 1]$, f is continuous on $[0, 1]$ except at only one point, 0. So f is integrable on $[0, 1]$. Also $f(x) > 0$ for all $x \in [0, 1]$.
$\frac{1}{f}$ is unbounded on $[0, 1]$ and therefore $\frac{1}{f}$ is not integrable on $[0, 1]$.

Note 2. If however, $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $f(x) > 0$ for all $x \in [a, b]$, then there exists a positive real number k such that $f(x) \geq k$ for all $x \in [a, b]$. [Ex.6, Exercises 13.]

In this case, $\frac{1}{f}$ is integrable on $[a, b]$, by the theorem.

Theorem 11.6.7. Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be both integrable on $[a, b]$ and there exists a positive real number k such that $g(x) \geq k$ for all $x \in [a, b]$. Then f/g is integrable on $[a, b]$.

Proof. Since $f \in \mathcal{R}[a, b]$, f is bounded on $[a, b]$. Therefore there exists a positive real number B such that $| f(x) | < B$ for all $x \in [a, b]$.
Since $g(x) \geq k$ for all $x \in [a, b]$, $\frac{1}{g(x)} \leq \frac{1}{k}$ for all $x \in [a, b]$.

Therefore $| \frac{f}{g}(x) | = | \frac{f(x)}{g(x)} | < \frac{B}{k}$ for all $x \in [a, b]$. This shows that f/g is bounded on $[a, b]$.

Since $g \in \mathcal{R}[a, b]$ and $g(x) \geq k > 0$ for all $x \in [a, b]$, $\frac{1}{g} \in \mathcal{R}[a, b]$.

Since f and $\frac{1}{g}$ are both integrable on $[a, b]$, $\frac{f}{g}$ is integrable on $[a, b]$ by Theorem 11.6.5.

Theorem 11.6.8. Let $I = [a, b] \subset \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $J = [c, d] \subset \mathbb{R}$ such that $f(I) \subset J$ and $\phi : [c, d] \to \mathbb{R}$ be continuous on $[c, d]$. Then the composite function $\phi \circ f$ is integrable on $[a, b]$.

Proof. Since ϕ is continuous on $[c, d]$, ϕ is bounded on $[c, d]$ and therefore there exists a positive real number k such that $| \phi(t) | \leq k$ for all $t \in [c, d]$.

Let us choose $\epsilon > 0$. Since ϕ is continuous on $[c, d]$, ϕ is uniformly continuous on $[c, d]$ and therefore there exists a positive δ such that for all $s, t \in [c, d]$, $| s - t | < \delta \Rightarrow | \phi(s) - \phi(t) | < \frac{\epsilon}{2(b-a)}$ ... ... (i)

Since f is integrable on $[a, b]$, there exists a partition $P = (a = x_0, x_1, \cdots, x_n = b)$ of $[a, b]$ such that $U(P, f) - L(P, f) < \frac{\epsilon}{4k}.\delta$ ... (ii)

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$;

$M'_r = \sup\limits_{x \in [x_{r-1}, x_r]} \phi \circ f(x)$, $m'_r = \inf\limits_{x \in [x_{r-1}, x_r]} \phi \circ f(x)$, for $r = 1, 2, \ldots, n$.

Let us consider two subsets A and B of the set $S = \{1, 2, \ldots, n\}$ (the indices of the points of partition P), where $A = \{r : M_r - m_r < \delta\}$, $B = \{r : M_r - m_r \geq \delta\}$.

Let $r \in A$. Then for $x, y \in [x_{r-1}, x_r]$, $| f(x) - f(y) | < \delta$ and by (i) this implies $| \phi(f(x)) - \phi(f(y)) | < \frac{\epsilon}{2(b-a)}$.

Therefore if $r \in A$, $M'_r - m'_r \leq \frac{\epsilon}{2(b-a)}$, since $M'_r - m'_r$ is the supremum of the set $\{\phi(f(x)) - \phi(f(y)) : x, y \in [x_{r-1}, x_r]\}$.

Consequently, $\sum\limits_{r \in A} (M'_r - m'_r)(x_r - x_{r-1}) \leq \dfrac{\epsilon}{2(b-a)}.(b-a)$

$$\text{i.e., } \le \frac{\epsilon}{2}. \quad \dots \text{ (iii)}$$

Let $r \in B$. Then $\sum_{r \in B}(M'_r - m'_r)(x_r - x_{r-1}) \le 2k(x_r - x_{r-1})$

$$\le 2k.(x_r - x_{r-1})\frac{(M_r - m_r)}{\delta}$$
$$\le \frac{2k}{\delta}[U(P,f) - L(P,f)]$$
$$< \frac{2k}{\delta}\cdot\frac{\epsilon\delta}{4k}, \text{ using (i)}$$
$$\text{i.e., } < \frac{\epsilon}{2} \dots \dots \text{ (iv)}$$

Therefore $U(P, \phi\circ f) - L(P, \phi\circ f) = \sum_{r=1}^{n}(M'_r - m'_r)(x_r - x_{r-1})$

$$< \epsilon, \text{ using (iii) and (iv).}$$

This proves that $\phi\circ f$ is integrable on $[a,b]$.

Note. If $f : I \to \mathbb{R}$ be integrable on I and $\phi : J \to \mathbb{R}$ be integrable on J, then $\phi\circ f$ may not be integrable on I. For example, let $\phi : [0,1] \to \mathbb{R}$ be defined by $\phi(x) = 1$, if $x \ne 0$
$= 0$, if $x = 0$.

Then ϕ is integrable on $[0, 1]$.

Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = 0$, if x be irrational
$= \frac{1}{n}$, if $x = \frac{m}{n}$ where m, n are positive integers and $gcd(m,n) = 1$.

Then f is integrable on $[0, 1]$.

$\phi\circ f : [0,1] \to \mathbb{R}$ is defined by $\phi\circ f(x) = 0$, if x be irrational
$= 1$, if x be rational.

$\phi\circ f$ is not integrable on $[0,1]$.

Theorem 11.6.9. Let a function $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and $c \in (a,b)$. Then f is integrable on $[a,c]$ and also integrable on $[c,b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.

Proof. Since $f \in \mathcal{R}[a,b]$, f is bounded on $[a,b]$. Therefore f is bounded on $[a,c]$ as well as on $[c,b]$.

Let us choose $\epsilon > 0$. Since f is integrable on $[a,b]$, there exists a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \epsilon$.

If $c \notin P$, let P_c be the refinement of P by adjoining the point c to P, i.e., $P_c = P \cup \{c\}$.

Then $L(P,f) \le L(P_c,f) \le U(P_c,f) \le U(P,f)$.

If however, $c \in P$, then $P_c = P$.

In any case, $U(P_c,f) - L(P_c,f) \le U(P,f) - L(P,f) < \epsilon$.

Let $P_c = P_1 \cup P_2$ where $P_1 = [a,c] \cap P_c, P_2 = [c,b] \cap P_c$.

Then P_1 is a partition of $[a,c]$ and P_2 is a partition of $[c,b]$ and

$U(P_c, f) = U(P_1, f) + U(P_2, f)$, $L(P_c, f) = L(P_1, f) + L(P_2, f)$.

Therefore $[U(P_1, f) - L(P_1, f)] + [U(P_2, f) - L(P_2, f)] < \epsilon$

Since $U(P_1, f) - L(P_1, f) \geq 0$ and $U(P_2, f) - L(P_2, f) \geq 0$, it follows that $U(P_1, f) - L(P_1, f) < \epsilon$ and $U(P_2, f) - L(P_2, f) < \epsilon$.

By the condition for integrability, f is integrable on $[a, c]$ and also integrable on $[c, b]$.

Second part.

For any partition P of $[a, b]$,

$U(P, f) \geq U(P_c, f)$, where $P_c = P \cup \{c\}$
$= U(P_1, f) + U(P_2, f)$, where $P_1 = [a, c] \cap P_c$ and $P_2 = [c, b] \cap P_c$
$\geq \overline{\int}_a^c f + \overline{\int}_c^b f$.

This shows that $\overline{\int}_a^c f + \overline{\int}_c^b f$ is a lower bound of the set $\{U(P, f) : P \in \mathcal{P}[a, b]\}$. Since the infimum of the set is $\overline{\int}_a^b f$, it follows that

$\overline{\int}_a^c f + \overline{\int}_c^b f \leq \overline{\int}_a^b f$ ... ... (i)

For any partition P of $[a, b]$,

$L(P, f) \leq L(P_c, f)$, where $P_c = P \cup \{c\}$
$= L(P_1, f) + L(P_2, f)$, where $P_1 = [a, c] \cap P_c$ and $P_2 = [c, b] \cap P_c$
$\leq \underline{\int}_a^c f + \underline{\int}_c^b f$.

This shows that $\underline{\int}_a^c f + \underline{\int}_c^b f$ is an upper bound of the set $\{L(P, f)$ $P \in \mathcal{P}[a, b]\}$. Since the supremum of the set is $\underline{\int}_a^b f$, it follows that

$\underline{\int}_a^c f + \underline{\int}_c^b f \geq \underline{\int}_a^b f$ ... ... (ii)

But $\underline{\int}_a^b f = \overline{\int}_a^b f = \int_a^b f$, $\underline{\int}_a^c f = \overline{\int}_a^c f = \int_a^c f$, $\underline{\int}_c^b f = \overline{\int}_c^b f = \int_c^b f$.

From (i) and (ii) we have $\int_a^c f + \int_c^b f \leq \int_a^b f \leq \int_a^c f + \int_c^b f$.

Consequently, $\int_a^b f = \int_a^c f + \int_c^b f$. This completes the proof.

Corollary. If $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $a < c < d < b$, then f is integrable on $[c, d]$.

Proof. $f \in \mathcal{R}[a, b] \Rightarrow f \in \mathcal{R}[c, b]$, since $a < c < b$
$\Rightarrow f \in \mathcal{R}[c, d]$, since $c < d < b$.

Theorem 11.6.10. Let $f : [a, b] \to \mathbb{R}$ and $c \in (a, b)$. If f is integrable on $[a, c]$ as well as on $[c, b]$ then f is integrable on $[a, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. Since $f \in \mathcal{R}[a, c]$, f is bounded on $[a, c]$. Since $f \in \mathcal{R}[c, b]$, f is bounded on $[c, b]$. Therefore f is bounded on $[a, b]$.

Let us choose $\epsilon > 0$. Since f is integrable on $[a, c]$, there exists a partition P_1 of $[a, c]$ such that $U(P_1, f) - L(P_1, f) < \frac{\epsilon}{2}$.

Since f is integrable on $[c, b]$, there exists a partition P_2 of $[c, b]$ such that $U(P_2, f) - L(P_2, f) < \frac{\epsilon}{2}$.

Let $P = P_1 \cup P_2$. Then P is a partition of $[a, b]$ and

$U(P, f) = U(P_1, f) + U(P_2, f)$, $L(P, f) = L(P_1, f) + L(P_2, f)$.

So $U(P, f) - L(P, f) = [U(P_1, f) - L(P_1, f)] + [U(P_2, f) - L(P_2, f)] < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Therefore for a chosen $\epsilon > 0$ there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$.

This being a sufficient condition for integrability, f is integrable on $[a, b]$.

Second part. (Same Previous)

For any partition P of $[a, b]$,

$U(P, f) \geq U(P_c, f)$, where $P_c = P \cup \{c\}$
$= U(P_1, f) + U(P_2, f)$, where $P_1 = [a, c] \cap P_c$ and $P_2 = [c, b] \cap P_c$
$\geq \overline{\int}_a^c f + \overline{\int}_c^b f$.

This shows that $\overline{\int}_a^c f + \overline{\int}_c^b f$ is a lower bound of the set $\{U(P, f) : P \in \mathcal{P}[a, b]\}$. Since the infimum of the set is $\overline{\int}_a^b f$, it follows that

$\overline{\int}_a^c f + \overline{\int}_c^b f \leq \overline{\int}_a^b f$ ... ... (i)

For any partition P of $[a, b]$,

$L(P, f) \leq L(P_c, f)$, where $P_c = P \cup \{c\}$
$= L(P_1, f) + L(P_2, f)$, where $P_1 = [a, c] \cap P_c$ and $P_2 = [c, b] \cap P_c$
$\leq \underline{\int}_a^c f + \underline{\int}_c^b f$.

This shows that $\underline{\int}_a^c f + \underline{\int}_c^b f$ is an upper bound of the set $\{L(P, f) : P \in \mathcal{P}[a, b]\}$. Since the supremum of the set is $\underline{\int}_a^b f$, it follows that

$\underline{\int}_a^c f + \underline{\int}_c^b f \geq \underline{\int}_a^b f$ ... ... (ii)

But $\underline{\int}_a^b f = \overline{\int}_a^b f$, $\underline{\int}_a^c f = \overline{\int}_a^c f$, $\underline{\int}_c^b f = \overline{\int}_c^b f$.

From (i) and (ii) we have $\int_a^c f + \int_c^b f \leq \int_a^b f \leq \int_a^c f + \int_c^b f$.

Consequently, $\int_a^b f = \int_a^c f + \int_c^b f$.

This completes the proof.

Theorem 11.6.11. Let $f : [a, b] \to \mathbb{R}$, $\phi : [a, b] \to \mathbb{R}$ be both bounded on $[a, b]$ such that $f(x) = \phi(x)$ except for a finite number of points in

$[a, b]$. If f be integrable on $[a, b]$ then ϕ is also integrable on $[a, b]$ and $\int_a^b \phi = \int_a^b f$.

Proof. Since ϕ is bounded on $[a, b]$, there exists a positive number B such that $|\phi(x)| < B$ for all $x \in [a, b]$. Let $f(x) = \phi(x)$ except at p points $x_1, x_2, \ldots, x_p$ such that $x_1 < x_2 < \cdots < x_p$.

Case 1. Let $a < x_1 < x_2 < \cdots < x_p < b$.

Let $\epsilon > 0$. Let the points $x_1, x_2, \ldots, x_p$ be enclosed by p non-overlapping subintervals $[x_1 - \frac{\delta_1}{2}, x_1 + \frac{\delta_1}{2}], [x_2 - \frac{\delta_2}{2}, x_2 + \frac{\delta_2}{2}], \cdots, [x_p - \frac{\delta_p}{2}, x_p + \frac{\delta_p}{2}]$ such that $\delta_1 + \delta_2 + \cdots + \delta_p < \frac{\epsilon}{4B}$.

Let $M^{(r)} = \sup\limits_{x \in [x_r - \frac{\delta_r}{2}, x_r + \frac{\delta_r}{2}]} \phi(x)$, $m^{(r)} = \inf\limits_{x \in [x_r - \frac{\delta_r}{2}, x_r + \frac{\delta_r}{2}]} \phi(x), r = 1, 2, \ldots, p$.

Then $M^{(r)} - m^{(r)} < 2B$, for $r = 1, 2, \ldots, p$.

On each of the remaining $p+1$ subintervals $[a, x_1 - \frac{\delta_1}{2}], [x_1 + \frac{\delta_1}{2}, x_2 - \frac{\delta_2}{2}], \cdots, [x_p + \frac{\delta_p}{2}, b]$, $f = \phi$ and since f is integrable on these subintervals, there exist partitions P_1 of $[a, x_1 - \frac{\delta_1}{2}], P_2$ of $[x_1 + \frac{\delta_1}{2}, x_2 - \frac{\delta_2}{2}], \cdots, P_{p+1}$ of $[x_p - \frac{\delta_p}{2}, b]$ such that

$U(P_k, \phi) - L(P_k, \phi) < \frac{\epsilon}{2(p+1)}$, for $k = 1, 2, \ldots, p+1$.

The partitions $P_1, P_2, \ldots, P_{p+1}$ are disjoint.

Let $P = P_1 \cup P_2 \cup \cdots \cup P_{p+1}$. Then P is a partition of $[a, b]$.

$$\begin{aligned} U(P, \phi) - L(P, \phi) &= [U(P_1, \phi) - L(P_1, \phi)] + [U(P_2, \phi) - L(P_2, \phi)] \\ &+ \cdots + [U(P_{p+1}, \phi) - L(P_{p+1}, \phi)] + (M^{(1)} - m^{(1)})\delta_1 + \cdots + (M^{(p)} - m^{(p)})\delta_p \\ &< \frac{\epsilon}{2(p+1)}.(p+1) + 2B.\frac{\epsilon}{4B} \\ &= \epsilon. \end{aligned}$$

Therefore there exists a patition P of $[a, b]$ such that $U(P, \phi) - L(P, \phi) < \epsilon$. Hence ϕ is integrable on $[a, b]$.

Case 2. Either $a = x_1$, or $x_p = b$, or both.

If $a = x_1$, the subinterval enclosing the point x_1 can be chosen as $[a, a + \delta_1]$.

If $x_p = b$, the subinterval enclosing the point x_p can be chosen as $[b - \delta_p, b]$.

In any case, proceeding with similar arguments it can be proved that ϕ is integrable on $[a, b]$.

Second part.

Let $g(x) = f(x) - \phi(x)$, $x \in [a, b]$.

Then g is bounded on $[a,b]$ and $g(x) = 0$ on $[a,b]$ except at p points. Hence g is integrable on $[a,b]$, by Theorem 11.5.3.

Let $g_+(x) = \max\{g(x),0\}$, $g_-(x) = \min\{g(x),0\}$, i.e., $g_+(x) = \frac{1}{2}[g(x) + |g(x)|]$ and $g_-(x) = \frac{1}{2}[g(x) - |g(x)|]$.

Then g_+ and g_- are both integrable on $[a,b]$. $g_+(x) \geq 0$ and $-g_-(x) \geq 0$ for all $x \in [a,b]$.

Let $P = (a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b)$ be a partition of $[a,b]$.

Let $m_r = \inf\limits_{x\in[x_{r-1},x_r]} g_+(x), r = 1, 2, \ldots, n.$

As $g_+(x) = 0$ in $[x_{r-1}, x_r]$ for all x except at p points at most, $m_r = 0$. This holds for $r = 1, 2, \ldots, n$.

Therefore $L(P, g_+) = 0$ and this holds for every partition $P \in P[a,b]$. Therefore $\underline{\int}_a^b g_+ = 0$.

Since g_+ is integrable on $[a,b]$, $\int_a^b g_+ = \underline{\int}_a^b g_+ = 0$.

By similar arguments, $\underline{\int}_a^b g_- = 0$ and hence $\int_a^b g_- = 0$.

As $g(x) = g_+(x) + g_-(x)$ for all $x \in [a,b]$, it follows that $\int_a^b g = 0$.

Since f and ϕ are both integrable on $[a,b]$, $\int_a^b f - \int_a^b \phi = \int_a^b g = 0$.

Therefore $\int_a^b f = \int_a^b \phi$.

This completes the proof.

Note. If f and ϕ be both bounded on $[a,b]$, $f(x) = \phi(x)$ except for an enumerable number of points on $[a,b]$ and f is integrable on $[a,b]$, then ϕ may not be integrable on $[a,b]$. For example,

let $f(x) = 1,\ x \in [0,1]$ and
$$\begin{aligned}\phi(x) &= 0,\ x \in [0,1] \cap \mathbb{Q}\\ &= 1,\ x \in [0,1] - \mathbb{Q}.\end{aligned}$$

Then f and ϕ are both bounded on $[0,1]$. $f = \phi$ on $[0,1]$ except for an enumerable number of points. f is integrable on $[0,1]$ but ϕ is not integrable on $[0,1]$.

Definition.

Piecewise continuous function. A function $f : [a,b] \to \mathbb{R}$ is said to be a *piecewise continuous* function on $[a,b]$ if there exists a partition $P = (x_0, x_1, \ldots, x_n)$ of $[a,b]$ where $a = x_0 < x_1 < \cdots < x_n = b$, such that f is continuous on the open interval (x_{k-1}, x_k) for $1 \leq k \leq n$ and each of $f(a+0), f(b-0), f(x_k+0), f(x_k-0)$ is finite for $1 \leq k \leq n-1$.

Clearly, a piecewise continuous function on $[a,b]$ is continuous on $[a,b]$ except for a finite number of points of jump discontinuity.

A step function on $[a,b]$ is an example of a piecewise continuous function on $[a,b]$. [page 100]

As a direct application of the Theorem 11.6.10, the evaluation of the definite integral $\int_a^b f$ becomes very simple when f is a step function or a piecewise continuous function on $[a, b]$.

Let $f : [a, b] \to \mathbb{R}$ be a step function on $[a, b]$ and $P = (x_0, x_1, \ldots, x_n)$ be a partition of $[a, b]$ where $a = x_0 < x_1 < \cdots < x_n = b$, such that $f(x) = c_k$ on (x_{k-1}, x_k) for $1 \le k \le n$ and $f(x_k) = \mu_k$ for $0 \le k \le n$.

Then f is bounded on $[a, b]$ and is continuous on $[a, b]$ except for the points $x_0, x_1, \ldots, x_n$ at most.

Let us define n functions $\phi_1, \phi_2, \ldots, \phi_n$ on $[a, x_1], [x_1, x_2], \cdots, [x_{n-1}, b]$ respectively by $\phi_k(x) = c_k, 1 \le k \le n$.

$$\begin{aligned}
\text{Then } \int_a^b f &= \int_a^{x_1} \phi_1 + \int_{x_1}^{x_2} \phi_2 + \cdots + \int_{x_{n-1}}^{b} \phi_n, \\
&\qquad \text{by Theorems 11.6.9 and 11.6.11} \\
&= c_1(x_1 - a) + c_2(x_2 - x_1) + \cdots + c_n(b - x_{n-1}) \\
&= \sum_{k=1}^{n} c_k(x_k - x_{k-1}).
\end{aligned}$$

Let $f : [a, b] \to \mathbb{R}$ be a piecewise continuous function on $[a, b]$ and $P = (x_0, x_1, \ldots, x_n)$ be a partition of $[a, b]$ where $a = x_0 < x_1 < \cdots < x_n = b$, such that f is continuous on (x_{k-1}, x_k) for $1 \le k \le n$ and each of $f(a+0), f(b-0), f(x_k - 0), f(x_k + 0)$ (for $1 \le k \le n-1$) is finite.

f is continuous on $[a, b]$ except for the points $x_0, x_1, \ldots, x_n$, at most.

Let us define n functions $\phi_1, \phi_2, \ldots, \phi_n$ on $[a, x_1]$, $[x_1, x_2]$, $\cdots$, $[x_{n-1}, b]$ respectively by

$$\begin{aligned}
&\phi_1(a) = f(a+0); \quad \phi_1(x) = f(x), x \in (a, x_1); \phi_1(x_1) = f(x_1 - 0) \\
&\phi_2(x_1) = f(x_1 + 0); \phi_2(x) = f(x), x \in (x_1, x_2); \phi_2(x_2) = f(x_2 - 0) \\
&\quad \cdots \qquad \cdots \qquad \cdots \\
&\phi_n(x_{n-1}) = f(x_{n-1} + 0); \quad \phi_n(x) = f(x), x \in (x_{n-1}, b); \phi_n(b) = f(b - 0).
\end{aligned}$$

Then $\int_a^b f = \int_a^{x_1} \phi_1 + \int_{x_1}^{x_2} \phi_2 + \cdots + \int_{x_{n-1}}^{b} \phi_n$, by Theorems 11.6.9 and 11.6.11.

Worked Examples.

1. A function $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$ and for every $c \in (a, b)$, f is integrable on $[c, b]$. Prove that f is integrable on $[a, b]$.

Let M be the supremum and m be the infimum of f on $[a, b]$.

Let us consider a sequence of points $\{c_n\}$ in (a, b) such that $\lim c_n = a$.

Let $\epsilon > 0$. Then there exists a natural number k such that $| c_n - a | < \frac{\epsilon}{2(M-m)}$ for all $n \ge k$.

Then $| c_k - a | < \frac{\epsilon}{2(M-m)}$.

Since f is integrable on $[c, b]$ for every $c \in (a, b)$, f is integrable on $[c_k, b]$. Therefore there exists a partition Q of $[c_k, b]$ such that $U(Q, f) - L(Q, f) < \frac{\epsilon}{2}$.

Let P be a partition of $[a, b]$ defined by $P = \{a\} \cup Q$.

Then $U(P, f) - L(P, f) < (M - m)(c_k - a) + [U(Q, f) - L(Q, f)]$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$, i.e., $< \epsilon$.

This proves that f is integrable on $[a, b]$.

Note 1. Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and for every $d \in (a, b)$, f is integrable on $[a, d]$. Then f is integrable on $[a, b]$.

The proof is similar.

Note 2. Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and for every c, d satisfying $a < c < d < b$, f is integrable on $[c, d]$. Then f is integrable on $[a, b]$.

Proof. Taking $d = \frac{a+b}{2}$, f is integrable on $[c, \frac{a+b}{2}]$ for every $c \in (a, \frac{a+b}{2})$. Therefore f is integrable on $[a, \frac{a+b}{2}]$.

Taking $c = \frac{a+b}{2}$, f is integrable on $[\frac{a+b}{2}, d]$ for every $d \in (\frac{a+b}{2}, b)$. Therefore f is integrable on $[\frac{a+b}{2}, b]$.

Consequently, f is integrable on $[a, b]$, by Theorem 11.6.10.

2. Let $f(x) = [x], x \in [0, 3]$. Prove that f is integrable on $[0, 3]$. Evaluate $\int_0^3 f$.

$$\begin{aligned} f(x) &= 0,\ 0 \le x < 1 \\ &= 1,\ 1 \le x < 2 \\ &= 2,\ 2 \le x < 3 \\ &= 3,\ x = 3. \end{aligned}$$

f is bounded on $[0, 3]$. f is continuous on $[0, 3]$ except for the points 1, 2, 3. So f is integrable on $[0, 3]$.

Let us define functions ϕ_1, ϕ_2, ϕ_3 on $[0, 1], [1, 2], [2, 3]$ respectively by $\phi_1(x) = 0, x \in [0, 1]$; $\phi_2(x) = 1, x \in [1, 2]$; $\phi_3(x) = 2, x \in [2, 3]$.

$$\begin{aligned} \text{Then } \int_0^3 f &= \int_0^1 f + \int_1^2 f + \int_2^3 f, \text{ by Theorem 11.6.9} \\ &= \int_0^1 \phi_1 + \int_1^2 \phi_2 + \int_2^3 \phi_3, \text{ by Theorem 11.6.11} \\ &= \int_0^1 0dx + \int_1^2 1dx + \int_2^3 2dx \\ &= 0 + (2 - 1) + 2.(3 - 2) = 3. \end{aligned}$$

3. $f(x) = \frac{1}{n}, \frac{1}{n+1} < x \le \frac{1}{n} (n = 1, 2, 3, \cdots)$
$= 0, x = 0.$

Prove that f is integrable on $[0, 1]$. Evaluate $\int_0^1 f$.

$$\begin{aligned} f(x) &= 1, \tfrac{1}{2} < x \le 1 \\ &= \tfrac{1}{2}, \tfrac{1}{3} < x \le \tfrac{1}{2} \\ &= \tfrac{1}{3}, \tfrac{1}{4} < x \le \tfrac{1}{3} \\ &\cdots\cdots \\ &= 0, x = 0. \end{aligned}$$

f is bounded on $[0,1]$ and is continuous on $[0,1]$ except at $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$

The set of points of discontnuity of f on $[0,1]$ is an infinite set having only one limit point. Therefore f is integrable on $[0,1]$.

Let us define functions $\phi_1, \phi_2, \ldots$ on $[\frac{1}{2}, 1], [\frac{1}{3}, \frac{1}{2}], \cdots$ respectively by $\phi_1(x) = 1, \frac{1}{2} \leq x \leq 1$; $\phi_2(x) = \frac{1}{2}, \frac{1}{3} \leq x \leq \frac{1}{2}, \cdots$

$\int_{\frac{1}{2}}^{1} \phi_1 = 1(1 - \frac{1}{2}) = \frac{1}{2}$, $\int_{\frac{1}{3}}^{\frac{1}{2}} \phi_2 = \frac{1}{2}(\frac{1}{2} - \frac{1}{3}), \cdots \int_{\frac{1}{n+1}}^{\frac{1}{n}} \phi_n = \frac{1}{n}(\frac{1}{n} - \frac{1}{n+1}), \cdots$

Now $\int_0^1 f$

$= \lim_{n\to\infty} [1(1-\frac{1}{2}) + \frac{1}{2}(\frac{1}{2} - \frac{1}{3}) + \frac{1}{3}(\frac{1}{3} - \frac{1}{4}) + \cdots + \frac{1}{n}(\frac{1}{n} - \frac{1}{n+1})]$

$= \lim_{n\to\infty} [(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}) - (\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)})]$

$= \lim_{n\to\infty} [(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}) - (1 - \frac{1}{n+1})]$

$= \frac{\pi^2}{6} - 1$, since $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$ and $\lim(1 - \frac{1}{n+1}) = 1$.

11.7. Inequalities.

Theorem 11.7.1. Let a function $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. If M be the supremum and m be the infimum of f on $[a,b]$ then

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

Proof. Let $P_0 = (a = x_0, x_1, x_2, \ldots, x_n = b)$ be a partition of $[a,b]$.

Let M_r be the supremum of f and m_r be the infimum of f on $[x_{r-1}, x_r]$, for $r = 1, 2, \ldots, n$. Then $m \leq m_r$, $M \geq M_r$, for $r = 1, 2, \ldots, n$.

$L(P_0, f) = m_1(x_1 - x_0) + \cdots + m_n(x_n - x_{n-1})$
$\qquad \geq m[(x_1 - x_0) + \cdots + (x_n - x_{n-1})] = m(b-a)$.

Similarly, $U(P_0, f) \leq M(b-a)$.

Since f is integrable on $[a,b]$, $L(P,f) \leq \int_a^b f \leq U(P,f)$ for all partitions P of $[a,b]$.

Therefore $m(b-a) \leq L(P_0, f) \leq \int_a^b f \leq U(P_0, f) \leq M(b-a)$.
or, $m(b-a) \leq \int_a^b f \leq M(b-a)$.

This completes the proof.

Corollary 1. Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Then there exists a real number μ satisfying $m \leq \mu \leq M$ such that $\int_a^b f = \mu(b-a)$.

Corollary 2. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Then there exists a point $c \in [a,b]$ such that $\int_a^b f = f(c)(b-a)$.

Proof. Since f is continuous on $[a,b]$, f is integrable on $[a,b]$ and f attains every real number μ satisfying $m \leq \mu \leq M$ at least at a point in $[a,b]$. Therefore there exists a point $c \in [a,b]$ such that $f(c) = \mu$ and $\int_a^b f = f(c)(b-a)$.

Theorem 11.7.2. Let a function $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and $f(x) \geq 0$ for all $x \in [a,b]$. Then $\int_a^b f \geq 0$.

Proof. Since f is integrable on $[a,b]$, f is bounded on $[a,b]$.

Let $P = (x_0, x_1, x_2, \ldots, x_n)$ be a partition of $[a,b]$.

Let m_r be the infimum of f on $[x_{r-1}, x_r]$, for $r = 1, 2, \ldots, n$.

Then $m_r \geq 0$, for $r = 1, 2, \ldots, n$.

$L(P,f) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1}) \geq 0$.

Therefore $\underline{\int_a^b} f = \sup\{L(P,f) : P \in \mathcal{P}[a,b]\} \geq 0$.

Since f is integrable on $[a,b]$, $\int_a^b f = \underline{\int_a^b} f$ and therefore $\int_a^b f \geq 0$.

Theorem 11.7.3. Let $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ be both integrable on $[a,b]$ and $f(x) \geq g(x)$ for all $x \in [a,b]$. Then $\int_a^b f \geq \int_a^b g$.

Proof. Let $\phi : [a,b] \to \mathbb{R}$ be defined by $\phi(x) = f(x) - g(x), x \in [a,b]$.

Since f and g are both integrable on $[a,b]$, ϕ is integrable on $[a,b]$ and $\int_a^b \phi = \int_a^b f - \int_a^b g$.

But $\int_a^b \phi \geq 0$ by the previous theorem. Therefore $\int_a^b f \geq \int_a^b g$.

Corollary. Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and $M = \sup\limits_{x \in [a,b]} f(x)$, $m = \inf\limits_{x \in [a,b]} f(x)$. Then $m(b-a) \leq \int_a^b f \leq M(b-a)$.

Proof. Let $\phi(x) = M$ and $\psi(x) = m, x \in [a,b]$. Then ϕ and ψ are both integrable on $[a,b]$ and $\int_a^b \phi = M(b-a)$ $\int_a^b \psi = m(b-a)$ [Ex.1, 11.2.]

Since $\psi(x) \leq f(x) \leq \phi(x)$ on $[a,b]$, $\int_a^b \psi \leq \int_a^b f \leq \int_a^b \phi$.

Theorem 11.7.4. Let a function $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and $f(x) \geq 0$ for all $x \in [a,b]$. Let there exist a point c in $[a,b]$ such that f is continuous at c and $f(c) > 0$, then $\int_a^b f > 0$.

Proof. **Case 1.** Let $a < c < b$.

Let $\epsilon = \frac{1}{2}f(c) > 0$. Since f is continuous at c, there exists a positive δ such that $f(c) - \epsilon < f(x) < f(c) + \epsilon$ for all $x \in [c-\delta, c+\delta] \subset [a,b]$.

$\int_a^b f = \int_a^{c-\delta} f + \int_{c-\delta}^{c+\delta} f + \int_{c+\delta}^b f$.

Since $f(x) \geq 0$ for all $x \in [a, c-\delta]$, $\int_a^{c-\delta} f \geq 0$.

Since $f(x) \geq 0$ for all $x \in [c+\delta, b]$, $\int_{c+\delta}^b f \geq 0$.

Since $f(x) > \frac{1}{2}f(c)$ for all $x \in [c-\delta, c+\delta]$, $\int_{c-\delta}^{c+\delta} f \geq \frac{1}{2}f(c).2\delta > 0$.

Consequently, $\int_a^b f > 0$.

Case 2. Let $c = a$.

Let $\epsilon = \frac{1}{2}f(a) > 0$. As f is continuous at a, there exists a positive δ such that $f(a) - \epsilon < f(x) < f(a) + \epsilon$ for all $x \in [a, a+\delta] \subset [a, b]$.

$\int_a^b f = \int_a^{a+\delta} f + \int_{a+\delta}^b f$.

Since $f(x) \geq 0$ for all $x \in [a+\delta, b]$, $\int_{a+\delta}^b f \geq 0$.

Since $f(x) > \frac{1}{2}f(a)$ for all $x \in [a, a+\delta]$, $\int_a^{a+\delta} f \geq \frac{1}{2}f(a).\delta > 0$.

Consequently, $\int_a^b f > 0$.

Case 3. Let $c = b$.

Proceeding as in case 2, $\int_a^b f > 0$. This completes the proof.

Note 1. If f is continuous on $[a, b]$ and $f(x) > 0$ on $[a, b]$ then $\int_a^b f > 0$.

Note 2. If f is integrable on $[a, b]$ and $f(x) > 0$ on $[a, b]$ then also $\int_a^b f > 0$, because there exists at least a point of continuity $c \in [a, b]$ of f. [Remark after Example 5, 11.5.].

Theorem 11.7.5. Let the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be both integrable on $[a, b]$ and $f(x) \geq g(x)$ for all $x \in [a, b]$. Let there exist a point c in $[a, b]$ such that f and g are both continuous at c and $f(c) > g(c)$. Then $\int_a^b f > \int_a^b g$.

Proof. Let $\phi : [a, b] \to \mathbb{R}$ be defined by $\phi(x) = f(x) - g(x), x \in [a, b]$. Then $\phi(= f - g)$ is integrable on $[a, b]$ and $\phi(x) \geq 0$ for all $x \in [a, b]$. Also ϕ is continuous at c and $\phi(c) > 0$.

Therefore by the previous theorem, $\int_a^b \phi > 0$.

But $\int_a^b \phi = \int_a^b (f - g) = \int_a^b f - \int_a^b g$. Therefore $\int_a^b f > \int_a^b g$.

This completes the proof.

Theorem 11.7.6. Let a function $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Then

$$\int_a^b | f | \geq | \int_a^b f |.$$

Proof. If $a \in \mathbb{R}$, then $- | a | \leq a \leq | a |$ ... ... (i)

If $a, b \in \mathbb{R}$, then $| a | \leq b \Leftrightarrow -b \leq a \leq b$ ... ... (ii)

For all $x \in [a, b]$, $- | f(x) | \leq f(x) \leq | f(x) |$, by (i)

That is, $- | f | (x) \leq f(x) \leq | f | (x)$, for all $x \in [a, b]$.

$f \in \mathcal{R}[a, b] \Rightarrow | f | \in \mathcal{R}[a, b]$. Therefore $- \int_a^b | f | \leq \int_a^b f \leq \int_a^b | f |$.

This implies $| \int_a^b f | \leq \int_a^b | f |$, by (ii). This completes the proof.

Worked Examples.

1. Prove that $\frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx < \frac{2\pi^2}{9}$.

$1 \leq \frac{1}{\sin x} \leq 2$ for all $x \in [\frac{\pi}{6}, \frac{\pi}{2}]$. Therefore $x \leq \frac{x}{\sin x} \leq 2x$ for all $x \in [\frac{\pi}{6}, \frac{\pi}{2}]$.

Let $f(x) = \frac{x}{\sin x}$, $\phi(x) = x$, $\psi(x) = 2x, x \in [\frac{\pi}{6}, \frac{\pi}{2}]$.

f and ϕ are both bounded and integrable on $[\frac{\pi}{6}, \frac{\pi}{2}]$ and $f(x) \geq \phi(x)$ for all $x \in [\frac{\pi}{6}, \frac{\pi}{2}]$. Also f and ϕ are both continuous at $\frac{\pi}{3}$ and $f(\frac{\pi}{3}) > \phi(\frac{\pi}{3})$.

Hence $\int_{\pi/6}^{\pi/2} f(x)dx > \int_{\pi/6}^{\pi/2} \phi(x)dx = \int_{\pi/6}^{\pi/2} xdx = \frac{\pi^2}{9}$.

f and ψ are both bounded and integrable on $[\frac{\pi}{6}, \frac{\pi}{2}]$ and $f(x) \leq \psi(x)$ for all $x \in [\frac{\pi}{6}, \frac{\pi}{2}]$. Also f and ψ are both continuous at $\frac{\pi}{3}$ and $f(\frac{\pi}{3}) < \psi(\frac{\pi}{3})$.

Hence $\int_{\pi/6}^{\pi/2} f(x)dx < \int_{\pi/6}^{\pi/2} \psi(x)dx = 2\int_{\pi/6}^{\pi/2} xdx = \frac{2\pi^2}{9}$.

Consequently, $\frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx < \frac{2\pi^2}{9}$.

2. If a function f is continuous on $[a, b]$, $f(x) \geq 0$ on $[a, b]$ and $\int_a^b f = 0$, prove that $f = 0$ on $[a, b]$ identically.

If f is not identically zero on $[a, b]$ then there exists a point $c \in [a, b]$ such that $f(c) > 0$.

Since f is continuous on $[a, b]$, f is integrable on $[a, b]$.

Since $f(x) \geq 0$ on $[a, b]$ and f is integrable on $[a, b]$, $\int_a^b f \geq 0$.

Since f is continuous at $c \in [a, b]$ and $f(c) > 0$, $\int_a^b f > 0$.

But by hypothesis, $\int_a^b f = 0$. Therefore there exists no point c in $[a, b]$ such that $f(c) > 0$. So f is identically zero on $[a, b]$.

3 A function g is continuous on $[a, b]$ and $g(x) = \int_a^x g(t)dt$. Prove that $g(x) = 0$ for all $x \in [a, b]$.

Since g is continuous on $[a, b], g$ is bounded on $[a, b]$. Therefore there exists a positive real number B such that $| g(x) | \leq B$ for all $x \in [a, b]$ ... ...(i)

$$| g(x) | = | \int_a^x g(t)dt | \leq \int_a^x | g(t) | dt \quad \ldots \quad \ldots\text{(ii)}$$

Since $| g(t) | \leq B$ for all $t \in [a, b]$, it follows from (ii) that $| g(x) | \leq B(x - a)$ for all $x \in [a, b]$ ... ...(iii)

Using (ii) and (iii), $| g(x) | \leq B\frac{(x-a)^2}{2!}$ for all $x \in [a, b]$ ...(iv)

$$\text{For } 0 < x \leq 1,\ F(x) = \int_{-1}^{x} f(t)dt = \int_{-1}^{0} f(t)dt + \int_{0}^{x} f(t)dt.$$
$$= 0 + \int_{0}^{x} 1dt = x.$$

$$\text{We have } F(x) = 0, -1 \leq x \leq 0$$
$$= x, 0 < x \leq 1.$$

Clearly, F is continuous on $[-1, 1]$.

Note. Although f is not continuous on $[-1, 1]$, F is continuous on $[-1, 1]$.

We observe that the function F is continuous on $[a, b]$ when f is integrable on $[a, b]$.

If, however, f be continuous on $[a, b]$ then F will be differentiable on $[a, b]$ as we shall see in the next theorem.

Theorem 11.8.2. If a function $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, then the function F defined by $F(x) = \int_a^x f(t)dt, x \in [a, b]$ is differentiable at any point $c \in [a, b]$ at which f is continuous and $F'(c) = f(c)$.

Proof. Let $c \in [a, b)$. Let $\epsilon > 0$. Since f is continuous at c there exists a positive δ such that $| f(x) - f(c) | < \epsilon$ for all $x \in [c, c + \delta)$.

Let us choose h satisfying $0 < h < \delta$. Then $f(c) - \epsilon < f(x) < f(c) + \epsilon$ for all $x \in [c, c + h]$.

Therefore $\int_c^{c+h} f(c) - \epsilon \leq \int_c^{c+h} f \leq \int_c^{c+h} f(c) + \epsilon$

or, $[f(c) - \epsilon].h \leq F(c + h) - F(c) \leq [f(c) + \epsilon].h$

or, $|\frac{F(c+h)-F(c)}{h} - f(c)| < \epsilon$. This holds for all h satisfying $0 < h < \delta$.

This implies $\lim\limits_{h \to 0+} \dfrac{F(c + h) - F(c)}{h} = f(c)$.

That is, $RF'(c) = f(c)$ ... ... (i)

Let $c \in (a, b]$. Let $\epsilon > 0$. Since f is continuous at c there exists a positive η such that $| f(x) - f(c) | < \epsilon$ for all $x \in (c - \eta, c]$.

Let us choose h satisfying $0 < h < \eta$. Then $f(c) - \epsilon < f(x) < f(c) + \epsilon$ for all $x \in [c - h, c]$.

Therefore $\int_{c-h}^{c} f(c) - \epsilon \leq \int_{c-h}^{c} f \leq \int_{c-h}^{c} f(c) + \epsilon$

or, $[f(c) - \epsilon].h \leq F(c) - F(c - h) \leq [f(c) + \epsilon].h$

or, $|\frac{F(c-h)-F(c)}{-h} - f(c)| < \epsilon$. This holds for all h satisfying $0 < h < \delta$.

This implies $\lim\limits_{h \to 0-} \dfrac{F(c + h) - F(c)}{h} = f(c)$.

That is, $LF'(c) = f(c)$ ... ... (ii)

From (i) and (ii) it follows that f is differentiable at any point $c \in [a, b]$ at which f is continuous and $F'(c) = f(c)$.

Using (ii) and (iv), $\mid g(x) \mid \leq B\frac{(x-a)^3}{3!}$ for all $x \in [a,b]$.

Continuing thus we have $\mid g(x) \mid \leq B\frac{(x-a)^n}{n!}$ for all $x \in [a,b]$ and for all $n \in \mathbb{N}$. Since $g(a) = 0, \mid g(x) \mid \geq 0$ for all $x \in [a,b]$.

Therefore $0 \leq \mid g(x) \mid \leq B\frac{(x-a)^n}{n!}$ for all $x \in [a,b]$ and for all $n \in \mathbb{N}$.

Since $\lim\limits_{n\to\infty} \frac{(x-a)^n}{n!} = 0$, it follows that $g(x) = 0$ for all $x \in [a,b]$.

11.8. Fundamental Theorem.

Let a function $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Then for each $x \in [a,b]$, f is integrable on $[a,x]$. $\int_a^x f(t)dt$ exists and it depends on x. Therefore we can define a function F on $[a,b]$ by

$$F(x) = \int_a^x f(t)dt.$$

Theorem 11.8.1. If $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ then the function F defined by $F(x) = \int_a^x f(t)dt, x \in [a,b]$ is continuous on $[a,b]$.

Proof. Let x_1, x_2 be any two points in $[a,b]$.
$F(x_2) - F(x_1) = \int_a^{x_2} f(t)dt - \int_a^{x_1} f(t)dt = \int_{x_1}^{x_2} f(t)dt.$
Therefore $\mid F(x_2) - F(x_1) \mid = \mid \int_{x_1}^{x_2} f(t)dt \mid$.

Since f is integrable on $[a,b]$, f is bounded on $[a,b]$. Therefore there exists a real number $k > 0$ such that $\mid f(x) \mid < k$ for all $x \in [a,b]$.

If $x_2 > x_1$, $\mid \int_{x_1}^{x_2} f(t)dt \mid \leq \int_{x_1}^{x_2} \mid f(t) \mid dt \leq (x_2 - x_1)k$.

If $x_1 > x_2$, $\mid \int_{x_1}^{x_2} f(t)dt \mid = \mid \int_{x_2}^{x_1} f(t)dt \mid \leq \int_{x_2}^{x_1} \mid f(t) \mid dt \leq (x_1 - x_2)k$.
Consequently, $\mid F(x_2) - F(x_1) \mid \leq \mid x_2 - x_1 \mid k$.

Let us choose $\epsilon > 0$. Then $\mid F(x_2) - F(x_1) \mid < \epsilon$ for all x_1, x_2 in $[a,b]$ satisfying $\mid x_2 - x_1 \mid < \frac{\epsilon}{k}$.
Let $\delta = \epsilon/k$. Then $\mid F(x_2) - F(x_1) \mid < \epsilon$ for all x_1, x_2 in $[a,b]$ satisfying $\mid x_2 - x_1 \mid < \delta$.

This proves that F is uniformly continuous on $[a,b]$ and therefore F is continuous on $[a,b]$.

Worked Example.

1. Let $f(x) = 0, \; -1 \leq x \leq 0$
$\qquad\;\; = 1, \; 0 < x \leq 1.$

Prove that f is integrable on $[-1,1]$. Show that the function F defined by $F(x) = \int_{-1}^x f(t)dt$ is continuous on $[-1,1]$.

f is bounded on $[-1,1]$ and is continuous on $[-1,1]$ except at only one point, 0. Therefore f is integrable on $[-1,1]$.
For $-1 \leq x \leq 0$, $F(x) = \int_{-1}^x f(t)dt = 0$.

Corollary. If $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ then F is differentiable on $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$.

An alternative proof for $F'(x) = f(x)$ on $[a, b]$, assuming continuity of f on $[a, b]$.

Theorem 11.8.3. If a function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ then the function F defined by $F(x) = \int_a^x f(t)dt, x \in [a, b]$ is differentiable on $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$.

Proof. **Case 1.** Let $c \in (a, b)$.

Let us choose h such that $c + h \in [a, b]$. Then

$F(c + h) - F(c) = \int_c^{c+h} f(t)dt.$

Let $h > 0$. Since f is continuous on $[c, c+h]$, f is bounded on $[c, c+h]$.

Let $M = \sup\limits_{t\in[c,c+h]} f(t)$, $m = \inf\limits_{t\in[c,c+h]} f(t)$.

Then $m \leq f(t) \leq M$ for all $t \in [c, c + h]$.

Therefore $mh \leq \int_c^{c+h} f(t)dt \leq Mh$

or, $\int_c^{c+h} f(t)dt = \mu h$ where $m \leq \mu \leq M$.

Since f is continuous on $[c, c + h]$, $\mu = f(c + \theta h)$ for some θ satisfying $0 \leq \theta \leq 1$. Then $\frac{F(c+h)-F(c)}{h} = f(c + \theta h)$.

Since f is continuous at c, $\lim\limits_{h\to 0+} f(c + \theta h) = f(c)$.

$$\text{Therefore we have } \lim_{h\to 0+} \frac{F(c+h) - F(c)}{h} = f(c) \qquad \text{(i)}$$

Let $h < 0$. Considering the interval $[c + h, c]$, we have

$-mh \leq \int_{c+h}^{c} f(t)dt \leq -Mh$ where $M = \sup\limits_{t\in[c+h,c]} f(t)$, $m = \inf\limits_{t\in[c+h,c]} f(t)$

or, $\frac{F(c+h)-F(c)}{h} = \mu$, where $m \leq \mu \leq M$.

Since f is continuous on $[c + h, c]$, $\mu = f(c + \theta h)$ for some θ satisfying $0 \leq \theta \leq 1$.

Taking limit as $h \to 0-$ and noting that $\lim\limits_{h\to 0-} f(c + \theta h) = f(c)$, we have

$$\lim_{h\to 0-} \frac{F(c+h) - F(c)}{h} = f(c) \quad \cdots \quad \text{(ii)}$$

From (i) and (ii) we have $F'(c) = f(c)$.

Case 2. Let $c = a$.

Let us choose h such that $a + h < b$. Then

$F(a + h) - F(a) = \int_a^{a+h} f(t)dt.$

Considering the interval $[a, a + h]$, we have

$mh \leq \int_a^{a+h} f(t)dt \leq Mh$ where $M = \sup\limits_{t\in[a,a+h]} f(t)$, $m = \inf\limits_{t\in[a,a+h]} f(t)$

or, $\frac{F(a+h)-F(a)}{h} = \mu$, where $m \leq \mu \leq M$.

Since f is continuous on $[a, a+h]$, $\mu = f(a+\theta h)$ for some θ satisfying $0 \leq \theta \leq 1$.

Taking limit as $h \to 0+$ and noting that $\lim_{h\to 0+} f(a+\theta h) = f(a)$, we have $\lim_{h\to 0+} \dfrac{F(a+h)-F(a)}{h} = f(a)$.

or, $F'(a) = f(a)$.

Case 3. Let $c = b$. Similar proof.

This completes the proof.

Definition. A function ϕ is called an *antiderivative* or a *primitive* of a function f on an interval I, if $\phi'(x) = f(x)$ for all $x \in I$.

If ϕ be an antiderivative of f on I, then $\phi + c$, where $c \in \mathbb{R}$, is obviously an antiderivative of f on I. This shows that if f admits of an antiderivative on I, then there exist many antiderivatives of f on I.

It follows from the previous theorem that if f be continuous on a closed interval $[a, b]$, then f possesses an antiderivative on $[a, b]$ given by F. Therefore continuity of f ensures the existence of an antiderivative of f.

It is worthwhile to note that continuity of f is not a necessary condition for the existence of an antiderivative of f.

For example, let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$\begin{aligned} f(x) &= 2x\sin\tfrac{1}{x} - \cos\tfrac{1}{x}, x \neq 0 \\ &= 0, \quad x = 0. \end{aligned}$$

f is not continuous on $[-1, 1]$, 0 being the point of discontinuity.

Let $\phi : [-1, 1] \to \mathbb{R}$ be defined by

$$\begin{aligned} \phi(x) &= x^2\sin\tfrac{1}{x}, x \neq 0 \\ &= 0, \quad x = 0. \end{aligned}$$

Then $\phi'(x) = f(x)$ for all $x \in [-1, 1]$.

Thus ϕ is an antiderivative of f on $[-1, 1]$ although f is not continuous on $[-1, 1]$.

Worked Example (continued).

2. Let $f(x) = 1, 0 \leq x \leq 1$
$\qquad\qquad = x, 1 < x \leq 2.$

Verify that the function F defined by $F(x) = \int_0^x f(t)dt, x \in [0, 2]$ is differentiable on $[0, 2]$ and $F'(x) = f(x), x \in [0, 2]$.

f is continuous on $[0, 2]$ and therefore f is integrable on $[0, 2]$.Hence $\int_0^x f(t)dt$ exists for all $x \in [0, 2]$.

For $0 \leq x \leq 1$, $F(x) = \int_0^x f(t)dt = \int_0^x dt = x$.

For $1 < x \le 2$, $F(x) = \int_0^x f(t)dt = \int_0^1 f(t)dt + \int_1^x f(t)dt = 1 + \int_1^x tdt = \frac{1}{2}(x^2+1)$.

We have $F(x) = x, 0 \le x \le 1$
$= \frac{1}{2}(x^2+1), 1 < x \le 2.$

$\lim_{h\to 0-} \frac{F(1+h)-F(1)}{h} = 1$, $\lim_{h\to 0+} \frac{F(1+h)-F(1)}{h} = 1$. Therefore $F'(1) = 1$.

So we have $F'(x) = 1, 0 \le x \le 1$
$= x, 1 < x \le 2.$

Therefore $F'(x) = f(x)$ for all $x \in [0,2]$.

Theorem 11.8.4. If $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and $\phi : [a,b] \to \mathbb{R}$ be an antiderivative of f on $[a,b]$, then

$$\int_a^b f = \phi(b) - \phi(a).$$

Proof. Since f is continuous on $[a,b]$, f is integrable on $[a,b]$.

Let $F(x) = \int_a^x f(t)dt, x \in [a,b]$.

Since f is continuous on $[a,b]$, F is differentiable on $[a,b]$ and $F'(x) = f(x)$ for all $x \in [a,b]$. So F is an antiderivative of f on $[a,b]$.

Since ϕ is an antiderivative of f on $[a,b]$, for all $x \in [a,b], \phi(x) = F(x) + c$, where c is a constant.

So $\phi(a) = F(a) + c = c$, since $F(a) = 0$.

Therefore $\phi(x) = F(x) + \phi(a)$, for all $x \in [a,b]$.

Consequently, $\int_a^b f = F(b) = \phi(b) - \phi(a)$.

Note. The theorem states that if f be *continuous* on $[a,b]$ then the integral $\int_a^b f$ can be evaluated in terms of an antiderivative of f on $[a,b]$.

Theorem 11.8.5. (Fundamental theorem of Integral calculus)

If (i) $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, and
(ii) f possesses an antiderivative ϕ on $[a,b]$, then

$$\int_a^b f = \phi(b) - \phi(a).$$

Proof. Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$.

Let $M_r = \sup_{x\in[x_{r-1},x_r]} f(x)$, $m_r = \inf_{x\in[x_{r-1},x_r]} f(x)$, for $r = 1,2,\ldots,n$.

Since $\phi'(x) = f(x)$ for all $x \in [a,b]$, ϕ satisfies all conditions of Lagrange's Mean value theorem on $[x_{r-1}, x_r]$, for $r = 1,2,\ldots,n$.

Therefore for $r = 1,2,\ldots,n$,

$$\begin{aligned} \phi(x_r) - \phi(x_{r-1}) &= \phi'(\xi_r)(x_r - x_{r-1}) \text{ for some } \xi_r \text{ in } (x_{r-1}, x_r) \\ &= f(\xi_r)(x_r - x_{r-1}). \end{aligned}$$

The summation gives $\sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1}) = \phi(b) - \phi(a)$.
But $m_r \le f(\xi_r) \le M_r$ for $r = 1, 2, \ldots, n$.
Therefore $\sum_{r=1}^{n} m_r(x_r - x_{r-1}) \le \phi(b) - \phi(a) \le \sum_{r=1}^{n} M_r(x_r - x_{r-1})$.

Therefore $L(P, f) \le \phi(b) - \phi(a) \le U(P, f)$.
This holds for all partitions P of $[a, b]$.
So $\phi(b) - \phi(a)$ is an upper bound of the set $\{L(P, f) : P \in \mathcal{P}[a, b]\}$.

As the supremum of the set is $\underline{\int}_a^b f$, it follows that

$$\underline{\int}_a^b f \le \phi(b) - \phi(a) \quad \ldots \ldots \text{(i)}$$

Also $\phi(b) - \phi(a)$ is a lower bound of the set $\{U(P, f) : P \in \mathcal{P}[a, b]\}$.

As the infimum of the set is $\overline{\int}_a^b f$, it follows that

$$\overline{\int}_a^b f \ge \phi(b) - \phi(a) \quad \ldots \ldots \text{(ii)}$$

From (i) and (ii) $\underline{\int}_a^b f \le \phi(b) - \phi(a) \le \overline{\int}_a^b f$.

Since f is integrable on $[a, b]$, $\overline{\int}_a^b f = \underline{\int}_a^b f = \int_a^b f$.

Consequently, $\int_a^b f = \phi(b) - \phi(a)$.

Corollary. Since $\phi(b) - \phi(a) = (b - a)f(\xi)$ for some $\xi \in (a, b)$, $\int_a^b f = (b - a)f(a + \theta(b - a))$ for some θ satisfying $0 < \theta < 1$.

Note. The evaluation of the integral $\int_a^b f$ in terms of the antiderivative of f is possible if f satisfies the conditions of the theorem. That these conditions are independent of each other can be seen from the following two examples.

(i) Let $f : [-1, 1] \to \mathbb{R}$ be defined by
$$\begin{aligned} f(x) &= 0, \ -1 \le x < 0 \\ &= 1, \ 0 \le x \le 1. \end{aligned}$$

f is bounded on $[-1, 1]$ and continuous on $[-1, 1]$ except at only one point, 0. Therefore f is integrable on $[-1, 1]$.

If possible, let g be an antiderivative of f on $[-1, 1]$.

$$\begin{aligned} \text{Then } g'(x) &= 0, \ -1 \le x < 0 \\ &= 1, \ 0 \le x \le 1. \end{aligned}$$

g is differentiable on $[-1, 1]$ and $g'(-1) \ne g'(1)$. By Darboux's theorem, g' must assume every real number lying between $g'(-1)$ and $g'(1)$, i.e., between 0 and 1. But it does not do so.

It follows that g does not exist. That is, f has no antiderivative on $[-1, 1]$.

Therefore f has no antiderivaitive on $[-1, 1]$ although f is integrable on $[-1, 1]$.

(ii) Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$\begin{aligned} f(x) &= 2x \sin \tfrac{1}{x^2} - \tfrac{2}{x} \cos \tfrac{1}{x^2}, x \neq 0 \\ &= 0, x = 0. \end{aligned}$$

f is unbounded on every neighbourhood of 0. Hence f is not integrable on $[-1, 1]$.

Let $\phi : [-1, 1] \to \mathbb{R}$ be defined by $\begin{aligned} \phi(x) &= x^2 \sin \tfrac{1}{x^2}, x \neq 0 \\ &= 0, x = 0. \end{aligned}$

Then $\phi'(x) = f(x)$ on $[-1, 1]$. So ϕ is an antiderivative of f on $[-1, 1]$.

Therefore f has an antiderivative on $[-1, 1]$ although f is not integrable on $[-1, 1]$.

Worked Examples (continued).

3. Let f be defined on $[-2, 2]$ by $\begin{aligned} f(x) &= 3x^2 \cos \tfrac{\pi}{x^2} + 2\pi \sin \tfrac{\pi}{x^2}, x \neq 0 \\ &= 0, \quad x = 0. \end{aligned}$

Show that f is integrable on $[-2, 2]$. Evaluate $\int_{-2}^{2} f$.

f is bounded on $[-2, 2]$. f is continuous on $[-2, 2]$ except at 0. Since f is continuous on $[-2, 2]$ except at only one point, f is integrable on $[-2, 2]$.

Let $\phi : [-2, 2] \to \mathbb{R}$ be defined by $\begin{aligned} \phi(x) &= x^3 \cos \tfrac{\pi}{x^2}, \quad x \neq 0. \\ &= 0, \quad x = 0. \end{aligned}$

Then $\phi'(x) = 3x^2 \cos \frac{\pi}{x^2} + 2\pi \sin \frac{\pi}{x^2}$, for all $x(\neq 0) \in [-2, 2]$.

$\lim\limits_{x \to 0} \dfrac{\phi(x) - \phi(0)}{x - 0} = \lim\limits_{x \to 0} x^2 \cos \dfrac{\pi}{x^2} = 0$. Therefore $\phi'(0) = 0$.

Hence ϕ is an antiderivative of f on $[-2, 2]$.

By the fundamental theorem,

$$\begin{aligned} \int_{-2}^{2} f(x)dx &= \phi(2) - \phi(-2) \\ &= 8 \cos \tfrac{\pi}{4} + 8 \cos \tfrac{\pi}{4} = 8\sqrt{2}. \end{aligned}$$

4. A function f be defined on $[0, 3]$ by $f(x) = [x], x \in [0, 3]$.

Show that f is integrable on $[0, 3]$ but $\int_0^3 f$ can not be evaluated by the fundamental theorem.

f is bounded on $[0, 3]$ and is continuous on $[0, 3]$, except at the points 1, 2, 3. f has jump discontinuity at 1, 2, 3.

Therefore f is integrable on $[0, 3]$. $\int_0^3 f(x)dx = 3$ [by worked Example 2, page 434]

Let g be an antiderivative of f on $[0, 3]$. Then $g' = f$ on $[0, 3]$ and g' must have jump discontinuity at 1, 2, 3. Since a derived function cannot have a jump discontinuity in its domain, g does not exist.

Since f has no antiderivative on $[0, 3]$, the fundamental theorem cannot be utilised to evaluate $\int_0^3 f$.

5. A function f is defined on $[0, 1]$ by

$$\begin{aligned} f(x) &= \sqrt{1 - x^2}, x \in [0, 1] \cap \mathbb{Q}. \\ &= 1 - x, x \in [0, 1] - \mathbb{Q}. \end{aligned}$$

Show that f is not integrable on $[0, 1]$.

f is bounded on $[0, 1]$. For all $x \in (0, 1)$, $\sqrt{1 - x^2} > 1 - x$.

Let $I = [0, 1]$. $f/(I \cap \mathbb{Q})$ is monotone decreasing on $I \cap \mathbb{Q}$, $f/1 - \mathbb{Q}$ is monotone decreasing on $I - \mathbb{Q}$.

Let us take a partition P_n of $[0, 1]$ defined by $P_n = (x_0, x_1, \ldots, x_n)$ where $x_r = \frac{r}{n}$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$

Since $f/(I \cap \mathbb{Q})$ is monotone decreasing on $[x_{r-1}, x_r] \cap \mathbb{Q}$,

$$\sup_{x \in [x_{r-1}, x_r] \cap \mathbb{Q}} f(x) = f(x_{r-1}) = \sqrt{1 - x_{r-1}{}^2} = \sqrt{1 - (\tfrac{r-1}{n})^2}.$$

Since $f/(I - \mathbb{Q})$ is monotone decreasing on $[x_{r-1}, x_r] - \mathbb{Q}$ and x_{r-1} is rational, $\sup\limits_{x \in [x_{r-1}, x_r] - \mathbb{Q}} f(x) = \lim\limits_{n \to \infty} f(u_n)$ where $\{u_n\}$ is a sequence of irrational points in $[x_{r-1}, x_r]$ converging to $x_{r-1} = 1 - x_{r-1} = 1 - \frac{r-1}{n}$.

Since $1 - \frac{r-1}{n} < \sqrt{1 - (\frac{r-1}{n})^2}$, $\sup\limits_{x \in [x_{r-1}, x_r]} f(x) = \sqrt{1 - (\frac{r-1}{n})^2}$.

That is, $M_r = \sqrt{1 - (\frac{r-1}{n})^2}$.

Since $f/(I \cap \mathbb{Q})$ is monotone decreasing on $[x_{r-1}, x_r] \cap \mathbb{Q}$,

$$\inf_{x \in [x_{r-1}, x_r] \cap Q} f(x) = f(x_r) = \sqrt{1 - (\tfrac{r}{n})^2}.$$

Since $f/(I - \mathbb{Q})$ is monotone decreasing on $[x_{r-1}, x_r] - \mathbb{Q}$ and x_r is rational, $\inf\limits_{x \in [x_{r-1}, x_r] - \mathbb{Q}} f(x) = \lim\limits_{n \to \infty} f(\nu_n)$ where $\{\nu_n\}$ is a sequence of irrational points in $[x_{r-1}, x_r]$ converging to $x_r = 1 - x_r = 1 - \frac{r}{n}$.

Since $1 - \frac{r}{n} \leq \sqrt{1 - (\frac{r}{n})^2}$, $\inf\limits_{x \in [x_{r-1}, x_r]} f(x) = 1 - \frac{r}{n}$.

That is, $m_r = 1 - \frac{r}{n}$.

$$\begin{aligned} U(P_n, f) &= M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1}) \\ &= \tfrac{1}{n}\{\sqrt{1 - (\tfrac{0}{n})^2} + \sqrt{1 - (\tfrac{1}{n})^2} + \cdots + \sqrt{1 - (\tfrac{n-1}{n})^2}\}. \end{aligned}$$

$$\begin{aligned} L(P_n, f) &= m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1}) \\ &= \tfrac{1}{n}\{(1 - \tfrac{1}{n}) + (1 - \tfrac{2}{n}) + \cdots + (1 - \tfrac{n}{n})\} \\ &= \tfrac{1}{n^2}[1 + 2 + \cdots + (n-1)] = \tfrac{n-1}{2n}. \end{aligned}$$

Let us consider the sequence of partitions $\{P_n\}$ of $[0,1]$.
Here $\|P_n\| = \frac{1}{n}$ and $\lim\limits_{n\to\infty} \|P_n\| = 0$.

$\overline{\int_0^1} f = \lim\limits_{n\to\infty} U(P_n, f) = \lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=0}^{n-1} \sqrt{1-(\frac{r}{n})^2}$
$= \int_0^1 \sqrt{1-x^2}dx$, since $\sqrt{1-x^2}$ is integarble on $[0,1]$
$= [\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x]_0^1$ [by the fundamental theorem]
$= \frac{\pi}{4}$;
and $\underline{\int_0^1} = \lim\limits_{n\to\infty} L(P_n, f) = \lim\limits_{n\to\infty} \frac{n-1}{2n} = \frac{1}{2}$.

Since $\overline{\int_0^1} f \neq \underline{\int_0^1} f$, f is not integrable on $[0,1]$.

6. Prove that $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \frac{\pi}{6}$.

In $0 \le x \le 1$, $4 - x^2 + x^3 = 4 - (x^2 - x^3) \le 4$;
$4 - x^2 + x^3 = (4 - x^2) + x^3 \ge 4 - x^2$.
Therefore $\frac{1}{2} \le \frac{1}{\sqrt{4-x^2+x^3}} \le \frac{1}{\sqrt{4-x^2}}$ in $0 \le x \le 1$.

Let $f(x) = \frac{1}{2}, x \in [0,1]$; $g(x) = \frac{1}{\sqrt{4-x^2+x^3}}, x \in [0,1]$.

f and g are both continuous on $[0,1]$ and therefore they are integrable on $[0,1]$. We have $f(x) \le g(x)$ on $[0,1]$.
Therefore $\int_0^1 f(x)dx \le \int_0^1 g(x)dx$. Also $f(\frac{1}{2}) < g(\frac{1}{2})$.

Therefore $\int_0^1 f(x)dx < \int_0^1 g(x)dx$, by Theorem 11.7.5
or, $\int_0^1 \frac{1}{2}dx < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}}$.
or, $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}}$ ... ... (i)

Let $h(x) = \frac{1}{\sqrt{4-x^2}}, x \in [0,1]$.
Then h is continuous on $[0,1]$ and therefore h is integrable on $[0,1]$.
We have $g(x) \le h(x)$ on $[0,1]$.
Therefore $\int_0^1 g(x)dx \le \int_0^1 h(x)dx$. Also $g(\frac{1}{2}) < h(\frac{1}{2})$.

Therefore $\int_0^1 g(x)dx < \int_0^1 h(x)dx$, by Theorem 11.7.5.
or, $\int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \int_0^1 \frac{dx}{\sqrt{4-x^2}}$
$= [\sin^{-1}\frac{x}{2}]_0^1$, by the fundamental theorem
$= \frac{\pi}{6}$ ... ... (ii)

From (i) and (ii) $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \frac{\pi}{6}$.

7. Let $I = [a,b] \subset \mathbb{R}$ and let $f : I \to \mathbb{R}$ be continuous on I. Let $J = [c,d] \subset \mathbb{R}$ and let $u : J \to \mathbb{R}$ be differentiable on J and $u(J) \subset I$;

$v : J \to \mathbb{R}$ be differentiable on J and $v(J) \subset I$. If $g : J \to \mathbb{R}$ be defined by $g(x) = \int_{u(x)}^{v(x)} f(t)dt$ for $x \in J$, then prove that

$g'(x) = (f \circ v)(x).v'(x) - (f \circ u)(x).u'(x)$ for all $x \in J$.

Let $u(x) = w, v(x) = z$ for $x \in J$. Then $w \in I, z \in I$.
Let $H(z) = \int_a^z f(t)dt, z \in v(J)$, $G(w) = \int_a^w f(t)dt, w \in u(J)$.
Then $g(x) = H(z) - G(w)$ for $x \in J$.

Since f is continuous on I, f is continuous on $[a, z]$ for all $z \in v(J)$ and f is continuous on $[a, w]$ for all $w \in u(J)$.

Therefore $H'(z) = f(z)$ for all $z \in v(J), G'(w) = f(w)$ for all $w \in u(J)$.

Since u is differentiable on J, $u'(x)$, i.e., $\frac{dw}{dx}$ exists for all $x \in J$.
Since v is differentiable on J, $v'(x)$, i.e., $\frac{dz}{dx}$ exists for all $x \in J$.

For all $x \in J$, $g'(x) = \frac{d}{dx}H(z) - \frac{d}{dx}G(w)$
$= H'(z)\frac{dz}{dx} - G'(w)\frac{dw}{dx}$
$= f(v(x)).v'(x) - f(u(x)).u'(x)$
$= (f \circ v)(x).v'(x) - (f \circ u)(x).u'(x)$.

Note. If, in particular, $u(x) = a$ for all $x \in J$, then $g : J \to \mathbb{R}$ is defined by $g(x) = \int_a^{v(x)} f(t)dt$ for $x \in J$ and in that case

$g'(x) = (f \circ v)(x).v'(x)$ for all $x \in J$.

8. Find ϕ' where ϕ is defined on $[0, 1]$ by $\phi(x) = \int_{x^3}^{x^2} \frac{1}{\sqrt[3]{1+t^2}}dt, x \in [0, 1]$.

Let $u = x^3, x \in [0, 1]$; $v = x^2, x \in [0, 1]$; $f(t) = \frac{1}{\sqrt[3]{1+t^2}}, t \in \mathbb{R}$.
Then $\phi(x) = \int_0^v f(t)dt - \int_0^u f(t)dt$.
Let $H(v) = \int_0^v f(t)dt, v \in [0, 1]$. As $x \in [0, 1], v \in [0, 1]$.
Let $G(u) = \int_0^u f(t)dt, u \in [0, 1]$. As $x \in [0, 1], u \in [0, 1]$.

Since f is continuous on $[0, v]$ for all $v \in [0, 1], H'(v) = f(v)$ for all $v \in [0, 1]$.
Since f is continuous on $[0, u]$ for all $u \in [0, 1], G'(u) = f(u)$ for all $u \in [0, 1]$.
For all $x \in [0, 1]$, $\phi'(x) = H'(v)\frac{dv}{dx} - G'(u)\frac{du}{dx} = \frac{2x}{\sqrt[3]{1+x^4}} - \frac{3x^2}{\sqrt[3]{1+x^6}}$.

9. $\phi(x) = \int_{x^2}^{x^3} \frac{1}{(1+t^2)^3}dt, x \in [1, \infty)$. Find $\phi'(x)$.

Let $u = x^2, x \in [1, \infty)$; $v = x^3, x \in [1, \infty)$; $f(t) = \frac{1}{(1+t^2)^3}, t \in \mathbb{R}$.
Then $\phi(x) = \int_0^v f(t)dt - \int_0^u f(t)dt$.
Let $H(v) = \int_0^v f(t)dt, v \in [1, \infty)$. As $x \in [1, \infty), v \in [1, \infty)$.
Let $G(u) = \int_0^u f(t)dt, u \in [1, \infty)$. As $x \in [1, \infty), u \in [1, \infty)$.

Since f is continuous on $[0, v]$ for all $v \in [1, \infty)$, $H'(v) = f(v)$ for all $v \in [1, \infty)$.

Since f is continuous on $[0, u]$ for all $u \in [1, \infty)$, $G'(u) = f(u)$ for all $u \in [1, \infty)$.

For all $x \in [1, \infty)$, $\phi'(x) = H'(v)\frac{dv}{dx} - G'(u)\frac{du}{dx} = \frac{3x^2}{(1+x^6)^3} - \frac{2x}{(1+x^4)^3}$.

10. Evaluate $\lim\limits_{x \to 0} \dfrac{\int_0^{x^2} e^{\sqrt{1+t}} dt}{x^2}$.

Let $y = \int_0^{x^2} e^{\sqrt{1+t}} dt$;
$u = x^2$; $F(x) = \int_0^x e^{\sqrt{1+t}} dt$; $f(t) = e^{\sqrt{1+t}}, t \in \mathbb{R}$.
Then $y = F(u)$, $\frac{dy}{dx} = F'(u)\frac{du}{dx} = 2xF'(u)$.

Since f is continuous on $[0, x]$, $F'(x) = f(x) = e^{\sqrt{1+x}}$.
So $\frac{dy}{dx} = 2xe^{\sqrt{1+x^2}}$.

$$\lim_{x \to 0} \frac{\int_0^{x^2} e^{\sqrt{1+t}} dt}{x^2} \text{ (form } \tfrac{0}{0}) = \lim_{x \to 0} \frac{2xe^{\sqrt{1+x^2}}}{2x} = \lim_{x \to 0} e^{\sqrt{1+x^2}} = e.$$

11. A function f is continuous on $\mathbb{R}$ and $\int_{-x}^{x} f(t)dt = 2\int_0^x f(t)dt$ for all $x \in \mathbb{R}$. Prove that f is an even function on $\mathbb{R}$.

$2\int_0^x f(t)dt = \int_{-x}^{x} f(t)dt = \int_{-x}^{0} f(t)dt + \int_0^x f(t)dt$
or, $\int_0^x f(t)dt = \int_{-x}^{0} f(t)dt = -\int_0^{-x} f(t)dt$. ... (i)

Let $F(x) = \int_0^x f(t)dt, x \in \mathbb{R}$.
Since f is continuous on $\mathbb{R}$, $F'(x) = f(x)$ for all $x \in \mathbb{R}$. ... (ii)

From (i) $F(x) = -\int_0^{-x} f(t)dt$
$= -\int_0^{u} f(t)dt$, where $u = -x$
$= -F(u) = -F(-x)$.

Therefore $F'(x) = -F'(-x).(-1) = F'(-x)$.

From (ii) $f(x) = f(-x)$ for all $x \in \mathbb{R}$, i.e., f is an even function on $\mathbb{R}$.

As an extension of the fundamenlal theorem we come to the following theorems with stricter conditions.

Theorem 11.8.6. If (i) $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, and

(ii) there exists a function $\phi : [a, b] \to \mathbb{R}$ such that ϕ is continuous on $[a, b]$ and $\phi'(x) = f(x)$ for all $x \in (a, b)$, then

$$\int_a^b f = \phi(b) - \phi(a).$$

Proof. Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < \cdots < x_n = b$, be a partition of $[a, b]$.

Let $M_r = \sup_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

ϕ satisfies all conditions of Lagrange's Mean value theorem on $[x_{r-1}, x_r]$, for $r = 1, 2, \ldots, n$.

$$\begin{aligned} \phi(x_r) - \phi(x_{r-1}) &= \phi'(\xi_r)(x_r - x_{r-1}) \text{ for some } \xi_r \text{ in } (x_{r-1}, x_r) \\ &= f(\xi_r)(x_r - x_{r-1}). \end{aligned}$$

Therefore $\sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1}) = \phi(b) - \phi(a)$.

But $m_r \leq f(\xi_r) \leq M_r$ for $r = 1, 2, \ldots, n$.

Therefore $\sum_{r=1}^{n} m_r(x_r - x_{r-1}) \leq \phi(b) - \phi(a) \leq \sum_{r=1}^{n} M_r(x_r - x_{r-1})$.

This gives $L(P, f) \leq \phi(b) - \phi(a) \leq U(P, f)$.

This holds for all partitions P of $[a, b]$.

Proceeding with the same arguments as in the fundamental theorem (Theorem 11.8.5), we have $\int_a^b f = \phi(b) - \phi(a)$.

Worked Example (continued).

12. A function f is defined on $[-2, 1]$ by $f(x) = \text{sgn}\, x$ and $\phi(x) = |x|$. Show that $\int_{-2}^{1} f(x)dx = \phi(1) - \phi(-2)$ although $\phi'(x) \neq f(x)$ on $[-2, 1]$.

ϕ is continuous on $[0, 1]$ and $\phi'(x) = \text{sgn}\, x$ on $(0, 1)$.

Therefore $\int_0^1 f(x)dx = \phi(1) - \phi(0)$, by the theorem
$= \phi(1)$, since $\phi(0) = 0$.

ϕ is continuous on $[-2, 0]$ and $\phi'(x) = \text{sgn}\, x$ on $(-2, 0)$.

Therefore $\int_{-2}^{0} f(x)dx = \phi(0) - \phi(-2)$, by the theorem
$= -\phi(-2)$, since $\phi(0) = 0$.

Since f is integrable on $[-2, 1]$, $\int_{-2}^{1} f(x)dx = \int_{-2}^{0} f(x)dx + \int_0^1 f(x)dx$ $= \phi(1) - \phi(-2)$.

Theorem 11.8.7. If (i) $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, and (ii) there exists a function $\phi : [a, b] \to \mathbb{R}$ such that ϕ is continuous on $[a, b]$ and $\phi'(x) = f(x)$ for all $x \in [a, b] - E$, where E is a finite set $\subset [a, b]$, then

$$\int_a^b f = \phi(b) - \phi(a).$$

Proof. Let $\phi'(x) \neq f(x)$ at the points $x_1, x_2, \ldots, x_m$.

Case 1. Let $a < x_1 < x_2 < \cdots < x_m < b$.

$\phi'(x) = f(x)$ on the open intervals $(a, x_1), (x_1, x_2), \ldots, (x_m, b)$.

Proceeding as in the proof of the previous theorem, we have $\int_{x_{r-1}}^{x_r} f = \phi(x_r) - \phi(x_{r-1})$ for $r = 1, 2, \ldots, m+1$, where $a = x_0, b = x_{m+1}$.

Since f is integrable on $[a,b]$, $\int_a^b f = \int_a^{x_1} f + \int_{x_1}^{x_2} f + \cdots + \int_{x_m}^b f$
$= \phi(b) - \phi(a)$.

Case 2. If $a = x_1$ then we consider the open intervals excluding (a, x_1) and if $b = x_m$ then we consider the open intervals excluding (x_m, b) and proceed along similar lines to obtain the result.

Worked Example (continued).

13. Let f be defined on $[0,3]$ by $f(x) = x$, if $1 < x \le 3$
$= -x$, if $0 \le x \le 1$

and ϕ be defined on $[0,3]$ by $\phi(x) = \frac{1}{2} \mid x^2 - 1 \mid$. Show that $\int_0^3 f = \phi(3) - \phi(0)$.

f is integrable on $[0,3]$. ϕ is continuous on $[0,3]$ and $\phi'(x) = f(x)$ for all $x \in [0,3] - \{1\}$.

Therefore $\int_0^3 f(x)dx = \phi(3) - \phi(0)$, by the theorem.

11.9. Another definition of integrability.

Let $f : [a,b] \to \mathbb{R}$ and let $P = (x_0, x_1, x_2, \ldots, x_n)$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a partition of $[a,b]$ and $\xi_1, \xi_2, \ldots, \xi_n$ are arbitrarily chosen points such that $x_{r-1} \le \xi_r \le x_r$, for $r = 1, 2, \ldots, n$. Then the sum

$$f(\xi_1)(x_1 - x_0) + f(\xi_2)(x_2 - x_1) + \cdots + f(\xi_n)(x_n - x_{n-1})$$

is called a *Riemann sum* for f corresponding to the partition P and the chosen intermediate points ξ_r. This is denoted by $S(P, f, \xi)$, or by $S(P, f)$.

Definition.

A function $f : [a,b] \to \mathbb{R}$ is said to be *Riemann integrable* on $[a,b]$ if there exists a real number B such that for each $\epsilon > 0$ there exists a positive $\delta(\epsilon)$ satisfying $\mid S(P,f) - B \mid < \epsilon$ for all partitions P of $[a,b]$ with $\|P\| < \delta$, where $S(P,f)$ is a Riemann sum for f corresponding to the partition P and to any choice of intermediate points.

In this case $B = \int_a^b f$.

This condition is expressed by the symbol $\lim\limits_{\|P\| \to 0} S(P,f) = B$.

Note. Since $S(P,f)$ is not a function of $\|P\|$, this limit is not of the type that we usually define.

Using this symbolic notation, the definition of integrability of a function f can be restated as the following:

A function $f : [a,b] \to \mathbb{R}$ is said to be integrable on $[a,b]$ if there exists a real number B such that $\lim_{\|P\|\to 0} S(P,f) = B$,

where $S(P,f)$ is a *Riemann sum* for f corresponding to the partition P of $[a,b]$ and to an arbitrary choice of intermediate points. In this case, $B = \int_a^b f$.

[[**Remark.** A partition $P = (x_0, x_1, \ldots, x_n)$ of $[a,b]$ divides the the interval $[a,b]$ into subintervals $[x_0,x_1],[x_1,x_2],\ldots,[x_{n-1},x_n]$. If a point ξ_r be selected at random in the subinterval $[x_{r-1},x_r]$ for $r = 1,2,\ldots,n$, then the chosen points $\xi_1,\xi_2,\ldots,\xi_n$ are called *tags* of the corresponding subintervals.

The ordered set of ordered pairs ($([x_0,x_1],\xi_1),([x_1,x_2],\xi_2)$, ..., $([x_{n-1},x_n],\xi_n)$), where each ordered pair contains a subinterval as the first element and the corresponding tag as the second element, is said to be a *tagged partition* of $[a,b]$ and it is denoted by $\dot{P}$ [P being the underlying partition of $[a,b]$.

Since each tag can be chosen in infinitely many ways, we can have infinitely many tagged partitions corresponding to a single partition P.

Evidently, the norm of a partition P is same as the norm of each tagged partition $\dot{P}$, since each of them have the same set of subintervals.

The definition of Riemann integrability of a function f can be rephrased in terms of tagged partitions.

A function $f : [a,b] \to \mathbb{R}$ is said to be *Riemann integrable* on $[a,b]$ if there exists a real number B such that for each $\epsilon > 0$ there exists a positive $\delta(\epsilon)$ satisfying $| S(\dot{P},f) - B | < \epsilon$ for all tagged partitions $\dot{P}$ of $[a,b]$ with $\|\dot{P}\| < \delta$.]]

Theorem 11.9.1. If $f : [a,b] \to \mathbb{R}$ be such that $\lim_{\|P\|\to 0} S(P,f) = B$, where $S(P,f)$ is a Riemann sum for f corresponding to the partition P of $[a,b]$, then B is unique.

Proof. Let $\lim_{\|P\|\to 0} S(P,f) = B_1$ and $\lim_{\|P\|\to 0} S(P,f) = B_2$.

Let us choose $\epsilon > 0$.

Since $\lim_{\|P\|\to 0} S(P,f) = B_1$, there exists a positive δ_1 such that if $\dot{P}$ be a tagged partition of $[a,b]$ with $\|\dot{P}\| < \delta_1$ then $|S(\dot{P},f) - B_1| < \frac{\epsilon}{2}$.

Since $\lim_{\|P\|\to 0} S(P,f) = B_2$, there exists a positive δ_2 such that if $\dot{P}$ be a tagged partition of $[a,b]$ with $\|\dot{P}\| < \delta_2$ then $|S(\dot{P},f) - B_2| < \frac{\epsilon}{2}$.

Let $\delta = \min\{\delta_1,\delta_2\}$. Then $|S(\dot{P},f) - B_1| < \frac{\epsilon}{2}$ and $|S(\dot{P},f) - B_2| < \frac{\epsilon}{2}$ for all tagged partitions $\dot{P}$ of $[a,b]$ with $\|\dot{P}\| < \delta$

$|B_1 - B_2| \le |B_1 - S(\dot{P}, f)| + |S(\dot{P}, f) - B_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$, i.e., $< \epsilon$ for all tagged partitions $\dot{P}$ of $[a, b]$ with $||\dot{P}|| < \delta$.

Since ϵ is arbitrary, it follows that $B_1 = B_2$ and therefore B is unique.

Theorem 11.9.2. If $f : [a, b] \to \mathbb{R}$ be such that $\lim_{||P|| \to 0} S(P, f)$ exists, where $S(P, f)$ is a Riemann sum for f corresponding to the partition P of $[a, b]$, then f is bounded on $[a, b]$.

Proof. Let $\lim_{||P|| \to 0} S(P, f) = B$. Let us choose $\epsilon = 1$.

Then there exists a positive δ such that $| S(P, f) - B | < 1$ for all partitions P of $[a, b]$ satisfying $||P|| < \delta$.

Let f be not bounded on $[a, b]$. Then there exists at least one subinterval. say $[u, v]$ of $[a, b]$ such that $|v - u| < \delta$ and f is not bounded on $[u, v]$.

Let $P_0 = (x_0, x_1, x_2, \ldots, x_n)$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a partition of $[a, b]$ such that $||P_0|| < \delta$ and $u = x_{r-1}, v = x_r$ for some $r = 1, 2, \ldots, n$.

$| S(P_0, f) - B | < 1$ for any choice of intermediate points ξ_r satisfying $x_{r-1} < \xi_r < x_r$.

Therefore $|f(\xi_1)(x_1 - x_0) + \cdots + f(\xi_r)(v - u) + \cdots + f(\xi_n)(x_n - x_{n-1}) - B| < 1$ for any intermediate points ξ_r satisfying $x_{r-1} < \xi_r < x_r$.

$|f(\xi_r)(v - u)| - |B - [f(\xi_1)(x_1 - x_0) + \cdots + f(\xi_{r-1})(x_{r-1} - x_{r-2}) + f(\xi_{r+1})(x_{r+1} - x_r) + \cdots + f(\xi_n)(x_n - x_{n-1})]|$

$\le |f(\xi_1)(x_1 - x_0) + \cdots + f(\xi_r)(v - u) + \cdots + f(\xi_n)(x_n - x_{n-1}) - B| < 1$, since $|b| - |a| \le |b - a|$ for all $a, b \in \mathbb{R}$.

Therefore $|f(\xi_r)(v - u)| < 1 + |[f(\xi_1)(x_1 - x_0) + \cdots + f(\xi_{r-1})(x_{r-1} - x_{r-2}) + f(\xi_{r+1})(x_{r+1} - x_r) + \cdots + f(\xi_n)(x_n - x_{n-1}) - B|$.

Let us keep $\xi_1, \xi_2, \ldots, \xi_{r-1}, \xi_{r+1}, \ldots, \xi_n$ fixed. Then $|f(\xi_r)(v - u)|$ is bounded for every $\xi_r \in [u, v]$ and we have a contradiction.

Therefore f is bounded on $[a, b]$. This completes the proof.

Note. The theorem says that a necessary condition for Riemann integrability of a function f on $[a, b]$ in the new approach is that f is bounded on $[a, b]$. Therefore in developing the theory of integrability we shall be confined to the functions in $\mathcal{B}[a, b]$ only.

Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and let $P = (x_0, x_1, x_2, \ldots, x_n)$ be a partition of $[a, b]$ and $\xi_1, \xi_2, \cdots, \xi_n$ are arbitrarily chosen points such that $x_{r+1} \le \xi_r \le x_r$, for $r = 1, 2, \ldots, n$.

Let $M_r = \sup_{x\in[x_{r-1},x_r]} f(x)$, $m_r = \inf_{x\in[x_{r-1},x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

Then $m_r \le f(\xi_r) \le M_r$, for $r = 1, 2, \ldots, n$. Therefore

$$\sum_{r=1}^{n} m_r(x_r - x_{r-1}) \le \sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1}) \le \sum_{r=1}^{n} M_r(x_r - x_{r-1})$$

or, $L(P,f) \le S(P,f) \le U(P,f)$.

Thus for a bounded function f, any Riemann sum corresponding to a partition P lies between the lower Darboux sum and the upper Darboux sum of f corresponding to P, no matter how we select the intermediate points.

If m_r and M_r be attained by f at some points in $[x_{r-1}, x_r]$ then for a particular choice of the intermediate points ξ_r, $S(P,f) = L(P,f)$ and for some other choice of the intermediate points ξ_r, $S(P,f) = U(P,f)$.

However, in general, the upper Darboux sum and the lower Darboux sum are not Riemann sums. But by proper choice of the intermediate points ξ_r, $S(P,f,\xi)$ can be made arbitrarily close to the upper and the lower Darboux sums.

The equivalence of two definitions of integrability of a function f, one in terms of exact bounds of the lower and the upper Darboux sums and the other in terms of limits of Riemann sums, can be established by the following theorems.

Theorem 11.9.3. Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ in the sense of definition 11.2. Then for each $\epsilon > 0$ there exists a positive δ such that $|S(P,f) - \int_a^b f| < \epsilon$ for all partitions P of $[a,b]$ satisfying $||P|| < \delta$, where $S(P,f)$ is a Riemann sum for f corresponding to P and to an arbitrary choice of intermediate points.

Proof. Since f is integrable on $[a,b]$ and $\epsilon > 0$, there exists a positive δ such that $U(P,f) - L(P,f) < \epsilon$ for all partitions P of $[a,b]$ satisfying $||P|| < \delta$.

For every partition P of $[a,b]$, $L(P,f) \le \int_a^b f \le U(P,f)$.

For every partition P of $[a,b]$, $L(P,f) \le S(P,f) \le U(P,f)$ where $S(P,f)$ is a Riemann sum for f corresponding to P and to any choice of intermediate points. Therefore for every partition P of $[a,b]$,

$| S(P,f) - \int_a^b f | \le U(P,f) - L(P,f)$, where $S(P,f)$ is any Riemann sum for f corresponding to P and to any choice of intermediate points.

Hence $| S(P,f) - \int_a^b f | < \epsilon$ for all partitions P of $[a,b]$ satisfying $||P|| < \delta$, where $S(P,f)$ is a Riemann sum for f corresponding to P and to any choice of intermediate points.

Theorem 11.9.4. Let $f : [a, b] \to \mathbb{R}$ be such that there exist a real number B such that $\lim_{\|P\| \to 0} S(P, f) = B$, where $S(P, f)$ is a Riemann sum for f corresponding to the partition P of $[a, b]$ and to any choice of intermediate points. Then f is integrable on $[a, b]$ in the sense of definition 11.2 and $\int_a^b f = B$.

Proof. Let $\epsilon > 0$. There exists a $\delta > 0$ such that

$B - \frac{\epsilon}{2} < S(P, f) < B + \frac{\epsilon}{2}$ for all partitions P of $[a, b]$ such that $\|P\| < \delta$ where $S(P, f)$ is any Riemann sum for f corresponding to P and to any choice of intermediate points.

Since $\lim_{\|P\| \to 0} S(P, f)$ exists, f is bounded on $[a, b]$. Let us take a partition P_0 of $[a, b]$ such that $\|P_0\| < \delta$.

Let $P_0 = (x_0, x_1, x_2, \ldots, x_n)$ where $a = x_0 < x_1 < \cdots < x_n = b$.

Let $M_r = \sup_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

There exist points α_r, β_r in $[x_{r-1}, x_r]$ such that

$M_r - \frac{\epsilon}{2(b-a)} < f(\alpha_r) \le M_r$, $m_r \le f(\beta_r) < m_r + \frac{\epsilon}{2(b-a)}$.

Selecting the points $\alpha_1, \alpha_2, \ldots, \alpha_n$ as intermediate points

$U(P_0, f) - \frac{\epsilon}{2} < S(P_0, f, \alpha) < U(P_0, f)$.

Selecting the points $\beta_1, \beta_2, \ldots, \beta_n$ as intermediate points

$L(P_0, f) \le S(P_0, f, \beta) < L(P_0, f) + \frac{\epsilon}{2}$

Since $B - \frac{\epsilon}{2} < S(P_0, f) < B + \frac{\epsilon}{2}$ where $S(P_0, f)$ is a Riemann sum for f corresponding to any choice of intermediate points, we have

$U(P_0, f) - \frac{\epsilon}{2} < B + \frac{\epsilon}{2}$ and $B - \frac{\epsilon}{2} < L(P_0, f) + \frac{\epsilon}{2}$.

Hence $B - \epsilon < L(P_0, f) \le U(P_0, f) < B + \epsilon$ ... ... (i)

Therefore $U(P_0, f) - L(P_0, f) < 2\epsilon$.

This proves that f is integrable on $[a, b]$ in the sense of definition 11.2.

As $L(P, f) \le \int_a^b f \le U(P, f)$ for every partition P of $[a, b]$, it follows from (i) that

$B - \epsilon < L(P, f) \le \int_a^b f \le U(P, f) < B + \epsilon$ for all partitions P satisfying $\|P\| < \delta$.

Therefore $\left| \int_a^b f - B \right| < \epsilon$. Since ϵ is arbitrary, $\int_a^b f = B$.

Theorem 11.9.5. Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and let $\{P_n\}$ be a sequence of partitions of $[a, b]$ such that the sequence $\{\|P_n\|\}$ converges to 0. If there exists a real number B such that corresponding to each $\epsilon > 0$ there exists a natural number k satisfying $| S(P_n, f) - B | < \epsilon$ for all $n \ge k$, where $S(P_n, f)$ is a Riemann sum for f corresponding to

P_n and to any choice of intermediate points, then f is integrable on $[a, b]$ and $\int_a^b f = B$.

Proof. Let $\epsilon > 0$. By the given condition there exists a natural number k_1 such that $| S(P_n, f) - B | < \frac{\epsilon}{2}$ for all $n \geq k_1$.

Let $\|P_{k_1}\| = \delta$. Since $\lim \|P_n\| = 0$, there exists a natural number k_2 such that $\|P_n\| < \delta$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $| S(P_n, f) - B | < \frac{\epsilon}{2}$ and $\|P_n\| < \delta$ for all $n \geq k$.

It follows that $| S(P_n, f) - B | < \frac{\epsilon}{2}$ for all partitions P_n satisfying $\|P_n\| < \delta$.

Let P_0 be a partition of $[a, b]$ such that $\|P_0\| < \delta$.

Then $| S(P_0, f) - B | < \frac{\epsilon}{2}$.

Let $P_0 = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < \cdots < x_n = b$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

There exist points α_r, β_r in $[x_{r-1}, x_r]$ such that
$M_r - \frac{\epsilon}{2(b-a)} < f(\alpha_r) \leq M_r$, $m_r \leq f(\beta_r) < m_r + \frac{\epsilon}{2(b-a)}$.

Selecting $\alpha_1, \alpha_2, \ldots, \alpha_n$ as intermediate points, we have
$U(P_0, f) - \frac{\epsilon}{2} < S(P_0, f, \alpha) < U(P_0, f)$.

Selecting $\beta_1, \beta_2, \ldots, \beta_n$ as intermediate points, we have
$L(P_0, f) < S(P_0, f, \beta) < L(P_0, f) + \frac{\epsilon}{2}$.

Since $B - \frac{\epsilon}{2} < S(P_0, f) < B + \frac{\epsilon}{2}$ where $S(P_0, f)$ is a Riemann sum for f corresponding to any choice of intermediate points, we have
$U(P_0, f) - \frac{\epsilon}{2} < B + \frac{\epsilon}{2}$ and $B - \frac{\epsilon}{2} < L(P_0, f) + \frac{\epsilon}{2}$.

Hence $B - \epsilon < L(P_0, f) \leq U(P_0, f) < B + \epsilon$ ... ... (i)

Therefore $U(P_0, f) - L(P_0, f) < 2\epsilon$.

This proves that f is integrable on $[a, b]$.

Since for every partition P of $[a, b]$ $L(P, f) \leq \int_a^b f \leq U(P, f)$, from (i) it follows that

$B - \epsilon < L(P_0, f) \leq \int_a^b f \leq U(P_0, f) < B + \epsilon$.

So we have $| \int_a^b f - B | < \epsilon$.

This holds for each $\epsilon > 0$. Hence $\int_a^b f = B$.

Note. The theorem says that if f be bounded on $[a, b]$ and $\{P_n\}$ be a sequence of partitions of $[a, b]$ such that the sequence $\{\|P_n\|\}$ converges to 0, then if $\lim\limits_{n \to \infty} S(P_n, f) = B$ where $S(P_n, f)$ is a Riemann sum for f corresponding to P_n and to any choice of intermediate points, then f is integrable on $[a, b]$ and $\int_a^b f = B$.

In particular, if $P_n = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < \cdots < x_n = b$ and $x_1 - x_0 = x_2 - x_1 = \cdots = x_n - x_{n-1}$ then $\|P_n\| = \frac{b-a}{n}$ and $\lim \|P_n\| = 0$; and if for every choice of intermediate points $\xi_1, \xi_2, \ldots, \xi_n$, $\lim\limits_{n\to\infty} \frac{b-a}{n} \sum\limits_{r=1}^{n} f(\xi_r) = B$ then $\int_a^b f$ exists and equals B.

Theorem 11.9.6. Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $\{P_n\}$ be a sequence of partitions of $[a, b]$ such that the sequence $\{\|P_n\|\}$ converges to 0. Then if $\epsilon > 0$ be given, there exists a natural number k such that $| S(P_n, f) - \int_a^b f | < \epsilon$ for all $n \geq k$ where $S(P, f)$ is a Riemann sum of f corresponding to P and any choice of intermediate points.

Proof. Since f is integrable on $[a, b]$ and $\epsilon > 0$ there exists a positive δ such that for all partitions P satisfying $\|P\| < \delta$

$$U(P, f) - L(P, f) < \epsilon.$$

Since $\lim \|P_n\| = 0$, there exists a natural number k such that

$$\|P_n\| < \delta \text{ for all } n \geq k.$$

Therefore $U(P_n, f) - L(P_n, f) < \epsilon$ for all $n \geq k$.

Since f is integrable on $[a, b]$, $L(P_n, f) \leq \int_a^b f \leq U(P_n, f)$ for all $n \in \mathbb{N}$.

Also for each P_n, $L(P_n, f) \leq S(P_n, f) \leq U(P_n, f)$ where $S(P_n, f)$ is a Riemann sum for f corresponding to P_n and to any choice of intermediate points.

Therefore $| S_n(P_n, f) - \int_a^b f | \leq U(P_n, f) - L(P_n, f) < \epsilon$ for all $n \geq k$.

Note. The theorem says that if f be integrable on $[a, b]$ and $\{P_n\}$ be a sequence of partitions of $[a, b]$ such that $\lim \|P_n\| = 0$, then $\lim\limits_{n\to\infty} S(P_n, f) = \int_a^b f$ where $S(P_n, f)$ is a Riemann sum for f corresponding to the partitions P_n and to any choice of intermediate points.

In particular, if $P_n = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < \cdots < x_n = b$ and $x_1 - x_0 = x_2 - x_1 = \cdots = x_n - x_{n-1}$ then $\|P_n\| = \frac{b-a}{n}$ and $\lim \|P_n\| = 0$. Then for an *integrable* function f, $\int_a^b f = \lim\limits_{n\to\infty} \frac{b-a}{n} \sum\limits_{r=1}^{n} f(\xi_r)$ for *any particular choice* of points $\xi_1, \xi_2, \ldots, \xi_n$.

Worked Examples.

1. A function f is defined on $[0, 1]$ by

$$f(x) = 1, \text{ if } x \text{ is rational}$$
$$= 0, \text{ if } x \text{ is irrational.}$$

Show that f is not integrable on $[0, 1]$.

f is bounded on $[0, 1]$.

Let P_n be the partition of $[0, 1]$ defined by $P_n = (x_0, x_1, x_2, \ldots, x_n)$ where $x_r = \frac{r}{n}, 0 \leq r \leq n$.

Let us choose α_r in $[x_{r-1}, x_r]$ by $\alpha_r = x_r$, for $r = 1, 2, \ldots, n$.
Then $S(P_n, f, \alpha) = \frac{1}{n}[f(\alpha_1) + f(\alpha_2) + \cdots + f(\alpha_n)] = 1$.

Let us choose β_r in $[x_{r-1}, x_r]$ by $\beta_r = x_r - \frac{1}{\sqrt{2}n}$, for $r = 1, 2, \ldots, n$.
Then $S(P_n, f, \beta) = \frac{1}{n}[f(\beta_1) + f(\beta_2) + \cdots + f(\beta_n)] = 0$.

Let us consider the sequence of partitions $\{P_n\}$. $\|P_n\| = \frac{1}{n}$. $\lim \|p_n\| = 0$.
$\lim\limits_{n \to \infty} S(P_n, f, \alpha) = 1$, $\lim\limits_{n \to \infty} S(P_n, f, \beta) = 0$.

Since for two different choices of intermediate points ξ_r, the Riemann sums $S(P_n, f, \xi)$ converge to two different limits, f is not integrable on $[0, 1]$.

2. A function f is defined on $[0, 1]$ by

$$\begin{aligned} f(x) &= x, \text{ if } x \text{ is rational} \\ &= 1 - x, \text{ if } x \text{ is irrational.} \end{aligned}$$

Show that f is not integrable on $[0, 1]$.

f is bounded on $[0, 1]$. Let P_n be the partition of $[0, 1]$ defined by $P_n = (x_0, x_1, x_2, \ldots, x_{2n})$ where $x_r = \frac{r}{2n}, 0 \leq r \leq 2n$.
Let us choose α_r in $[x_{r-1}, x_r]$ by $\alpha_r = x_r$, for $r = 1, 2, \ldots, n$ and $\alpha_r = x_r - \frac{1}{\sqrt{5}n}$, for $r = n + 1, \ldots, 2n$.

$$\begin{aligned} \text{Then } S(P_n, f, \alpha) &= \tfrac{1}{2n}[(x_1 + x_2 + \cdots + x_n) + (1 - x_{n+1} + \tfrac{1}{\sqrt{5}n}) \\ &\quad + \cdots + (1 - x_{2n} + \tfrac{1}{\sqrt{5}n})] \\ &= \tfrac{1}{2n}[\tfrac{1+2+\cdots+n}{2n} + n + \tfrac{1}{\sqrt{5}} - \tfrac{(n+1)+\cdots+2n}{2n}] \\ &= \tfrac{1}{2n}[\tfrac{2n}{4} + \tfrac{1}{\sqrt{5}}]. \end{aligned}$$

Let us choose β_r in $[x_{r-1}, x_r]$ by $\beta_r = x_r - \frac{1}{\sqrt{5}n}$, for $r = 1, \ldots, n$ and $\beta_r = x_r$, for $r = n + 1, \ldots, 2n$.

$$\begin{aligned} \text{Then } S(P_n, f, \beta) &= \tfrac{1}{2n}[(1 - x_1 + \tfrac{1}{\sqrt{5}n} + \cdots + (1 - x_n + \tfrac{1}{\sqrt{5}n}) \\ &\quad + (x_{n+1} + \cdots + x_{2n})] \\ &= \tfrac{1}{2n}[n - \tfrac{1+2+\cdots+n}{2n} + \tfrac{1}{\sqrt{5}} + \tfrac{(n+1)+\cdots+2n}{2n}] \\ &= \tfrac{1}{2n}[\tfrac{6n}{4} + \tfrac{1}{\sqrt{5}}]. \end{aligned}$$

Let us consider the sequence of partitions $\{P_n\}$. $\|P_n\| = \frac{1}{2n}$. $\lim \|p_n\| = 0$. $\lim\limits_{n \to \infty} S(P_n, f, \alpha) = \frac{1}{4}$, $\lim\limits_{n \to \infty} S(P_n, f, \beta) = \frac{3}{4}$.

Since for two different choices of intermediate points ξ_r, the Riemann sums $S(P_n, f, \xi)$ converge to two different limits, f is not integrable on $[0, 1]$.

3. Let $f : [0,1] \to \mathbb{R}$ be integrable on $[0, 1]$. Show that

$$\int_0^1 f = \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n} f\left(\frac{r}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n-1} f\left(\frac{r}{n}\right).$$

Let $P_n = (0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1)$.
Then $\|P_n\| = \frac{1}{n}$ and $\{P_n\}$ is a sequence of partition of $[0,1]$ such that $\lim \|P_n\| = 0$.

Since f is integrable on $[0, 1]$, $\lim S(P_n, f) = \int_0^1 f$ where $S(P_n, f)$ is the Riemann sum $\frac{1}{n}\{f(\xi_1) + f(\xi_2) + \cdots + f(\xi_n)\}$ for any particular choice of intermediate points $\xi_1, \xi_2, \ldots, \xi_n$.

Let $\xi_r = \frac{r}{n}$, $r = 1, 2, \ldots, n$. Then $\int_0^1 f = \lim \frac{1}{n}[\ldots + f(\frac{n}{n})]$

Let $\xi_r = \frac{r-1}{n}$. Then $\int_0^1 f = \lim \frac{1}{n}[\ldots + f(\frac{n-1}{n})]$

Note. If f be integrable ...

5. Evaluate the limit
$\lim_{n\to\infty} \{(1+\frac{1^2}{n^2})(1+\frac{2^2}{n^2})\cdots(1+\frac{n^2}{n^2})\}^{1/n}$ as an integral.

Let $P = \{(1+\frac{1^2}{n^2})(1+\frac{2^2}{n^2})\cdots(1+\frac{n^2}{n^2})\}^{1/n}$.
Then $\log P = \frac{1}{n}\sum_{r=1}^{n} \log(1+\frac{r^2}{n^2})$.

$\lim_{n\to\infty} \log P = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \log(1+\frac{r^2}{n^2})$

$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f(\frac{r}{n})$, where $f(x) = \log(1+x^2)$

$= \int_0^1 \log(1+x^2)dx$, since f is integrable on $[0,1]$.

Hence $\lim_{n\to\infty} P = e^{\int_0^1 \log(1+x^2)dx}$.

6. Evaluate the limit
$\lim_{n\to\infty} [\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+3n}]$ as an integral.

Let $S = \lim_{n\to\infty} [\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+3n}]$.
Then $S = \lim_{n\to\infty} \frac{1}{n}[\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{3n}{n}}]$

$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{3n} \frac{1}{1+\frac{r}{n}} = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{3n} f(\frac{r}{n})$, where $f(x) = \frac{1}{1+x}$.

Let $P_n = (0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{3n}{n})$ be a partition of $[0,3]$ dividing $[0,3]$ into $3n$ subintervals of equal length. $\lim \|P_n\| = \lim \frac{1}{n} = 0$. Let us choose $\xi_r = \frac{r}{n}$, $r = 1, 2, \ldots, 3n$.

Then $\frac{1}{n}\sum_{r=1}^{3n} f(\frac{r}{n})$ is the Riemann sum for f on the interval $[0,3]$ corresponding to the partition P_n and the chosen intermediate points $\xi_1, \xi_2, \ldots, \xi_n$.

As f is continuous on $[0,3]$, f is integrable on $[0,3]$.

Therefore $\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{3n} f(\frac{r}{n}) = \int_0^3 f(x)dx$ and $S = \int_0^3 f(x)dx = \int_0^3 \frac{1}{1+x}dx = [\log(1+x)]_0^1 = \log 2$.

7. Evaluate $\int_a^b x^2 dx$.

Let $f(x) = x^2, x \in [a,b]$. Then f is continuous on $[a,b]$ and therefore f is integrable on $[a,b]$.

Let $P_n = (a, a+h, \ldots, a+nh)$, where $h = \frac{b-a}{n}$. Then $\|P_n\| = h$. $\{P_n\}$ is a sequence of partitions of $[a,b]$ such that $\lim \|P_n\| = 0$.

Since f is integrable on $[a, b]$,

$\int_a^b f = \lim_{n\to\infty} h \sum_{r=1}^{n} f(\xi_r)$ for any particular choice of intermediate points $\xi_1, \xi_2, \ldots, \xi_n$, where $a + \overline{r-1}h \le \xi_r \le a + rh$.

Let $\xi_r = a + rh$. Then $\int_a^b x^2 dx$

$= \lim_{n\to\infty} h[f(a+h) + f(a+2h) + \cdots + f(a+nh)]$

$= \lim_{h\to 0} h[(a+h)^2 + (a+2h)^2 + \cdots + (a+nh)^2]$

$= \lim_{h\to 0} h[na^2 + ah.n(n+1) + h^2.\dfrac{n(n+1)(2n+1)}{6}]$

$= \lim_{h\to 0}[a^2(b-a) + a.(b-a)(b-a+h) + \frac{(b-a)(b-a+h)(2b-2a+h)}{6}]$

$= (b-a)[a^2 + a(b-a) + \frac{(b-a)^2}{3}]$

$= \frac{(b-a)}{3}(b^2 + ab + a^2) = \frac{b^3-a^3}{3}$.

8. Evaluate $\int_a^b x^{99} dx$ where $0 < a < b$.

Let $f(x) = x^{99}$, $x \in [a, b]$. Then f is continuous on $[a, b]$ and therefore f is integrable on $[a, b]$.

Let $P_n = (a, ah, ah^2, \ldots, ah^{n-1}, b)$ where $h^n = b/a$. Then $\|P_n\| = ah^{n-1}(h-1) = a.(\frac{b}{a})^{\frac{n-1}{n}}[(\frac{b}{a})^{\frac{1}{n}} - 1]$.

$\{P_n\}$ is a sequence of partitions of $[a, b]$ such that $\lim \|P_n\| = 0$.

Since f is integrable on $[a, b]$,

$\int_a^b f = \lim_{n\to\infty}[a(h-1)f(\xi_1) + ah(h-1)f(\xi_2) + \cdots + ah^{n-1}(h-1)f(\xi_n)]$ for any patticular choice of intermediate points $\xi_1, \xi_2, \ldots, \xi_n$, where $ah^{r-1} \le \xi_r \le ah^r$.

Let $\xi_r = ah^{r-1}$.

Then $\int_a^b x^{99} dx = \lim_{h\to 1} a^{100}(h-1)[1 + h^{100} + h^{2.100} + \cdots + h^{(n-1)100}]$

$= \lim_{h\to 1} a^{100}(h-1)[\dfrac{h^{100n} - 1}{h^{100} - 1}]$

$= \lim_{h\to 1} \dfrac{h-1}{h^{100} - 1}[(\frac{b}{a})^{100} - 1].a^{100}$

$= \frac{1}{100}.(b^{100} - a^{100})$.

9. Evaluate $\int_a^b \frac{1}{x} dx, 0 < a < b$.

Let $f(x) = \frac{1}{x}, x \in [a, b]$. Then f is continuous on $[a, b]$. Therefore f is integrable on $[a, b]$.

Let $P_n = (a, ah, ah^2, \ldots, ah^{n-1}, b)$ where $h^n = b/a$ be a partition of $[a, b]$. Then $\|P_n\| = ah^{n-1}(h-1) = a.(\frac{b}{a})^{\frac{n-1}{n}}[(\frac{b}{a})^{\frac{1}{n}} - 1]$.

$\{P_n\}$ is a sequence of partitions of $[a, b]$ such that $\lim \|P_n\| = 0$.

Since f is integrable on $[a, b]$,

$\int_a^b f = \lim\limits_{\|P_n\| \to 0} [a(h-1)f(\xi_1) + ah(h-1)f(\xi_2) + \cdots + ah^{r-1}(h-1)f(\xi_r) + \cdots + ah^{n-1}(h-1)f(\xi_n)]$ for any particular choice of intermediate points $\xi_1, \xi_2, \ldots, \xi_n$ where $ah^{r-1} \leq \xi_r \leq ah^r$.

Let $\xi_r = ah^{r-1}$. Then $\int_a^b (\frac{1}{x})dx$

$$= \lim_{h \to 1} [a(h-1).\tfrac{1}{a} + ah(h-1).\tfrac{1}{ah} + \cdots + ah^{n-1}(h-1).\tfrac{1}{ah^{n-1}}]$$

$$= \lim_{h \to 1} n(h-1) = \lim_{h \to 1} n\{(\tfrac{b}{a})^{\frac{1}{n}} - 1\}$$

$$= \lim_{n \to \infty} \frac{e^{\frac{1}{n}\log\frac{b}{a}} - 1}{\frac{1}{n}} = \log \tfrac{b}{a}.$$

11.10. Integration by substitution.

Theorem 11.10.1.

Let (i) $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$,

(ii) $\phi : [\alpha, \beta] \to \mathbb{R}$ be differentiable on $[\alpha, \beta]$ such that $\phi(\alpha) = a, \phi(\beta) = b$, and

(iii) $f \circ \phi$ and ϕ' are integrable on $[\alpha, \beta]$ and $\phi'(t) \neq 0$ for all $t \in [\alpha, \beta]$.

Then $\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(t))\phi'(t)dt$.

Since $\phi'(t) \neq 0$ on $[\alpha, \beta]$, it follows from Darboux's theorem that either $\phi'(t) > 0$ for all $t \in [\alpha, \beta]$ or $\phi'(t) < 0$ for all $t \in [\alpha, \beta]$, i.e., either ϕ is strictly increasing on $[\alpha, \beta]$ or ϕ is strictly decreasing on $[\alpha, \beta]$.

Accordingly, the theorem can be stated in two parts.

First part.

Let (i) $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$,

(ii) $\phi : [\alpha, \beta] \to \mathbb{R}$ be differentiable and strictly increasing on $[\alpha, \beta]$ such that $\phi(\alpha) = a, \phi(\beta) = b$, and

(iii) $f \circ \phi$ and ϕ' are integrable on $[\alpha, \beta]$.

Then $\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(t))\phi'(t)dt$.

Proof. Since ϕ is differentiable on $[\alpha, \beta], \phi$ is continuous on $[\alpha, \beta]$.

Since ϕ is continuous and strictly increasing on $[\alpha, \beta]$ and $\phi(\alpha) = a, \phi(\beta) = b, \phi^{-1}$ is continuous and strictly increasing on $[a, b]$.

Let $P = (x_0, x_1, \ldots, x_n)$ be any partition of $[a, b]$ and $Q = \{y_0, y_1, \ldots, y_n\}$ where $y_i = \phi^{-1}(x_i)$ be the corresponding partition of $[\alpha, \beta]$.

By Lagrange's Mean value theorem for the function ϕ on $[y_{r-1}, y_r]$, $\phi(y_r) - \phi(y_{r-1}) = (y_r - y_{r-1})\phi'(\eta_r)$ for some $\eta_r \in (y_{r-1}, y_r)$.

That is, $x_r - x_{r-1} = (y_r - y_{r-1})\phi'(\eta_r)$, $r = 1, 2, \ldots, n$ ... ... (i)

Let $\phi(\eta_r) = \xi_r$, $r = 1, 2, \ldots, n$.

Now $S(P, f, \xi) = f(\xi_1)(x_1 - x_0) + f(\xi_2)(x_2 - x_1) + \cdots + f(\xi_n)(x_n - x_{n-1})$

$= f(\phi(\eta_1))\phi'(\eta_1)(y_1 - y_0) + \cdots + f(\phi(\eta_n))\phi'(\eta_n)(y_n - y_{n-1})$

$= S(Q, (f \circ \phi).\phi', \eta)$.

Since f is integrable on $[a, b]$, $\lim\limits_{\|P\| \to 0} S(P, f, \xi) = \int_a^b f$.

Since ϕ' is integrable on $[a, b]$, ϕ' is bounded on $[a, b]$. It follows from (i) that $\|Q\| \to 0$ as $\|P\| \to 0$.

Therefore $\int_a^b f(x)dx = \lim\limits_{\|Q\| \to 0} S(Q, (f \circ \phi).\phi', \eta)$

$= \int_\alpha^\beta (f \circ \phi).\phi'(t)dt$, since $f_0\phi$ and ϕ' are both integrable on $[\alpha, \beta]$

$= \int_\alpha^\beta f(\phi(t))\phi'(t)dt$.

Second part

Let (i) $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$,

(ii) $\phi : [\beta, \alpha] \to \mathbb{R}$ be differentiable and strictly decreasing on $[\beta, \alpha]$ such that $\phi(\alpha) = a$, $\phi(\beta) = b$, and

(iii) $f \circ \phi$ and ϕ' are integrable on $[\beta, \alpha]$.

Then $\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(t))\phi'(t)dt$.

Similar proof.

Note 1. The theorem is called *substitution theorem* because under the stated conditions, the integral $\int_a^b f(x)dx$ can be evaluated by the substitution $x = \phi(t)$.

Note 2. If $\phi'(t) = 0$ at a finite number of points in $[\alpha, \beta]$ and all the other conditions remain same, the theorem still holds.

Another version of the theorem with wider conditions is given below.

Theorem 11.10.2. Let $I = [\alpha, \beta]$ be a closed and bounded interval and a function $\phi : I \to \mathbb{R}$ be such that ϕ' is continuous and $\neq 0$ on I.

Let $\phi(\alpha) = a$, $\phi(\beta) = b$ and a function f be continuous on $\phi([\alpha, \beta])$.

Then $\int_\alpha^\beta f(\phi(t))\phi'(t)dt = \int_a^b f(x)dx$.

Proof. Since ϕ' is continuous on I, ϕ is continuous on I.

Since $\phi' \neq 0$ on I, it follows from Darboux's theorem that either $\phi'(t) > 0$ for all $t \in [\alpha, \beta]$, or $\phi'(t) < 0$ for all $t \in [\alpha, \beta]$, i.e., either ϕ is strictly increasing on I, or strictly decreasing on I. Therefore $\phi[\alpha, \beta] = [a, b]$ or$[b, a]$ according as

$$\phi'(t) > 0 \text{ on } I, \text{ or } \phi'(t) < 0 \text{ on } I.$$

Case 1. Let $a < b$.

Let $F(x) = \int_a^x f(u)du, a \leq x \leq b$. Since f is continuous on $[a, b], F'(x) = f(x)$ for all $x \in [a, b]$.

Since ϕ is continuous on $[\alpha, \beta]$ and f is continuous on $[\phi(\alpha).\phi(\beta)]$, $f_o\phi$ is continuous on $[\alpha, \beta]$.

Let $G(x) = \int_\alpha^x f(\phi(t))\phi'(t)dt, x \in [\alpha, \beta]$.

Since both $f_o\phi$ and ϕ' are continuous on $[\alpha.\beta]$, $G'(x) = f(\phi(x)).\phi'(x)$, for all $x \in [\alpha, \beta]$.

Since ϕ is differentiable on $[\alpha, \beta]$ and F is differentiable on $[\phi(\alpha), \phi(\beta)]$,

$$\begin{aligned}[F(\phi(x))]' &= F'(\phi(x))\phi'(x) \\ &= f(\phi(x))\phi'(x) \text{ for all } x \in [\alpha, \beta].\end{aligned}$$

Therefore $F(\phi(x)) - G(x)$ is a constant for all $x \in [\alpha, \beta]$.

But $F(\phi(\alpha)) = G(\alpha) = 0$. Therefore $F(\phi(x)) = G(x)$ for all $x \in [\alpha, \beta]$.

In particular, $F[\phi(\beta)] = G(\beta)$, i.e., $\int_a^b f = \int_\alpha^\beta f(\phi(t))\phi'(t)dt$.

Case II. $a > b$.

Similar proof.

Examples.

1. Evaluate $\int_{-1}^1 \frac{1}{1+x^2}dx$ by the substitution $x = \tan t$.

Let $\phi(t) = \tan t, t \in [-\frac{\pi}{4}, \frac{\pi}{4}]$.

ϕ is differentiable and strictly increasing on $[-\frac{\pi}{4}, \frac{\pi}{4}]$. $\phi'(t)$ is integrable on $[-\frac{\pi}{4}, \frac{\pi}{4}]$. $\phi(-\frac{\pi}{4}) = -1, \phi(\frac{\pi}{4}) = 1$.

Let $f(x) = \frac{1}{1+x^2}, x \in [-1, 1]$.

f is continuous and therefore integrable on $[-1. 1]$.

$$\begin{aligned}\text{Then } \int_{-1}^1 f(x)dx &= \int_{-\pi/4}^{\pi/4} f(\phi(t))\phi'(t)dt \\ &= \int_{-\pi/4}^{\pi/4} \frac{1}{1+\tan^2 t} . \sec^2 t dt \\ &= \int_{-\pi/4}^{\pi/4} dt = \frac{\pi}{2}.\end{aligned}$$

2. Evaluate $\int_0^3 \frac{tdt}{\sqrt{1+t^2}}$.

Let $\phi(t) = 1+t^2, t \in [0,3]$. ϕ is differentiable on $[0,3], \phi'$ is continuous and therefore integrable on $[0,3], \phi'(t) > 0$ on $[0,3]$. $\phi(0) = 1,\ \phi(3) = 10$.

Let $f(x) = \frac{1}{\sqrt{x}}, x \in [1,10]$. Then f is integrable on $[1,10]$.

Then $\int_0^3 f(\phi(t))\phi'(t)dt = \int_1^{10} f(x)dx$
or, $\int_0^3 \frac{1}{\sqrt{1+t^2}} 2tdt = \int_1^{10} \frac{1}{\sqrt{x}}dx$
$= [2\sqrt{x}]_1^{10}$, by the fundamental theorem 11.8.5.
$= 2(\sqrt{10} - 1)$.

So $\int_0^3 \frac{t}{\sqrt{1+t^2}}dt = (\sqrt{10} - 1)$.

3. Evaluate $\int_{-1}^1 \frac{e^{2\tan^{-1}t}}{1+t^2}dt$.

Let $\phi(t) = \tan^{-1} t,\ t \in [-1,1]$.
ϕ is differentiable on $[-1,1], \phi'$ is continuous and therefore integrable on $[-1,1]$. $\phi'(t) > 0$ on $[-1,1]$. $\phi(-1) = -\frac{\pi}{4}, \phi(1) = \frac{\pi}{4}$.
Let $f(x) = e^{2x}, x \in [-\frac{\pi}{4}, \frac{\pi}{4}]$. f is integrable on $[-\frac{\pi}{4}, \frac{\pi}{4}]$.
Then $\int_{-1}^1 f(\phi(t))\phi'(t)dt = \int_{-\pi/4}^{\pi/4} f(x)dx$

or, $\int_{-1}^1 \frac{e^{2\tan^{-1}t}}{1+t^2}dt = \int_{-\pi/4}^{\pi/4} e^{2x}dx$
$= [\frac{e^{2x}}{2}]_{-\pi/4}^{\pi/4}$, by the fundamental theorem
$= \sinh \frac{\pi}{2}$.

11.11. Integration by parts.

Theorem 11.11.1. Let $f : [a,b] \to \mathbb{R},\ g : [a,b] \to \mathbb{R}$ be both differentiable on $[a,b]$ and f', g' are both integrable on $[a,b]$. Then
$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$.

Proof. Since f and g are both differentiable on $[a,b]$, fg is differentiable on $[a,b]$.

Since f and g are differentiable on $[a,b]$, f and g are continuous on $[a,b]$ and therefore they are both integrable on $[a,b]$.

Therefore $fg' + f'g$ is integrable on $[a,b]$, i.e., $(fg)'$ is integrable on $[a,b]$.

So by the fundamental theorem, $\int_a^b (fg)' = [fg]_a^b = f(b)g(b) - f(a)g(a)$. Also $\int_a^b (fg)' = \int_a^b (fg' + f'g) = \int_a^b fg' + \int_a^b f'g$.

Therefore $\int_a^b f(x)g'(x)dx + \int_a^b f'(x)g(x)dx = f(b)g(b) - f(a)g(a)$
or, $\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$.

11.12. Mean value theorems.

Theorem 11.12.1. (First Mean value theorem)

If (i) $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be both integrable on $[a,b]$, and
(ii) $g(x)$ has the same sign for all $x \in [a,b]$,
then there is a number μ such that
$\int_a^b f(x)g(x)dx = \mu \int_a^b g(x)dx$ where $m \le \mu \le M$ and
$$m = \inf_{x\in[a,b]} f(x),\ M = \sup_{x\in[a,b]} g(x).$$

If further, f is continuous on $[a,b]$ then there exists a point ξ in $[a,b]$ such that $\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$.

Proof. **Case 1.** Let $g(x) > 0, x \in [a,b]$.

Since $m = \inf\limits_{x\in[a,b]} f(x)$ and $M = \sup\limits_{x\in[a,b]} f(x)$, $m \le f(x) \le M$ for all $x \in [a,b]$. Therefore $mg(x) \le f(x)g(x) \le Mg(x)$ for all $x \in [a,b]$.

Since f and g are both integrable on $[a,b]$, mg, fg and Mg are integrable on $[a,b]$, and
$$\int_a^b mg(x)dx \le \int_a^b f(x)g(x)dx \le \int_a^b Mg(x)dx$$
$$\text{or, } m\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le M\int_a^b g(x)dx.$$
Therefore $\int_a^b f(x)g(x)dx = \mu\int_a^b g(x)dx$, where $m \le \mu \le M$.

Case II. Let $g(x) < 0, x \in [a,b]$.

The proof is similar.

Second part. If f be continuous on $[a,b]$ there exists a point ξ in $[a,b]$ such that $f(\xi) = \mu$, where $m \le \mu \le M$.

It follows that $\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$, where $a \le \xi \le b$.

Corollary. If, in particular, $g(x) = 1$ for all $x \in [a,b]$, then
$\int_a^b f(x)dx = \mu\int_a^b dx = \mu(b-a)$, where $m \le \mu \le M$.

If, moreover, f is continuous on $[a,b]$, there exists a point ξ in $[a,b]$ such that $\int_a^b f(x)dx = f(\xi)(b-a)$.

Since $\xi \in [a,b], \xi = a + \theta(b-a)$ for some θ satisfying $0 \le \theta \le 1$.

Therefore $\int_a^b f(x)dx = (b-a)f(a+\theta(b-a)), 0 \le \theta \le 1$.

Examples.

1. If $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and $\int_a^b f(x)dx = 0$, prove that there exists at least a point $c \in [a,b]$ such that $f(c) = 0$.

Since f is continuous on $[a,b]$, f is integrable on $[a,b]$

By the first Mean value theorem there exists a point c in $[a, b]$ such that $\int_a^b f(x)dx = f(c)(b-a)$.

Since $\int_a^b f(x)dx = 0$, it follows that $f(c) = 0$.

2. Use first Mean value theorem to prove that
$\frac{\pi}{6} \leq \int_0^{1/2} \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}dx \leq \frac{\pi}{6}.\frac{1}{\sqrt{1-k^2/4}}$, $k^2 < 1$.

Let $f(x) = \frac{1}{\sqrt{1-k^2x^2}}, x \in [0, \frac{1}{2}]$; $g(x) = \frac{1}{\sqrt{1-x^2}}, x \in [0, \frac{1}{2}]$. Then f and g are integrable on $[0, \frac{1}{2}]$ and $g(x) > 0$ for all $x \in [0, \frac{1}{2}]$.

Since f is continuous on $[0, \frac{1}{2}]$, by the first Mean value theorem there exists a point ξ in $[0, \frac{1}{2}]$ such that

$$\int_0^{1/2} f(x)g(x)dx = f(\xi) \int_0^{1/2} g(x)dx.$$

or, $\int_0^{1/2} \frac{1}{\sqrt{(1-k^2x^2)(1-x^2)}}dx = \frac{1}{\sqrt{1-k^2\xi^2}}.\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}dx = \frac{\pi}{6}.\frac{1}{\sqrt{1-k^2\xi^2}}$.

Since $0 \leq \xi \leq \frac{1}{2}, 1 \leq \frac{1}{\sqrt{1-k^2\xi^2}} \leq \frac{1}{\sqrt{1-k^2/4}}$

Therefore $\frac{\pi}{6} \leq \int_0^{1/2} \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}dx \leq \frac{\pi}{6}.\frac{1}{\sqrt{1-k^2/4}}$.

Lemma 11.12.2. Abel's Inequality.

If (i) $a_1, a_2, \ldots, a_n$ be n positive real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n$,

(ii) $\nu_1, \nu_2, \ldots, \nu_n \in \mathbb{R}$ and

(iii) there exist $h, H \in \mathbb{R}$ such that $h < \nu_1 + \nu_2 + \cdots + \nu_p < H$ for $1 \leq p \leq n$,

then $a_1h < a_1\nu_1 + a_2\nu_2 + \cdots + a_n\nu_n < a_1H$.

Proof. Let $s_p = \nu_1 + \nu_2 + \cdots + \nu_p, 1 \leq p \leq n$.

Then $a_1\nu_1 + a_2\nu_2 + \cdots + a_n\nu_n$
$= a_1s_1 + a_2(s_2 - s_1) + \cdots + a_n(s_n - s_{n-1})$
$= (a_1 - a_2)s_1 + (a_2 - a_3)s_2 + \cdots + (a_{n-1} - a_n)s_{n-1} + a_ns_n$.

Since $a_r - a_{r-1} \geq 0$ for $r = 2, 3, \ldots, n$ and $h < s_r < H$ for $r = 1, 2, \ldots, n$, we have $a_1\nu_1 + a_2\nu_2 + \cdots + a_n\nu_n$
$< H[(a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n-1} - a_n) + a_n] = a_1H$ and
$> h[(a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n-1} - a_n) + a_n] = a_1h$.

Combining, we have $a_1h < a_1\nu_1 + a_2\nu_2 + \cdots + a_n\nu_n < a_1H$.

This completes the proof.

Theorem 11.12.3. Second Mean value theorem (Bonnet's form)

If (i) $f : [a, b] \to \mathbb{R}$ and $\phi : [a, b] \to \mathbb{R}$ be both integrable on $[a, b]$, and

(ii) f is monotone decreasing and non-negative on $[a, b]$,

then there exists a point ξ in $[a, b]$ such that

$$\int_a^b f(x)\phi(x)dx = f(a) \int_a^\xi \phi(x)dx.$$

Proof. Let $P = (x_0, x_1, \ldots, x_n)$ be a partition of $[a, b]$. Then $a = x_0 < x_1 < \cdots < x_n = b$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} \phi(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_t]} \phi(x)$, for $r = 1, 2, \ldots, n$.

Let $\xi_1 = a, \xi_r$ be an arbitrary point in $[x_{r-1}, x_r]$, for $r = 1, 2, \ldots, n$

Then $m_r(x_r - x_{r-1}) \leq \int_{x_{r-1}}^{x_r} \phi(x)dx \leq M_r(x_r - x_{r-1})$

and $m_r(x_r - x_{r-1}) \leq \phi(\xi_r)(x_r - x_{r-1}) \leq M_r(x_r - x_{r-1})$.

We have $\sum\limits_{r=1}^{p} m_r(x_r - x_{r-1}) \leq \int_a^p \phi(x)dx \leq \sum\limits_{r=1}^{p} M_r(x_r - x_{r-1})$

and $\sum\limits_{r=1}^{p} m_r(x_r - x_{r-1}) \leq \sum\limits_{r=1}^{p} \phi(\xi_r)(x_r - x_{r-1}) \leq \sum\limits_{r=1}^{p} M_r(x_r - x_{r-1})$.

Therefore $\left| \sum\limits_{r=1}^{p} \phi(\xi_r)(x_r - x_{r-1}) - \int_a^{x_p} \phi(x)dx \right| \leq \sum\limits_{r=1}^{p} (M_r - m_r)(x_r - x_{r-1}) \leq \sum\limits_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1})$

or, $\int_a^{x_p} \phi(x)dx - \sum\limits_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1}) \leq \sum\limits_{r=1}^{p} \phi(\xi_r)(x_r - x_{r-1}) \leq \int_a^{x_p} \phi(x)dx + \sum\limits_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1})$.

As ϕ is integrable on $[a, b]$, $\int_a^x \phi(x)dx$ is continuous on $[a, b]$ and therefore $\int_a^x \phi(x)dx$ is bounded on $[a, b]$.

Let $M = \sup\limits_{x \in [a,b]} \int_a^x \phi(x)dx$, $m = \inf\limits_{x \in [a,b]} \int_a^x f(x)dx$.

Then $m - \sum\limits_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1}) \leq \sum\limits_{r=1}^{p} \phi(\xi_r)(x_r - x_{r-1}) \leq M + \sum\limits_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1})$. This inequality holds for $p = 1, 2, \ldots, n$.

Let $\nu_r = \phi(\xi_r)(x_r - x_{r-1}), a_r = f(\xi_r)$. Then

(i) $a_1, a_2, \ldots, a_n$ are positive numbers and $a_1 \geq a_2 \geq \cdots \geq a_n$,

(ii) $\nu_1, \nu_2, \ldots, \nu_n$ are n numbers such that $h \leq \nu_1 + \nu_2 + \cdots + \nu_p \leq H$ for $1 \leq p \leq n$, where

$h = m - \sum\limits_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1}), H = M + \sum\limits_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1})$.

By Abel's Inequality,

$f(a).h \le \sum\limits_{r=1}^{n} f(\xi_r)\phi(\xi_r)(x_r - x_{r-1}) \le f(a).H.$

Let $\|P\| \to 0$. Then $h \to m$, since $\lim\limits_{\|P\|\to 0}[U(P,\phi) - L(P,\phi)] = 0$;

$H \to M$, since $\lim\limits_{\|P\|\to 0}[U(P,\phi) - L(P,\phi)] = 0$;

and $\sum\limits_{r=1}^{n} f(\xi_r)\phi(\xi_r)(x_r - x_{r-1}) \to \int_a^b f(x)\phi(x)dx$, since $f\phi \in \mathcal{R}[a,b]$.

It follows that $mf(a) \le \int_a^b f(x)\phi(x)dx \le Mf(a)$

or, $\int_a^b f(x)\phi(x)dx = \mu f(a)$ where $m \le \mu \le M$.

But M, m are the supremum and the infimum of the continuous function $\int_a^x \phi(t)dt$ on $[a,b]$. Therefore there exists a point ξ in $[a,b]$ such that $\int_a^\xi \phi(t)dt = \mu$.

Consequently, $\int_a^b f(x)\phi(x)dx = f(a)\int_a^\xi \phi(x)dx, a \le \xi \le b$.

This completes the proof.

Theorem 11.12.4. Second Mean value theorem (Weierstrass' form)

If (i) $f : [a,b] \to \mathbb{R}$ and $\phi : [a,b] \to \mathbb{R}$ be both integrable on $[a,b]$, and

(ii) f is monotonic on $[a,b]$,

then there exists a point ξ in $[a,b]$ such that

$\int_a^b f(x)\phi(x)dx = f(a)\int_a^\xi \phi(x)dx + f(b)\int_\xi^b \phi(x)dx.$

Proof. **Case 1.** Let f be monotone decreasing on $[a,b]$.

Let $\psi(x) = f(x) - f(b), x \in [a,b]$.

Then ψ is monotone decreasing on $[a,b]$ and $\psi(x) \ge 0$ on $[a,b]$.

By Bonnet's theorem there exists a point ξ in $[a,b]$ such that

$\int_a^b \psi(x)\phi(x)dx = \psi(a)\int_a^\xi \phi(x)dx$

or, $\int_a^b [f(x) - f(b)]\phi(x)dx = [f(a) - f(b)]\int_a^\xi \phi(x)dx$

or, $\int_a^b f(x)\phi(x)dx = f(a)\int_a^\xi \phi(x)dx + f(b)[\int_a^b \phi(x)dx - \int_a^\xi \phi(x)dx]$

$= f(a)\int_a^\xi \phi(x)dx + f(b)\int_\xi^b \phi(x)dx.$

Case 2. Let f be monotone increasing on $[a,b]$.

Let $\psi(x) = f(b) - f(x), x \in [a,b]$.

Then ψ is monotone decreasing on $[a,b]$ and $\psi(x) \ge 0$ on $[a,b]$.

By Bonnet's theorem there exists a point ξ in $[a,b]$ such that

$\int_a^b \psi(x)\phi(x)dx = \psi(a)\int_a^\xi \phi(x)dx$

or, $\int_a^b [f(b)-f(x)]\phi(x)dx = [f(b)-f(a)]\int_a^\xi \phi(x)dx$

or, $\int_a^b f(x)\phi(x)dx = f(a)\int_a^\xi \phi(x)dx + f(b)[\int_a^b \phi(x)dx - \int_a^\xi \phi(x)dx]$

$$= f(a)\int_a^\xi \phi(x)dx + f(b)\int_\xi^b \phi(x)dx.$$

This completes the proof.

Examples (continued).

3. Show that the second Mean value theorem (Weierstrass' form) is applicable to $\int_a^b \frac{\sin x}{x}dx$ where $0 < a < b < \infty$. Also prove that

$$\left| \int_a^b \frac{\sin x}{x}dx \right| < 4/a.$$

Let $f(x) = \frac{1}{x}, x \in [a,b], \phi(x) = \sin x \in [a,b]$.

Then f and ϕ are both integrable on $[a,b]$ and f is monotone decreasing on $[a,b]$.

By the Mean value theorem (Weierstrass' form) there exists a point ξ in $[a,b]$ such that

$\int_a^b f(x)\phi(x)dx = f(a)\int_a^\xi \phi(x)dx + f(b)\int_\xi^b \phi(b)dx$

or, $\int_a^b \frac{\sin x}{x}dx = (1/a)\int_a^\xi \sin x dx + (1/b)\int_\xi^b \sin x dx$

$= (1/a)[-\cos\xi + \cos a] + (1/b)[-\cos b + \cos\xi]$

Therefore $\left| \int_a^b \frac{\sin x}{x}dx \right| \leq (\frac{1}{a}) \mid -\cos\xi + \cos a \mid + (\frac{1}{b}) \mid -\cos b + \cos\xi \mid$

$\leq (\frac{1}{a})\{\mid -\cos\xi \mid + \mid \cos a \mid\} + (\frac{1}{b})\{\mid -\cos b \mid + \mid \cos\xi \mid\}$

$\leq \frac{1}{a}(1+1) + (\frac{1}{b})(1+1)$

$< 4/a$, since $a < b$.

4. Show that the second Mean value theorem (Bonnet's form) is applicable to $\int_a^b \frac{\sin x}{x}dx$ where $0 < a < b < \infty$. Also prove that $\left| \int_a^b \frac{\sin x}{x}dx \right| \leq \frac{2}{a}$.

Let $f(x) = \frac{1}{x}, x \in [a,b], \phi(x) = \sin x, x \in [a,b]$.

Then f and ϕ are both integrable on $[a,b]$ and f is monotone decreasing on $[a,b]$ and $f(x) > 0$ for all $x \in [a,b]$.

By the Mean value theorem (Bonnet's form) there exists a point ξ in $[a,b]$ such that $\int_a^b \frac{\sin x}{x}dx = (\frac{1}{a})\int_a^\xi \sin x dx = (\frac{1}{a})\{-\cos\xi + \cos a\}$.

Therefore $\left| \int_a^b \frac{\sin x}{x}dx \right| \leq \frac{2}{a}$.

11.13. Logarithmic function.

Definition. The logarithmic function L (or log) is defined by

$L(x) = \int_1^x \frac{1}{t} dt$, for $x > 0$.

Property 1. $L(1) = 0$.

From definition it follows that $L(1) = \int_1^1 \frac{1}{t} dt = 0$.

Property 2. $L(x) < 0$ if $0 < x < 1$
$= 0$ if $x = 1$
> 0 if $x > 1$.

Proof. If $0 < x < 1$, the function f defined by $f(t) = 1/t, t \in [x, 1]$ is continuous on $[x, 1]$ and $f(t) > 0$ for all $t \in [x, 1]$.

Therefore $\int_x^1 f(t)dt > 0$. That is, $L(x) < 0$.

If $x > 1$, the fuction f defined by $f(t) = 1/t, t \in [1, x]$ is continuous on $[1, x]$ and $f(t) > 0$ for all $t \in [1, x]$.

Therefore $\int_1^x f(t)dt > 0$. That is, $L(x) > 0$.

$$\text{Thus } L(x) \begin{array}{ll} < & 0 \text{ if } 0 < x < 1 \\ = & 0 \text{ if } x = 1 \\ > & 0 \text{ if } x > 1. \end{array}$$

Property 3. For $x > 0, y > 0, L(xy) = L(x) + L(y)$.

Proof. Since $xy > 0, L(xy) = \int_1^{xy} \frac{1}{t} dt = \int_1^x (\frac{1}{t})dt + \int_x^{xy} (\frac{1}{t})dt$

$= L(x) + \int_x^{xy} (\frac{1}{t})dt = L(x) + \int_1^y \frac{1}{u} du$ [putting $t = xu$ in the second]

$= L(x) + L(y)$.

Corollary 1. In particular, if $y = \frac{1}{x}$, then $L(x) + L(\frac{1}{x}) = L(1) = 0$. Therefore $L(\frac{1}{x}) = -L(x), x > 0$.

Corollary 2. For $x > 0, y > 0, L(\frac{x}{y}) = L(x.\frac{1}{y}) = L(x) + L(\frac{1}{y}) = L(x) - L(y)$.

Property 4. For $x > 0, L(x^n) = nL(x), n$ being an integer.

Proof. **Case 1.** $n = 0$.

In this case, $L(x^n) = L(1) = 0$ and $nL(x) = 0$. Therefore $L(x^n) = nL(x)$.

Case 2. n is a positive integer.

When $n = 1$, the property holds.

Let the property hold for $n = m$, where m is a positive integer.

$$\begin{aligned} \text{Then } L(x^m) &= mL(x). \\ \text{So } L(x^{m+1}) &= L(x^m) + L(x), \text{ by property 3} \\ &= mL(x) + L(x) \\ &= (m+1)L(x). \end{aligned}$$

This shows that the property holds for $n = m+1$ if it holds for $n = m$. Also the property holds for $n = 1$.

By the principle of induction, the property holds for all positive integers n.

Case 3. n is a negative integer.

Let $n = -m$, where m is a positive integer.

$$\begin{aligned} \text{Then } L(x^n) = L(x^{-m}) &= L\{(\tfrac{1}{x})^m\} \\ &= mL(\tfrac{1}{x}), \text{ by case 2} \\ &= -mL(x) \\ &= nL(x). \end{aligned}$$

Combining all cases, the proof is complete.

Property 5. For $x > 0, L(x^\alpha) = \alpha L(x), \alpha$ being a rational number.

Proof. **Case 1.** α is an integer.

This is property 3.

Case 2. α is a positive fraction.

Let $\alpha = p/q, p$ and q are positive integers, $q > 1$.

$$\begin{aligned} L(x^\alpha) = L(x^{p/q}) &= L\{(x^{\frac{1}{q}})^p\} \\ &= pL(x^{\frac{1}{q}}), \text{ by property 3.} \end{aligned}$$

$$\begin{aligned} \text{Also } L(x) &= L\{(x^{\frac{1}{q}})^q\} \\ &= qL(x^{\frac{1}{q}}), \text{ by property 3.} \end{aligned}$$

$$\begin{aligned} \text{Therefore } L(x^\alpha) &= \tfrac{p}{q}L(x) \\ &= \alpha L(x). \end{aligned}$$

Case 3. α is a negative fraction.

Let $\alpha = -\beta$ where β is a positive fraction.

$$\begin{aligned} \text{Then } L(x^\alpha) &= L(x^{-\beta}) \\ &= L\{(\tfrac{1}{x})^\beta\} \\ &= \beta L(\tfrac{1}{x}), \text{ by case 2} \\ &= -\beta L(x) \\ &= \alpha L(x). \end{aligned}$$

Combining all cases, the proof is complete.

Corollary. If $x > 0$ and α be a real number, $L(x^\alpha) = \alpha L(x)$.

If α be irrational, let us consider a sequence $\{\alpha_n\}$ of rational points converging to α, Then $L(x^{\alpha_n}) = \alpha_n L(x)$ for all $n \in \mathbb{N}$.

Taking limit as $n \to \infty$ and noting that L is continuous, we have $L(x^\alpha) = \alpha L(x)$.

Property 6. The function L defined by $L(x) = \int_1^x \frac{1}{t} dt,\ x > 0$ is strictly increasing on $(0, \infty)$.

Also (i) $\lim\limits_{x \to \infty} L(x) = \infty$, (ii) $\lim\limits_{x \to 0+} L(x) = -\infty$.

Proof. Let $0 < x_1 < x_2$.

Then $L(x_2) - L(x_1) = \int_{x_1}^{x_2} \frac{1}{t} dt > 0$, since $\frac{1}{t}$ is continuous on $[x_1, x_2]$ and $\frac{1}{t} > 0$ on $[x_1, x_2]$.

Therefore $0 < x_1 < x_2 \Rightarrow L(x_2) > L(x_1)$.

This proves that the function L is strictly increasing on $(0, \infty)$.

(i) Let us choose $G > 0$.

Since $0 < 1/G$ and $L(2) > 0$, there exists a natural number m such that $0 < \frac{1}{mL(2)} < \frac{1}{G}$, by Archimedean property of $\mathbb{R}$.

Therefore $L(2^m) > G$.

Since L is strictly increasing on $(0, \infty)$, $L(x) > G$ for all $x > 2^m$.

This proves that $\lim\limits_{x \to \infty} L(x) = \infty$.

(ii) Let us choose $G > 0$. As in case (i), we have $L(2^m) > G$.

or, $L(\frac{1}{2^m}) < -G$.

Since L is strictly increasing on $(0, \infty)$, $L(x) < -G$ for all $x < \frac{1}{2^m}$.

This proves that $\lim\limits_{x \to 0+} L(x) = -\infty$.

Property 7. The function L defined by $L(x) = \int_1^x \frac{1}{t} dt, x > 0$ is continuous on $(0, \infty)$.

As $L(x)$ is defined as an integral, if follows from the Theorem 11.8.1 that L is continuous on $(0, \infty)$.

Property 8. $\frac{d}{dx} L(x) = \frac{1}{x}, x > 0$.

Proof. Let $x_0 > 0$ and let us choose $h > 0$.

Then $\frac{L(x_0+h) - L(x_0)}{h} = \frac{1}{h} \int_{x_0}^{x_0+h} \frac{1}{t} dt$.

For all $t \in [x_0, x_0 + h]$, $\frac{1}{x_0+h} \le \frac{1}{t} \le \frac{1}{x_0}$.

So we have $\int_{x_0}^{x_0+h} \frac{1}{x_0+h} dt \le \int_{x_0}^{x_0+h} \frac{1}{t} dt \le \int_{x_0}^{x_0+h} \frac{1}{x_0} dt$

or, $\frac{1}{x_0+h} \le \frac{L(x_0+h) - L(x_0)}{h} \le \frac{1}{x_0}$.

By sandwich theorem, $\lim\limits_{h \to 0+} \frac{L(x_0+h) - L(x_0)}{h} = \frac{1}{x_0}$ ... (i)

Let us choose $h < 0$ such that $x_0 + h > 0$.

For all $t \in [x_0 + h, x_0]$, $\frac{1}{x_0} \le \frac{1}{t} \le \frac{1}{x_0+h}$.

So we have $\int_{x_0+h}^{x_0} \frac{1}{x_0} dt \leq \int_{x_0+h}^{x_0} \frac{1}{t} dt \leq \int_{x_0+h}^{x_0} \frac{1}{x_0+h} dt$

or, $\frac{-h}{x_0} \leq \int_{x_0+h}^{x_0} \frac{1}{t} dt \leq \frac{-h}{x_0+h}$

or, $\frac{1}{x_0} \leq \frac{1}{h} \int_{x_0}^{x_0+h} \frac{1}{t} dt \leq \frac{1}{x_0+h}$.

By sandwich theorem, $\lim\limits_{h \to 0-} \frac{L(x_0+h)-L(x_0)}{h} = \frac{1}{x_0}$ ... *(ii)*

From (i) and (ii) $\lim\limits_{h \to 0} \dfrac{L(x_0+h) - L(x_0)}{h} = \frac{1}{x_0}$.

This implies $\frac{d}{dx} L(x) = \frac{1}{x}, x > 0$.

Corollary. $\lim\limits_{x \to 0} \frac{L(1+x)}{x} = 1$.

Property 9. *The logarithmic function L defined by*

$$L(x) = \int_1^x \frac{1}{t} dt, x > 0$$

is a bijective function from $(0, \infty)$ to $(-\infty, \infty)$.

Proof. As L is a strictly increasing function on $(0, \infty)$, it is one-to-one on $(0, \infty)$.

As $L(x) \to \infty$ as $x \to \infty$ and $L(x) \to -\infty$ as $x \to 0+$ and L is continuous and strictly increasing on $(0, \infty)$, the function L assumes every real number in $(-\infty, \infty)$ exactly once.

This proves that L is a bijective function with domain $(0, \infty)$ and range $(-\infty, \infty)$.

Definition. The unique real number x satisfying $L(x) = 1$ is denoted by e, i.e., $L(e) = 1$. Therefore e is defined by

$$1 = \int_1^e \frac{1}{t} dt.$$

Property 10. $e = \lim\limits_{n \to \infty} (1 + \frac{1}{n})^n$.

Proof. We have $\frac{d}{dx} L(x) = \frac{1}{x}, x > 0$. So $L'(1) = 1$.

That is, $\lim\limits_{h \to 0} \frac{L(1+h)-L(1)}{h} = 1$

or, $\lim\limits_{h \to 0} \frac{L(1+h)}{h} = 1$.

Let us consider the sequence $\{h_n\}$ where $h_n = \frac{1}{n}$. $\lim h_n = 0$.

By sequential criterion, $\lim\limits_{h_n \to 0} \frac{L(1+h_n)}{h_n} = 1$

or, $\lim\limits_{n \to \infty} nL(1 + \frac{1}{n}) = 1$

or, $\lim\limits_{n \to \infty} L\{(1 + \frac{1}{n})^n\} = 1$

or, $L\{\lim\limits_{n \to \infty} (1 + \frac{1}{n})^n\} = 1$, since L is continuous.

Since $L(e) = 1$ and and L is a bijective function on $(0, \infty)$ it follows that $\lim\limits_{n\to\infty}(1 + \frac{1}{n})^n = e$.

Property 11. For $x > -1$ and $x \neq 0$, $\frac{x}{1+x} < L(1 + x) < x$.

Proof. **Case 1.** Let $x_0 > 0$.

Then $1 + x_0 > 1$. For all $t \in [1, 1 + x_0]$, $\frac{1}{1+x_0} \leq \frac{1}{t} \leq 1$.

Let $f(t) = \frac{1}{t}, t \in [1, 1 + x_0]$. Then f is continuous on $[1, 1 + x_0]$ and $\frac{1}{1+x_0} < f(t) < 1$ at $t = 1 + \frac{x_0}{2}$.

Therefore $\int_1^{1+x_0} \frac{1}{1+x_0} dt < \int_1^{1+x_0} \frac{1}{t} dt < \int_1^{1+x_0} dt$

or, $\frac{x_0}{1+x_0} < L(1 + x_0) < x_0$.

As $x_0 > 0$ is arbitrary, $\frac{x}{1+x} < L(1 + x_0) < x$ for all $x > 0$.

Case 2. Let $-1 < x_0 < 0$.

Then $0 < x_0 + 1 < 1$.

For all $t \in [x_0 + 1, 1]$, $1 \leq \frac{1}{t} \leq \frac{1}{1+x_0}$.

Let $f(t) = \frac{1}{t}, t \in [x_0 + 1, 1]$. Then f is continuous on $[x_0 + 1, 1]$ and $1 < f(t) < \frac{1}{1+x_0}$ at $t = \frac{x_0}{2} + 1$.

Therefore $\int_{1+x_0}^1 dt < \int_{1+x_0}^1 \frac{1}{t} dt < \int_{1+x_0}^1 \frac{1}{1+x_0} dt$

or, $-x_0 < -L(1 + x_0) < \frac{-x_0}{1+x_0}$.

or, $\frac{x_0}{1+x_0} < L(1 + x_0) < x_0$.

As $x_0 \in (-1, 0)$ is arbitrary, $\frac{x}{1+x} < L(1 + x) < x$ for all $x \in (-1, 0)$.

Hence for all $x > -1$ and $x \neq 0$, $\frac{x}{1+x} < L(1 + x) < x$.

Note. For all $x \in N'(0, \frac{1}{2})$ $\quad \frac{1}{1+x} < \frac{L(1+x)}{x} < 1$. But $\lim\limits_{x\to 0} \frac{1}{1+x} = 1$.

Therefore by sandwich theorem, $\lim\limits_{x\to 0} \frac{L(1+x)}{x} = 1$.

Property 12. $2 < e < 3$.

Proof. $L(2) = \int_1^2 \frac{1}{t} dt$.

For all $t \in [1, 2]$, $\frac{1}{t} \leq 1$. So we have $\int_1^2 \frac{1}{t} dt \leq \int_1^2 1 dt$.

The function $\frac{1}{t}$ is continuous on $[1, 2]$ and also $\frac{1}{t} < 1$ at $t = 2$.

Therefore $\int_1^2 \frac{1}{t} dt < \int_1^2 1 dt = 1$, by Theorem 11.7.5.

Since L is a strictly increasing function on $(0, \infty)$ and $L(2) < 1$ and $L(e) = 1$, it follows that $2 < e$ ... ... (A)

Again $L(3) = \int_1^3 \frac{1}{t} dt = \int_1^2 \frac{1}{t} dt + \int_2^3 \frac{1}{t} dt$.

Now $\int_1^2 \frac{1}{t} dt = \int_0^1 \frac{du}{2-u}$, by the substitution $t = 2 - u$

and $\int_2^3 \frac{1}{t} dt = \int_0^1 \frac{du}{2+u}$, by the substitution $t = 2 + u$.

Hence $\int_1^3 \frac{1}{t}dt = \int_0^1 \frac{du}{2-u} + \int_0^1 \frac{du}{2+u} = 4\int_0^1 \frac{du}{4-u^2}$.

In $0 \le u \le 1$, $\frac{1}{4-u^2}$ is continuous and $\frac{1}{4-u^2} \ge \frac{1}{4}$.

It follows that $\int_0^1 \frac{du}{4-u^2} \ge \int_0^1 \frac{1}{4}du$.

Both the functions f and g where $f(u) = \frac{1}{4-u^2}, u \in [0,1]$; $g(u) = \frac{1}{4}, u \in [0,1]$ are continuous on $[0,1]$ and $f(\frac{1}{2}) > g(\frac{1}{2})$.

Therefore $\int_0^1 \frac{1}{4-u^2} > \frac{1}{4}$, by Theorem 11.7.5.

Consequently, $\int_1^3 \frac{1}{t}dt > 1$. That is, $L(3) > 1$.

Since L is a strictly increasing function on $(0,\infty)$ and $L(3) > 1$ and $L(e) = 1$, it follows that $e < 3$ ... ... (B)

From (A) and (B) it follows that $2 < e < 3$.

11.14. Exponential function.

Since the logarithmic function is a bijective function on $(0,\infty)$ with the range $(-\infty,\infty)$, it admits of an inverse function.

The inverse function of the logarithmic function is said to be the *exponential function* and it is denoted by $E(x)$ or e^x.

The domain of the exponential function is $\mathbb{R}$ and the range is $(0,\infty)$.

Since the logarithmic function is continuous and strictly increasing on $(0,\infty)$ with its range $(-\infty,\infty)$, the exponential function is continuous and strictly increasing on $(-\infty,\infty)$ with its range $(0,\infty)$.

Therefore for all $x > 0$, $EL(x) = x$ and for all $x \in \mathbb{R}$, $LE(x) = x$.

Property 1. $E(0) = 1$.

Since $EL(x) = x$ for all $x > 0$, $EL(1) = 1$ and since $L(1) = 0$, it follows that $E(0) = 1$.

Property 2. $E(x)E(y) = E(x+y)$ for all $x, y \in \mathbb{R}$.

Proof. For all $x, y \in \mathbb{R}, LE(x+y) = x + y = LE(x) + LE(y)$.

Since $E(x) > 0$ for all real x,

$L(E(x)) + L(E(y)) = L(E(x) + E(y))$, by property 3, 11.13.

or, $LE(x+y) = L(E(x) + E(y))$.

Since logarithmic function is one-to-one, it follows that

$E(x+y) = E(x).E(y)$.

Corollary. In particular, if $y = -x$, then $E(x-x) = E(x).E(-x)$ or, $E(x).E(-x) = E(0) = 1$. i.e., $E(-x) = \frac{1}{E(x)}$ for all real x.

Property 3. $E(nx) = \{E(x)\}^n, n$ being an integer.

Proof. **Case 1.** $n = 0$. The property holds trivially.

Case 2. n is a positive integer.

The property holds for $n = 1$.

Let us assume that the property holds for $n = m$, a positive integer. Then $E(mx) = \{E(x)\}^m$.

$$E\{(m+1)x\} = E(mx+x) = E(mx)E(x) = \{E(x)^m\}E(x) = \{E(x)\}^{m+1}.$$

This shows that the property holds for $n = m+1$ if it holds for $n = 1$.

By the principle of induction, $E(nx) = \{E(x)\}^n$ for all positive integers n.

Case 3. n is a negative integer, say $n = -m$.

$$E(nx) = E(-mx) = \frac{1}{E(mx)} = \frac{1}{\{E(x)\}^m} = \{E(x)\}^{-m} = \{E(x)\}^n.$$

Combining all case, the proof is complete

Property 4. $E(\alpha x) = \{E(x)\}^\alpha, \alpha$ being a rational number.

Proof. **Case 1.** $n = 0$. The property holds trivially.

Case 2. α is a positive rational number, say $\alpha = \frac{p}{q}$, where p and q are positive integers.

$E(px) = E(q.(\frac{p}{q})x) = [E((\frac{p}{q})x)]^q$, by case 1.

Also $E(px) = [E(x)]^p$. Therefore $[E(x)]^p = [E((\frac{p}{q})x)]^q$.

Since $E(x) > 0$ for all $x, [E(x)]^{\frac{p}{q}} = E((\frac{p}{q})x)$

or, $E(\alpha x) = \{E(x)\}^\alpha$.

Case 3. α is a negative rational number.

Let $\alpha = -\beta$, where β is a positive rational number.

$$E(\alpha x) = E(-\beta x) = \frac{1}{E(\beta x)} = \frac{1}{\{E(x)\}^\beta}, \text{ by case 1}$$
$$= \{E(x)\}^{-\beta} = \{E(x)\}^\alpha.$$

Combining the cases, the proof is complete.

Property 5. $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1.$

Proof. Let $y = e^x - 1$. Then $x = \log(1+y)$.

Since the exponential function is continuous on $(-\infty, \infty), x \to 0$ implies $e^x \to e^0 = 1$. Therefore as $x \to 0, y \to 0$.

Now $\lim\limits_{x\to 0} \frac{e^x-1}{x} = \lim\limits_{y\to 0} \frac{y}{\log(1+y)}$.

But $\lim\limits_{y\to 0} \frac{\log(1+y)}{y} = 1$, by the corollary of property 9, 11.13.

Hence $\lim\limits_{x\to 0} \frac{e^x-1}{x} = 1$.

Property 6. $\frac{d}{dx}(e^x) = e^x$ for all $x \in \mathbb{R}$.

Proof. Let $x_0 \in \mathbb{R}$. Then

$\lim\limits_{h \to 0} \frac{e^{x_0+h} - e^{x_0}}{h} = \lim\limits_{h \to 0} \frac{e^{x_0} e^h - e^{x_0}}{h} = e^{x_0} \lim\limits_{h \to 0} \frac{e^h - 1}{h} = e^{x_0}$, by property 5.

Hence $\frac{d}{dx}(e^x) = e^x$ for all $x \in \mathbb{R}$.

Worked Examples.

1. Prove that $\lim\limits_{x \to \infty} \frac{L(x)}{x^\alpha} = 0$ for all real $\alpha > 0$.

$\lim\limits_{x \to \infty} L(x) = \infty$ and $\lim\limits_{x \to \infty} x^\alpha = \infty$, for all real $\alpha > 0$.

Let us choose β such that $0 < \beta < \alpha$.

If $t > 1$, then $t^\beta > 1$ and $0 < t^{-1} < t^{-1+\beta}$.

If $x > 1$, then $0 \le \int_1^x t^{-1} dt \le \int_1^x t^{-1+\beta} dt$.

t^{-1} and $t^{-1+\beta}$ are both continuous on $[1, x]$ and $0 < t^{-1} < t^{-1+\beta}$ for $1 < t \le x$.

Therefore if $x > 1$, $0 < \int_1^x t^{-1} dt < \int_1^x t^{-1+\beta} dt$.

or, $0 < L(x) < \frac{x^\beta - 1}{\beta} < \frac{x^\beta}{\beta}$.

Therefore if $x > 1$, $0 < \frac{L(x)}{x^\alpha} < \frac{1}{\beta x^{\alpha - \beta}}$, since $x^\alpha > 0$.

$\lim\limits_{x \to \infty} \frac{1}{x^{\alpha - \beta}} = 0$, since $\alpha > \beta$ and therefore $\lim\limits_{x \to \infty} \frac{L(x)}{x^\alpha} = 0$.

Note. This result says that if $\alpha > 0$ then x^α tends to ∞ with x more rapidly than $L(x)$ does.

2. Prove that $\lim\limits_{x \to \infty} \frac{x^\alpha}{e^x} = 0$ for all real α.

Case 1. $\alpha > 0$.

We have $\lim\limits_{y \to \infty} \frac{L(y)}{y^\beta} = 0$ for all $\beta > 0$.

Let $\alpha = \frac{1}{\beta}$. Then $\lim\limits_{y \to \infty} \frac{L(y)^\alpha}{y} = 0$ for all $\alpha > 0$.

Let $L(y) = x$. Then $x \to \infty$ as $y \to \infty$ and $\lim\limits_{x \to \infty} \frac{x^\alpha}{e^x} = 0$ for all $\alpha > 0$.

Case 2. $\alpha = 0$.

$\lim\limits_{x \to \infty} \frac{x^\alpha}{e^x} = \lim\limits_{x \to \infty} \frac{1}{e^x} = \lim\limits_{x \to \infty} e^{-x} = \lim\limits_{y \to -\infty} e^y = 0$.

Case 3. $\alpha < 0$. Let $\beta = -\alpha$. Then $\beta > 0$.

$\lim\limits_{x \to \infty} \frac{x^\alpha}{e^x} = \lim\limits_{x \to \infty} \frac{1}{x^\beta e^x} = 0$, since $\lim\limits_{x \to \infty} \frac{1}{e^x} = 0$ and $\lim\limits_{x \to \infty} \frac{1}{x^\beta} = 0$.

Combining all the cases, we have the result.

Note. This result says that $E(x)$ tends to ∞ with x more rapidly than any power of x does.

Exercises 21

1. Take the partition $P_n = (0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n})$ of $[0,1]$ and show that $\sup\{L(P_n, f) : n \in \mathbb{N}\} = \inf\{U(P_n, f) : n \in \mathbb{N}\}$ for the function f.

   (i) $f(x) = x^2, x \in [0,1]$, (ii) $f(x) = x^3, x \in [0,1]$.

   Deduce that f is integrable on $[0,1]$.

2. Let $f : [a,b] \to \mathbb{R}$ be bounded and monotone increasing on $[a,b]$. If P_n be the partition of $[a,b]$ dividing into n sub-intervals of equal length prove that

   $\int_a^b f \leq U(P_n, f) \leq \int_a^b f + \frac{b-a}{n}[f(b) - f(a)]$.

   Consider the sequence of partitions $\{P_n\}$ and deduce that $\lim\limits_{n\to\infty} U(P_n, f) = \int_a^b f$.

   Utilise this result to evaluate

   (i) $\int_0^1 x dx$, (ii) $\int_0^1 x^2 dx$, (iii) $\int_0^1 e^x dx$.

   [**Hint.** $U(P_n, f) - L(P_n, f) = \frac{b-a}{n}[f(b) - f(a)]$, $L(P_n, f) \leq \int_a^b f \leq U(P_n, f)$.]

3. Let $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$. Let P_n be the partition of $[a,b]$ dividing into n sub intervals of equal length. Consider the sequence of partitions $\{P_n\}$ and prove that $\lim\limits_{n\to\infty} L(P_n, f) = \underline{\int}_a^b f$, $\lim\limits_{n\to\infty} U(P_n, f) = \overline{\int}_a^b f$.

   Evaluate $\underline{\int}_0^1 f$ and $\overline{\int}_0^1 f$ when

   (i) $f(x) = 2x, x \in [0,1]$, (ii) $f(x) = \cos x, x \in [0, \frac{\pi}{2}$,

4. Let $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ $(0 < a < b)$. Let $P_n = (a, ar, ar^2, \ldots, ar^n)$ where $r^n = b/a$. Consider the sequence of partitions $\{P_n\}$ and prove that $\lim\limits_{n\to\infty} L(P_n, f) = \underline{\int}_a^b f$, $\lim\limits_{n\to\infty} U(P_n, f) = \overline{\int}_a^b f$.

   Evaluate $\underline{\int}_1^2 f$ and $\overline{\int}_1^2 f$ when

   (i) $f(x) = x^9, x \in [1,2]$, (ii) $f(x) = x^{99}, x \in [1,2]$.

5. A function f is defined on [0,1] by $f(x) = x^2$, x is rational
   $= x^3$, x is irrational.

   (i) Evaluate $\underline{\int}_0^1, \overline{\int}_0^1 f$; (ii) Show that f is not integrable on $[0,1]$.

6. A function f is defined on [0,1] by $f(x) = x^2 + x^3$, x is rational
   $= x + x^2$, x is irrational.

   (i) Evaluate $\underline{\int}_0^1 f, \overline{\int}_0^1 f$; (ii) Show that f is not integrable on $[0,1]$.

7. A function f is defined on $[0,1]$ by $f(x) = \sin x$, x is rational
$= x$, x is irrational.

(i) Evaluate $\underline{\int}_0^{\frac{\pi}{2}} f, \overline{\int}_0^{\frac{\pi}{2}} f$; (ii) Show that f is not integrable on $[0, \frac{\pi}{2}]$.

8. Let $a, b \in \mathbb{R}$ and $a < b$. If $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$, prove that f is integrable on $[a,b]$.

Give an example of a function f integrable on $[0,1]$, but f is not a function of bounded variation on $[0,1]$.

9. Let $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Prove that

(i) $\max(f,g) : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$;

(ii) $\min(f,g) : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$.

[**Hint.** $\max(f,g) = \frac{1}{2}(f+g) + \frac{1}{2} \mid f - g \mid$; $\min(f,g) = \frac{1}{2}(f+g) - \frac{1}{2} \mid f - g \mid$.]

10. Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Prove that

(i) $f^+ : [a,b] \to \mathbb{R}$ defined by $f^+(x) = \sup\{f(x), 0\}, x \in [a,b]$ is integrable on $[a,b]$.

(ii) $f^- : [a,b] \to \mathbb{R}$ defined by $f^-(x) = -\inf\{f(x), 0\}, x \in [a,b]$ is integrable on $[a,b]$.

[**Hint.** $f^+ = \frac{1}{2}(\mid f \mid + f)$; $f^- = \frac{1}{2}(\mid f \mid - f)$.]

11. Let $f(x) = x[x], x \in [0,3]$. Show that f is integrable on $[0,3]$. Evaluate $\int_0^3 f$.

12. Let $f(x) = x - [x], x \in [0,3]$. Show that f is integrable on $[0,3]$. Evaluate $\int_0^3 f$.

13. A function f is defined on $I = [0,10]$ by $f(x) = 0$ when $x \in I \cap \mathbb{Z}$.
$= 1$ when $x \in I - \mathbb{Z}$.

Prove that f is integrable on I. Evaluate $\int_0^{10} f$.

14. A function f is defined on $[0,1]$ by $f(0) = 1$,

$f(x) = (-1)^{n-1}$ when $\frac{1}{n+1} < x \leq \frac{1}{n}$ $(n = 1, 2, 3, \ldots)$.

Prove that (i) f is integrable on $[0,1]$, (ii) $\int_0^1 f = \log(4/e)$.

15. A function f is defined on $[0,1]$ by $f(0) = 0$,

$f(x) = \frac{1}{2^n}, \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}$ $(n = 0, 1, 2, \ldots)$.

Prove that (i) f is integrable on $[0,1]$, (ii) $\int_0^1 f = \frac{2}{3}$.

16. A function f is defined on $[0,1]$ by $f(0) = 0$,

$f(x) = \lim\limits_{n\to\infty} \frac{(1+\sin\frac{\pi}{x})^n - 1}{(1+\sin\frac{\pi}{x})^n + 1}$, $x \in (0,1]$.

Prove that (i) f is integrable on $[0, 1]$.

[**Hint.** $\frac{1}{2} < x < 1 \Rightarrow \pi < \frac{\pi}{x} < 2\pi \Rightarrow 0 < 1 + \sin\frac{\pi}{x} < 1 \Rightarrow f(x) = -1$.
$\frac{1}{3} < x < \frac{1}{2} \Rightarrow 2\pi < \frac{\pi}{x} < 3\pi \Rightarrow 1 < 1 + \sin\frac{\pi}{x} < 2 \Rightarrow f(x) = 1$]

17. Show that

(i) $-\frac{1}{2} < \int_0^1 \frac{x^3 \cos 5x}{2+x^2} dx < \frac{1}{2}$ (ii) $\frac{1}{3\sqrt{2}} < \int_0^1 \frac{x^2}{\sqrt{1+x}} dx < \frac{1}{3}$

(iii) $\frac{\pi^3}{24\sqrt{2}} < \int_0^{\pi/2} \frac{x^2}{\sin x + \cos x} dx < \frac{\pi^3}{24}$ (iv) $\frac{1}{3} < \int_0^1 \frac{dx}{1+x+x^2} < \frac{\pi}{4}$

(v) $\frac{\pi^3}{96} < \int_{-\pi/2}^{\pi/2} \frac{x^2}{5+3\sin x} dx < \frac{\pi^3}{24}$.

18. (i) If a function f is continuous on a closed interval $[a, b]$ and $\int_a^b fg = 0$ for every continuous function g on $[a, b]$, prove that $f(x) = 0$ for all $x \in [a, b]$.

[**Hint.** Take, in particular, $g = f$.]

(ii) A function f is integrable on $[a, b]$ and $\int_a^b f^2(x)dx = 0$. Prove that $f(x) = 0$ at every point of continuity in $[a, b]$.

19. A function f is continuous for all $x \geq 0$ and $f(x) \neq 0$ for all $x > 0$.
If $\{f(x)\}^2 = 2\int_0^x f(t)dt$ prove that $f(x) = x$ for all $x \geq 0$.

[**Hint.** $f(0) = 0$ and $f'(x) = 1$ for all $x > 0$. Use Lagrange's mean value theorem to f on $[0, x]$.]

20. The functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are both continuous on $[a, b]$ and $\int_a^b | f - g | = 0$. Prove that $f = g$.

Give an example of functions f and g both integrable on $[a, b]$ such that $\int_a^b | f - g | = 0$, but $f \neq g$.

21. A function f is defined on $[0, 2]$ by
$$f(x) = 0, 0 \leq x \leq 1$$
$$= 1, 1 < x \leq 2$$
$$= 2, 2 < x \leq 3.$$

Let $F(x) = \int_0^x f(t)dt, x \in [0, 2]$. Find F. Show that F is continuous on $[0, 2]$.

22. A function f is defined on $[0, 3]$ by
$$f(x) = x, 0 \leq x \leq 1$$
$$= 1, 1 < x \leq 2$$
$$= x - 1, 2 < x \leq 3.$$

Show that f is integrable on $[0, 3]$.

Let $F(x) = \int_0^x f(t)dt, x \in [0, 3]$. Find F. Show that $F'(x) = f(x), x \in [0, 3]$.

23. A function f is defined on $[0, 3]$ by
$$f(x) = x, 0 \leq x < 1$$
$$= 1, 1 \leq x \leq 2$$
$$= x, 2 < x \leq 3.$$

Show that f is integrable on $[0, 3]$.

Let $F(x) = \int_0^x f(t)dt, 0 \leq x \leq 3$. Find F. Find $F'(x)$ at all points where F is differentiable.

24. For $x \geq 0$, let $\phi(x) = \lim\limits_{n\to\infty} \frac{x^n+2}{x^n+1}$; and $f(x) = \int_0^x \phi(t)dt$.

Show that f is continuous at 1 but not differentiable at 1.

25. Find F' where F is defined on $[1, \infty)$ by

(i) $F(x) = \int_x^{e^x} \sqrt{1+t^2}dt$, (ii) $F(x) = \int_x^{x^2} \sin\sqrt{t}dt$,

26. Prove that

(i) $\lim\limits_{x\to 2} \dfrac{\int_2^x e^{\sqrt{1+t^2}}dt}{x-2} = e^{\sqrt{5}}$, (ii) $\lim\limits_{x\to 0} \dfrac{\int_0^{x^2} \sin\sqrt{t}dt}{x^3} = \frac{2}{3}$,

(iii) $\lim\limits_{x\to 0} \dfrac{\int_{-x}^{x} f(t)dt}{\int_0^{2x} f(t+1)dt} = \dfrac{f(0)}{f(1)}$, where f is continuous on $\mathbb{R}$.

27. A function f is continuous on $\mathbb{R}$ and $\int_{-x}^{x} f(t)dt = 0$ for all $x \in \mathbb{R}$. Prove that f is an odd function on $\mathbb{R}$.

28. A function f is defined on $[-3, 3]$ by $f(x) = 2x\sin\frac{\pi}{x} - \pi\cos\frac{\pi}{x}, x \neq 0$.
$= 0, \quad x = 0$.

Show that (i) f is not continuous on $[-3, 3]$; (ii) f is integrable on $[-3, 3]$;

(iii) f has an antiderivative ϕ on $[-3, 3]$; (iv) $\int_{-3}^{3} f = \phi(3) - \phi(-3)$.

29. Let $f(x) = \operatorname{sgn} x, x \in [-1, 3]$.

(i) Show that f is integrable on $[-1, 3]$. (ii) Evaluate $\int_{-1}^{3} f$.

(iii) Show that the evaluation of $\int_{-1}^{3} f$ cannot be done by the fundamental theorem of Integral calculus.

30. Let $f(x) = x[x], x \in [0, 3]$.

(i) Show that f is integrable on $[0, 3]$. (ii) Evaluate $\int_0^3 f$.

(iii) Show that the evaluation of $\int_0^3 f$ cannot be done by the fundamental theorem of Integral calculus.

31. Let $f(x) = [x]$, $x \in [1, 3]$; $\phi(x) = x$, $x \in [1, 2]$
$= 2x - 2$, $x \in (2, 3]$.

(i) Show that f is integrable on $[1, 3]$. (ii) Evaluate $\int_1^3 f$.

(iii) Without evaluating the integral show that $\int_1^3 f = \phi(3) - \phi(1)$.

32. A function f is defined on $[0,1]$ by $f(x) = 2x$, x is rational
$= 1 - x$, x is irrational.

Take the partition $P_n = (0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n})$ of $[0,1]$. Choose intermediate points α_r, β_r in $[x_{r-1}, x_r]$ to show that $\lim\limits_{n\to\infty} S(P_n, f, \alpha) \neq \lim\limits_{n\to\infty} S(P_n, f, \beta)$.

Deduce that f is not integrable on $[0,1]$.

33. Evaluate the limits

(i) $\lim\limits_{n\to\infty} [\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+3n}]$

(ii) $\lim\limits_{n\to\infty} [\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{n\pi}{n}]$

(iii) $\lim\limits_{n\to\infty} [\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+4n^2}]$

(iv) $\lim\limits_{n\to\infty} [(1+\frac{1}{n})(1+\frac{2}{n})\cdots(1+\frac{n}{n})]^{\frac{1}{n}}$

(v) $\lim\limits_{n\to\infty} [(1+\frac{1}{n^2})(1+\frac{2^2}{n^2})^2\cdots(1+\frac{n^2}{n^2})^n]^{\frac{1}{n}}$.

34. Let $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ be both continuous on $[a,b]$ and $\int_a^b f = \int_a^b g$. Prove that there exists a point c in $[a,b]$ such that $f(c) = g(c)$.

35. A function $f : [0,1] \to \mathbb{R}$ is continuous on $[0,1]$ and $\int_0^x f(t)dt = \int_x^1 f(t)dt$ for all $x \in [0,1]$. Prove that $f(x) = 0$ for all $x \in [0,1]$.

36. (a) Discuss the applicability of the second Mean value theorem to the integral $\int_{-\pi/2}^{\pi/2} x^2 \cos x\, dx$.

(b) Verify second Mean value theorem (Weierstrass form) for the function f on the indicated intervals.

(i) $f(x) = x\sin x, x \in [-\pi/2, \pi/2]$, (ii) $f(x) = xe^x, x \in [-1,1]$, (iii) $f(x) = x\sin x, x \in [\pi, 2\pi]$.

37. Use Bonnet's form of second Mean value theorem to prove that $|\int_a^b \sin x^2 dx| \leq \frac{1}{a}$ if $0 < a < b < \infty$,

[**Hint.** Take $f(x) = \frac{1}{2x}$, $x \in [a,b]$, $\phi(x) = 2x\sin x^2$, $x \in [a,b]$ in the theorem 11.12.3.]

38. If f_0 is continuous on $[0,\infty)$ and for all $n \in \mathbb{N}$, $f_n(x) = \int_0^x f_{n-1}(t)dt$, $x \geq 0$. Prove that $f_n(x) = \frac{1}{(n-1)!}\int_0^x f_0(t)(x-t)^{n-1}dt$.

[**Hint.** f_n is continuous on $[0,\infty)$ for all $n \in \mathbb{N}$. On integration by parts,
$f_n(x) = xf_{n-1}(x) - \int_0^x tf_{n-2}(t)dt = \int_0^x (x-t)f_{n-2}(t)dt$.
Integrate by parts successively.]

39. A function f is continuous on $[0, \infty)$ and $\phi(x) = \frac{1}{3!}\int_0^x (x-t)^3 f(t)dt$, $x \geq 0$. Show that $\phi^{iv}(x) = f(x)$ for all $x \geq 0$.

40. Justifying each step evaluate the integrals by substitution

(i) $\int_0^2 t\sqrt{1+t^2}dt$, (ii) $\int_0^2 t^2\sqrt{1+t^3}dt$,

(iii) $\int_0^2 te^{t^2}dt$, (iv) $\int_0^{\frac{\pi}{2}} \sin^3 t \cos t dt$.

41. If $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $g(x) = \int_a^x f(t)dt$ for all $x \in [a, b]$, prove that g is a function of bounded variation on $[a, b]$.

[**Hint.** Using Theorem 11.8.1, $|g(x_2) - g(x_1)| \leq k|x_2 - x_1|$ for any two points x_1, x_2 in $[a, b]$.]

42. If $n \in \mathbb{N}$ and $n \geq 2$, prove that $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \log n < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$.

[**Hint.** If $k \in \mathbb{N}$ and $k \geq 2$, $\frac{1}{k} < \int_{k-1}^{k} \frac{1}{t}dt < \frac{1}{k-1}$.]

43. If $f : [a, b] \to \mathbb{R}$ be bounded and integrable on $[a, b]$, prove that
(i) $\lim\limits_{n\to\infty} \int_a^b f(x)\sin nx\, dx = 0$ (ii) $\lim\limits_{n\to\infty} \int_a^b f(x)\cos nx\, dx = 0$.

[Riemann-Lebesgue theorem]

[**Hint.**(i) Let $\epsilon > 0$. Since f is integrable on $[a, b]$, there exists a partition $P = (a = x_0, x_1, ..., x_{p-1}, x_p = b)$ of $[a, b]$ such that $U(P, f) - L(P, f) < \frac{\epsilon}{2}$.

Let M_r, m_r be the supremum and the infimum of f on $[x_{r-1}, x_r]$, $r = 1, 2, ..., p$.

$\int_a^b f(x)\sin nx\, dx = \int_a^{x_1} f(x)\sin nx\, dx + \cdots + \int_{x_{p-1}}^b f(x)\sin nx\, dx$
$= \{\int_a^{x_1}[f(x) - f(x_1)]\sin nx\ \ dx + \cdots + \int_{x_{p-1}}^b [f(x) - f(b)]\sin nx\ \ dx\} + \{\int_a^{x_1} f(x_1)\sin nx\, dx + \cdots + \int_{x_{p-1}}^b f(b)\sin nx\, dx\} = s_1 + s_2$, say.

$|s_1| \leq \int_a^{x_1} |f(x) - f(x_1)|dx + \cdots + \int_{x_{p-1}}^b |f(x) - f(b)|dx \leq (M_1 - m_1)(x_1 - a) + \cdots + (M_p - m_p)(b - x_{p-1}) < \frac{\epsilon}{2}$.

$|s_2| \leq |f(x_1)| \int_a^{x_1} \sin nx\ dx + \cdots + |f(x_p)| \int_{x_{p-1}}^b \sin nx\ dx \leq \frac{2}{n}[|f(x_1)| + \cdots + |f(x_p)|]$, since $|\int_{x_{r-1}}^{x_r} \sin nx\, dx| \leq |\frac{\cos nx_{r-1} - \cos nx_r}{n}| \leq \frac{2}{n}$.

Since $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, there exists a natural number k such that $|s_2| < \frac{\epsilon}{2}$ for all $n > k$.

Therefore $|\int_a^b f(x)\sin nx\, dx| \leq |s_1| + |s_2| < \epsilon$ for all $n > k$.]

12. IMPROPER INTEGRALS

12.1. Introduction. In the preceding chapter on Riemann Integral the theory of integration was developed under two assumptions –

(i) the interval of integration was required to be a closed and bounded interval, and

(ii) the integrand was required to be bounded on the interval.

The scope of the theory of integration may be widened by relaxing these restrictions. If these restrictions are relaxed we have the following two types of integrals, called *improper integrals* or *infinite integrals* –

(a) *improper integrals on a finite interval where the integrand is unbounded;*

(b) *improper integrals on an unbounded interval.*

We define *convergence* of improper integrals and discuss the properties of each type separately.

A. Improper integrals on a closed and bounded interval, the integrand having infinite discontinuities.

12.2. Definitions.

I. Convergence of the improper integral $\int_a^b f(x)dx$ when a is the *only* point of infinite discontinuity of f in $[a, b]$.

Let the left end point a of the closed and bounded interval $[a, b]$ be the only point of infinite discontinuity of a function f which is bounded and integrable on $[a + \epsilon, b]$ for every ϵ satisfying $0 < \epsilon < b - a$.

Let $\phi(\epsilon) = \int_{a+\epsilon}^b f(x)dx, 0 < \epsilon < b - a$.

If $\lim\limits_{\epsilon \to 0+} \phi(\epsilon)$ exists (finitely) then the improper integral $\int_a^b f(x)dx$ is said to be *convergent*. If the limit be l, we write $\int_a^b f(x)dx = l$.

If $\lim\limits_{\epsilon \to 0+} \phi(\epsilon)$ does not exist (finitely) then the improper integral $\int_a^b f(x)dx$ is said to be *divergent*.

Note. If a be the *only* point of infinite discontinuity of a function f which is bounded and integrable on $[a+\epsilon, b]$ for every ϵ satisfying $0 < \epsilon < b-a$ and $\int_a^b f(x)dx$ is convergent, then $\int_a^c f(x)dx$ is also convergent for all $c \in (a, b)$.

Examples.

1. The integral $\int_0^1 \frac{1}{x}dx$ is improper, since 0 is a point of infinite discontinuity of the integrand.

The integrand is bounded and integrable on $[0+\epsilon, 1]$ for all ϵ satisfying $0 < \epsilon < 1$.

$\lim\limits_{\epsilon \to 0+} \int_{0+\epsilon}^1 \frac{1}{x}dx = \lim\limits_{\epsilon \to 0+} [-\log \epsilon] = \infty$.

Therefore the improper integral $\int_0^1 \frac{1}{x}dx$ is divergent.

2. The integral $\int_0^1 \frac{1}{\sqrt{x}}dx$ is improper, since 0 is a point of infinite discontinuity of the integrand.

The integrand is bounded and integrable on $[0+\epsilon, 1]$ for all ϵ satisfying $0 < \epsilon < 1$.

$\lim\limits_{\epsilon \to 0+} \int_{0+\epsilon}^1 \frac{1}{\sqrt{x}}dx = \lim\limits_{\epsilon \to 0+} [2 - 2\sqrt{\epsilon}] = 2$.

Therefore the improper integral $\int_0^1 \frac{1}{\sqrt{x}}dx$ is convergent and $\int_0^1 \frac{1}{\sqrt{x}}dx = 2$.

II. Convergence of the improper integral $\int_a^b f(x)dx$ when b is the *only* point of infinite discontinuity of f in $[a, b]$.

Let the right end point b of the closed and bounded interval $[a, b]$ be the only point of infinite discontinuity of a function f which is bounded and integrable on $[a, b-\epsilon]$ for every ϵ satisfying $0 < \epsilon < b-a$.

Let $\phi(\epsilon) = \int_a^{b-\epsilon} f(x)dx, 0 < \epsilon < b-a$.

If $\lim\limits_{\epsilon \to 0+} \phi(\epsilon)$ exists (finitely) then the improper integral $\int_a^b f(x)dx$ is said to be *convergent*. If the limit be l, we write $\int_a^b f(x)dx = l$.

If $\lim\limits_{\epsilon \to 0+} \phi(\epsilon)$ does not exist (finitely) then the improper integral $\int_a^b f(x)dx$ is said to be *divergent*.

Note. If b be the *only* point of infinite discontinuity of a function f which is bounded and integrable on $[a, b-\epsilon]$ for every ϵ satisfying $0 < \epsilon < b-a$ and $\int_a^b f(x)dx$ is convergent, then $\int_c^b f(x)dx$ is also convergent for all $c \in (a, b)$.

Examples (continued).

3. The integral $\int_0^1 \frac{1}{1-x}dx$ is improper, since 1 is a point of infinite discontinuity of the integrand.

The integrand is bounded and integrable on $[0, 1-\epsilon]$ for all ϵ satisfying $0 < \epsilon < 1$.

$\lim\limits_{\epsilon \to 0+} \int_0^{1-\epsilon} \frac{1}{1-x}dx = \lim\limits_{\epsilon \to 0+} [-\log \epsilon] = \infty.$

Therefore the improper integral $\int_0^1 \frac{1}{1-x}dx$ is divergent.

4. The integral $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$ is improper, since 1 is a point of infinite discontinuity of the integrand.

The integrand is bounded and integrable on $[0, 1-\epsilon]$ for all ϵ satisfying $0 < \epsilon < 1$.

$\lim\limits_{\epsilon \to 0+} \int_0^{1-\epsilon} \frac{1}{\sqrt{1-x^2}}dx = \lim\limits_{\epsilon \to 0+} [\sin^{-1}(1-\epsilon)] = \frac{\pi}{2}.$

Therefore the improper integral $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$ is convergent and $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \frac{\pi}{2}$.

III. Convergence of the improper integral $\int_a^b f(x)dx$ when a and b are the *only* points of infinite discontinuity of f in $[a, b]$.

Let the end points a, b of the closed and bounded interval $[a, b]$ be the only points of infinite discontinuity of a function f which is bounded and integrable on $[a+\epsilon, b-\epsilon']$ for every ϵ, ϵ' satisfying $0 < \epsilon < b-a$, $0 < \epsilon' < b-a$.

Let $c \in (a, b)$.

If the improper integrals $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ be both convergent according to the definitions given above, then the improper integral $\int_a^b f(x)dx$ is said to be *convergent* and we write

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Note. If the end points a and b of the closed and bounded interval $[a, b]$ be the *only* points of infinite discontinuity of a function f and the improper integral $\int_a^b f(x)dx$ be convergent, then for *any* point $d \in (a, b)$

$$\int_a^b f(x)dx = \int_a^d f(x)dx + \int_d^b f(x)dx.$$

Examples (continued).

5. The integral $\int_0^2 \frac{1}{\sqrt{x(2-x)}}dx$ is improper, since 0 and 2 are points of infinite discontinuity of the integrand.

The integrand is bounded and integrable on $[0+\epsilon, 2-\epsilon']$ for all ϵ, ϵ' satisfying $0<\epsilon<2$, $0<\epsilon'<2$.

Let us examine if $\lim\limits_{\epsilon\to 0+}\int_\epsilon^1 \frac{1}{\sqrt{x(2-x)}}dx$ and $\lim\limits_{\epsilon'\to 0+}\int_1^{2-\epsilon'} \frac{1}{\sqrt{x(2-x)}}dx$ exist.

$\lim\limits_{\epsilon\to 0+}\int_{0+\epsilon}^1 \frac{1}{\sqrt{x(2-x)}}dx = \lim\limits_{\epsilon\to 0+}[\sin^{-1}(x-1)]_\epsilon^1 = \frac{\pi}{2}$,

$\lim\limits_{\epsilon'\to 0+}\int_1^{2-\epsilon'} \frac{1}{\sqrt{x(2-x)}}dx = \lim\limits_{\epsilon'\to 0+}[\sin^{-1}(x-1)]_1^{2-\epsilon'} = \frac{\pi}{2}$.

Therefore $\int_0^2 \frac{1}{\sqrt{x(2-x)}}dx$ is convergent and $\int_0^2 \frac{1}{\sqrt{x(2-x)}}dx = \pi$.

IV. Convergence of the improper integral $\int_a^b f(x)dx$ when an interior point c is the *only* point of infinite discontinuity of f in $[a,b]$.

If the improper integrals $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ be both convergent according to the definitions given above, then the improper integral $\int_a^b f(x)dx$ is said to be *convergent* and we write

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

Therefore if both the limits $\lim\limits_{\epsilon\to 0+}\int_a^{c-\epsilon} f(x)dx$ and $\lim\limits_{\epsilon'\to 0+}\int_{c+\epsilon'}^b f(x)dx$ exist then $\int_a^b f(x)dx = \lim\limits_{\epsilon\to 0+}\int_a^{c-\epsilon} f(x)dx + \lim\limits_{\epsilon'\to 0+}\int_{c+\epsilon'}^b f(x)dx$.

If the improper integral $\int_a^b f(x)dx$ is convergent, its value is also equal to the symmetric limit $[\lim\limits_{\epsilon\to 0+}\int_a^{c-\epsilon} f(x)dx + \lim\limits_{\epsilon\to 0+}\int_{c+\epsilon}^b f(x)dx]$.

It may happen that the improper integral $\int_a^b f(x)dx$ is divergent but the limit $\lim\limits_{\epsilon\to 0+}[\int_a^{c-\epsilon} f(x)dx + \int_{c+\epsilon}^b f(x)dx]$ exists, then this symmetric limit is called the *Cauchy principal value* of the improper integral $\int_a^b f(x)dx$ and it is denoted by $P\int_a^b f(x)dx$.

For example, let us consider the improper integral $\int_{-1}^1 f(x)dx$, where
$f(x) = \frac{1}{x}$, $x \neq 0$
$= 0$, $x = 0$.

Here $\lim\limits_{\epsilon\to 0+}\int_{-1}^{0-\epsilon} f(x)dx + \lim\limits_{\epsilon'\to 0+}\int_{0+\epsilon'}^1 f(x)dx$
$= \lim\limits_{\epsilon\to 0+}\log\epsilon + \lim\limits_{\epsilon'\to 0+}(-\log\epsilon')$

and this limit does not exist if $\epsilon\to 0+$, $\epsilon'\to 0+$ independently.

But $\lim\limits_{\epsilon\to 0+}[\int_{-1}^{0-\epsilon} f(x)dx + \int_{0+\epsilon}^1 f(x)dx] = \lim\limits_{\epsilon\to 0+}(\log\epsilon - \log\epsilon) = 0$.

Therefore the integral $\int_{-1}^1 f(x)dx$ is divergent but $P\int_{-1}^1 f(x)dx = 0$.

V. Convergence of the improper integral $\int_a^b f(x)dx$ when a finite number of points $c_1, c_2, \ldots, c_m$ are the only points of infinite discontinuity of f in $[a, b]$.

Case 1. Let $a < c_1 < c_2 < \cdots < c_m < b$.

If the improper integrals $\int_a^{c_1} f(x)dx$, $\int_{c_1}^{c_2} f(x)dx, \cdots, \int_{c_m}^b f(x)dx$ be all convergent according to the definitions given above, then the improper integral $\int_a^b f(x)dx$ is said to be *convergent* and we write

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \cdots + \int_{c_m}^b f(x)dx.$$

Case 2. Either $a = c_1$ or $b = c_m$ or both.

If $a = c_1$, then $\int_a^b f(x)dx = \int_a^{c_2} f(x)dx + \int_{c_2}^{c_3} f(x)dx + \cdots + \int_{c_m}^b f(x)dx$, provided each integral in the right hand side is convergent according to the definitions given.

If $b = c_m$, then $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \cdots + \int_{c_{m-1}}^b f(x)dx$, provided each integral in the right hand side is convergent according to the definitions given.

12.3. Tests for convergence, positive integrand.

Theorem 12.3.1. Let a be the *only* point of infinite discontinuity of a function f which is integrable on $[a+\epsilon, b]$ for all ϵ satisfying $0 < \epsilon < b-a$ and $f(x) > 0$ for all $x \in (a, b]$.

A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)dx$ is that there exists a positive real number k such that

$$\int_{a+\epsilon}^b f(x)dx < k \text{ for all } \epsilon \text{ satisfying } 0 < \epsilon < b - a.$$

Proof. Let $\phi(\epsilon) = \int_{a+\epsilon}^b f(x)dx$, $0 < \epsilon < b - a$.

Let $0 < \epsilon_1 < \epsilon_2 < b - a$. Then $\phi(\epsilon_1) - \phi(\epsilon_2) = \int_{a+\epsilon_1}^b f(x)dx - \int_{a+\epsilon_2}^b f(x)dx = \int_{a+\epsilon_1}^{a+\epsilon_2} f(x)dx \geq 0$, since $f(x) > 0$ on $[a+\epsilon_1, a+\epsilon_2]$.

$0 < \epsilon_1 < \epsilon_2 \Rightarrow \phi(\epsilon_1) \geq \phi(\epsilon_2)$. This shows that ϕ is a monotone decreasing function on $(0, b-a)$. Therefore $\phi(\epsilon)$ will tend to a finite limit as $\epsilon \to 0+$ if and only if ϕ is bounded above.

In other words, the improper integral $\int_a^b f(x)dx$ is convergent if and only if there exists a positive real number k such that

$$\int_{a+\epsilon}^b f(x)dx < k \text{ for all } \epsilon \text{ satisfying } 0 < \epsilon < b - a.$$

Note. If ϕ be not bounded above, then $\phi(\epsilon)$ tends to ∞ as $\epsilon \to 0+$ and the improper integral $\int_a^b f(x)dx$ diverges to ∞.

Theorem 12.3.2. Let b be the *only* point of infinite discontinuity of a function f which is integrable on $[a, b-\epsilon]$ for all ϵ satisfying $0 < \epsilon < b-a$ and $f(x) > 0$ for all $x \in [a, b)$.

A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)dx$ is that there exists a positive real number k such that

$$\int_a^{b-\epsilon} f(x)dx < k \text{ for all } \epsilon \text{ satisfying } 0 < \epsilon < b - a.$$

Similar proof.

Theorem 12.3.3. Comparison test.

Let a be the *only* point of infinite discontinuity of the functions f and g which are both integrable on $[a + \epsilon, b]$ for all ϵ satisfying $0 < \epsilon < b - a$ and $0 < f(x) \leq kg(x)$ for all $x \in (a, b]$, where $k > 0$. Then

(i) $\int_a^b g(x)dx$ is convergent $\Rightarrow \int_a^b f(x)dx$ is convergent;

(ii) $\int_a^b f(x)dx$ is divergent $\Rightarrow \int_a^b g(x)dx$ is divergent.

Proof. Since f and g are both integrable on $[a+\epsilon, b]$ and $0 < f(x) \leq kg(x)$ for all $x \in [a + \epsilon, b]$, we have $\int_{a+\epsilon}^b f(x)dx \leq k \int_{a+\epsilon}^b g(x)dx$.

This holds for all ϵ satisfying $0 < \epsilon < b - a$.

(i) If $\int_a^b g(x)dx$ be convergent then there exists a positive real number k' such that $\int_{a+\epsilon}^b g(x)dx < k'$ for all ϵ satisfying $0 < \epsilon < b - a$.

Then $\int_{a+\epsilon}^b f(x)dx < kk'$ for all ϵ satisfying $0 < \epsilon < b - a$ and this proves that $\int_a^b f(x)dx$ is convergent.

(ii) Let $\phi(\epsilon) = \int_{a+\epsilon}^b f(x)dx$, $\psi(\epsilon) = \int_{a+\epsilon}^b g(x)dx$ for $0 < \epsilon < b - a$. Then $\phi(\epsilon) \leq k\psi(\epsilon)$ for all $\epsilon \in (0, b - a)$.

If $\int_a^b f(x)dx$ be divergent, then ϕ is not bounded above on $(0, b - a)$ and therefore ψ is not bounded above on $(0, b - a)$.

This proves that $\int_a^b g(x)dx$ is divergent.

This completes the proof.

Theorem 12.3.4. Comparison test (limit form).

Let a be the *only* point of infinite discontinuity of the functions f and g which are both integrable on $[a + \epsilon, b]$ for all ϵ satisfying $0 < \epsilon < b - a$ and $f(x) > 0$, $g(x) > 0$ for all $x \in (a, b]$.

If $\lim_{x \to a+} \frac{f(x)}{g(x)} = l$, where l is a *non-zero finite* number, then the two improper integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converge or diverge together.

Proof. Since $\frac{f(x)}{g(x)} > 0$ for all $x \in (a, b]$, $l \geq 0$. Since l is non-zero, $l > 0$.

Let us choose a positive δ such that $l - \delta > 0$. Since $\lim_{x \to a+} \frac{f(x)}{g(x)} = l$, there exists a point c in (a, b) such that $l - \delta < \frac{f(x)}{g(x)} < l + \delta$ for all $x \in (a, c]$

or, $(l - \delta)g(x) < f(x) < (l + \delta)g(x)$ for all $x \in (a, c]$.

(i) Let $\int_a^b f(x)dx$ be convergent. Then $\int_a^c f(x)dx$ is convergent.

Since $(l - \delta)g(x) < f(x)$ for all $x \in (a, c]$ and $l - \delta > 0$, $\int_a^c g(x)dx$ is convergent by comparison test and therefore $\int_a^b g(x)dx$ is convergent.

(ii) Let $\int_a^b g(x)dx$ be convergent. Then $\int_a^c g(x)dx$ is convergent.

Since $f(x) < (l + \delta)g(x)$ for all $x \in (a, c]$ and $l + \delta > 0$, $\int_a^c f(x)dx$ is convergent by comparison test and therefore $\int_a^b f(x)dx$ is convergent

(iii) Let $\int_a^b f(x)dx$ be divergent. Then $\int_a^c f(x)dx$ is divergent.

Since $(l + \delta)g(x) > f(x)$ for all $x \in (a, c]$ and $l + \delta > 0$, $\int_a^c g(x)dx$ is divergent by comparison test and therefore $\int_a^b g(x)dx$ is divergent.

(iv) Let $\int_a^b g(x)dx$ be divergent. Then $\int_a^c g(x)dx$ is divergent.

Since $f(x) > (l - \delta)g(x)$ for all $x \in (a, c]$ and $l - \delta > 0$, $\int_a^c f(x)dx$ is divergent by comparison test and therefore $\int_a^b f(x)dx$ is divergent.

Thus the improper integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converge or diverge together.

Note 1. If $\lim_{x \to a+} \frac{f(x)}{g(x)} = 0$ then for a pre-assigned positive ϵ, there exists a positive $\delta < b - a$ such that $f(x) < \epsilon g(x)$ for all x satisfying $a < x < a + \delta < b$. Then $\int_a^b g(x)dx$ converges $\Rightarrow \int_a^b f(x)dx$ converges.

2. If $\lim_{x \to a+} \frac{f(x)}{g(x)} = \infty$, then for a pre-assigned positive G, there exists a positive $\delta < b - a$ such that $f(x) > Gg(x)$ for all x satisfying $a < x < a + \delta < b$. Then $\int_a^b g(x)dx$ diverges $\Rightarrow \int_a^b f(x)dx$ diverges.

12.3.5. A useful comparison integral.

The integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is convergent if and only if $\mu < 1$.

Proof. The integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is proper if $\mu \leq 0$.

Let $\mu > 0$. Let $f(x) = \frac{1}{(x-a)^\mu}, a < x \leq b$. a is the only point of infinite discontinuity of f. f is integrable on $[a + \epsilon, b]$ for $0 < \epsilon < b - a$ and $f(x) > 0$ for all $x \in (a, b]$.

Let $\phi(\epsilon) = \int_{a+\epsilon}^{b} \frac{dx}{(x-a)^\mu}$, $0 < \epsilon < b - a$.

If $\mu \neq 1$, $\phi(\epsilon) = \int_{a+\epsilon}^{b} (x-a)^{-\mu} dx = \frac{1}{1-\mu}\left[\frac{1}{(b-a)^{\mu-1}} - \frac{1}{\epsilon^{\mu-1}}\right]$.

If $0 < \mu < 1$, $\lim\limits_{\epsilon \to 0+} \phi(\epsilon) = \dfrac{1}{1-\mu} \cdot \dfrac{1}{(b-a)^{\mu-1}}$ and if $\mu > 1$, $\lim\limits_{\epsilon \to 0+} \phi(\epsilon) = \infty$.

If $\mu = 1$, $\phi(\epsilon) = \int_{a+\epsilon}^{b} \frac{dx}{(x-a)} = \log|b-a| - \log|\epsilon|$ and $\lim\limits_{\epsilon \to 0+} \phi(\epsilon) = \infty$.

Since $\lim\limits_{\epsilon \to 0+} \phi(\epsilon)$ exists finitely when $0 < \mu < 1$ and $\lim\limits_{\epsilon \to 0+} \phi(\epsilon) = \infty$ when $\mu \geq 1$, the improper integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is convergent if and only if $0 < \mu < 1$.

Since the integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is proper if $\mu \leq 0$ and the improper integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is convergent if and only if $0 < \mu < 1$, it follows that the integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is convergent if and only if $\mu < 1$.

12.3.6. μ test. (A practical test)

Let a be the *only* point of infinite discontinuity of a function f which is integrable on $[a+\epsilon, b]$ for $0 < \epsilon < b - a$ and $f(x) > 0$ for all $x \in (a, b]$.

If $\lim\limits_{x \to a+} f(x)(x-a)^\mu = l$ where l is a *non-zero finite* number, then the integral $\int_a^b f(x)dx$ is convergent if and only if $\mu < 1$.

Proof. Since $f(x)(x-a)^\mu > 0$ for all $x \in (a, b]$, $l > 0$.

Let us choose a positive δ such that $l - \delta > 0$.

Since $\lim\limits_{x \to a+} f(x)(x-a)^\mu = l$, there exists a point c in (a, b) such that

$l - \delta < f(x)(x-a)^\mu < l + \delta$ for all $x \in (a, c]$

or, $\frac{l-\delta}{(x-a)^\mu} < f(x) < \frac{l+\delta}{(x-a)^\mu}$ for all $x \in (a, c]$.

If $\mu < 1$, the integral $\int_a^c \frac{1}{(x-a)^\mu} dx$ is convergent. Since $l + \delta > 0$, it follows from the right hand inequality that $\int_a^c f(x)dx$ is convergent.

If $\mu \geq 1$, the integral $\int_a^c \frac{1}{(x-a)^\mu} dx$ is divergent. Since $l - \delta > 0$, it follows from the left hand inequality that $\int_a^c f(x)dx$ is divergent.

Therefore $\int_a^c f(x)dx$ is convergent if and only if $\mu < 1$ and therefore $\int_a^b f(x)dx$ is convergent if and only if $\mu < 1$.

Note. If $\lim\limits_{x \to b-} f(x)(b-x)^\mu = l$ where l is a *non-zero finite* number, then the integral $\int_a^b f(x)dx$ is convergent if and only if $\mu < 1$.

Worked Examples.

1. Examine the convergence of $\int_0^1 \frac{x^{p-1}}{1+x}dx$.

The integral is a proper one if $p-1 \geq 0$. If $p < 1$, 0 is the only point of infinite discontinuity of the integrand.

Let $f(x) = \frac{x^{p-1}}{1+x}$, $x \in (0,1]$, $g(x) = x^{p-1}$, $x \in (0,1]$. Then $f(x) > 0, g(x) > 0$ for all $x \in (0,1]$.

$\lim\limits_{x \to 0+} \frac{f(x)}{g(x)} = 1$ (a non-zero finite number) and $\int_0^1 g(x)dx$ is convergent if and only if $1-p < 1$, i.e., if and only if $p > 0$.

By comparison test, $\int_0^1 f(x)dx$ is convergent if and only if $p > 0$.

Therefore $\int_0^1 \frac{x^{p-1}}{1+x}dx$ is convergent if and only if $p > 0$.

2. Show that $\int_0^1 \frac{1}{(x+1)(x+2)\sqrt{x(1-x)}}dx$ is convergent.

Let the given integral be $\int_0^1 f(x)dx$. 0 and 1 are the only points of infinite discontinuity of f. $f(x) > 0$ for all $x \in (0,1)$.

Let us examine the convergence of the improper integrals $\int_0^{\frac{1}{2}} f(x)dx$ and $\int_{\frac{1}{2}}^1 f(x)dx$.

Convergence of $\int_0^{\frac{1}{2}} f(x)dx$ at 0.

$\lim\limits_{x \to 0+} \sqrt{x}f(x) = \frac{1}{2}$. By μ test, $\int_0^{\frac{1}{2}} f(x)dx$ is convergent ... (i)

Convergence of $\int_{\frac{1}{2}}^1 f(x)dx$ at 1.

$\lim\limits_{x \to 1-} \sqrt{1-x}f(x) = \frac{1}{6}$. By μ test, $\int_{\frac{1}{2}}^1 f(x)dx$ is convergent... (ii)

From (i) and (ii) it follows that $\int_0^1 \frac{1}{(x+1)(x+2)\sqrt{x(1-x)}}dx$ is convergent.

3. Examine the convergence of $\int_0^1 \frac{x^{p-1}}{1-x}dx$.

1 is a point of infinite discontinuity of the integrand.
If $p < 1$, 0 is a point of infinite discontinuity of the integrand.

Let us examine the convergence of $\int_0^{\frac{1}{2}} \frac{x^{p-1}}{1-x}dx$ and of $\int_{\frac{1}{2}}^1 \frac{x^{p-1}}{1-x}dx$.

Convergence of $\int_0^{\frac{1}{2}} \frac{x^{p-1}}{1-x}dx$ at 0 when $p < 1$.

Let $f(x) = \frac{x^{p-1}}{1-x}$, $x \in (0, \frac{1}{2}]$; $g(x) = x^{p-1}$, $x \in (0, \frac{1}{2}]$. Then $f(x) > 0, g(x) > 0$ for all $x \in (0, \frac{1}{2}]$.

$\lim\limits_{x \to 0+} \frac{f(x)}{g(x)} = 1$ and $\int_0^{\frac{1}{2}} g(x)dx$ is convergent if and only if $p > 0$.

By comparison test, $\int_0^{\frac{1}{2}} f(x)dx$ is convergent if and only if $p > 0$, i.e., $\int_0^{\frac{1}{2}} \frac{x^{p-1}}{1-x}dx$ is convergent if $p > 0$ and divergent if $p \leq 0$ ...(i)

Convergence of $\int_{\frac{1}{2}}^1 \frac{x^{p-1}}{1-x}dx$ at 1.

Let $f(x) = \frac{x^{p-1}}{1-x}$, $g(x) = \frac{1}{1-x}$, $x \in [\frac{1}{2}, 1)$. Then $f(x) > 0, g(x) > 0$ for all $x \in [\frac{1}{2}, 1)$. $\lim\limits_{x \to 1-} \frac{f(x)}{g(x)} = 1$ and $\int_{\frac{1}{2}}^1 g(x)dx$ is divergent.

By comparison test, $\int_{\frac{1}{2}}^1 f(x)dx$ is divergent, i.e., $\int_{\frac{1}{2}}^1 \frac{x^{p-1}}{1-x}dx$ is divergent ...(ii)

From (i) and (ii) it follows that $\int_0^1 \frac{x^{p-1}}{1-x}dx$ is divergent.

4. Show that $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is convergent if and only if m, n are both positive.

Let the given integral be $\int_0^1 f(x)dx$. It is a proper integral if $m \geq 1$ and $n \geq 1$.

0 is the only point of infinite discontinuity of f if $m < 1$ and 1 is the only point of infinite discontinuity of f if $n < 1$.

Let us examine the convergence of $\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}dx$ when $m < 1$ and the convergence of $\int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}dx$ when $n < 1$.

Convergence of $\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}dx$ at 0 when $m < 1$.

$f(x) > 0$ for all $x \in (0, \frac{1}{2}]$. $\lim\limits_{x \to 0+} f(x)x^{1-m} = \lim\limits_{x \to 0+} (1-x)^{n-1} = 1$ (a non-zero finite number).

By μ test, $\int_0^{\frac{1}{2}} f(x)dx$ is convergent if and only if $1 - m < 1$, i.e., if and only if $m > 0$.

Convergence of $\int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}dx$ at 1 when $n < 1$.

$f(x) > 0$ for all $x \in [\frac{1}{2}, 1)$. $\lim\limits_{x \to 1-} f(x)(1-x)^{1-n} = \lim\limits_{x \to 1-} x^{m-1} = 1$ (a non-zero finite number).

By μ test, $\int_{\frac{1}{2}}^1 f(x)dx$ is convergent if and only if $1 - n < 1$, i.e., if and only if $n > 0$.

Therefore both the integrals $\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}dx$ and $\int_{\frac{1}{2}}^0 x^{m-1}(1-x)^{n-1}dx$ are convergent if and only if $m > 0$ and $n > 0$.

Hence $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is convergent if and only if $m > 0$ and $n > 0$.

Note. The integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m > 0, n > 0$ is called the *Beta function* and it is denoted by $B(m, n)$.

5. Show that $\int_0^{\frac{\pi}{2}} \frac{x^m}{\sin^n x} dx$ is convergent if and only if $n < 1 + m$.

Let the given integral be $\int_0^{\frac{\pi}{2}} f(x)dx$.

If $m - n \geq 0$, it is a proper integral since $\lim\limits_{x \to 0+} (\frac{x}{\sin x})^n = 1$.

If $m - n < 0$, 0 is the only point of infinite discontinuity of f. $f(x) > 0$ for all $x \in (0, \frac{\pi}{2})$.

$f(x) > 0$ for all $x \in (0, \frac{\pi}{2}]$. $\lim\limits_{x \to 0+} x^{n-m} f(x) = \lim\limits_{x \to 0+} (\frac{x}{\sin x})^n = 1$ (a non-zero finite number).

By μ test, $\int_0^{\frac{\pi}{2}} f(x)dx$ is convergent if and only if $n - m < 1$, i.e., if and only if $n < 1 + m$.

Therefore the given integral is convergent if and only if $n < 1 + m$.

6. Examine the convergence of $\int_0^1 x^{n-1} \log x \, dx$.

0 is the only possible point of infinite discontinuity of the integrand.

Let us examine the convergence of $\int_0^{\frac{1}{2}} x^{n-1} \log x dx$. The integrand is negative in $(0, \frac{1}{2}]$.

Let $f(x) = -x^{n-1} \log x$, $x \in (0, \frac{1}{2}]$. Then $f(x) > 0$ for all $x \in (0, \frac{1}{2}]$.

If $n - 1 > 0$, the integral $\int_0^{\frac{1}{2}} f(x)dx$ is a proper one, since $\lim\limits_{x \to 0+} x^r \log x = 0$, for all $r > 0$.

If $n - 1 \leq 0$, 0 is the only point of infinite discontinuity of f.

Let m be a positive number such that $m + n - 1 > 0$. Then $\lim\limits_{x \to 0+} x^{m+n-1} \log x = 0$. Therefore $\lim\limits_{x \to 0+} x^m f(x) = 0$.

Let $g(x) = \frac{1}{x^m}$, $x \in (0, \frac{1}{2}]$. Then $g(x) > 0$ for all $x \in (0, \frac{1}{2}]$.

Since $\lim\limits_{x \to 0+} \frac{f(x)}{g(x)} = 0$ and $\int_0^{\frac{1}{2}} g(x)dx$ is convergent if $m < 1$, it follows that $\int_0^{\frac{1}{2}} f(x)dx$ is convergent if $m < 1$.

Therefore $\int_0^1 f(x)dx$ is convergent if $m < 1$ and $m + n - 1 > 0$, i.e., if $1 - n < m < 1$, i.e., if $n > 0$.

If $n = 0$, the integral reduces to $\int_0^1 \frac{\log x}{x} dx$.

$\int_\epsilon^1 \frac{\log x}{x} dx = -\frac{1}{2}(\log \epsilon)^2 \to -\infty$ as $\epsilon \to 0+$ and therefore $\int_0^1 f(x)dx$ is divergent if $n = 0$.

If $n < 0$, then $x^{n-1} \geq x^{-1}$ for all $x \in (0, 1]$. Since the integral $\int_0^1 \frac{\log x}{x} dx$ is divergent, it follows that $\int_0^1 f(x)dx$ is divergent.

Hence the given integral is convergent if and only if $n > 0$.

7. Examine the convergence of $\int_0^1 x^{m-1}(1 - x)^{n-1} \log x \, dx$.

0 and 1 are the only possible points of infinite discontinuity of the integrand.

Let $f(x) = -x^{m-1}(1-x)^{n-1}\log\, x$, $x \in (0, \frac{1}{2}]$. Then $f(x) > 0$ for all $x \in (0, \frac{1}{2}]$.

Convergence of $\int_0^{\frac{1}{2}} f(x)dx$ at 0.

If $m - 1 > 0$ then $\lim\limits_{x \to 0+} -x^{m-1}\log\, x = 0$ and therefore the integral $\int_0^{\frac{1}{2}} f(x)dx$ is proper.

When $m - 1 \le 0$, let p be a positive number such that $p + m - 1 > 0$. Then $\lim\limits_{x \to 0+} -x^{p+m-1}\log\, x = 0$ and therefore $\lim\limits_{x \to 0+} x^p f(x) = 0$.

Let $g(x) = \frac{1}{x^p}$, $x \in (0, \frac{1}{2}]$. Then $g(x) > 0$ for all $x \in (0, \frac{1}{2}]$ and $\lim\limits_{x \to 0+} \frac{f(x)}{g(x)} = 0$. Therefore the convergence of the integral $\int_0^{\frac{1}{2}} g(x)dx$ will imply convergence of the integral $\int_0^{\frac{1}{2}} f(x)dx$.

$\int_0^{\frac{1}{2}} g(x)dx$ is convergent if $p < 1$. Therefore $\int_0^{\frac{1}{2}} f(x)dx$ is convergent if $p < 1$ and $p + m - 1 > 0$, i.e., if $1 - m < p < 1$, i.e., if $m > 0$.

Let us examine the convergence of $\int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\log\, xdx$. The integrand is negative in $[\frac{1}{2}, 1)$.

Let $f(x) = -x^{m-1}(1-x)^{n-1}\log\, x$, $x \in [\frac{1}{2}, 1)$. Then $f(x) > 0$ for all $x \in [\frac{1}{2}, 1)$.

Convergence of $\int_{\frac{1}{2}}^1 f(x)dx$ at 1.

$\lim\limits_{x \to 1-} (1-x)^{n-1}\log\, x = \lim\limits_{x \to 1-} \frac{\log\, x}{(1-x)^{1-n}} = \lim\limits_{x \to 1-} \frac{(1-x)^n}{x(n-1)}$ is finite if $n \ge 0$.

Therefore the integral $\int_{\frac{1}{2}}^1 -x^{m-1}(1-x)^{n-1}\log\, xdx$ is proper if $n \ge 0$.

When $n < 0$, let q be a positive number such that $q + n > 0$. Then $\lim\limits_{x \to 1-} -(1-x)^{q+n-1}\log\, x = \lim\limits_{x \to 1-} \frac{-\log\, x}{(1-x)^{1-q-n}} = \lim\limits_{x \to 1-} \frac{-(1-x)^{q+n}}{x(1-q-n)} = 0.$ and therefore $\lim\limits_{x \to 1-} (1-x)^q f(x) = 0$.

Let $h(x) = \frac{1}{(1-x)^q}$, $x \in [\frac{1}{2}, 1)$. Then $h(x) > 0$ for all $x \in [\frac{1}{2}, 1)$ and $\lim\limits_{x \to 1-} \frac{f(x)}{h(x)} = 0$. Therefore the convergence of the integral $\int_{\frac{1}{2}}^1 h(x)dx$ will imply convergence of the integral $\int_{\frac{1}{2}}^1 f(x)dx$.

$\int_{\frac{1}{2}}^1 h(x)dx$ is convergent if $q < 1$. Therefore $\int_{\frac{1}{2}}^1 f(x)dx$ is convergent if $q < 1$ and $q + n > 0$, i.e., if $n > -q$ and $-q > -1$, i.e., if $n > -1$.

Hence the given integral is convergent if $m > 0$ and $n > -1$.

8. Examine the convergence of $\int_0^{\frac{\pi}{2}} \log \sin x \, dx$.

Let $f(x) = \log \sin x$, $x \in (0, \frac{\pi}{2}]$. 0 is a point of infinite discontinuity of f. $f(x) > 0$ for all $x \in (0, \frac{\pi}{2}]$.

We have $\lim\limits_{x \to 0+} \sqrt{x}(\log x) = 0$ and $\lim\limits_{x \to 0+} \sqrt{x} \log \frac{\sin x}{x} = 0$.

Therefore $\lim\limits_{x \to 0+} \sqrt{x}[\log x + \log \frac{\sin x}{x}] = 0$.

or, $\lim\limits_{x \to 0+} \sqrt{x} \log (\sin x) = 0$.

Let $g(x) = \frac{1}{\sqrt{x}}$, $x \in (0, \frac{\pi}{2}]$. Then $g(x) > 0$ for all $x \in (0, \frac{\pi}{2}]$.

$\lim\limits_{x \to 0+} \dfrac{f(x)}{g(x)} = 0$ and $\int_0^{\frac{\pi}{2}} g(x)dx$ is convergent. By comparison test, $\int_0^{\frac{\pi}{2}} f(x)dx$ is convergent, i.e., $\int_0^{\frac{\pi}{2}} \log \sin x \, dx$ is convergent.

9. Examine the convergence of $\int_0^{\frac{\pi}{2}} \sin^{m-1} x \cos^{n-1} x \, dx$.

The integral is a proper one if $m \geq 1$ and $n \geq 1$.

If $m < 1$, 0 is the only point of infinite discontinuity of the integrand and if $n < 1$, 1 is the only point of infinite discontinuity of the integrand.

Let us examine the convergence of $\int_0^{\frac{\pi}{4}} \sin^{m-1} x \cos^{n-1} x \, dx$ when $m < 1$ and the convergence of $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^{m-1} x \cos^{n-1} x \, dx$ when $n < 1$.

Convergence of $\int_0^{\frac{\pi}{4}} \sin^{m-1} x \cos^{n-1} x \, dx$ when $m < 1$.

Let $f(x) = \sin^{m-1} x \cos^{n-1} x$, $x \in (0, \frac{\pi}{4}]$. $f(x) > 0$ for all $x \in (0, \frac{\pi}{4}]$.

$\lim\limits_{x \to 0+} x^{1-m} f(x) = \lim\limits_{x \to 0+} (\frac{\sin x}{x})^{m-1} \cos^{n-1} x = 1$ (a non-zero finite number).

By μ test, $\int_0^{\frac{\pi}{4}} f(x)dx$ is convergent if and only if $1 - m < 1$, i.e., if and only if $m > 0$ ... (i)

Convergence of $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^{m-1} x \cos^{n-1} x \, dx$ when $n < 1$.

Let $f(x) = \sin^{m-1} x \cos^{n-1} x$, $x \in [\frac{\pi}{4}, \frac{\pi}{2})$. $f(x) > 0$ for all $x \in [\frac{\pi}{4}, \frac{\pi}{2})$.

$\lim\limits_{x \to \frac{\pi}{2}-} (\frac{\pi}{2} - x)^{1-n} f(x) = \lim\limits_{x \to \frac{\pi}{2}-} \sin^{m-1} x \, (\frac{\cos x}{\frac{\pi}{2} - x})^{n-1} = 1$ (a non-zero finite number).

By μ test, $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} f(x)dx$ is convergent if and only if $1 - n < 1$, i.e., if and only if $n > 0$ ... (ii)

From (i) and (ii) it follows that $\int_0^{\frac{\pi}{2}} \sin^{m-1} x \cos^{n-1} x \, dx$ is convergent if and only if $m > 0$ and $n > 0$.

12.4. Tests for convergence of an improper integral when the integrand does not necessarily keep the same sign on a bounded interval.

Theorem 12.4.1. (Cauchy)

Let a be the *only* point of infinite discontinuity of a function f which is integrable on $[a+\epsilon, b]$ for all ϵ satisfying $0 < \epsilon < b - a$ and $f(x)$ may not keep the same sign on $(a, b]$.

A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)dx$ is that for a pre-assigned positive ϵ there exists a positive $\delta < b - a$ such that

$$\left| \int_{a+\lambda_1}^{a+\lambda_2} f(x)dx \right| < \epsilon \text{ for all } \lambda_1, \lambda_2 \text{ satisfying } 0 < \lambda_1 < \lambda_2 < \delta.$$

Proof. Let $F(\lambda) = \int_{a+\lambda}^b f(x)dx$, $0 < \lambda < b - a$.

The improper integral $\int_a^b f(x)dx$ is convergent if $\lim_{\lambda \to 0+} F(\lambda)$ exists finitely. By Cauchy's criterion for the existence of finite limits, $\lim_{\lambda \to 0+} F(\lambda)$ exists finitely if and only if for a pre-assigned positive ϵ there corresponds a positive $\delta < b - a$ such that

$|F(\lambda_1) - F(\lambda_2)| < \epsilon$ for all λ_1, λ_2 satisfying $0 < \lambda_1 < \lambda_2 < \delta$

or, $\left| \int_{a+\lambda_1}^{a+\lambda_2} f(x)dx \right| < \epsilon$ for all λ_1, λ_2 satisfying $0 < \lambda_1 < \lambda_2 < \delta$.

This completes the proof.

Definition.

The improper integral $\int_a^b f(x)dx$ is said to be *absolutely convergent* if $\int_a^b |f|(x)dx$ be convergent.

Theorem 12.4.2. An absolutely convergent improper integral $\int_a^b f(x)dx$ (where a is the only point of infinite discontinuity of f in $[a, b]$ and f is integrable on $[a+\epsilon, b]$ for all ϵ satisfying $0 < \epsilon < b - a$) is convergent.

Proof. Here the integral $\int_a^b |f|(x)dx$ is convergent and a is the only point of infinite discontinuity of f in $[a, b]$.

Let $\epsilon > 0$. Then there exists a positive $\delta < b - a$ such that

$$\left| \int_{a+\lambda_1}^{a+\lambda_2} |f|(x)dx \right| < \epsilon \text{ for all } \lambda_1, \lambda_2 \text{ satisfying } 0 < \lambda_1 < \lambda_2 < \delta.$$

We also have $\left| \int_{a+\lambda_1}^{a+\lambda_2} f(x)dx \right| \leq \int_{a+\lambda_1}^{a+\lambda_2} |f|(x)dx$.

Therefore $\left| \int_{a+\lambda_1}^{a+\lambda_2} f(x)dx \right| < \epsilon$ for all λ_1, λ_2 satisfying $0 < \lambda_1 < \lambda_2 < \delta$.

This proves that the integral $\int_a^b f(x)dx$ is convergent.

This completes the proof.

Note 1. The converse of the theorem is not true. We shall establish this by some examples.

Note 2. Since $|f(x)|$ is always positive, comparison tests can be applied to establish the convergence of the improper integral $\int_a^b |f|(x)dx$.

Worked Examples (continued).

10. Show that the improper integral $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}dx$ is convergent.

Let $f(x) = \frac{\sin\frac{1}{x}}{\sqrt{x}}$, $x \in (0,1]$.

0 is the only point of infinite discontinuity of f. $f(x)$ does not keep the same sign in the interval $(0,1]$.

$|\frac{\sin\frac{1}{x}}{\sqrt{x}}| \leq \frac{1}{\sqrt{x}}$ for all $x \in (0,1]$ and $\int_0^1 \frac{1}{\sqrt{x}}dx$ is convergent.

Therefore $\int_0^1 |\frac{\sin\frac{1}{x}}{\sqrt{x}}|dx$ is convergent, i.e., $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}dx$ is absolutely convergent and therefore $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}dx$ is convergent.

11. A function f is defined on $[0,1]$ by $f(0) = 0$,

$f(x) = (-1)^{n+1}(n+1)$, for $\frac{1}{n+1} < x \leq \frac{1}{n}$ $(n = 1, 2, 3, \ldots)$

Examine convergence of the integrals (i) $\int_0^1 f(x)dx$, (ii) $\int_0^1 |f|(x)dx$.

(i) f is bounded and integrable on $[\epsilon, 1]$ for every $\epsilon > 0$. 0 is the only point of infinite discontinuity of f in $[0,1]$

Let us choose $\epsilon > 0$. There exists a natural number p such that $\frac{1}{p+1} < \epsilon \leq \frac{1}{p}$.

$$\int_\epsilon^1 f(x)dx = \int_{\frac{1}{2}}^1 f(x)dx + \int_{\frac{1}{3}}^{\frac{1}{2}} f(x)dx + \cdots + \int_{\frac{1}{p}}^{\frac{1}{p-1}} f(x)dx + \int_\epsilon^{\frac{1}{p}} f(x)dx$$

$$= \int_{\frac{1}{2}}^1 2dx + \int_{\frac{1}{3}}^{\frac{1}{2}}(-3)dx + \cdots + \int_{\frac{1}{p}}^{\frac{1}{p-1}}(-1)^p p dx + \int_\epsilon^{\frac{1}{p}}(-1)^{p+1}(p+1)dx$$

$$= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \cdots + (-1)^p \tfrac{1}{p-1} + (-1)^{p+1} \int_\epsilon^{\frac{1}{p}} p + 1dx.$$

$$|\int_\epsilon^1 f(x)dx - [1 - \tfrac{1}{2} + \tfrac{1}{3} - \cdots + (-1)^p \tfrac{1}{p-1}]| = |(p+1)\int_\epsilon^{\frac{1}{p}} dx|$$

$$< \tfrac{1}{p}, \text{ since } \int_\epsilon^{\frac{1}{p}} dx < \int_{\frac{1}{p+1}}^{\frac{1}{p}} dx.$$

As $\epsilon \to 0, p \to \infty$.

Therefore $\lim\limits_{\epsilon\to 0}\int_\epsilon^1 f(x)dx = \lim\limits_{p\to\infty}[1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^p \frac{1}{p-1}]$ (i)

Since the series $[1 - \frac{1}{2} + \frac{1}{3} - \cdots]$ is a convergent series, it follows from (i) that $\lim\limits_{\epsilon\to 0}\int_\epsilon^1 f(x)dx$ is finite and therefore $\int_0^1 f(x)dx$ is convergent.

(ii) $|f|$ is bounded and integrable on $[\epsilon, 1]$ for every $\epsilon > 0$. 0 is the only point of infinite discontinuity of $|f|$ in $[0, 1]$.

Let us choose $\epsilon > 0$. There exists a natural number p such that $\frac{1}{p+1} < \epsilon \leq \frac{1}{p}$.

$\int_\epsilon^1 |f|(x)dx$

$= \int_{\frac{1}{2}}^1 |f|(x)dx + \int_{\frac{1}{3}}^{\frac{1}{2}} |f|(x)dx + \cdots + \int_{\frac{1}{p}}^{\frac{1}{p-1}} |f|(x)dx + \int_\epsilon^{\frac{1}{p}} |f|(x)dx$

$= \int_{\frac{1}{2}}^1 2dx + \int_{\frac{1}{3}}^{\frac{1}{2}} 3dx + \cdots + \int_{\frac{1}{p}}^{\frac{1}{p-1}} pdx + \int_\epsilon^{\frac{1}{p}} (p+1)dx$

$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1} + (p+1)(\frac{1}{p} - \epsilon)$

$> 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}]$, since $\frac{1}{p} > \epsilon$ (ii)

As $\epsilon \to 0, p \to \infty$.

Since the series $[1 + \frac{1}{2} + \frac{1}{3} + \cdots]$ is a divergent series, it follows from (ii) that $\lim_{\epsilon \to 0} \int_\epsilon^1 f(x)dx$ is not finite and consequently, the integral $\int_0^1 |f|(x)dx$ is divergent.

Note. This example establishes that the converse of the theorem 15.4.2 is not true.

B. Improper integrals on an unbounded interval.

12.5. Definitions.

I. Convergence of the improper integral $\int_a^\infty f(x)dx$ where f is integrable on $[a, X]$ for all $X > a$.

Let $\phi(X) = \int_a^X f(x)dx, X > a$.

If $\lim_{X \to \infty} \phi(X)$ exists (finitely) then the improper integral $\int_a^\infty f(x)dx$ is said to be *convergent*. If the limit be l, we write $\int_a^\infty f(x)dx = l$.

If $\lim_{X \to \infty} \phi(X)$ does not exist (finitely) then the improper integral $\int_a^\infty f(x)dx$ is said to be *divergent*.

Examples.

1. Let us consider the integral $\int_0^\infty e^{-x}dx$. The integrand is integrable on any closed interval $[0, X]$, $X > 0$. The integral is improper.

Let $\phi(X) = \int_0^X e^{-x}dx, X > 0$. Then $\phi(X) = 1 - e^{-X}$. $\lim_{X \to \infty} \phi(X) = 1$.

Therefore the integral $\int_0^\infty e^{-x}dx$ is convergent and $\int_0^\infty e^{-x}dx = 1$.

2. Let us consider the integral $\int_0^\infty \frac{1}{1+x}dx$. The integrand is integrable on any closed interval $[0, X]$, $X > 0$. The integral is improper.

Let $\phi(X) = \int_0^X \frac{1}{1+x}dx, X > 0$. Then $\phi(X) = \log(1+X)$.

$\lim\limits_{X\to\infty} \phi(X) = \infty$. Therefore the integral $\int_0^\infty \frac{1}{1+x}dx$ is divergent.

II. Convergence of the improper integral $\int_{-\infty}^b f(x)dx$ where f is integrable on $[X, b]$ for all $X < b$.

Let $\phi(X) = \int_X^b f(x)dx, X < b$.

If $\lim\limits_{X\to-\infty} \phi(X)$ exists (finitely) then the improper integral $\int_{-\infty}^b f(x)dx$ is said to be *convergent*. If the limit be l, we write $\int_{-\infty}^b f(x)dx = l$.

III. Convergence of the improper integral $\int_{-\infty}^\infty f(x)dx$ where f is integrable on $[X_1, X_2]$ for all $X_1, X_2 \in \mathbb{R}$ satisfying $X_1 < X_2$.

Let $c \in \mathbb{R}$. If both the integrals $\int_{-\infty}^c f(x)dx$ and $\int_c^\infty f(x)dx$ be convergent according to the definitions I and II above, then the improper integral $\int_{-\infty}^\infty f(x)dx$ is said to be *convergent* and we write

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx.$$

Example (continued).

3. Let us consider the integral $\int_{-\infty}^\infty \frac{dx}{1+x^2}$. The integrand is integrable on any closed interval $[X_1, X_2]$, $X_2 > X_1$. The integral is improper.

Let us consider the integrals $\int_{-\infty}^a \frac{dx}{1+x^2}$ and $\int_a^\infty \frac{dx}{1+x^2}$, where $a \in \mathbb{R}$.
Let $\phi(X) = \int_X^a \frac{dx}{1+x^2}dx$, $X < a$. Then $\phi(X) = \tan^{-1} a - \tan^{-1} X$.

$\lim\limits_{X\to-\infty} \phi(X) = \tan^{-1} a + \frac{\pi}{2}$.
Therefore the improper integral $\int_{-\infty}^a \frac{dx}{1+x^2}$ is convergent.

Let $\psi(X) = \int_a^X \frac{dx}{1+x^2}dx$, $X > a$. Then $\psi(x) = \tan^{-1} X - \tan^{-1} a$.

$\lim\limits_{X\to\infty} \psi(X) = \frac{\pi}{2} - \tan^{-1} a$.
Therefore the improper integral $\int_a^\infty \frac{dx}{1+x^2}$ is convergent.

Consequently, the integral $\int_{-\infty}^\infty \frac{dx}{1+x^2}$ is convergent and $\int_{-\infty}^\infty \frac{dx}{1+x^2} = (\tan^{-1} a + \frac{\pi}{2}) + (\frac{\pi}{2} - \tan^{-1} a) = \pi$.

IV. Convergence of the improper integral $\int_{-\infty}^\infty f(x)dx$ where f has a finite number of points of infinite discontinuity $c_1, c_2, \ldots, c_m$.

Let $c_1 < c_2 < \cdots < c_m$. If each of the integrals $\int_{-\infty}^{c_1} f(x)dx$, $\int_{c_1}^{c_2} f(x)dx$, $\ldots$, $\int_{c_{m-1}}^{c_m} f(x)dx$ and $\int_{c_m}^{\infty} f(x)dx$ be convergent according to the definitions given in 12.2 and 12.5, then the improper integral $\int_{-\infty}^{\infty} f(x)dx$ is said to be *convergent* and we write

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \cdots + \int_{c_{m-1}}^{c_m} f(x)dx + \int_{c_m}^{\infty} f(x)dx.$$

12.6. Tests for convergence, positive integrand.

Theorem 12.6.1. Let a function f be integrable on $[a, X]$ for all $X > a$ and $f(x) > 0$ for all $x \geq a$.

A necessary and sufficient condition for the convergence of the improper integral $\int_a^{\infty} f(x)dx$ is that there exists a positive real number k such that

$$\int_a^X f(x)dx < k \text{ for all } X > a.$$

Proof. Let $\phi(X) = \int_a^X f(x)dx$, $X > a$.

Let $a < X_1 < X_2$. Then $\phi(X_2) - \phi(X_1) = \int_a^{X_2} f(x)dx - \int_a^{X_1} f(x)dx = \int_{X_1}^{X_2} f(x)dx \geq 0$, since $f(x) > 0$ on $[X_1, X_2]$.

$a < X_1 < X_2 \Rightarrow \phi(X_1) < \phi(X_2)$. This shows that ϕ is a monotone increasing function on (a, ∞). Therefore $\phi(X)$ will tend to a finite limit as $X \to \infty$ if and only if ϕ is bounded above.

In other words, the improper integral $\int_a^{\infty} f(x)dx$ is convergent if and only if there exists a positive real number k such that

$$\int_a^X f(x)dx < k \text{ for all } X > a.$$

Note. If ϕ be not bounded above, then $\phi(X)$ tends to ∞ as $X \to \infty$ and the improper integral $\int_a^{\infty} f(x)dx$ diverges to ∞.

Theorem 12.6.2. Comparison test.

Let the functions f and g be both integrable on $[a, X]$ for all $X > a$ and $0 < f(x) \leq kg(x)$ for all $x \geq a$, where $k > 0$. Then

(i) $\int_a^{\infty} g(x)dx$ is convergent $\Rightarrow \int_a^{\infty} f(x)dx$ is convergent;

(ii) $\int_a^{\infty} f(x)dx$ is divergent $\Rightarrow \int_a^{\infty} g(x)dx$ is divergent.

Proof. Since f and g are both integrable on $[a, X]$ and $0 < f(x) \leq kg(x)$ for all $x \in [a, X]$, we have $\int_a^X f(x)dx \leq k \int_a^X g(x)dx$.

i) If $\int_a^{\infty} g(x)dx$ be convergent then there exists a positive real number k' such that $\int_a^X g(x)dx < k'$ for all $X > a$.

Then $\int_a^X f(x)dx < kk'$ for all $X > a$ and this proves that $\int_a^\infty f(x)d$ is convergent.

(ii) Let $\int_a^\infty f(x)dx$ be divergent. Then $\lim\limits_{X\to\infty} \int_a^X f(x)dx = \infty$.

Since $\int_a^X f(x)dx \leq k\int_a^X g(x)dx$ and $k > 0$ it follows tha $\lim\limits_{X\to\infty} \int_a^X g(x)dx = \infty$. Consequently, $\int_a^\infty g(x)dx$ is divergent.

This completes the proof.

Theorem 12.6.3. Comparison test (limit form).

Let the functions f and g be both integrable on $[a, X]$ for all $X >$ and $f(x) > 0, g(x) > 0$ for all $x \geq a$.

If $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = l$, where l is a *non-zero* finite number, then the two im proper integrals $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converge or diverge togethe

Proof. Since $\frac{f(x)}{g(x)} > 0$ for all $x \geq a$, $l \geq 0$. Since l is non-zero, $l > 0$.

Let us choose a positive ϵ such that $l - \epsilon > 0$. Since $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} =$ there exists a positive real number $k > a$ such that $l - \epsilon < \frac{f(x)}{g(x)} < l +$ for all $x > k$

or, $(l - \epsilon)g(x) < f(x) < (l + \epsilon)g(x)$ for all $x > k > a$.

(i) Let $\int_a^\infty f(x)dx$ be convergent.

Since $(l - \epsilon)g(x) < f(x)$ for all $x > k > a$ and $l - \epsilon > 0$, $\int_a^\infty g(x)d$ is convergent, by comparison test.

(ii) Let $\int_a^b g(x)dx$ be convergent.

Since $f(x) < (l + \epsilon)g(x)$ for all $x > k > a$ and $l + \epsilon > 0$, $\int_a^\infty f(x)d$ is convergent, by comparison test.

(iii) Let $\int_a^\infty f(x)dx$ be divergent.

Since $(l + \epsilon)g(x) > f(x)$ for all $x > k > a$ and $l + \epsilon > 0$, $\int_a^\infty g(x)d$ is divergent, by comparison test.

(iv) Let $\int_a^\infty g(x)dx$ be divergent.

Since $(l - \epsilon)g(x) < f(x)$ for all $x > k > a$ and $l - \epsilon > 0$, $\int_a^\infty f(x)d$ is divergent, by comparison test.

Thus the improper integrals $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converge o diverge together.

This completes the proof.

Note 1. If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$, then for a pre-assigned positive ϵ there exists a positive real number $b > a$ such that $f(x) < \epsilon g(x)$ for all $x > b$.

Then $\int_a^\infty g(x)dx$ is convergent implies that $\int_a^\infty f(x)dx$ is convergent.

2. If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$ then for a pre-assigned positive G there exists a positive real number $b > a$ such that $f(x) > Gg(x)$ for all $x > b > a$.

Then $\int_a^\infty g(x)dx$ is divergent implies that $\int_a^\infty f(x)dx$ is divergent.

12.6.4. A useful comparison integral.

The improper integral $\int_a^\infty \frac{dx}{x^\mu}$, where $a > 0$, is convergent if and only if $\mu > 1$.

Proof. Let $\phi(X) = \int_a^X \frac{dx}{x^\mu}$, $X > a$.

If $\mu \neq 1$, we have $\phi(X) = \int_a^X \frac{dx}{x^\mu} = [\frac{x^{1-\mu}}{1-\mu}]_a^X = \frac{1}{1-\mu}[X^{1-\mu} - a^{1-\mu}]$;

and if $\mu = 1$, $\phi(X) = \int_a^X \frac{dx}{x} = \log X - \log a$.

If $\mu = 1$, $\lim_{X \to \infty} \phi(X) = \lim_{X \to \infty} \log X - \log a = \infty$.

If $\mu \neq 1$, $\lim_{X \to \infty} \phi(X) = \lim_{X \to \infty} \frac{1}{1-\mu}[X^{1-\mu} - a^{1-\mu}] = \infty$, if $\mu < 1$

$= \frac{1}{(\mu-1)a^{\mu-1}}$, if $\mu > 1$.

Since $\lim_{X \to \infty} \phi(X)$ exists finitely when $\mu > 1$ and $\lim_{X \to \infty} \phi(X) = \infty$ when $\mu \leq 1$, the improper integral $\int_a^\infty \frac{dx}{x^\mu}$ is convergent if $\mu > 1$ and divergent if $\mu \leq 1$.

Hence the improper integral $\int_a^\infty \frac{dx}{x^\mu}$, where $a > 0$, is convergent if and only if $\mu > 1$. This completes the proof.

12.6.5. μ test. (A practical test)

Let $f(x) > 0$ for all $x \geq a$. If $\lim_{x \to \infty} x^\mu f(x) = l$, where l is a *non-zero finite* number, the improper integral $\int_a^\infty f(x)dx$ is convergent if and only if $\mu > 1$.

Proof. Let $\lim_{x \to \infty} x^\mu f(x) = l$. Then $l > 0$.

Let us choose a positive ϵ such that $l - \epsilon > 0$. Since $\lim_{X \to \infty} x^\mu f(x) = l$, there exists a positive real number $k > a$ such that

$|x^\mu f(x) - l| < \epsilon$ for all $x \geq k$

or, $l - \epsilon < x^\mu f(x) < l + \epsilon$ for all $x \geq k$

or, $\frac{l-\epsilon}{x^\mu} < f(x) < \frac{l+\epsilon}{x^\mu}$ for all $x \geq k > a$.

$\int_a^\infty \frac{l}{x^\mu}dx$ is convergent if $\mu > 1$. Since $l + \epsilon > 0$, it follows from the last inequality that $\int_a^\infty f(x)dx$ is convergent if $\mu > 1$.

$\int_a^\infty \frac{l}{x^\mu}dx$ is divergent if $\mu \leq 1$. Since $l - \epsilon > 0$, it follows from the first inequality that $\int_a^\infty f(x)dx$ is divergent if $\mu \leq 1$.

Therefore $\int_a^\infty f(x)dx$ is convergent if and only if $\mu > 1$.

Worked Examples.

1. Examine the convergence of the improper integral $\int_1^\infty \frac{1}{x(1+x^2)}dx$.

Let the given integral be $\int_1^\infty f(x)dx$. Then $f(x) > 0$ for all $x \geq 1$.

Let $g(x) = \frac{1}{x^3}$. Then $\int_1^\infty g(x)dx$ is convergent and $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$, a non-zero finite number.

By comparison test, $\int_1^\infty f(x)dx$ is convergent.

2. Examine the convergence of the improper integral $\int_0^\infty x^{-\frac{1}{2}}e^{-x}dx$.

Let the given integral be $\int_0^\infty f(x)dx$. Then 0 is a point of infinite discontinuity of f.

We are to examine the convergence at 0 as well as at ∞.

Convergence at 0.

Let us consider the integral $\int_0^1 f(x)dx$. $f(x) > 0$ for all x in $(0, 1]$.

Let $g(x) = \frac{1}{\sqrt{x}}$. Then $\lim\limits_{x\to 0} \frac{f(x)}{g(x)} = 1$, a non-zero finite number and $\int_0^1 g(x)dx$ is convergent.

By comparison test, $\int_0^1 f(x)dx$ is convergent (i)

Convergence at ∞.

Let us consider the integral $\int_1^\infty f(x)dx$. $f(x) > 0$ for all $x \geq 1$ and $e^x > x$ for all $x \geq 1$. Therefore $f(x) < \frac{1}{x^{\frac{3}{2}}}$ and $\int_1^\infty \frac{1}{x^{\frac{3}{2}}}dx$ is convergent.

By comparison test, $\int_1^\infty f(x)dx$ is convergent ... (ii)

From (i) and (ii) it follows that $\int_0^\infty f(x)dx$ is convergent.

3. Examine the convergence of the improper integral $\int_1^\infty \frac{1}{x^{\frac{1}{2}}(1+x)^{\frac{1}{4}}}dx$.

Let the given integral be $\int_1^\infty f(x)dx$. Then $f(x) > 0$ for all $x \geq 1$.

Let $g(x) = \frac{1}{x^{\frac{3}{4}}}$. Then $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$, a non-zero finite number and $\int_1^\infty g(x)dx$ is divergent

By comparison test, $\int_1^\infty f(x)dx$ is divergent.

4. Prove that the integral $\int_0^\infty x^{m-1}e^{-x}dx$ is convergent if and only if $m > 0$.

Let the given integral be $\int_0^\infty f(x)dx$. If $m \geq 1$, 0 is not a point of infinite discontinuity of f. f has an infinite discontinuity at 0 if $m < 1$.

Convergence at 0. $(m < 1)$

$f(x) > 0$ for all $x \in (0, 1]$. Let $g(x) = x^{m-1}$, $x \in (0, 1]$. Then $g(x) > 0$ for all $x \in (0, 1]$ and $\lim\limits_{x \to 0} \frac{f(x)}{g(x)} = 1$, a non-zero finite number.

$\int_0^1 g(x)dx$ is convergent if and only if $1 - m < 1$, i.e., if and only if $m > 0$.

By comparison test, $\int_0^1 f(x)dx$ is convergent if and only if $m > 0$.

Convergence at ∞.

$f(x) > 0$ for all $x \geq 1$. Let $g(x) = \frac{1}{x^2}$, $x \geq 1$. Then $g(x) > 0$ for all $x \geq 1$ and $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = \lim\limits_{x \to \infty} \frac{x^{m+1}}{e^x} = 0$ for all m.

As the integral $\int_1^\infty g(x)dx$ is convergent, therefore the integral $\int_1^\infty x^{m-1}e^{-x}dx$ is convergent for all m.

Hence the given integral is convergent if and only if $m > 0$.

Note. The integral $\int_0^1 x^{m-1}e^{-x}dx$, $m > 0$ is called the *Gamma function* and is denoted by $\Gamma(m)$.

5. Prove that the integral $\int_0^\infty (\frac{1}{1+x} - \frac{1}{e^x})\frac{1}{x}dx$ is convergent.

Let the given integral be $\int_0^\infty f(x)dx$.

Since $e^x > 1+x$ for all $x > 0$, $f(x) > 0$ for all $x > 0$. Since $\lim\limits_{x \to 0}(\frac{1}{1+x} - \frac{1}{e^x})\frac{1}{x} = \lim\limits_{x \to 0} \frac{e^x - 1 - x}{x(1+x)e^x} = 1$, 0 is not a point of infinite discontinuity of f.

We are to examine the convergence of the integral at ∞.

Let $g(x) = \frac{1}{x^2}, x > 0$. Then $g(x) > 0$ for all $x > 0$ and $\int_0^\infty g(x)dx$ is convergent.

$\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = \lim\limits_{x \to \infty} \frac{x(e^x - (1+x))}{(1+x)e^x} = 1$, a non-zero finite number.

By comparison test, $\int_0^\infty f(x)dx$ is convergent.

Hence the given integral is convergent.

6. Examine the convergence of the improper integral $\int_0^\infty \frac{x^{p-1}}{1+x}\, dx$.

Let us examine the convergence of the integrals $\int_0^1 \frac{x^{p-1}}{1+x}\, dx$ and $\int_1^\infty \frac{x^{p-1}}{1+x}\, dx$.

If $p \geq 1$ the integral $\int_0^1 \frac{x^{p-1}}{1+x}\, dx$ is a proper one. If $p < 1$, 0 is the only point of infinite discontinuity of the integrand.

Let $f(x) = \frac{x^{p-1}}{1+x}, 0 < x \le 1$. Let $g(x) = x^{p-1}, 0 < x \le 1$. Then $f(x) > 0$ and $g(x) > 0$ for all $x \in (0,1]$ and $\lim\limits_{x \to 0+} \frac{f(x)}{g(x)} = 1$.

$\int_0^1 g(x)dx$ is convergent if $p > 0$ and divergent if $p \le 0$.

By comparison test, $\int_0^1 \frac{x^{p-1}}{1+x}\, dx$ is convergent if $p > 0$ and divergent if $p \le 0$... (i)

Let us consider $\int_1^\infty \frac{x^{p-1}}{1+x}\, dx$.

Let $f(x) = \frac{x^{p-1}}{1+x}, x > 1$. Let $\phi(x) = x^{p-2}$, $x > 1$. Then $\lim\limits_{x \to \infty} \frac{f(x)}{\phi(x)} = \lim\limits_{x \to \infty} \frac{x}{1+x} = 1$.

$\int_1^\infty \phi(x)\, dx$ is convergent if $2 - p > 1$, i.e., if $p < 1$ and divergent if $p \ge 1$.

By comparison test, $\int_1^\infty \frac{x^{p-1}}{1+x}\, dx$ is convergent if $p < 1$ and divergent if $p \ge 1$... (ii)

It follows from (i) and (ii) that the improper integral $\int_0^\infty \frac{x^{p-1}}{1+x}\, dx$ is convergent if $0 < p < 1$ and divergent otherwise.

Theorem 12.6.6. (Cauchy-Maclaurin integral test)

If f is a monotone decreasing function on $[1, \infty)$ and $f(x) > 0$ for all $x \in [1, \infty)$, then the improper integral $\int_1^\infty f(x)dx$ and the infinite series $\sum\limits_1^\infty f(n)$ converge or diverge together.

Proof. Let $F(X) = \int_1^X f(x)dx$, $X \ge 1$.

Since $f(x) > 0$ for all $x \in [1, \infty)$, F is a monotone increasing function on $[1, \infty)$ and the improper integral $\int_1^\infty f(x)dx$ is convergent or divergent according as F is bounded above or unbounded above on $[1, \infty)$.

Let $s_n = f(1) + f(2) + \cdots + f(n)$.

Since $f(n) > 0$ for all $n \in \mathbb{N}$, the sequence $\{s_n\}$ is a monotone increasing sequence and the series $\sum\limits_1^\infty f(n)$ is convergent or divergent according as the sequence $\{s_n\}$ is bounded above or unbounded above.

Let $X > 1$. Then there exists a natural number n such that $n \le X < n + 1$. Since $f(x) > 0$ for all $x \in [1, \infty)$, $\int_1^n f(x)dx \le \int_1^X f(x)dx \le \int_1^{n+1} f(x)dx$ ... (i)

Let r be a positive integer. Then for all $x \in [r, r + 1]$, $f(r) \ge f(x) \ge f(r + 1)$.

Therefore $\int_r^{r+1} f(r)dx \ge \int_r^{r+1} f(x)dx \ge \int_r^{r+1} f(r + 1)dx$

or, $f(r) \geq \int_r^{r+1} f(x)dx \geq f(r+1)$ ... (ii)

From (ii) $f(1) + f(2) + \cdots + f(n) \geq \int_1^{n+1} f(x)dx \geq \int_1^X f(x)dx$
or, $s_n \geq F(X)$ ... (iii)

From (ii) $f(2) + f(3) + \cdots + f(n) \leq \int_1^n f(x)dx \leq \int_1^X F(X)dx$
or, $s_n - f(1) \leq F(X)$ (iv)

Let the series $\sum\limits_1^\infty f(n)$ be convergent. Then the sequence $\{s_n\}$ is bounded above and since $F(X) \leq s_n$, it follows that F is bounded above on $[1, \infty)$. Consequently, the improper integral $\int_1^\infty f(x)dx$ is convergent.

Let the series $\sum\limits_1^\infty f(n)$ be divergent. Then the sequence $\{s_n\}$ is unbounded above and since $F(X) \geq s_n - f(1)$, it follows that F is unbounded above on $[1, \infty)$. Consequently, the improper integral $\int_1^\infty f(x)dx$ is divergent.

Let the improper integral $\int_1^\infty f(x)dx$ be convergent. Then F is bounded above on $[1, \infty)$ and since $s_n \leq F(X) + f(1)$, it follows that the sequence $\{s_n\}$ is bounded above. Consequently, the series $\sum\limits_1^\infty f(n)$ is convergent.

Let the improper integral $\int_1^\infty f(x)dx$ be divergent. Then F is unbounded above on $[1, \infty)$ and since $s_n \geq F(X)$, it follows that the sequence $\{s_n\}$ is unbounded above. Consequently, the series $\sum\limits_1^\infty f(n)$ is divergent.

Therefore the integral $\int_1^\infty f(x)dx$ and the series $\sum\limits_1^\infty f(n)$ converge or diverge together. This completes the proof.

Worked Examples (continued).

7. Prove that for $s > 0$, the improper integral $\int_1^\infty \frac{x^{-s}}{1+x}dx$ is convergent.

Let $f(x) = \frac{x^{-s}}{1+x}$ for $x \geq 1$, where $s > 0$. Then $f(x) > 0$ for all $x \geq 1$.
For $1 \leq x_1 < x_2$, we have $f(x_2) - f(x_1) = \frac{1}{x_2^s(1+x_2)} - \frac{1}{x_1^s(1+x_1)} < 0$.
Therefore f is a monotone decreasing function on $[1, \infty)$.

By Cauchy-Maclaurin theorem, the improper integral $\int_1^\infty \frac{x^{-s}}{1+x}dx$ and the infinite series $\sum\limits_1^\infty f(n)$ converge or diverge together.

The series $\sum\limits_1^\infty f(n) = \sum\limits_1^\infty \frac{1}{n^s(1+n)}$ is convergent, since $s > 0$.

Consequently, $\int_1^\infty \frac{x^{-s}}{1+x}dx$ is convergent.

8. Use Cauchy-Maclaurin's theorem to test the convergence of the series $\sum_1^\infty \frac{1}{n^p}$ for $p > 0$.

Let $f(x) = \frac{1}{x^p}, p > 0,\ x \in [1, \infty)$. $f(x) > 0$ for all $x \geq 1$ and f is a monotone decreasing function on $[1, \infty)$.

By Cauchy-Maclaurin theorem, the improper integral $\int_1^\infty \frac{1}{x^p} dx$ and the infinite series $\sum_1^\infty f(n)$ converge or diverge together.

The integral $\int_1^\infty \frac{1}{x^p} dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Therefore the series $\sum_1^\infty \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $0 < p \leq 1$.

12.7. Tests for convergence of the improper integral on an infinite range of integration, where the integrand may not keep the same sign.

Theorem 12.7.1. (Cauchy)
Let $a \in \mathbb{R}$ and a function f be integrable on $[a, X]$ for every $X > a$.

A necessary and sufficient condition for the convergence of the improper integral $\int_a^\infty f(x)dx$ is that for a pre-assigned positive ϵ there exists a positive number X_0 such that

$$\left| \int_{X_1}^{X_2} f(x)dx \right| < \epsilon \text{ for all } X_1, X_2 > X_0.$$

Proof. Let $F(X) = \int_a^X f(x)dx,\ X > a$.

The improper integral $\int_a^\infty f(x)dx$ is convergent if $\lim_{X\to\infty} F(X)$ exists finitely.

By Cauchy's criterion for the existence of finite limits, $\lim_{X\to\infty} F(X)$ exists finitely if and only if for a pre-assigned positive ϵ there corresponds a positive X_0 such that

$|F(X_1) - F(X_2)| < \epsilon$ for all X_1, X_2 satisfying $X_1, X_2 > X_0$,

i.e., $\left| \int_a^{X_1} f(x)dx - \int_a^{X_2} f(x)dx \right| < \epsilon$ for all X_1, X_2 satisfying $X_1, X_2 > X_0$

or, $\left| \int_{X_1}^{X_2} f(x)dx \right| < \epsilon$ for all X_1, X_2 satisfying $X_1, X_2 > X_0$.

This completes the proof.

Definitions.

The improper integral $\int_a^\infty f(x)dx$ is said to be *absolutely convergent* if the integral $\int_a^\infty |f|(x)dx$ be convergent.

The improper integral $\int_{-\infty}^\infty f(x)dx$ is said to be *absolutely convergent* if the integral $\int_{-\infty}^\infty |f|(x)dx$ be convergent.

Theorem 12.7.2. An absolutely convergent improper integral $\int_a^\infty f(x)dx$ (where f is bounded and integrable on $[a, X]$ for every $X > a$) is convergent.

Proof. Here f is integrable on $[a, X]$ for every $X > a$ and the improper integral $\int_a^\infty f(x)dx$ is absolutely convergent.

Then the integral $\int_a^\infty |f|(x)dx$ is convergent. Therefore for a pre-assigned positive ϵ there exists a positive X_0 such that

$$|\int_{X_1}^{X_2} |f|(x)dx| < \epsilon \text{ for all } X_1, X_2 \text{ satisfying } X_1, X_2 > X_0.$$

We also have $|\int_{X_1}^{X_2} f(x)dx| \leq \int_{X_1}^{X_2} |f|(x)dx$.

Therefore for a pre-assigned positive ϵ there exists a positive X_0 such that $|\int_{X_1}^{X_2} f(x)dx| < \epsilon$ for all X_1, X_2 satisfying $X_1, X_2 > X_0$.

This implies that the integral $\int_a^\infty f(x)dx$ is convergent.

This completes the proof.

Note 1. The converse of the theorem is not true. We shall establish this by some examples.

Note 2. Since $|f(x)|$ is always positive, comparison tests can be applied to establish the convergence of the improper integral $\int_a^\infty |f|(x)dx$.

Worked Examples (continued).

9. Examine the convergence of the improper integral $\int_1^\infty f(x)dx$, where

$$f(x) = \tfrac{1}{x^2}, \text{ if } x \text{ be rational} \geq 1$$
$$= -\tfrac{1}{x^2}, \text{ if } x \text{ be irrational} > 1.$$

$|f|(x) = \frac{1}{x^2}$, $x \geq 1$. $\int_1^\infty |f|(x)dx$ is convergent and therefore $\int_1^\infty f(x)dx$ is absolutely convergent.

Consequently, the integral $\int_1^\infty f(x)dx$ is convergent.

10. Examine the convergence of the improper integral $\int_0^\infty \frac{\cos mx}{x^2+a^2}dx$, $m > 0, a > 0$.

Let the given integral be $\int_0^\infty f(x)dx$. Let $g(x) = \frac{1}{x^2+a^2}$, $x \geq 0$. Then $|f|(x) \leq g(x)$ for all $x \geq 0$.

$$\lim_{X\to\infty} \int_0^X g(x)dx = \lim_{X\to\infty} \tan^{-1} \tfrac{X}{a} = \tfrac{\pi}{2}.$$

Therefore $\int_0^\infty g(x)dx$ is convergent.

By comparison test, $\int_0^\infty |f|(x)dx$ is convergent and therefore $\int_0^\infty f(x)dx$ is convergent.

11. Examine the convergence of the improper integral $\int_0^\infty \frac{\cos x}{\sqrt{x^3+x}}dx$.

Let the given integral be $\int_0^\infty f(x)dx$. Then $|f|(x) \le \frac{1}{\sqrt{x^3+x}}$. Let $g(x) = \frac{1}{\sqrt{x^3+x}}$, $x > 0$.

0 is a point of infinite discontinuity of g.

Convergence of $\int_0^1 g(x)dx$.

$g(x) > 0$ for all $x \in (0,1]$. Let $u(x) = \frac{1}{\sqrt{x}}$, $x \in (0,1]$. Then $u(x) > 0$ for all $x \in (0,1]$.

$\lim\limits_{x\to 0} \frac{g(x)}{u(x)} = 1$, a non-zero finite number and $\int_0^1 u(x)dx$ is convergent.

By comparison test, $\int_0^1 g(x)dx$ is convergent ... (i)

Convergence of $\int_1^\infty g(x)dx$.

$g(x) > 0$ for all $x > 1$. Let $v(x) = \frac{1}{x^{\frac{3}{2}}}$, $x > 1$. Then $v(x) > 0$ for all $x > 1$.

$\lim\limits_{x\to\infty} \frac{g(x)}{v(x)} = 1$, a non-zero finite number and $\int_1^\infty v(x)dx$ is convergent.

By comparison test, $\int_1^\infty g(x)dx$ is convergent ...(ii)

From (i) and (ii) it follows that $\int_0^\infty g(x)dx$ is convergent.

Since $|f|(x)$ and $g(x)$ are both positive for all $x > 0$ and $|f|(x) \le g(x)$ for all $x > 0$, $\int_0^\infty |f|(x)dx$ is convergent, by comparison test.

Therefore $\int_0^\infty f(x)dx$ is absolutely convergent and hence the given integral is convergent.

12. A function f is defined on $[1,\infty)$ by

$f(x) = \frac{(-1)^{n-1}}{n}$, for $n \le x < n+1$ $(n = 1,2,3,\ldots)$.

Examine convergence of the integrals (i) $\int_1^\infty f(x)dx$, (ii) $\int_1^\infty |f|(x)dx$.

(i) Let us choose $X > 1$. There exists a natural number n such that $n \le X < n+1$.

Let $F(X) = \int_1^X f(x)dx$.

Then $F(X) = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-2}}{n-1} + \int_n^X \frac{(-1)^{n-1}}{n}dx$

$= 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-2}}{n-1} + \frac{(-1)^{n-1}}{n}(X-n)$.

So $|F(X) - [1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-2}}{n-1}]| < \frac{1}{n}$ ... (i)

As $X \to \infty, n \to \infty$.

From (i) it follows that $\lim\limits_{x\to\infty} F(X) = \lim\limits_{n\to\infty}[1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-2}}{n-1}]$.

Since the series $[1 - \frac{1}{2} + \frac{1}{3} - \cdots]$ is a convergent series, it follows that the integral $\int_1^\infty f(x)dx$ is convergent.

Note. Since the series $[1 - \frac{1}{2} + \frac{1}{3} - \cdots]$ converges to $\log 2$, $\int_1^\infty f(x)dx = \log 2$.

(ii) Let us choose $X > 1$. There exists a natural number n such that $n \le X < n+1$.

Let $F(X) = \int_1^X |f|(x)dx$.

Then $F(X) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \int_n^X \frac{1}{n}dx$

$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}(X - n).$

$\ge 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$, since $X - n \ge 0$.

As $X \to \infty, n \to \infty$. As the series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is a divergent series, it follows that the integral $\int_1^\infty |f|(x)dx$ is divergent.

Note. This example establishes that the converse of the theorem 12.7.2 is not true.

12.8. Tests for convergence of the integral of a product.

Theorem 12.8.1. Let a function f be integrable on $[a, X]$ for every $X > a$ and the integral $\int_a^\infty f(x)dx$ is absolutely convergent and a function ϕ be bounded on $[a, \infty)$ and integrable on $[a, X]$ for every $X > a$.

Then the integral $\int_a^\infty f(x)\phi(x)dx$ is absolutely convergent.

Proof. Since the function ϕ is bounded on $[a, \infty)$, there exists a positive real number k such that $|\phi(x)| < k$ for all $x \ge a$.

Since $\int_a^\infty |f|(x)dx$ (with positive integrand) is convergent, there exists a positive real number k_1 such that $\int_a^X |f|(x)dx < k_1$ for all $X > a$.

$\int_a^X |f(x)\phi(x)|dx < k \int_a^X |f(x)|dx < kk_1$ for all $X > a$.

This implies $\int_a^X |f(x)\phi(x)|dx$ is bounded for all $X > a$ and therefore $\int_a^X |f(x)\phi(x)|dx$ is convergent.

Consequently, the integral $\int_a^\infty f(x)\phi(x)dx$ is absolutely convergent.

Theorem 12.8.2. (Abel's test)

Let (i) a function ϕ be monotonic and bounded on $[a, \infty)$ and

(ii) the integral $\int_a^\infty f(x)dx$ be convergent.

Then the integral $\int_a^\infty f(x)\phi(x)dx$ is convergent.

Proof. Since the function ϕ is monotonic on $[a, \infty)$, ϕ is integrable on $[a, X]$ for all $X > a$.

By the second Mean value theorem, $\int_{t_1}^{t_2} f(x)\phi(x)dx =$

$\phi(t_1) \int_{t_1}^{\xi} f(x)dx + \phi(t_2) \int_{\xi}^{t_2} f(x)dx$, where $a < t_1 \le \xi \le t_2$ ... ...(i)

Since ϕ is bounded on $[a, \infty)$, there exists a positive real number k such that $|\phi(x)| < k$ for all $x \geq a$. Therefore it follows that $|\phi(t_1)| < k$, $|\phi(t_2)| < k$.

Let us choose $\epsilon > 0$. Since $\int_a^\infty f(x)dx$ is convergent, there exists a positive real number X such that $|\int_{t_1}^{t_2} f(x)dx| < \frac{\epsilon}{2k}$ for all $t_1, t_2 > X$. Since $t_1 \leq \xi \leq t_2$, it follows that $|\int_{t_1}^{\xi} f(x)dx| < \frac{\epsilon}{2k}$, $|\int_{\xi}^{t_2} f(x)dx| < \frac{\epsilon}{2k}$.

From (i) we have

$|\int_{t_1}^{t_2} f(x)\phi(x)dx| < |\phi(t_1)||\int_{t_1}^{\xi} f(x)dx| + |\phi(t_2)||\int_{\xi}^{t_2} f(x)\phi(x)dx| < k.\frac{\epsilon}{2k} + k.\frac{\epsilon}{2k}$, i.e., $< \epsilon$ for all $t_1, t_2 > X$.

Therefore the integral $\int_a^\infty f(x)\phi(x)dx$ is convergent.

This completes the proof.

Theorem 12.8.3. (Dirichlet's test)

Let (i) a function ϕ be monotonic and bounded on $[a, \infty)$ and $\lim_{x \to \infty} \phi(x) = 0$ and

(ii) the integral $\int_a^X f(x)dx$ be bounded on $[a, X]$ for all $X > a$.

Then the integral $\int_a^\infty f(x)\phi(x)dx$ is convergent.

Proof. Since the function ϕ is monotonic on $[a, \infty)$, ϕ is integrable on $[a, X]$ for all $X > a$.

By the second Mean value theorem, $\int_{t_1}^{t_2} f(x)\phi(x)dx$

$= \phi(t_1) \int_{t_1}^{\xi} f(x)dx + \phi(t_2) \int_{\xi}^{t_2} f(x)dx$, where $a < t_1 \leq \xi \leq t_2$ ... ...(i)

Since the integral $\int_a^X f(x)dx$ is bounded on $[a, X]$ for all $X > a$, there exists a positive real number k such that $|\int_a^X f(x)dx| < k$ for all $X > a$.

Therefore $|\int_{t_1}^{\xi} f(x)dx| = |\int_a^{\xi} f(x)dx - \int_a^{t_1} f(x)dx|$

$\leq |\int_a^{\xi} f(x)dx| + |\int_a^{t_1} f(x)dx| < 2k$.

Similarly, $|\int_{\xi}^{t_2} f(x)dx| < 2k$.

Let us choose $\epsilon > 0$. Since $\lim_{x \to \infty} \phi(x) = 0$, there exists a positive real number X such that $|\phi(x)| < \frac{\epsilon}{4k}$ for all $x > X$.

Let $t_1, t_2 > X$. Then $|\phi(t_1)| < \frac{\epsilon}{4k}$ and $|\phi(t_2)| < \frac{\epsilon}{4k}$.

From (i) we have

$|\int_{t_1}^{t_2} f(x)\phi(x)dx| \leq |\phi(t_1)||\int_{t_1}^{\xi} f(x)dx| + |\phi(t_2)||\int_{\xi}^{t_2} f(x)dx| < \frac{\epsilon}{4k}.2k + \frac{\epsilon}{4k}.2k$, i.e., $< \epsilon$ for all $t_1, t_2 > X$.

Therefore the integral $\int_a^\infty f(x)\phi(x)dx$ is convergent.

This completes the proof.

Worked Examples (continued).

13. Show that the improper integral $\int_0^\infty \frac{\sin x}{x}\, dx$ is convergent.

Since $\lim\limits_{x \to 0+} \frac{\sin x}{x} = 1$, 0 is not a point of infinite discontinuity of the integrand. Therefore $\int_0^1 \frac{\sin x}{x}\, dx$ is convergent. ... (i)

Let us consider the improper integral $\int_1^\infty \frac{\sin x}{x}\, dx$.

Let $f(x) = \sin x$, $x \geq 1$; $g(x) = \frac{1}{x}$, $x \geq 1$. Then g is a bounded and monotone decreasing function on $[1, \infty)$ and $\lim\limits_{x \to \infty} g(x) = 0$.

$|\int_1^X f(x)dx| = |-\cos X + \cos 1| < 2$. Therefore $\int_1^X f(x)dx$ is bounded on $[1, X]$ for all $X > 1$.

By Dirichlet's test, $\int_1^\infty \frac{\sin x}{x}\, dx$ is convergent. ... (ii)

From (i) and (ii) it follows that $\int_0^\infty \frac{\sin x}{x}\, dx$ is convergent.

14. Show that the improper integral $\int_0^\infty |\frac{\sin x}{x}|\, dx$ is not convergent.

Let $f(x) = |\frac{\sin x}{x}|$, $x > 0$
$\quad = 1$, $x = 0$.

Then f is continuous and hence integrable on $[0, X]$ for all $X > 0$.
Therefore $|\frac{\sin x}{x}|$ is integrable on $[0, X]$ for all $X > 0$.

Let us consider the integral $\int_0^{n\pi} |\frac{\sin x}{x}|dx$, where n is a positive integer.

$$\int_0^{n\pi} |\frac{\sin x}{x}|dx = \int_0^{n\pi} \frac{|\sin x|}{x}\, dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\, dx.$$

Now $\int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\, dx = \int_0^\pi \frac{|\sin u|}{(r-1)\pi + u}\, du$, $\quad [\, x = (r-1)\pi + u]$
$$= \int_0^\pi \frac{\sin u}{(r-1)\pi + u}\, du.$$

For all $u \in [0, \pi], (r-1)\pi + u \leq r\pi$.
Therefore $\int_0^\pi \frac{\sin u}{(r-1)\pi + u}\, du \geq \frac{1}{r\pi} \int_0^\pi \sin u\, du = \frac{2}{r\pi}$.

Hence $\int_0^{n\pi} |\frac{\sin x}{x}|dx \geq \frac{2}{\pi}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n})$ ... (i)

As $n \to \infty$, the R.H.S. of (i) gives the series $\sum\limits_{n=1}^{\infty} \frac{2}{\pi n}$ which is a divergent series.

Hence $\lim\limits_{n \to \infty} \int_0^{n\pi} f(x)dx = \infty$. This implies that the improper integral $\int_0^\infty |\frac{\sin x}{x}|dx$ is divergent.

Note. These two examples establish that the converse of the theorem 12.7.2 is not true.

15. Show that the improper integral $\int_0^\infty e^{-ax}\frac{\sin x}{x}dx$ is convergent if $a \geq 0$.

If $a = 0$ the integral reduces to $\int_0^\infty \frac{\sin x}{x}dx$ and it is convergent. [Ex.1]

Let $a > 0$ and let $\phi(x) = e^{-ax}$, $x \geq 0$.

Then $\phi'(x) = -ae^{-ax} < 0$ for all $x \geq 0$.

Therefore ϕ is a bounded monotone function on $[0, \infty)$.
And $\int_0^\infty \frac{\sin x}{x}dx$ is convergent, by Dirichlet's test. [Ex.1]

By Abel's test, $\int_0^\infty \phi(x)\frac{\sin x}{x}dx$ is convergent.

Therefore the integral $\int_0^\infty e^{-ax}\frac{\sin x}{x}dx$ is convergent if $a \geq 0$.

16. Prove that $\int_0^\infty \frac{\sin mx}{x^n}dx$ $(m > 0, n > 0)$ is convergent if $0 < n < 2$ and absolutely convergent if $1 < n < 2$.

Let us choose a positive real number a such that $am < \pi$. Let us examine the convergence of the integrals $\int_0^a \frac{\sin mx}{x^n}dx$ and $\int_a^\infty \frac{\sin mx}{x^n}dx$.

Convergence of the integral $\int_0^a \frac{\sin mx}{x^n}dx$.

Let $f(x) = \frac{\sin mx}{x^n}$, $g(x) = \frac{1}{x^{n-1}}$, $x \in (0, a]$. Then $f(x) > 0, g(x) > 0$ for all $x \in (0, a]$.

$\lim\limits_{x\to 0}\frac{f(x)}{g(x)} = m$. $\int_0^a g(x)dx$ is convergent if $n - 1 < 1$, i.e., if $n < 2$.

By comparison test, $\int_0^a f(x)dx$ is convergent if $n < 2$ ...(i)

Since $f(x) > 0$ for all $x \in (0, a]$, it follows that $\int_0^a f(x)dx$ is absolutely convergent if $n < 2$ ...(ii)

Convergence of the integral $\int_a^\infty \frac{\sin mx}{x^n}dx$.

$\int_a^X \sin mx dx$ is bounded for all $X > a$; and for $n > 0$, $\frac{1}{x^n}$ is a monotone decreasing function, bounded below, on $[a, \infty)$ and $\lim\limits_{x\to\infty}\frac{1}{x^n} = 0$.

By Dirichlet's test, $\int_a^\infty \frac{\sin mx}{x^n}dx$ $(n > 0)$ is convergent ... (iii)

From (i) and (iii) it folloes that $\int_0^\infty \frac{\sin mx}{x^n}dx$ is convergent if $0 < n < 2$.

Absolute convergence of the integral $\int_a^\infty \frac{\sin mx}{x^n}dx$.

Let $f(x) = \frac{\sin mx}{x^n}$ and $g(x) = \frac{1}{x^n}$, $x \geq a$. Then $|f(x)| \leq g(x)$ for all $x \geq a$.

$\int_a^\infty g(x)dx$ is convergent if $n > 1$. By comparison test, $\int_a^\infty |f|(x)dx$ is convergent if $n > 1$, i.e., $\int_a^\infty f(x)dx$ is absolutely convergent if $n > 1$ ... (iv).

From (ii) and (iv) it follows that $\int_0^\infty \frac{\sin mx}{x^n}dx$ is absolutely convergent if $1 < n < 2$.

17. Show that the improper integral $\int_1^\infty \frac{x}{1+x^2} \sin x \, dx$ is convergent.

Let $f(x) = \sin x$, $\phi(x) = \frac{x}{1+x^2}$, $x \geq 1$. Then $\int_1^\infty f(x)dx$ is bounded.

$\phi'(x) = -\frac{1}{1+x^2} < 0$ for all $x \geq 1$. Therefore ϕ is monotone decreasing on $[1, \infty)$. $|\frac{x}{1+x^2}| \leq \frac{1}{2}$ for all $x \geq 1$. Therefore ϕ is bounded on $[1, \infty)$. $\lim\limits_{x \to \infty} \phi(x) = \lim\limits_{x \to \infty} \frac{x}{1+x^2} = 0$.

By Dirichlet's theorem, $\int_1^\infty f(x).\phi(x)dx$ is convergent, i.e., $\int_1^\infty \frac{x}{1+x^2} \sin x \, dx$ is convergent.

18. Show that the improper integral $\int_0^\infty \frac{1}{1+x^2 \sin^2 x} dx$ is divergent.

Let $f(x) = \frac{1}{1+x^2 \sin^2 x}$, $x \geq 0$. Then f is continuous and hence integrable on $[0, X]$ for all $X > 0$.

Let us consider the integral $\int_0^{n\pi} \frac{1}{1+x^2 \sin^2 x} dx$, $n \in \mathbb{N}$.

$\int_0^{n\pi} \frac{1}{1+x^2 \sin^2 x} dx = \sum\limits_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{1}{1+x^2 \sin^2 x} dx$.

For all $x \in [(r-1)\pi, r\pi]$, we have $\frac{1}{1+x^2 \sin^2 x} \geq \frac{1}{1+r^2\pi^2 \sin^2 x}$.

$\int_{(r-1)\pi}^{r\pi} \frac{1}{1+r^2\pi^2 \sin^2 x} dx = \int_0^{\pi} \frac{1}{1+r^2\pi^2 \sin^2 u} du \quad [x = (r-1)\pi + u]$

$= 2\int_0^{\frac{\pi}{2}} \frac{1}{1+r^2\pi^2 \sin^2 u} du = 2\int_0^{\frac{\pi}{2}} \frac{1}{1+k^2 \sin^2 u} du$, where $k^2 = r^2\pi^2$.

We have

$\int \frac{1}{1+k^2 \sin^2 u} \, du = \int \frac{\sec^2 u}{1+(k^2+1)\tan^2 u} du = \frac{1}{\sqrt{k^2+1}} \tan^{-1}(\sqrt{k^2+1} \tan u)$.

Therefore $2\int_0^{\frac{\pi}{2}} \frac{1}{1+k^2 \sin^2 u} du$

$= 2 \lim\limits_{\epsilon \to 0} [\frac{1}{\sqrt{k^2+1}} \tan^{-1}(\sqrt{k^2+1} \tan u)]_0^{\frac{\pi}{2}-\epsilon} = \frac{\pi}{\sqrt{1+k^2}}$.

Hence $\int_0^{n\pi} \frac{1}{1+x^2 \sin^2 x} dx \geq \sum\limits_{r=1}^{n} \frac{\pi}{\sqrt{1+r^2\pi^2}} \qquad \ldots$ (i)

Let $t_n = \frac{\pi}{\sqrt{1+n^2\pi^2}}$. As $n \to \infty$, the R.H.S. of (i) becomes the infinite series $\sum\limits_{n=1}^{\infty} t_n$ which is a divergent series.

Hence $\lim\limits_{n \to \infty} \int_0^{n\pi} f(x)dx = \infty$. This implies that the improper integral $\int_0^\infty \frac{1}{1+x^2 \sin^2 x} dx$ is divergent.

19. Show that the improper integral $\int_0^\infty \frac{1}{1+x^4 \sin^2 x} dx$ is convergent.

Let $f(x) = \frac{1}{1+x^4 \sin^2 x}$, $x \geq 0$. Then f is continuous and hence integrable on $[0, X]$ for all $X > 0$.

Let us consider the integral $\int_0^{n\pi} \frac{1}{1+x^4 \sin^2 x} dx$, $n \in \mathbb{N}$.

$\int_0^{n\pi} \frac{1}{1+x^4\sin^2 x}dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{1}{1+x^4\sin^2 x}dx.$

For all $x \in [(r-1)\pi, r\pi]$, we have $\frac{1}{x^4\sin^2 x} \leq \frac{1}{1+(r-1)^4\pi^4\sin^2 x}$.

Now $\int_{(r-1)\pi}^{r\pi} \frac{1}{1+(r-1)^4\pi^4\sin^2 x}dx = \int_0^{\pi} \frac{1}{1+(r-1)^4\pi^4\sin^2 u}du$

[by the substitution $x = (r-1)\pi + u$]

$= 2\int_0^{\frac{\pi}{2}} \frac{1}{1+(r-1)^4\pi^4\sin^2 u}du$

$= 2\int_0^{\frac{\pi}{2}} \frac{1}{1+k^2\sin^2 u}du$, where $k^2 = (r-1)^4\pi^4$.

We have

$\int \frac{1}{1+k^2\sin^2 u}\, du = \int \frac{\sec^2 u}{1+(k^2+1)\tan^2 u}dx = \frac{1}{\sqrt{k^2+1}} \tan^{-1}(\sqrt{k^2+1}\tan\ u).$

Therefore $2\int_0^{\frac{\pi}{2}} \frac{1}{1+k^2\sin^2 u}du$

$= 2\lim_{\epsilon\to 0}[\frac{1}{\sqrt{k^2+1}} \tan^{-1}(\sqrt{k^2+1}\tan\ u)]_0^{\frac{\pi}{2}-\epsilon} = \frac{\pi}{\sqrt{1+k^2}}.$

Hence $\int_0^{n\pi} \frac{1}{1+(n-1)^4\pi^4\sin^2 x}dx \leq \sum_{r=1}^{n} \frac{\pi}{\sqrt{1+(r-1)^4\pi^4}}$ ...(i)

Let $t_n = \frac{\pi}{\sqrt{1+\pi^4 n^4}}$. As $n \to \infty$, the R.H.S. of (i) becomes the infinite series $\sum_{n=0}^{\infty} t_n$. $\sum_{n=1}^{\infty} t_n$ is a convergent series, by comparison test. $[v_n = \frac{1}{n^2}]$

Hence $\lim_{n\to\infty} \int_0^{n\pi} f(x)dx$ is finite. This implies that the improper integral $\int_0^{\infty} \frac{1}{1+x^4\sin^2 x}dx$ is convergent.

12.9. Some theorems.

Theorem 12.9.1. Let a function f be bounded and integrable on the interval $[a, X]$ for every $X > a$ and the improper integral $\int_a^{\infty} f(x)dx$ be convergent. If $\lim_{x\to\infty} f(x) = l$, then $l = 0$.

Proof. Let $l > 0$. Let us choose a positive ϵ such that $l - \epsilon > 0$.

Since $\lim_{x\to\infty} f(x) = l$, there exists a positive real number B_1 such that $l - \epsilon < f(x) < l + \epsilon$ for all $x > B_1$.

Let $l - \epsilon = k$. Then $k > 0$ and $f(x) > k > 0$ for all $x > B_1$.

Since $\int_a^{\infty} f(x)dx$ is convergent, for the same chosen ϵ there exists a positive real number B_2 such that $|\int_{X_1}^{X_2} f(x)dx| < \epsilon$ for all X_1, X_2 satisfying $X_2 > X_1 > B_2$.

Let $B = \max\{B_1, B_2\}$. Then $f(x) > k > 0$ for all $x > B$ (i)

and $|\int_{X_1}^{X_2} f(x)dx| < \epsilon$ for all X_1, X_2 satisfying $X_2 > X_1 > B$ (ii)

Now $f(x) > k > 0$ for all $x > B \Rightarrow \int_{X_1}^{X_2} f(x)dx \geq k(X_2 - X_1) > k$ for all X_1, X_2 satisfying $X_2 > X_1 > B$.

This contradicts the condition (ii). Therefore $l \not> 0$ ... (iii)

Let $l < 0$. Let us choose a positive ϵ such that $l + \epsilon < 0$.

Since $\lim_{x\to\infty} f(x) = l$, there exists a positive real number G_1 such that $l - \epsilon < f(x) < l + \epsilon$ for all $x > G_1$.

Let $l + \epsilon = k$. Then $k < 0$ and $f(x) < k < 0$ for all $x > G_1$.

Since $\int_a^\infty f(x)dx$ is convergent, for the same chosen ϵ there exists a positive real number G_2 such that $|\int_{X_1}^{X_2} f(x)dx| < \epsilon$ for all X_1, X_2 satisfying $X_2 > X_1 > G_2$.

Let $G = \max\{G_1, G_2\}$. Then $f(x) < k < 0$ for all $x > G$ (iv)
and $|\int_{X_1}^{X_2} f(x)dx| < \epsilon$ for all X_1, X_2 satisfying $X_2 > X_1 > G$ ... (v)

Now $f(x) < k < 0$ for all $x > G \Rightarrow \int_{X_1}^{X_2} f(x)dx \leq k(X_2 - X_1) < k$ for all X_1, X_2 satisfying $X_2 > X_1 > G$.
Therefore $|\int_{X_1}^{X_2} f(x)dx| > |k|$ for all X_1, X_2 satisfying $X_2 > X_1 > G$.

This contradicts the condition (v). Therefore $l \not< 0$ (vi)

From (iii) and (vi) it follows that $l = 0$.

Note. An important property of a convergent infinite series $\sum_1^\infty u_n$ is that $\lim_{n\to\infty} u_n = 0$. But this property does not hold in case of a convergent improper integral $\int_a^\infty f(x)dx$.

If f be bounded and integrable on $[a, X]$ for every $X > a$, then the convergence of the improper integral $\int_a^\infty f(x)dx$ does not necessarily imply that $\lim_{x\to\infty} f(x) = 0$.

For example, let $f(x) = \cos x^2$, $x \in [1, \infty)$. f is bounded and integrable on $[1, X]$ for all $X > 1$ and $\int_1^\infty f(x)dx$ is convergent. But $\lim_{x\to\infty} f(x)$ does not exist.

Theorem 12.9.2. Let a function ϕ be continuous on $(0, \infty)$ and $\lim_{x\to 0+} \phi(x) = \phi_0$ (finite), $\lim_{x\to\infty} \phi(x) = \phi_1$ (finite). Then

$$\int_0^\infty \frac{\phi(ax) - \phi(bx)}{x} dx = (\phi_0 - \phi_1) \log \frac{b}{a}, \text{ where } a > 0,\ b > 0 \text{ and } b > a.$$

Proof. Let $\epsilon > 0$ and let $X > \epsilon$.

Let us consider the integral $\int_\epsilon^X \frac{\phi(ax) - \phi(bx)}{x} dx$.

We have $\int_\epsilon^X \frac{\phi(ax) - \phi(bx)}{x} dx = \int_\epsilon^X \frac{\phi(ax)}{x} dx - \int_\epsilon^X \frac{\phi(bx)}{x} dx$.

Let $ax = u$, $bx = v$. Then we have $\int_\epsilon^X \frac{\phi(ax)}{x}dx = \int_{a\epsilon}^{aX} \frac{\phi(u)}{u}du$ and $\int_\epsilon^X \frac{\phi(bx)}{x}dx = \int_{b\epsilon}^{bX} \frac{\phi(v)}{v}dv = \int_{b\epsilon}^{bX} \frac{\phi(u)}{u}du.$

Therefore $\int_\epsilon^X \frac{\phi(ax)-\phi(bx)}{x}dx = \int_{a\epsilon}^{aX} \frac{\phi(u)}{u}du - \int_{b\epsilon}^{bX} \frac{\phi(u)}{u}du$

$= [\int_{a\epsilon}^{b\epsilon} \frac{\phi(u)}{u}du + \int_{b\epsilon}^{bX} \frac{\phi(u)}{u}du + \int_{bX}^{aX} \frac{\phi(u)}{u}du] - \int_{b\epsilon}^{bX} \frac{\phi(u)}{u}du$

$= \int_{a\epsilon}^{b\epsilon} \frac{\phi(u)}{u}du - \int_{aX}^{bX} \frac{\phi(u)}{u}du.$

By the first mean value theorem, there exists a point $\xi \in [a\epsilon, b\epsilon]$ and a point $\eta \in [aX, bX]$ such that

$\int_{a\epsilon}^{b\epsilon} \frac{\phi(u)}{u}du = \phi(\xi) \int_{a\epsilon}^{b\epsilon} \frac{1}{u}du = \phi(\xi)\log\frac{b}{a}$ and

$\int_{aX}^{bX} \frac{\phi(u)}{u}du = \phi(\eta) \int_{aX}^{bX} \frac{1}{u}du = \phi(\eta)\log\frac{b}{a}.$

Therefore $\int_\epsilon^X \frac{\phi(ax)-\phi(bx)}{x}dx = [\phi(\xi) - \phi(\eta)]\log\frac{b}{a}.$

Let $\epsilon \to 0+$. Then $\phi(\xi) \to \phi_0$

Let $X \to \infty$. Then $\phi(\eta) \to \phi_1$.

Therefore $\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx$ is convergent and equals to $[\phi_0 - \phi_1]\log\frac{b}{a}$.

Note. If $\int_0^1 \frac{\phi(x)}{x}dx$ be convergent at 0, then by the general principle of convergence, $\lim_{\epsilon\to 0}\int_{a\epsilon}^{b\epsilon} \frac{\phi(x)}{x}dx = 0$.

If $\int_1^\infty \frac{\phi(x)}{x}dx$ be convergent at ∞, then by the general principle of convergence, $\lim_{X\to\infty}\int_{aX}^{bX} \frac{\phi(x)}{x}dx = 0$.

Therefore (i) if $\int_0^1 \frac{\phi(x)}{x}dx$ be convergent at 0 and $\lim_{x\to\infty}\phi(x) = \phi_1$, then $\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx = -\phi_1\log\frac{b}{a}$, where $a > 0$, $b > 0$ and $b > a$;

(ii) if $\int_1^\infty \frac{\phi(x)}{x}dx$ be convergent at ∞ and $\lim_{x\to 0+}\phi(x) = \phi_0$, then $\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx = \phi_0\log\frac{b}{a}$, where $a > 0$, $b > 0$ and $b > a$;

(iii) if $\int_0^1 \frac{\phi(x)}{x}dx$ be convergent at 0 and $\int_1^\infty \frac{\phi(x)}{x}dx$ be convergent at ∞, then $\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx = 0$, where $a > 0$, $b > 0$ and $b > a$.

Worked Examples (continued).

20. Show that $\int_0^\infty \frac{\tan^{-1}(ax)-\tan^{-1}(bx)}{x} = \frac{\pi}{2}\log(a/b)$, $0 < b < a$.

Let $\phi(x) = \tan^{-1} x$, $x \geq 0$. Then ϕ is continuous on $[0,\infty)$.

$\lim_{x\to 0+}\phi(x) = \phi(0) = 0$, $\lim_{x\to\infty}\phi(x) = \frac{\pi}{2}$.

Therefore $\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx = [0-\frac{\pi}{2}].\log(b/a)$

or, $\int_0^\infty \frac{\tan^{-1}(ax)-\tan^{-1}(bx)}{x} = \frac{\pi}{2}\log(a/b)$.

21. Show that $\int_0^\infty \frac{\cos ax - \cos bx}{x}dx = \log\frac{b}{a}$, $0 < b < a$.

Let $\phi(x) = \cos x$, $x \geq 0$. Then ϕ is continuous on $[0, \infty)$.

Let $\epsilon > 0$ and let $X > \epsilon$.

$\int_\epsilon^X \frac{\cos ax - \cos bx}{x}dx = \int_\epsilon^X \frac{\cos ax}{x}dx - \int_\epsilon^X \frac{\cos bx}{x}dx$

$= \int_{a\epsilon}^{aX} \frac{\cos x}{x}dx - \int_{b\epsilon}^{bX} \frac{\cos x}{x} = \int_{a\epsilon}^{b\epsilon} \frac{\cos x}{x}dx - \int_{aX}^{bX} \frac{\cos x}{x}dx.$

By the first Mean value theorem, there exists a real number $\xi \in [a\epsilon, b\epsilon]$ such that $\int_{a\epsilon}^{b\epsilon} \frac{\cos x}{x}dx = \cos\xi \int_{a\epsilon}^{b\epsilon} \frac{1}{x}dx = \cos\xi \log\frac{b}{a}$.

Therefore $\lim_{\epsilon\to 0} \int_{a\epsilon}^{b\epsilon} \frac{\cos x}{x}dx = \log\frac{b}{a}$, since $\xi \to 0$ as $\epsilon \to 0$.

Since $\int_1^\infty \frac{\cos x}{x}dx$ is convergent at ∞, $\lim_{X\to\infty} \int_{aX}^{bX} \frac{\cos x}{x} = 0$.

Therefore $\int_0^\infty \frac{\cos ax - \cos bx}{x}dx = \log\frac{b}{a}$.

Deduction.

Prove that $\int_0^\infty \frac{\sin ax \sin bx}{x}\,dx = \frac{1}{2}\log\frac{a+b}{a-b}$, $0 < b < a$.

We have $\int_0^\infty \frac{\cos px - \cos qx}{x}dx = \log\frac{q}{p}$, $0 < q < p$.

or, $\int_0^\infty \frac{2\sin\frac{1}{2}(p+q)x \sin\frac{1}{2}(q-p)x}{x}dx = \log\frac{q}{p}$.

Let $p + q = 2a, q - p = 2b$. Then the result follows.

12.10. Evaluation of some improper integrals.

1. Evaluate $\int_0^\infty \frac{\sin x}{x}dx$.

$\int_0^\infty \frac{\sin x}{x}\,dx$ is a convergent improper integral. [page 28]

Let us consider the integral $\int_0^{\frac{\pi}{2}} \phi(x)\sin(2n+1)x\,dx$, where $\phi(x) = \frac{1}{x} - \frac{1}{\sin x}$, $x > 0$.

ϕ is continuous, and therefore integrable, on $[\epsilon, \frac{\pi}{2}]$, where $0 < \epsilon < \frac{\pi}{2}$ and $\lim_{x\to 0}\phi(x) = 0$. Therefore ϕ is integrable on $[0, \frac{\pi}{2}]$.

By Riemann-Lebesgue theorem; $\lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \phi(x)\sin(2n+1)x\,dx = 0$. [page 484]

Let $I_n = \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{x}\,dx$, $J_n = \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x}\,dx$.

Then $J_n - J_{n-1} = \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x}\,dx = \int_0^{\frac{\pi}{2}} 2\cos 2nx\,dx = 0.$

Therefore $J_n = J_{n-1} = \cdots = J_1 = \frac{\pi}{2}$.

As $\lim\limits_{n\to\infty}(I_n - J_n) = 0$ and $J_n = \frac{\pi}{2}$, it follows that $\lim\limits_{n\to\infty} I_n = \frac{\pi}{2}$, i.e., $\lim\limits_{n\to\infty} \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{x}\,dx = \frac{\pi}{2}$.

Now $\lim\limits_{n\to\infty} \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{x}\,dx = \lim\limits_{n\to\infty} \int_0^{\frac{(2n+1)\pi}{2}} \frac{\sin u}{u}\,du$. $[u = (2n+1)x]$

$= \int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx$, since $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx$ is convergent.

Consequently, $\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

2. Prove that $\int_0^{\infty} \frac{\sin mx}{x}\,dx = \frac{\pi}{2}$ or $-\frac{\pi}{2}$ according as $m > 0$ or $m < 0$.

Let $m > 0$. Let us choose $\epsilon > 0$ and $X > \epsilon$.

$\int_\epsilon^X \frac{\sin mx}{x}\,dx = \int_{m\epsilon}^{mX} \frac{\sin u}{u}\,du$ [by the substitution $mx = u$].

$\lim\limits_{X\to\infty,\epsilon\to 0} \int_{m\epsilon}^{mX} \frac{\sin u}{u}\,du = \int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

Therefore $\lim\limits_{X\to\infty,\epsilon\to 0} \int_\epsilon^X \frac{\sin mx}{x}\,dx = \frac{\pi}{2}$ and therefore $\int_0^{\infty} \frac{\sin mx}{x}\,dx = \frac{\pi}{2}$.

Similar proof when $m < 0$.

3. Show that $\int_0^{\infty} \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{\pi}{2}(b-a)$, $a > 0, b > 0$.

$\int_\epsilon^X \frac{\cos ax - \cos bx}{x^2}\,dx = \left[\frac{\cos ax - \cos bx}{-x}\right]_\epsilon^X + \int_\epsilon^X \frac{-a\sin ax + b\sin bx}{x}\,dx$

$= -\frac{\cos aX - \cos bX}{X} + \frac{\cos a\epsilon - \cos b\epsilon}{\epsilon} + \int_\epsilon^X \frac{-a\sin ax}{x}\,dx + \int_\epsilon^X \frac{b\sin bx}{x}\,dx.$

$\lim\limits_{X\to\infty} \frac{\cos aX - \cos bX}{-X} = 0$, $\lim\limits_{\epsilon\to 0} \frac{\cos a\epsilon - \cos b\epsilon}{\epsilon} = 0$,

$\lim\limits_{X\to\infty,\epsilon\to 0} \int_\epsilon^X \frac{-a\sin ax}{x}\,dx = -a\int_0^{\infty} \frac{\sin ax}{x} = -a\frac{\pi}{2}$,

$\lim\limits_{X\to\infty,\epsilon\to 0} \int_\epsilon^X \frac{b\sin bx}{x}\,dx = b\int_0^{\infty} \frac{\sin bx}{x} = b\frac{\pi}{2}$.

Therefore $\int_0^{\infty} \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{\pi}{2}(b-a)$, $a > 0, b > 0$.

4. Evaluate $\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx$.

Let us choose $\epsilon > 0$ and $X > \epsilon$.

$\int_\epsilon^X \frac{\sin^2 x}{x^2}\,dx = \frac{\sin^2 \epsilon}{\epsilon} - \frac{\sin^2 X}{X} + \int_\epsilon^X \frac{\sin 2x}{x}\,dx.$

As $\lim\limits_{\epsilon\to 0} \frac{\sin^2 \epsilon}{\epsilon} = 0$ and $\lim\limits_{X\to\infty} \frac{\sin^2 X}{X} = 0$,

$\lim\limits_{X\to\infty\epsilon\to 0} \int_\epsilon^X \frac{\sin^2 x}{x^2}\,dx = \lim\limits_{X\to\infty,\epsilon\to 0} \int_\epsilon^X \frac{\sin 2x}{x}\,dx$

$= \int_0^{\infty} \frac{\sin 2x}{x}\,dx$, since $\int_0^{\infty} \frac{\sin 2x}{x}\,dx$ is convergent

$= \frac{\pi}{2}$.

Therefore $\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$.

Evaluate $\int_0^{\frac{\pi}{2}} \log \sin x \, dx$.

$\int_0^{\frac{\pi}{2}} \log \sin x \, dx$ is a convergent improper integral. [worked out Ex.8,

Let $I = \int_0^{\frac{\pi}{2}} \log \sin x \, dx$. Let $\phi(\epsilon) = \int_\epsilon^{\frac{\pi}{2}} \log \sin x \, dx$, $0 < \epsilon < \frac{\pi}{2}$.

Then $I = \lim\limits_{\epsilon \to 0} \phi(\epsilon)$.

Now $\phi(\epsilon) = \int_\epsilon^{\frac{\pi}{2}} \log \sin x \, dx = \int_0^{\frac{\pi}{2}-\epsilon} \log \cos y \, dy \quad [x = \frac{\pi}{2} - y]$

$= \int_0^{\frac{\pi}{2}-\epsilon} \log \cos x \, dx.$

Then $2\phi(\epsilon) = \int_\epsilon^{\frac{\pi}{2}} \log \sin x \, dx + \int_0^{\frac{\pi}{2}-\epsilon} \log \cos x \, dx$

$= \int_\epsilon^{\frac{\pi}{2}-\epsilon} [\log \sin x + \log \cos x] \, dx + \int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} \log \sin x \, dx + \int_0^\epsilon \log \cos x \, dx$

$= \int_\epsilon^{\frac{\pi}{2}-\epsilon} \log \frac{\sin 2x}{2} dx + 2 \int_0^\epsilon \log \cos x dx$, since $\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} \log \sin x \, dx =$ $\int_0^\epsilon \log \cos x \, dx$, by the substitution $y = \frac{\pi}{2} - x$.

Therefore $2I = \lim\limits_{\epsilon \to 0} 2\phi(\epsilon)$

$= \lim\limits_{\epsilon \to 0} [\int_\epsilon^{\frac{\pi}{2}-\epsilon} \log \frac{\sin 2x}{2} \, dx + 2 \int_0^\epsilon \log \cos x \, dx]$

$= \lim\limits_{\epsilon \to 0} [\int_\epsilon^{\frac{\pi}{2}-\epsilon} \log \sin 2x \, dx - \int_\epsilon^{\frac{\pi}{2}-\epsilon} \log 2 \, dx + 2 \int_0^\epsilon \log \cos x \, dx]$

$= \lim\limits_{\epsilon \to 0} [\frac{1}{2} \int_{2\epsilon}^{\pi-2\epsilon} \log \sin u \, du - (\frac{\pi}{2} - 2\epsilon) \log 2 + 2 \int_0^\epsilon \log \cos x \, dx] \quad [u = 2x]$

$= \lim\limits_{\epsilon \to 0} [\frac{1}{2} \int_{2\epsilon}^{\frac{\pi}{2}} \log \sin u \, du + \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi-2\epsilon} \log \sin u \, du - (\frac{\pi}{2} - 2\epsilon) \log 2 +$ $2 \int_0^\epsilon \log \cos x \, dx]$

$= \lim\limits_{\epsilon \to 0} [\frac{1}{2} \int_{2\epsilon}^{\frac{\pi}{2}} \log \sin u \, du + \frac{1}{2} \int_{2\epsilon}^{\frac{\pi}{2}} \log \sin t \, dt - (\frac{\pi}{2} - 2\epsilon) \log 2$ $+ 2 \int_0^\epsilon \log \cos x \, dx] \quad [t = \pi - u]$

$= \lim\limits_{\epsilon \to 0} [\int_{2\epsilon}^{\frac{\pi}{2}} \log \sin x \, dx - (\frac{\pi}{2} - 2\epsilon) \log 2 + 2 \int_0^\epsilon \log \cos x \, dx].$

$= \lim\limits_{\epsilon \to 0} [\phi(2\epsilon) - (\frac{\pi}{2} - 2\epsilon) \log 2 + 2 \int_0^\epsilon \log \cos x \, dx]$ ... (i)

Let $f(\epsilon) = \int_0^\epsilon \log \cos x \, dx$, $0 \le \epsilon < \frac{\pi}{2}$. Then f is a continuous function on $[0, \frac{\pi}{4}]$, since $\log \cos x$ is integrable on $[0, \frac{\pi}{4}]$. Therefore $\lim\limits_{\epsilon \to 0} f(\epsilon) = f(0) = 0$.

$\lim\limits_{\epsilon \to 0} \phi(2\epsilon) = \lim\limits_{\epsilon \to 0} \phi(\epsilon) = I$, and $\lim\limits_{\epsilon \to 0} [\frac{\pi}{2} - 2\epsilon] = \frac{\pi}{2}$.

From (i) it follows that $2I = I - \frac{\pi}{2} \log 2$

or, $I = -\frac{\pi}{2} \log 2$, i.e., $\int_0^{\frac{\pi}{2}} \log \sin x \, dx = \frac{\pi}{2} \log \frac{1}{2}$.

6. Assuming that the integral $\int_0^{\frac{\pi}{2}} \log\sin x\, dx$ is convergent, prove that $\int_0^{\frac{\pi}{2}} \log\cos x\, dx$ is convergent and evaluate $\int_0^{\frac{\pi}{2}} \log\cos x\, dx$.

$\int_0^{\frac{\pi}{2}} \log\cos x\, dx$ is an improper integral. The integrand is continuous, and therefore integrable, on $[0, \frac{\pi}{2} - \epsilon]$ for all ϵ satisfying $0 < \epsilon < \frac{\pi}{2}$.

Let $\psi(\epsilon) = \int_0^{\frac{\pi}{2}-\epsilon} \log\cos x\, dx$.
Then $\psi(\epsilon) = \int_\epsilon^{\frac{\pi}{2}} \log\sin y\, dy \quad [x = \frac{\pi}{2} - y]$.
$= \int_\epsilon^{\frac{\pi}{2}} \log\sin x\, dx$.

Since $\int_0^{\frac{\pi}{2}} \log\sin x\, dx$ is convergent, $\lim\limits_{\epsilon\to 0}\int_\epsilon^{\frac{\pi}{2}} \log\sin x\, dx$ is finite, i.e., $\lim\limits_{\epsilon\to 0}\psi(\epsilon)$ is finite and this proves that $\int_0^{\frac{\pi}{2}} \log\cos x\, dx$ is convergent.

Let $I = \int_0^{\frac{\pi}{2}} \log\cos x\, dx$. Then $I = \lim\limits_{\epsilon\to 0}\psi(\epsilon) = \lim\limits_{\epsilon\to 0}\int_\epsilon^{\frac{\pi}{2}} \log\sin x\, dx$
$= \int_0^{\frac{\pi}{2}} \log\sin x\, dx = \frac{\pi}{2}\log\frac{1}{2}$.

7. Prove that $\int_0^{\frac{\pi}{2}} \cos 2nx\, \log\sin x\, dx$ is convergent. Evaluate the integral when n is a positive integer.

Let $f(x) = \cos 2nx\, \log\sin x, 0 < x \leq \frac{\pi}{2}$. Then f is bounded and integrable on $[\epsilon, \frac{\pi}{2}]$ for all ϵ satisfying $0 < \epsilon < \frac{\pi}{2}$.
We have $\lim\limits_{x\to 0+}\frac{\log\sin x}{1/\sqrt{x}} = 0$. [using L'Hospital's rule]

Let $g(x) = \frac{1}{\sqrt{x}}, 0 < x \leq \frac{\pi}{2}$. Then $g(x) > 0$ for all $x \in (0, \frac{\pi}{2}]$ and $\lim\limits_{x\to 0+}\frac{|f(x)|}{g(x)} = 0$. Therefore $\int_0^{\frac{\pi}{2}} |f(x)|\, dx$ is convergent, since $\int_0^{\frac{\pi}{2}} g(x)\, dx$ is convergent.

Consequently, $\int_0^{\frac{\pi}{2}} f(x)\, dx$ is convergent.

Second part.
$\int_0^{\frac{\pi}{2}} f(x)\, dx = \lim\limits_{\epsilon\to 0+}\int_\epsilon^{\frac{\pi}{2}} f(x)\, dx$, since $\int_0^{\frac{\pi}{2}} f(x)\, dx$ is convergent.

Integrating by parts, we have $\lim\limits_{\epsilon\to 0+}\int_\epsilon^{\frac{\pi}{2}} \cos 2nx \log\sin x\, dx$
$= \lim\limits_{\epsilon\to 0+}[\frac{\sin n\pi}{2n}\log\sin\frac{\pi}{2} - \frac{\sin 2n\epsilon}{2n}\log\sin\epsilon]$- $\lim\limits_{\epsilon\to 0+}\frac{1}{2n}\int_\epsilon^{\frac{\pi}{2}}\frac{\sin 2nx\cos x}{\sin x}\, dx$

$=- \lim\limits_{\epsilon\to 0+}\frac{1}{n}\int_\epsilon^{\frac{\pi}{2}}\cos x[\cos x + \cos 3x + \cdots + \cos(2n-1)x]\, dx$, since n is a positive integer and $\frac{\sin 2nx}{2\sin x} = \cos x + \cos 3x + \cdots + \cos(2n-1)x$ and $\lim\limits_{\epsilon\to 0+} - \frac{\sin 2n\epsilon}{2n}\log\sin\epsilon = 0$
$= - \lim\limits_{\epsilon\to 0+}\frac{1}{n}\int_\epsilon^{\frac{\pi}{2}}(\frac{1}{2}+2\cos 2x+2\cos 4x+\cdots+2\cos(2n-1)x+\cos 2nx)\, dx$
$= -\frac{\pi}{4n}$.

8. Assuming that $\int_0^{\frac{\pi}{2}} \cos 2nx \ \log \sin x \ dx = -\frac{\pi}{4n}$, when n is a positive integer, show that

$\int_0^{\pi} \cos nx \ \log 2(1-\cos x) \ dx = -\frac{\pi}{n}$, when n is a positive integer.

Let $f(x) = \cos nx \log 2(1-\cos x), 0 < x \leq \pi$. Then f is bounded and integrable on $[\epsilon, \pi]$ for every ϵ satisfying $0 < \epsilon < \pi$.

Let us evaluate $\lim\limits_{\epsilon \to 0+} \int_\epsilon^\pi \cos nx \log 2(1-\cos x) \ dx$.

$\lim\limits_{\epsilon \to 0+} \int_\epsilon^\pi \cos nx \log 2(1-\cos x) \ dx$

$= \lim\limits_{\epsilon \to 0+} \int_\epsilon^\pi \cos nx \ 2\log(2\sin\frac{x}{2}) \ dx$

$= \lim\limits_{\epsilon \to 0+} 2\int_{\frac{\epsilon}{2}}^{\frac{\pi}{2}} \cos 2nu \ 2\log(2\sin u) \ du$, by the substitution $x = 2u$.

$= \lim\limits_{\epsilon \to 0+} 4\int_{\frac{\epsilon}{2}}^{\frac{\pi}{2}} \cos 2nx[\log 2 + \log\sin x] \ dx$.

$= \lim\limits_{\epsilon \to 0+} [4\log 2\int_{\frac{\epsilon}{2}}^{\frac{\pi}{2}} \cos 2nx \ dx + 4\int_{\frac{\epsilon}{2}}^{\frac{\pi}{2}} \cos 2nx \ \log\sin x \ dx]$

$= \lim\limits_{\epsilon \to 0+} [4\log 2.\frac{1}{2n}(\sin n\pi - \sin n\epsilon)] - 4.\frac{\pi}{4n}$, since

$\lim\limits_{\epsilon \to 0+} \int_{\frac{\epsilon}{2}}^{\frac{\pi}{2}} \cos 2nx \log\sin x \ dx = \int_0^{\frac{\pi}{2}} \cos 2nx \ \log\sin x \ dx = -\frac{\pi}{4n}$

$= -\frac{\pi}{n}$.

9. Evaluate $\int_0^\infty \frac{x^{p-1}}{1+x} \ dx, \quad 0 < p < 1$.

The improper integral $\int_0^\infty \frac{x^{p-1}}{1+x} \ dx$ is convergent if $0 < p < 1$. [worked out Ex.6, page 506]

Let $I = \int_0^\infty \frac{x^{p-1}}{1+x} \ dx$.

Then $I = \int_0^1 \frac{x^{p-1}}{1+x} \ dx + \int_1^\infty \frac{x^{p-1}}{1+x} \ dx$

$= \int_0^1 \frac{x^{p-1}}{1+x} \ dx - \int_1^0 \frac{u^{-p+1}}{u(1+u)} \ du \qquad [u = \frac{1}{x}]$

$= \int_0^1 \frac{x^{p-1}}{1+x} \ dx + \int_0^1 \frac{u^{-p}}{(1+u)} \ du$

$= \int_0^1 \frac{x^{p-1}+x^{-p}}{1+x} dx.$

We have $\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n + \frac{(-1)^{n+1}x^{n+1}}{1+x}$ for all real $x \neq -1$ and for all $n \in \mathbb{N}$.

Therefore $\int_0^1 \frac{x^{p-1}+x^{-p}}{1+x} \ dx$

$= \int_0^1 (x^{p-1} + x^{-p})[\sum\limits_{r=0}^{n}(-1)^r x^r]dx + \int_0^1 (-1)^{n+1}(x^{p-1}+x^{-p})(\frac{x^{n+1}}{1+x}) \ dx$

$= \int_0^1 [\sum_{r=0}^{n} (-1)^r (x^{p+r-1} + x^{-p+r})]\, dx + R_n$, where $R_n =$

$$(-1)^{n+1} \int_0^1 \frac{x^{n+p}+x^{n-p+1}}{1+x} dx$$

$= \sum_{r=0}^{n} (-1)^r [\frac{1}{p+r} + \frac{1}{-p+r+1}] + R_n$

$= (\frac{1}{p} + \frac{1}{-p+1}) - (\frac{1}{p+1} + \frac{1}{-p+2}) + (\frac{1}{p+2} + \frac{1}{-p+3}) - \cdots + (-1)^n (\frac{1}{p+n} + \frac{1}{-p+n+1}) + R_n$.

$|R_n| = |\int_0^1 \frac{x^{n+p}+x^{n-p+1}}{1+x}\, dx| \leq \int_0^1 |\frac{x^{n+p}+x^{n-p+1}}{1+x}|\, dx \leq 2\int_0^1 x^n\, dx \leq \frac{2}{n+1}$, since $0 < p < 1 \Rightarrow 0 < x^p < 1$ and $0 < x^{1-p} < 1$ and $\frac{1}{1+x} < 1$ for all $x \in [0,1]$.

Therefore $\lim |R_n| = 0$ and this implies $\lim R_n = 0$.

Therefore $\int_0^1 \frac{x^{p-1}+x^{-p}}{1+x}\, dx = \frac{1}{p} - (\frac{1}{p-1} + \frac{1}{p+1}) + (\frac{1}{p-2} + \frac{1}{p+2}) - \cdots$

Let us recall $\operatorname{cosec} \theta = \frac{1}{\theta} - \frac{1}{\theta-\pi} - \frac{1}{\theta+\pi} + \frac{1}{\theta-2\pi} + \frac{1}{\theta+2\pi} - \frac{1}{\theta-3\pi} - \frac{1}{\theta+3\pi} + \cdots$, if $\theta \neq n\pi$, n being an integer.

Therefore $\pi \operatorname{cosec} p\pi = \frac{1}{p} - \frac{1}{p-1} - \frac{1}{p+1} + \frac{1}{p-2} + \frac{1}{p+2} - \frac{1}{p-3} - \frac{1}{p+3} + \cdots$

Hence $\int_0^1 \frac{x^{p-1}+x^{-p}}{1+x}\, dx = \frac{\pi}{\sin p\pi}$, $0 < p < 1$.

That is, $\int_0^\infty \frac{x^{p-1}}{1+x}\, dx = \frac{\pi}{\sin p\pi}$, $0 < p < 1$.

Exercises 22

1. Examine the convergence of the improper integrals:

(i) $\int_0^1 \log x dx$, (ii) $\int_0^1 \frac{\log x}{\sqrt{1-x}} dx$, (iii) $\int_0^1 \frac{\log(1-x)}{\sqrt{x}} dx$,

(iv) $\int_0^1 \frac{1}{\sqrt{x(1-x)}} dx$, (v) $\int_0^{\frac{\pi}{2}} \frac{1}{e^x - \cos x} dx$, (vi) $\int_0^\pi \frac{\tan x}{x}\, dx$,

(vii) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{x^n} dx$, (viii) $\int_0^1 \frac{x^{p-1}+x^{-p}}{1+x} dx$, (ix) $\int_0^1 \frac{x^p \log x}{1+x^2} dx$,

(x) $\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{\tan x}} dx$, (xi) $\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{\tan x}} dx$, (xii) $\int_0^\pi \frac{dx}{\cos\alpha - \cos x}$, $0 \leq \alpha \leq \pi$.

[**Hint.** (xii) Consider the cases $\alpha = 0$, $\alpha = \pi$, $0 < \alpha < \pi$.]

2. Examine the convergence of the improper integrals:

(i) $\int_0^\infty \frac{1}{(1+x)\sqrt{x}} dx$, (ii) $\int_0^\infty \frac{1}{x \log x} dx$, (iii) $\int_0^\infty \frac{1}{(x+\sin^2 x) \log x} dx$,

(iv) $\int_0^\infty \left(\frac{1}{x^2} - \frac{1}{x\sinh x}\right)dx$, (v) $\int_0^\infty \frac{\cosh 2x}{\cosh 3x}dx$, (vi) $\int_0^\infty \log(1+\text{sech}\, x)dx$.

[**Hint.** (iv) 0 is not a point of infinite discontinuity. $\frac{1}{x\sinh x} > 0$ for $x > 0$. (v) $\frac{\cosh 2x}{\cosh 3x} = \frac{e^{2x}+e^{-2x}}{e^{3x}+e^{-3x}} < \frac{2e^{2x}}{e^{3x}}$. (vi) For $x > 0, \log(1+x) < x$.]

3. Show that the following improper integrals are absolutely convergent.

(i) $\int_0^\infty \frac{\sin x}{1+x^2}dx$, (ii) $\int_0^\infty \frac{\cos x}{1+x^2}dx$, (iii) $\int_0^\infty e^{-ax}\cos bx dx, (a > 0)$,

(iv) $\int_0^\infty \frac{\sin x}{\sqrt{x+x^3}}dx$, (v) $\int_0^\infty \frac{x\sin x}{1+x^3}dx$, (vi) $\int_0^\infty e^{-a^2x^2}\cos bx dx$.

4. Assuming the result $\int_0^\infty \frac{\sin mx}{x}dx = \frac{\pi}{2}$ if $m > 0$, prove that

(i) $\int_0^\infty \frac{\sin^3 x}{x^3}dx = \frac{3\pi}{8}$, (ii) $\int_0^\infty \frac{\sin^3 x}{x}dx = \frac{\pi}{4}$, (iii) $\int_0^\infty \frac{\sin^4 x}{x^2}dx = \frac{\pi}{4}$.

[**Hint.** (iii) Integrate by parts and note that $4\sin^3 x\cos x = \sin 2x - \frac{1}{2}\sin 4x$.]

5. Prove that

(i) $\int_0^\infty \frac{x^m(1+x^n)}{1+x^p}dx$ $(m > 0, n > 0)$ is convergent if $p > 1+m+n$;

(ii) $\int_0^\infty \frac{x^{2m}}{1+x^{2n}}dx$ $(m > 0, n > 0)$ is convergent if $n - m > \frac{1}{2}$;

(iii) $\int_0^\infty \frac{x^{m-1}-x^{n-1}}{1-x}dx$ is convergent if $0 < m < 1$ and $0 < n < 1$;

(iv) $\int_0^\infty x^m(\log x)^n\, dx$ is convergent if $m < -1$;

(v) $\int_0^\infty \frac{\sin x(1-\cos x)}{x^n}dx$ is convergent if $0 < n < 4$, and absolutely convergent if $1 < n < 4$.

6. Prove that

(i) $\int_0^\infty \frac{1}{1+x^4\cos^2 x}dx$ is convergent; (ii) $\int_0^\infty \frac{1}{1+x^2\cos^2 x}dx$ is divergent;

(iii) $\int_0^\infty \frac{x}{1+x^4\sin^2 x}dx$ is divergent; (iv) $\int_0^\infty \frac{x}{1+x^6\sin^2 x}dx$ is convergent.

7. Assuming convergence of the integral $\int_0^{\frac{\pi}{2}} \log\sin x\; dx$ to $-\frac{\pi}{2}\log 2$, prove that

(i) $\int_0^\pi \log(1+\cos x)\, dx$ converges to $-\pi\log 2$,

(ii) $\int_0^\pi \log(1-\cos x)\, dx$ converges to $-\pi\log 2$,

(iii) $\int_0^\infty \log(x+\frac{1}{x})\frac{1}{1+x^2}\, dx$ converges to $-\pi\log 2$,

(iv) $\int_0^\infty \frac{\log(1+x^2)}{1+x^2}dx$ converges to $\pi \log 2$.

8. Assuming convergence of the integral $\int_0^{\frac{\pi}{2}} \cos 2nx \log \sin x\, dx$ to $-\frac{\pi}{4n}$, when n is a positive integer, prove that

(i) $\int_0^{\frac{\pi}{2}} \cos 2nx \log \cos x\, dx$ converges to $(-1)^{n+1}\frac{\pi}{4n}$, when n is a positive integer;

(ii) $\int_0^{\pi} \cos nx \log 2(1+\cos x)\, dx$ converges to $(-1)^{n+1}\frac{\pi}{n}$, when n is a positive integer.

12.11. Beta function and Gamma function.

The improper integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is convergent if $m > 0, n > 0$. The integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m > 0, n > 0$ is called the *Beta function* and it is denoted by $B(m,n)$.

Thus $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m > 0, n > 0$.

The improper integral $\int_0^\infty e^{-x}x^{n-1}dx$ is convergent if $n > 0$. The integral $\int_0^\infty e^{-x}x^{n-1}dx$, $n > 0$ is called the *Gamma function* and it is denoted by $\Gamma(n)$.

Thus $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$, $n > 0$.

Properties.

1. $B(1,1) = 1$.

Proof. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m > 0, n > 0$.

Therefore $B(1,1) = \int_0^1 dx = 1$.

2. $B(m,n) = B(n,m)$.

Proof. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m > 0, n > 0$.

$$= \lim_{\epsilon\to 0\,\delta\to 0}\int_\epsilon^{1-\delta} x^{m-1}(1-x)^{n-1}\,dx.$$

Let $x = 1-y$. Then $dx = -dy$.

$$\lim_{\epsilon\to 0\,\delta\to 0}\int_\epsilon^{1-\delta} x^{m-1}(1-x)^{n-1}\,dx = \lim_{\epsilon\to 0\,\delta\to 0}\int_\delta^{1-\epsilon}(1-y)^{m-1}y^{n-1}\,dy.$$

$$= \lim_{\delta\to 0\,\epsilon\to 0}\int_\delta^{1-\epsilon} y^{n-1}(1-y)^{m-1}\,dy = \int_0^1 y^{n-1}(1-y)^{m-1}dy = B(n,m).$$

Therefore $B(m,n) = B(n,m)$.

3. $B(m+1,n) = \frac{m}{m+n}B(m,n)$, $m > 0, n > 0$.

Proof. $B(m+1,n) = \int_0^1 x^m(1-x)^{n-1}dx$

$= \left[\frac{x^m(1-x)^n}{-n}\right]_0^1 + \frac{m}{n}\int_0^1 x^{m-1}(1-x)^n dx$

$= \frac{m}{n}\int_0^1 (1-x)x^{m-1}(1-x)^{n-1}dx$

$= \frac{m}{n}\int_0^1 x^{m-1}(1-x)^{n-1}dx - \frac{m}{n}\int_0^1 x^m(1-x)^{n-1}dx$

$= \frac{m}{n}B(m,n) - \frac{m}{n}B(m+1,n).$

Therefore $(1+\frac{m}{n})B(m+1,n) = \frac{m}{n}B(m,n)$

or, $B(m+1,n) = \frac{m}{m+n}B(m,n)$.

4. $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta, m > 0, n > 0.$

Proof. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\ m > 0, n > 0.$

Let $x = \sin^2\theta$. Then $dx = 2\sin\theta\cos\theta\, d\theta$.

As $x \to 0+, \theta \to 0+$; as $x \to 1-, \theta \to \frac{\pi}{2}-$.

Therefore $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\cos^{2n-2}\theta\sin\theta\cos\theta\, d\theta$

$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta, m > 0, n > 0.$

Deductions.

(i) $\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\, d\theta = \frac{1}{2}B(\frac{m+1}{2}, \frac{n+1}{2}), m > -1, n > -1.$

(ii) $\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta = \int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta = \frac{1}{2}B(\frac{n+1}{2}, \frac{1}{2}), n > -1.$

(iii) $B(\frac{1}{2}, \frac{1}{2}) = 2\int_0^{\frac{\pi}{2}} d\theta = \pi.$

5. $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx,\ m > 0, n > 0.$

Proof. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\ m > 0, n > 0.$

Let $x = \frac{t}{1+t}$. Then $dx = \frac{1}{(1+t)^2}$.

As $x \to 0+, t \to 0+$; as $x \to 1-, t \to \infty$.

Therefore $B(m,n) = \int_0^\infty (\frac{t}{1+t})^{m-1}(\frac{1}{1+t})^{n-1}\frac{1}{(1+t)^2}dt$

$= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}dt$

$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx.$

6. $B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx.$

Proof. We have $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx.$$

Let $x = \frac{1}{t}$ in the second integral. Then $dx = -\frac{1}{t^2}$.
As $x \to 1+, t \to 1-$; as $x \to \infty, t \to 0+$.

$$\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^1 \frac{1}{t^{m-1}} \frac{t^{m+n}}{(1+t)^{m+n}} \frac{1}{t^2} dt$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}} dt = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx.$$

Therefore $B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$

$$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx.$$

7. $\Gamma(1) = 1$.

Proof. $\Gamma(1) = \int_0^\infty e^{-x} dx = \lim_{X\to\infty} \int_0^X e^{-x} dx = \lim_{X\to\infty} [1 - e^{-X}]_0^X = 1.$

8. $\Gamma(n+1) = n\Gamma(n)$, $n > 0$.

Proof. $\int_\epsilon^X x^n e^{-x} dx = [\frac{x^n e^{-x}}{-1}]_\epsilon^X + n \int_\epsilon^X x^{n-1} e^{-x} dx$

$$= -X^n e^{-X} + \epsilon^n e^\epsilon + n \int_\epsilon^X x^{n-1} e^{-x} dx.$$

Proceeding to limit as $X \to \infty$ and $\epsilon \to 0$, we have

$\int_0^\infty e^{-x} x^n dx = n \int_0^\infty e^{-x} x^{n-1} dx$

or, $\Gamma(n+1) = n\Gamma(n)$, $n > 0$.

Corollary. If n be a positive integer then $\Gamma(n+1) = n!$.

$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)...2.1\Gamma(1) = n!$.

9. $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, $m > 0, n > 0$.

The proof of the property is beyond the scope of this book.

Deductions.

(i). $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

$B(\frac{1}{2}, \frac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} = \Gamma(\frac{1}{2})\Gamma(\frac{1}{2})$.

Therefore $(\Gamma(\frac{1}{2}))^2 = B(\frac{1}{2}, \frac{1}{2}) = \pi$ and this gives $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(ii). If m, n be positive integers, $B(m+1, n+1) = \frac{m!n!}{(m+n+1)!}$.

$B(m+1, n+1) = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}$, $m > -1, n > -1$.

If m, n are postive integers, $\Gamma(m+1) = m!$, $\Gamma(n+1) = n!$ and therefore $B(m+1, n+1) = \frac{m!n!}{(m+n+1)!}$.

10. Legender's Duplication formula.

$\sqrt{\pi}\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma(n+\frac{1}{2}),\ n > 0.$

Proof. $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m,n)$

$= 2\int_0^{\pi} 2\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta,\ m > 0, n > 0\ \ldots$ (i)

Taking $m = n$, we have $\frac{(\Gamma(n))^2}{\Gamma(2n)} = 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\cos^{2n-1}\theta\, d\theta$

$= \frac{1}{2^{2n-2}}\int_0^{\frac{\pi}{2}} \sin^{2n-1} 2\theta\, d\theta$

$= \frac{1}{2^{2n-1}}\int_0^{\pi} \sin^{2n-1}\phi\, d\phi$ $[2\theta = \phi]$

$= \frac{1}{2^{2n-1}}.2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\phi\, d\phi \ldots$ (ii)

Taking $m = \frac{1}{2}$ in (i), we have $\frac{\Gamma(n)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})} = 2\int_0^{\frac{\pi}{2}} \cos^{2n-1}\theta\ d\theta = 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\, d\theta\ \ldots$ (iii)

From (ii) and (iii) we have $\frac{(\Gamma(n))^2}{\Gamma(2n)} = \frac{1}{2^{2n-1}}\frac{\Gamma(n)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})},\ n > 0$

or, $\sqrt{\pi}\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma(n+\frac{1}{2})$, since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

11. $\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi},\ 0 < m < 1.$

Proof. We have $B(m, 1-m) = \frac{\Gamma(m)\Gamma(1-m)}{\Gamma(1)} = \Gamma(m)\Gamma(1-m)$.

Since $B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}dx,\ m > 0, n > 0,\ B(m, 1-m) = \int_0^{\infty} \frac{x^{m-1}}{1+x}dx,\ 0 < m < 1.$

Therefore $\Gamma(m)\Gamma(1-m) = \int_0^{\infty} \frac{x^{m-1}}{1+x}dx = \frac{\pi}{\sin m\pi},\ 0 < m < 1.$ [worked Ex.9, page 524]

12. (i) $\int_0^{\infty} e^{-kt}t^{n-1}dt = \frac{\Gamma(n)}{k^n},\ k > 0, n > 0;$

(ii) $\int_1^{\infty} \frac{(\log y)^{n-1}}{y^{k+1}} = \frac{\Gamma(n)}{k^n},\ k > 0, n > 0.$

Proof. (i) $\int_0^{\infty} e^{-kt}t^{n-1}dt$

$= \int_0^{\infty} e^{-y}(\frac{y}{k})^{n-1}\frac{1}{k}dy$ [Let $y = kt$. As $t \to \infty, t \to \infty$ since $k > 0$.]

$= \frac{1}{k^n}\int_0^{\infty} e^{-y}y^{n-1}dy,\ n > 0$

$= \frac{\Gamma(n)}{k^n}.$

(ii) $\int_1^{\infty} \frac{(\log y)^{n-1}}{y^{k+1}}dy$

$= \int_0^\infty t^{n-1}e^{-kt}dt$ [Let $\log y = t$. Then $y = e^t$. $y = 1 \Rightarrow t = 0$]
$= \frac{\Gamma(n)}{k^n}$, since $k > 0, n > 0$. [using (i)]

Worked Examples.

1. Prove that (i) $\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$; (ii) $\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$;

(i) Let $x^2 = t$. Then $dx = \frac{1}{2\sqrt{t}}dt$. As $x \to \infty, t \to \infty$.

$\int_0^\infty e^{-x^2}dx = \frac{1}{2}\int_0^\infty e^{-t}t^{-\frac{1}{2}}dt = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$.

(ii) Let $f(x) = e^{-x^2}, x \in \mathbb{R}$. Then f is an even function on $\mathbb{R}$.

Therefore $\int_{-\infty}^\infty e^{-x^2}dx = 2\int_0^\infty e^{-x^2}dx$, assuming convergence of the integral on the right

$= \sqrt{\pi}$.

2. Prove that $\int_0^{\frac{\pi}{2}} \sin^p x\, dx \times \int_0^{\frac{\pi}{2}} \sin^{p+1} x\, dx = \frac{\pi}{2(p+1)}, p > -1$.

We have $\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\, d\theta = \frac{1}{2}B(\frac{m+1}{2}, \frac{n+1}{2}), m > -1, n > -1$.

Therefore $\int_0^{\frac{\pi}{2}} \sin^p x\, dx = \frac{1}{2}B(\frac{p+1}{2}, \frac{1}{2}) = \frac{1}{2}\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{p+2}{2})}, p > -1$

and $\int_0^{\frac{\pi}{2}} \sin^{p+1} x\, dx = \frac{1}{2}B(\frac{p+2}{2}, \frac{1}{2}) = \frac{1}{2}\frac{\Gamma(\frac{p+2}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{p+3}{2})}, p > -2$.

$$\int_0^{\frac{\pi}{2}} \sin^p x\, dx \times \int_0^{\frac{\pi}{2}} \sin^{p+1} x\, dx = \frac{1}{4}\,\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{p+2}{2})}\cdot\frac{\Gamma(\frac{p+2}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{p+3}{2})}, p > -1$$

$$= \frac{1}{4}\frac{\pi\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p+3}{2})}, \text{ since } \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$= \frac{1}{4}\frac{2\pi}{p+1}, \text{ since } \Gamma(\tfrac{p+3}{2}) = \tfrac{p+1}{2}\Gamma(\tfrac{p+1}{2})$$

$$= \frac{\pi}{2(p+1)}.$$

3. Prove that $\int_a^b (x-a)^{m-1}(b-x)^{n-1}dx = (b-a)^{m+n-1}B(m,n)$, $m > 0, n > 0$.

We have $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m > 0, n > 0$.

$(x-a) + (b-x) = b - a \Rightarrow \frac{x-a}{b-a} + \frac{b-x}{b-a} = 1$.

Let $\frac{x-a}{b-a} = y$. Then $\frac{b-x}{b-a} = 1 - y$, $dx = (b-a)dy$.
As $x \to a, y \to 0$; as $x \to b, y \to 1$.

$$\int_a^b (x-a)^{m-1}(b-x)^{n-1}dx = \int_0^1 (b-a)^{m+n-1}y^{m-1}(1-y)^{n-1}dy$$
$$= (b-a)^{m+n-1}B(m,n).$$

4. Prove that $\int_0^1 \frac{1}{(1-x^n)^{\frac{1}{n}}} dx = \frac{\pi}{n} \operatorname{cosec} \frac{\pi}{n}, n > 1$.

Let $x^n = t$. Then $dx = \frac{1}{nt^{\frac{n-1}{n}}} dt$.

$$\int_0^1 \frac{1}{(1-x^n)^{\frac{1}{n}}} dx = \int_0^1 (1-t)^{-\frac{1}{n}} . \frac{1}{nt^{\frac{n-1}{n}}} dt$$
$$= \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} (1-t)^{(1-\frac{1}{n})-1} dt$$
$$= \frac{1}{n} B(\tfrac{1}{n}, 1 - \tfrac{1}{n}), \text{ since } 0 < \tfrac{1}{n} < 1$$
$$= \frac{1}{n} \frac{\Gamma(\frac{1}{n})\Gamma(1-\frac{1}{n})}{\Gamma(1)}$$
$$= \frac{1}{n} \frac{\pi}{\sin \frac{\pi}{n}} = \frac{\pi}{n} \operatorname{cosec} \frac{\pi}{n}.$$

5. If n be a positive integer, prove that $\Gamma(\frac{1}{n})\Gamma(\frac{2}{n})\Gamma(\frac{3}{n}) \ldots \Gamma(\frac{n-1}{n}) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$.

Let $P = \Gamma(\frac{1}{n})\Gamma(\frac{2}{n})\Gamma(\frac{3}{n}) \ldots \Gamma(\frac{n-1}{n})$.

Then $P = \Gamma(1 - \frac{1}{n})\Gamma(1 - \frac{2}{n}) \ldots \Gamma(\frac{2}{n})\Gamma(\frac{1}{n})$. [taking the factors in the reverse order]

$$P^2 = [\Gamma(\tfrac{1}{n})\Gamma(1 - \tfrac{1}{n})][\Gamma(\tfrac{2}{n})\Gamma(1 - \tfrac{2}{n}] \ldots [\Gamma(\tfrac{n-1}{n})\Gamma(\tfrac{1}{n})]$$
$$= \frac{\pi}{\sin \frac{\pi}{n}} . \frac{\pi}{\sin \frac{2\pi}{n}} \cdots \frac{\pi}{\sin \frac{(n-1)\pi}{n}} = \frac{\pi^{n-1}}{\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \ldots \sin \frac{(n-1)\pi}{n}} \ldots \text{(i)}$$

We prove the following lemma.

Lemma. $\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \ldots \sin \frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$.

Proof. $x^{2n} - 2x^n \cos 2n\theta + 1 = 0$ gives $x^n = \cos 2n\theta + i \sin 2n\theta$, i.e., $x = \cos(2\theta + \frac{2k\pi}{n}) + i \sin(2\theta + \frac{2k\pi}{n})$, where $k = 0, 1, \ldots, n-1$.

Therefore $x^{2n} - 2x^n \cos 2n\theta + 1 = \prod_{k=0}^{n-1} [x^2 - 2x \cos(2\theta + \dfrac{2k\pi}{n}) + 1]$.

Taking $x = 1$, we have $4 \sin^2 n\theta = \prod_{k=0}^{n-1} 4 \sin^2(\theta + \dfrac{k\pi}{n})$.

$\sin^2 n\theta = 4^{n-1} \sin^2 \theta \sin^2(\theta + \frac{\pi}{n}) \ldots \sin^2(\theta + \frac{(n-1)\pi}{n})$

or, $\sin n\theta = 2^{n-1} \sin \theta \sin (\theta + \frac{\pi}{n}) \sin (\theta + \frac{2\pi}{n}) \ldots \sin(\theta + \frac{(n-1)\pi}{n})$

or, $\frac{\sin n\theta}{\sin \theta} = 2^{n-1} \sin (\theta + \frac{\pi}{n}) \sin (\theta + \frac{2\pi}{n}) \ldots \sin(\theta + \frac{(n-1)\pi}{n})$.

Proceeding to limit as $\theta \to 0$, we have

$n = 2^{n-1} \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \ldots \sin \frac{(n-1)\pi}{n}$

or, $\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \ldots \sin \frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$.

This proves the lemma.

Using the lemma, we have from (i) $P^2 = \frac{\pi^{n-1}2^{n-1}}{n}$.

Therefore $P = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$.

6. Show that $\int_0^1 \log\Gamma(x)dx$ is convergent and evaluate it.

$\Gamma(x) = \int_0^\infty e^{t}t^{x-1}dt,\ x > 0.$

Let $f(t) = e^{t}t^{x-1},\ t > 0$. If $x > 0$ then f is a continuous function of t on $(0,\infty)$ and for $x > 0$, $f(t) > 0$ for all $t > 0$.

For $x > 0$, $\int_0^\infty f(t)dt$ is a convergent integral and for $x > 0$, $\int_0^\infty f(t)dt > 0$, i.e., $\Gamma(x) > 0$ for $x > 0$. So $\log\Gamma(x)$ is defined for $x > 0$.

We have $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$.

Therefore $\log\Gamma(x+1) = \log x + \log\Gamma(x)$ for $x > 0$
or, $\log\Gamma(x) = \log\Gamma(x+1) - \log x$ for $x > 0$ ... (i)

$\Gamma(x+1)$ is defined for all $x > -1$ and $\Gamma(x+1) > 0$ for all $x > -1$. Therefore the integral $\int_0^1 \log\Gamma(x+1)dx$ is a proper one.

The integral $\int_0^1 \log x dx$ is an improper integral, since 0 is a point of infinite discontinuity.

$\lim\limits_{\epsilon\to 0+}\int_\epsilon^1 \log x dx = \lim\limits_{\epsilon\to 0+}[x\log x - x]_\epsilon^1 = \lim\limits_{\epsilon\to 0+}[\epsilon - \epsilon\log\epsilon - 1] = -1.$

Therefore $\int_0^1 \log x dx$ is convergent.

From (i) it follows that $\int_0^1 \log\Gamma(x)dx$ is convergent.

Let $\phi(\epsilon) = \int_\epsilon^1 \log\Gamma(x)dx$.

Let $x = 1-y$. Then $\phi(\epsilon) = \int_0^{1-\epsilon}\log\Gamma(1-y)dy = \int_0^{1-\epsilon}\log\Gamma(1-x)dx$.

Since $\int_0^1 \log\Gamma(x)dx$ is convergent, $\lim\limits_{\epsilon\to 0+}\phi(\epsilon)$ is finite.

Therefore $\int_0^1 \log\Gamma(x)dx = \int_0^1 \log\Gamma(1-x)dx$.

Let $I = \int_0^1 \log\Gamma(x)dx$. Then

$2I = \int_0^1 \log\Gamma(x)dx + \int_0^1 \log\Gamma(1-x)dx$

$= \int_0^1 \log(\Gamma(x).\Gamma(1-x))\,dx = \int_0^1 \log\frac{\pi}{\sin\pi x}\,dx$

$= \log\pi - \int_0^1 \log\sin\pi x\,dx = \log\pi - \frac{1}{\pi}\int_0^\pi \log\sin$

$= \log\pi - \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \log\sin y\,dy = \log\pi - \frac{2}{\pi}.\frac{\pi}{}$

$= \log\pi + \log 2 = \log 2$

Therefore $I = \int_0^1 \log\Gamma$ ... $= \frac{1}{}$

Exercises 23

1. Prove that (i) $\int_0^{\frac{\pi}{2}} \tan^p \theta \, d\theta = \frac{\pi}{2} \sec \frac{p\pi}{2}$, if $-1 < p < 1$;

(ii) $\int_0^{\frac{\pi}{2}} \cot^p \theta \, d\theta = \frac{\pi}{2} \sec \frac{p\pi}{2}$, if $-1 < p < 1$.

[**Hint.** (i) $\tan^p \theta = \sin^p \theta \cos^{-p} \theta$.]

2. Prove that (i) $\int_0^1 x^{m-1}(1-x^p)^{n-1} = \frac{1}{p} B(\frac{m}{p}, n)$, if $m > 0, n > 0, p > 0$.

(ii) $\int_0^1 x^{m-1}(\log \frac{1}{x})^{n-1} \, dx = \dfrac{\Gamma(n)}{m^n}$, if $m > 0, n > 0$.

(iii) $\int_{-1}^1 (1+x)^{p-1}(1-x)^{q-1} dx = 2^{p+q-1} B(p,q)$, if $p > 0, q > 0$.

[**Hint.** (iii) Let $1 + x = 2y$.]

3. Prove that (i) $\int_0^1 \frac{1}{(1-x^6)^{\frac{1}{6}}} \, dx = \frac{\pi}{3}$; (ii) $\int_0^1 \frac{1}{(1-x^3)^{\frac{1}{3}}} \, dx = \frac{2\pi}{3\sqrt{3}}$;

4. Prove that (i) $\int_0^1 \frac{x^2}{\sqrt{1-x^4}} dx \times \int_0^1 \frac{1}{\sqrt{1+x^4}} dx = \frac{\pi}{4\sqrt{2}}$;

(ii) $\int_0^\infty x^2 e^{-x^4} dx \times \int_0^\infty e^{-x^4} dx = \frac{\pi}{8\sqrt{2}}$.

5. Prove that (i) $\Gamma(\frac{1}{9})\Gamma(\frac{2}{9})...\Gamma(\frac{8}{9}) = \frac{16\pi^4}{3}$;

(ii) $\Gamma(\frac{1+n}{2})\Gamma(\frac{1-n}{2}) = \pi \sec \frac{n\pi}{2}$, $-1 < n < 1$;

(iii) $2^n \Gamma(\frac{n+1}{2})\Gamma(\frac{n+2}{2}) = \sqrt{\pi}\Gamma(n+1)$, $n > -1$.

[**Hint.** (iii) Use Duplication formula.]

6. Prove that (i) $B(m,m) = 2^{1-2m} B(m, \frac{1}{2})$, $m > 0$;

(ii) $B(m,m).B(m+\frac{1}{2}, m+\frac{1}{2}) = \frac{\pi}{m}.2^{1-4m}$, $m > 0$.

[**Hint.** (i) $B(m,m) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2m-1} \theta d\theta$, $m > 0$.]

7.(a) Prove that $\int_0^\pi \dfrac{\sin^{m-1} x}{(2+\cos x)^m} dx = \dfrac{2^{m-1}}{3^{\frac{m}{2}}} B(\frac{m}{2}, \frac{m}{2})$, $m > 0$.

(b) Prove that $\int_0^\infty x^{2n} e^{-ax^2} dx = \dfrac{1.3.5...(2n-1)\sqrt{\pi}}{2^{n+1} a^{n+\frac{1}{2}}}$, $a > 0$.

8.(a) Show that $\int_0^1 x^{m-1}(\log x)^{n-1} dx$ is convergent if $m > 0, n > 0$.

(b) Show that (i) $\int_0^1 \sqrt{x}(\log x)^2 dx = \frac{16}{27}$, (ii) $\int_0^1 \frac{(\log x)^2}{\sqrt{x}} dx = 16$.

[**Hint.** (a) Let $x = e^{-y}$.]

9. Show that (i) $\int_0^{\frac{\pi}{2}} \sqrt{\cot x} \, dx = \frac{\pi}{\sqrt{2}}$, (ii) $\int_0^{\frac{\pi}{2}} \sqrt[3]{\cot x} \, dx = \frac{\pi}{\sqrt{3}}$,

(iii) $\int_0^\infty \frac{x^2(x^3-1)}{(1+x)^9} dx = 0$, (iv) $\int_0^\infty e^{-3x^2} x^3 dx = \frac{1}{9}$,

(v) $\int_0^\infty e^{-x^2} x^3 dx = 1$, (vi) $\int_{-1}^1 \frac{1}{(1+x)^{\frac{1}{3}}(1-x)^{\frac{2}{3}}} dx = \frac{2\pi}{\sqrt{3}}$.

13. SEQUENCE OF FUNCTIONS

13.1. Sequence of functions.

Let D be a subset of $\mathbb{R}$ and for each $n \in \mathbb{N}$, let $f_n : D \to \mathbb{R}$ be a function. Then $\{f_n\}$ is a sequence of functions on D to $\mathbb{R}$. D is said to be the domain of the sequence of functions $\{f_n\}$.

In particular, D may be a closed interval $[a, b]$ (or $[a, \infty)$), or an open interval (a, b) (or (a, ∞)).

To each $x_0 \in D$ the sequence $\{f_n\}$ gives rise to a sequence of real numbers $\{f_n(x_0)\}$, which is obtained by evaluating each f_n at x_0.

For some $x \in D$, the sequence $\{f_n(x)\}$ may converge to a limit and for some other $x \in D$, the sequence $\{f_n(x)\}$ may not converge.

13.2. Pointwise convergence.

Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}$, let $f_n : D \to \mathbb{R}$ be a function.

The sequence $\{f_n\}$ is said to be *pointwise convergent* on D if for each $x \in D$, the sequence $\{f_n(x)\}$ converges.

Let the sequence $\{f_n\}$ be pointwise convergent on D and let $c \in D$.

Then the sequence $\{f_n(c)\}$ is convergent. Let $\lim f_n(c) = l_c$. Since for all $x \in D, \{f_n(x)\}$ converges to a limit, l_x exists for all $x \in D$.

Let us define a function $f : D \to \mathbb{R}$ by $f(x) = l_x, x \in D$. Then f is said to be the *limit function* of the sequence $\{f_n\}$ on D.

In this case we also say that the sequence $\{f_n\}$ converges to f on D and we write $f = \lim f_n$ on D, or $f_n \to f$ on D.

Examples.

1. For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = x^n, x \in \mathbb{R}$.

Then $\{f_n\}$ is a sequence of functions on $\mathbb{R}$. For each $x \in (-1, 1)$ the sequence $\{f_n(x)\}$ converges to 0 and for $x = 1$, the sequence $\{f_n(x)\}$ converges to 1. For all other $x \in \mathbb{R}$, the sequence $\{f_n(x)\}$ is divergent.

Therefore the sequence $\{f_n\}$ is pointwise convergent on $(-1, 1]$ and the limit function f is defined by

$$\begin{aligned} f(x) &= 0, -1 < x < 1 \\ &= 1, x = 1. \end{aligned}$$

Note. Here we observe that although the domain of the sequence $\{f_n\}$ is $\mathbb{R}$, the domain of pointwise convergence of the sequence is a proper subset of $\mathbb{R}$.

2. For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{n}, x \in \mathbb{R}$. Then $\{f_n\}$ is a sequence of functions on $\mathbb{R}$. For each $x \in \mathbb{R}$, the sequence $\{f_n(x)\}$ converges to 0.

Therefore the sequence $\{f_n\}$ is pointwise convergent on $\mathbb{R}$ and the limit function f is defined by $f(x) = 0, x \in \mathbb{R}$.

3. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and for each $n \in \mathbb{N}$, let $f_n : D \to \mathbb{R}$ be defined by $f_n(x) = \frac{n}{x+n}, x \geq 0$. Then $\{f_n\}$ is a sequence of functions on D. For each $x \in D$, $\lim_{n\to\infty} f_n(x) = 1$.

Therefore the sequence is pointwise convergent on D to the function f defined by $f(x) = 1, x \geq 0$.

4. For each natural number n, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{nx}{1+n^2x^2}, x \in \mathbb{R}$. Then $\{f_n\}$ is a sequence of functions on $\mathbb{R}$.

For $x = 0$, the sequence is $\{0, 0, 0, \ldots\}$. This converges to 0.
For $x \neq 0$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = 0$.

Therefore the sequence $\{f_n\}$ is pointwise convergent on $\mathbb{R}$ to the function f defined by $f(x) = 0, x \in \mathbb{R}$.

5. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and for each natural number n, let $f_n : D \to \mathbb{R}$ be defined by $f_n(x) = \frac{nx}{1+nx}, x \geq 0$. Then $\{f_n\}$ is a sequence of functions on D.

For $x = 0$, the sequence is $\{0, 0, 0, \ldots\}$. This converges to 0.
For $x > 0$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+nx} = 1$.

Therefore the sequence $\{f_n\}$ is pointwise convergent on D to the function f defined by
$$\begin{aligned} f(x) &= 0, x = 0 \\ &= 1, x > 0. \end{aligned}$$

6. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and for each natural number n, let $f_n : D \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{1+nx}, x \geq 0$. Then $\{f_n\}$ is a sequence of functions on D.

For $x = 0$, the sequence is $\{0, 0, 0, \cdots\}$. This converges to 0.
For $x > 0$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1+nx} = 0$.

Therefore the sequence $\{f_n\}$ is pointwise convergent on D to the function f defined by $f(x) = 0, x > 0$.

7. Let $f_n(x) = \tan^{-1} nx, x \in \mathbb{R}$.

$$\text{Then } \lim_{n\to\infty} f_n(x) = \tfrac{\pi}{2}, \text{ if } x > 0$$
$$= 0 \text{ if } x = 0$$
$$= -\tfrac{\pi}{2} \text{ if } x < 0.$$

Therefore the sequence $\{f_n\}$ is pointwise convergent on $\mathbb{R}$ to the function f where $f(x) = \frac{\pi}{2}\text{sgn } x, x \in \mathbb{R}$.

8. For each natural number n let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{\sin nx}{n}, x \in \mathbb{R}$. Then $\{f_n\}$ is a sequence of functions on $\mathbb{R}$.

For each $x \in \mathbb{R}$, $\lim\limits_{n\to\infty} f_n(x) = 0$, since $\lim \frac{1}{n} = 0$ and $|\sin nx| \leq 1$.

Therefore the sequence $\{f_n\}$ is pointwise convergent on $\mathbb{R}$ to the function f defined by $f(x) = 0, x \in \mathbb{R}$.

9. Let $f_n(x) = xe^{-nx}, x \geq 0$.

For all $x \geq 0$, $0 \leq xe^{-nx} < \frac{1}{n}$, since $e^{nx} > nx$ for all $x > 0$.
By Sandwich theorem 5.5.5, $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \geq 0$.

Therefore the sequence $\{f_n\}$ is pointwise convergent on $[0, \infty)$ to the function f defined by $f(x) = 0, x \geq 0$.

10. Let $f_n(x) = x^2 e^{-nx}, x \geq 0$.

For all $x \geq 0$, $0 \leq x^2 e^{-nx} < \frac{2}{n^2}$, since $e^{nx} > \frac{n^2x^2}{2}$ for all $x \geq 0$.
By Sandwich theorem 5.5.5, $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \geq 0$.

Therefore the sequence $\{f_n\}$ is pointwise convergent on $[0, \infty)$ to the function f defined by $f(x) = 0, x \geq 0$.

Let $\{f_n\}$ be a sequence of functions that converges pointwise on a domain $D \subset \mathbb{R}$ to the function f.

Let $x' \in D$. Let us choose $\epsilon > 0$.

Since the sequence $\{f_n(x')\}$ converges to $f(x')$, there exists a natural number k' such that $|f_n(x') - f(x')| < \epsilon$ for all $n \geq k'$.

This k' depends on ϵ as well as on x'.

Let $x'' \in D$. Since the sequence $\{f_n(x'')\}$ converges to $f(x'')$, there exists a natural number k'' such that $|f_n(x'') - f(x'')| < \epsilon$ for all $n \geq k''$.

It is quite natural that k'' is different from k'.

If it is possible that for a pre-assigned positive ϵ, there exists a natural number k such that

for all $x \in D$, $|f_n(x) - f(x)| < \epsilon$ for all $n \geq k$

then we say that the convergence of the sequence $\{f_n\}$ is uniform on D.

13.3. Uniform convergence.

Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}$, let $f_n : D \to \mathbb{R}$ be a function. The sequence $\{f_n\}$ is said to be *uniformly convergent* on D to a function f if corresponding to a pre-assigned positive ϵ there exists a natural number $k(\epsilon)$ (depending on ϵ but not on $x \in D$) such that

for all $x \in D$, $| f_n(x) - f(x) | < \epsilon$ for all $n \geq k$.

In this case we write $\lim f_n = f$ uniformly on D, or $f_n \to f$ uniformly on D.

f is said to be the *uniform limit* of the sequence $\{f_n\}$ on D.

It follows that if the sequence $\{f_n\}$ is uniformly convergent on D to the function f then the sequence $\{f_n\}$ also converges pointwise on D to f. But that the converse is not true is discussed in the following examples.

Examples.

1. In Example 1 of 13.2, the sequence $\{f_n\}$ converges on $(-1, 1]$ to the function f where $f(x) = 0, -1 < x < 1$.
$= 1, x = 1$.

Let us examine if the convergence of the sequence $\{f_n\}$ is uniform on $(0, 1)$.

Let $c \in (0, 1)$. Then $| f_n(c) - f(c) | = c^n$.

Let $0 < \epsilon < 1$. Then $| f_n(c) - f(c) | < \epsilon$ whenever $c^n < \epsilon$,
i.e., whenever $n \log(1/c) > \log(1/\epsilon)$,
i.e., whenever $n > \log(1/\epsilon)/\log(1/c)$.

Let $k = [\log(1/\epsilon)/\log(1/c)] + 1$. Then k is a natural number and $| f_n(c) - f(c) | < \epsilon$ for all $n \geq k$.

Therefore for all $x \in (0, 1)$, $| f_n(x) - f(x) | < \epsilon$ for all $n \geq k$, where $k = [\log(1/\epsilon)/\log(1/x)] + 1$.
This k depends on ϵ as well as on x. As $x \to 1-, k \to \infty$.

It follows that there does not exist a natural number k such that for all $x \in (0, 1)$, $| f_n(x) - f(x) | < \epsilon$ holds for all $n \geq k$.

Consequently, $\{f_n\}$ is not uniformly convergent on $(0, 1)$.

Let $a \in \mathbb{R}$ such that $0 < a < 1$.
In $[0, a]$, the greatest value of $\log(1/\epsilon)/\log(1/x)$ is $\log(1/\epsilon)/\log(1/a)$.

Let $k = [\log(1/\epsilon)/\log(1/a)] + 1$. Then k is a natural number and for all $x \in [0, a]$, $| f_n(x) - f(x) | < \epsilon$ for all $n \geq k$.

This proves that the sequence $\{f_n\}$ is uniformly convergent on $[0, a]$.

2. In Example 2 of 13.2, the sequence $\{f_n\}$ converges on $\mathbb{R}$ to the function f where $f(x) = 0, x \in \mathbb{R}$.

Let us examine if the convergence of the sequence is uniform on $[0, \infty)$.

For all $x \geq 0, \mid f_n(x) - f(x) \mid = \frac{x}{n}$.

Let $\epsilon > 0$. If $k = [\frac{x}{\epsilon}] + 1$, then k is a natural number and for all $x \geq 0, \mid f_n(x) - f(x) \mid < \epsilon$ for all $n \geq k$.

This k depends on ϵ as well as on x. As $x \to \infty, k \to \infty$.

It follows that there does not exist a natural number k (depending only on the chosen ϵ) such that

for all $x \geq 0, \mid f_n(x) - f(x) \mid < \epsilon$ holds for all $n \geq k$.

This proves that the convergence of the sequence $\{f_n\}$ is not uniform on $[0, \infty)$.

Let $a \in \mathbb{R}$ such that $a > 0$. In $[0, a]$ the greatest value of $(\frac{x}{\epsilon})$ is $(\frac{a}{\epsilon})$.

Let $k = [\frac{a}{\epsilon}] + 1$. Then k is a natural number and for all $x \in [0, a]$, $\mid f_n(x) - f(x) \mid < \epsilon$ for all $n \geq k$.

This proves that the sequence $\{f_n\}$ is uniformly convergent on $[0, a]$.

3. In Example 4 of 13.2, the sequence $\{f_n\}$ converges on $\mathbb{R}$ to the function f where $f(x) = 0, x \in \mathbb{R}$.

Let us examine if the convergence of the sequence is uniform on $[0, \infty)$.

For all $x \geq 0, \mid f_n(x) - f(x) \mid = \frac{nx}{1+n^2x^2}$.

Let $u(x) = \frac{nx}{1+n^2x^2}$ for $x > 0$. Then $u'(x) = \frac{n(1-n^2x^2)}{(1+n^2x^2)^2}$.

$u'(x) > 0$ for $x < \frac{1}{n}$, $u'(x) = 0$ for $x = \frac{1}{n}$, $u'(x) < 0$ for $x > \frac{1}{n}$.

u is a maximum at $x = \frac{1}{n}$ and $u(\frac{1}{n}) = \frac{1}{2}$, i.e., $|f_n(\frac{1}{n}) - f(\frac{1}{n})| = \frac{1}{2}$.

Let $\epsilon = \frac{1}{4}$. If the sequence $\{f_n\}$ be uniformly convergent on $[0, \infty)$ to the function f, then for the chosen ϵ there must exist a natural number k such that for all $x \geq 0, \mid f_n(x) - f(x) \mid < \frac{1}{4}$ holds for all $n \geq k$.

But for every natural number k, $|f_k(\frac{1}{k}) - f(\frac{1}{k})| = \frac{1}{2} \not< \frac{1}{4}$.

This shows that no natural number k can be found so that for all $x \in [0, \infty), \mid f_n(x) - f(x) \mid < \frac{1}{4}$ holds for all $n \geq k$.

Therefore the sequence $\{f_n\}$ is not uniformly convergent on $[0, \infty)$.

Let $a \in \mathbb{R}$ such that $a > 0$.

For all $x > 0, \mid f_n(x) - f(x) \mid = \frac{nx}{1+n^2x^2} < \frac{1}{nx}$.

Then for all $x \geq a, \mid f_n(x) - f(x) \mid < \frac{1}{na}$.

Let us choose $\epsilon > 0$. If $k = [\frac{1}{a\epsilon}] + 1$, then k is a natural number and for all $x \geq a, \mid f_n(x) - f(x) \mid < \epsilon$ for all $n \geq k$.

This proves that the sequence $\{f_n\}$ is uniformly convergent on $[a, \infty)$.

4. In Example 6 of 13.2, the sequence $\{f_n\}$ converges to the function f where $f(x) = 0, x \geq 0$.

For all $x \geq 0, | f_n(x) - f(x) |= \frac{x}{1+nx} < \frac{1}{n}$.

Let $\epsilon > 0$. Then $| f_n(x) - f(x) |< \epsilon$ for all $n > \frac{1}{\epsilon}$.

Let $k = [\frac{1}{\epsilon}] + 1$. Then k is a natural number and for all $x \geq 0$, $| f_n(x) - f(x) |< \epsilon$ for all $n \geq k$.

This proves that the sequence $\{f_n\}$ is uniformly convergent on $[0, \infty)$.

5. In example 8 of 13.2, the sequence $\{f_n\}$ converges on $\mathbb{R}$ to the function f where $f(x) = 0, x \in \mathbb{R}$.

For all $x \in \mathbb{R}, | f_n(x) - f(x) |=| \frac{\sin nx}{n} |\leq \frac{1}{n}$.

Let $\epsilon > 0$. Then $| f_n(x) - f(x) |< \epsilon$ for all $n > \frac{1}{\epsilon}$.

Let $k = [\frac{1}{\epsilon}] + 1$. Then k is a natural number and for all $x \in \mathbb{R}$, $| f_n(x) - f(x) |< \epsilon$ for all $n \geq k$.

This proves that the sequence $\{f_n\}$ is uniformly convergent on $\mathbb{R}$.

6. In Example 9 of 13.2, the sequence $\{f_n\}$ converges to the function f where $f(x) = 0, x \geq 0$.

For all $x \geq 0, | f_n(x) - f(x) |= xe^{-nx} < \frac{1}{n}$, since $e^{nx} > nx$ for all $x \geq 0$.

Let $\epsilon > 0$. Then $| f_n(x) - f(x) |< \epsilon$ for all $n > \frac{1}{\epsilon}$.

Let $k = [\frac{1}{\epsilon}] + 1$. Then k is a natural number and for all $x \geq 0$, $| f_n(x) - f(x) |< \epsilon$ for all $n \geq k$.

This proves that the sequence $\{f_n\}$ is uniformly convergent on $[0, \infty)$.

Theorem 13.3.1. (Cauchy criterion)

Let $D \subset \mathbb{R}$ and let $\{f_n\}$ be a sequence of functions on D to $\mathbb{R}$.

A necessary and sufficient condition for uniform convergence of the sequence $\{f_n\}$ on D is that for a pre-assigned positive ϵ there exists a natural number k such that for all $x \in D$,
$| f_{n+p}(x) - f_n(x) |< \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

Proof. Let the sequence $\{f_n\}$ be uniformly convergent on D and let the limit function be f. Then for a pre-assigned positive ϵ there exists a natural number k (depending only on ϵ) such that

for all $x \in D, | f_n(x) - f(x) |< \frac{\epsilon}{2}$ for all $n \geq k$.

Therefore if $p = 1, 2, 3, \ldots$ then for all $x \in D$,
$| f_{n+p}(x) - f(x) |< \frac{\epsilon}{2}$ holds for all $n \geq k$.

Thus for all $x \in D$,

$$\begin{aligned} | f_{n+p}(x) - f_n(x) | &\leq | f_{n+p}(x) - f(x) | + | f_n(x) - f(x) | \\ &< \epsilon \quad \text{for all } n \geq k \text{ and } p = 1, 2, 3, \ldots. \end{aligned}$$

Conversely, let the condition be satisfied. Then for a chosen $\epsilon > 0$ there exists a natural number k such that for all $x \in D$,
$| f_{n+p}(x) - f_n(x) | < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

Let $x_0 \in D$. Then $| f_{n+p}(x_0) - f_n(x_0) | < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

It follows that the sequence $\{f_n(x_0)\}$ is Cauchy sequence in $\mathbb{R}$ and therefore it is convergent. Consequently, the sequence $\{f_n\}$ is pointwise convergent on D. Let the limit function be f.

Let us choose $\epsilon > 0$. Then by the condition, there exists a natural number k (depending only on ϵ) such that for all $x \in D$,
$| f_{n+p}(x) - f_n(x) | < \frac{\epsilon}{2}$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

Therefore for all $x \in D, f_k(x) - \frac{\epsilon}{2} < f_{k+p}(x) < f_k(x) + \frac{\epsilon}{2}$ for $p = 1, 2, 3, \ldots$

Since $\lim\limits_{p \to \infty} f_{k+p}(x) = f(x)$, taking limit as $p \to \infty$ we have
for all $x \in D, f_k(x) - \frac{\epsilon}{2} \leq f(x) \leq f_k(x) + \frac{\epsilon}{2}$
or, $| f_k(x) - f(x) | \leq \frac{\epsilon}{2} < \epsilon$ for all $x \in D$.
Similar inequalities hold for $k + 1, k + 2, \ldots$

Therefore for all $x \in D, | f_n(x) - f(x) | < \epsilon$ for all $n \geq k$.

This proves that the sequence $\{f_n\}$ is uniformly convergent on D.

Equivalent statement for Cauchy criterion.

A necessary and sufficient condition for uniform convergence of a sequence $\{f_n\}$ on D is that for a pre-assigned positive ϵ there exists a natural number k such that
for all $x \in D, | f_m(x) - f_n(x) | < \epsilon$ for all $m, n \geq k$.

Worked Examples.

1. A sequence of functions $\{f_n\}$ is defined on $[0, a], 0 < a < 1$, by $f_n(x) = x^n, x \in [0, a]$. Show that the sequence $\{f_n\}$ converges uniformly on $[0, a]$.

Let us choose $\epsilon > 0$ such that $0 < \epsilon < 2$.
For all $x \in [0, a]$ and for all $m, n \in \mathbb{N}$,

$$\begin{aligned} | f_m(x) - f_n(x) | &= | x^m - x^n | \\ &\leq | x |^m + | x |^n \\ &\leq a^m + a^n \\ &\leq 2a^m \text{ if } m \leq n. \end{aligned}$$

Now $| f_m(x) - f_n(x) | < \epsilon$ holds if $a^m < \frac{\epsilon}{2}$

i.e., if $m \log a < \log \frac{\epsilon}{2}$

i.e., if $m > \frac{\log \frac{\epsilon}{2}}{\log a}$, since $\log a < 0$.

Let $k = [\frac{\log \frac{\epsilon}{2}}{\log a}] + 1$. Then k is a natural number and for all $x \in [0, a]$, $| f_m(x) - f_n(x) | < \epsilon$ for all natural numbers m, n satisfying $n \geq m \geq k$.

By Cauchy's criterion, the sequence $\{f_n\}$ is uniformly convergent on $[0, a], 0 < a < 1$.

2. Let $r_1, r_2, r_3, \ldots$ be an enumeration of the set of all rational points in $[0, 1]$ and a sequence of functions $\{f_n\}$ is defined on $[0, 1]$ by

$$\begin{aligned} f_n(x) &= 0, x = r_1, r_2, \ldots, r_n \\ &= 1, x \in [0, 1] - \{r_1, r_2, \ldots, r_n\}. \end{aligned}$$

Show that the sequence $\{f_n\}$ is not uniformly convergent on $[0, 1]$.

Let us take $\epsilon = \frac{1}{2}$.

It is sufficient to establish that for all $k \in \mathbb{N}$ there exist natural numbers m and n such that $m, n \geq k$ and

$| f_m(x_0) - f_n(x_0) | \nless \frac{1}{2}$ for some $x_o \in [0, 1]$.

$$\begin{aligned} \text{For all } k \in \mathbb{N}, f_k(x) &= 0 \text{ if } x \in \{r_1, r_2, \ldots, r_k\} \\ &= 1 \text{ if } x \in [0, 1] - \{r_1, r_2, \ldots, r_k\}. \end{aligned}$$

For every natural number k, there exists a point $r_{k+1} \in [0, 1]$ such that $| f_k(r_{k+1}) - f_{k+1}(r_{k+1}) | = | 1 - 0 | = 1$.

Therefore no natural number k can be found such that for all $x \in [0, 1], | f_m(x) - f_n(x) | < \frac{1}{2}$ holds for all $m, n \geq k$.

By Cauchy criterion for uniform convergence of a sequence of functions, the sequence $\{f_n\}$ is not uniformly convergent on $[0, 1]$.

Theorem 13.3.2. Let $D \subset \mathbb{R}$ and let $\{f_n\}$ be a sequence of functions pointwise convergent on D to a function f. Let $M_n = \sup\limits_{x \in D} | f_n(x) - f(x) |$.

Then $\{f_n\}$ is uniformly convergent on D to f if and only if $\lim M_n = 0$.

Proof. Let the sequence $\{f_n\}$ be uniformly convergent on D to f.

Let $\epsilon > 0$. Then there exists a natural number k (depending only on ϵ) such that for all $x \in D$, $| f_n(x) - f(x) | < \frac{\epsilon}{2}$ for all $n \geq k$.

This implies $\sup\limits_{x \in D} | f_n(x) - f(x) | \leq \dfrac{\epsilon}{2} < \epsilon$ for all $n \geq k$

or, $| M_n | < \epsilon$ for all $n \geq k$. This proves that $\lim M_n = 0$.

Conversely, let $\lim M_n = 0$.

Let $\epsilon > 0$. Then there exists a natural number k such that $| M_n | < \epsilon$ for all $n \geq k$.

or, $\sup\limits_{x \in D} | f_n(x) - f(x) | < \epsilon$ for all $n \geq k$.

$$\text{Therefore for all } x \in D, |f_n(x) - f(x)| \le \sup_{x\in D} |f_n(x) - f(x)| < \epsilon \text{ for all } n \ge k.$$

This proves that the sequence $\{f_n\}$ is uniformly convergent to f on D.

Worked Examples (continued).

3. A sequence of functions $\{f_n\}$ is defined by $f_n(x) = \frac{nx}{1+n^2x^2}, 0 \le x \le 1$. Show that the sequence $\{f_n\}$ is not uniformly convergent on $[0, 1]$.

For $x = 0$, the sequence is $\{0, 0, 0, \ldots\}$. This converges to 0.
For $0 < x \le 1$, $\lim\limits_{n\to\infty} f_n(x) = 0$.

Thus the sequence $\{f_n\}$ is convergent on $[0, 1]$ and the limit function f is defined by $f(x) = 0, 0 \le x \le 1$.

Let $M_n = \sup\limits_{x\in[0,1]} |f_n(x) - f(x)|$. Then $M_n = \sup\limits_{x\in[0,1]} \dfrac{nx}{1+n^2x^2}$.

For $x > 0$, $\frac{\frac{1}{nx}+nx}{2} \ge \sqrt{\frac{1}{nx}.nx}$, the equality occurs when $x = \frac{1}{n}$.
That is, for $x > 0$, $\frac{nx}{1+n^2x^2} \le \frac{1}{2}$ and $\frac{nx}{1+n^2x^2} = \frac{1}{2}$ at $x = \frac{1}{n}$.
For $x = 0$, $\frac{nx}{1+n^2x^2} = 0$.
Therefore for $0 < x \le 1$, $\frac{nx}{1+n^2x^2} \le \frac{1}{2}$ and $\frac{nx}{1+n^2x^2} = \frac{1}{2}$ at $x = \frac{1}{n}$.
Clearly, $\sup\limits_{x\in[0,1]} \frac{nx}{1+n^2x^2} = \frac{1}{2}$. Therefore $M_n = \frac{1}{2}$ for all $n \in \mathbb{N}$.

Since $\lim M_n = \frac{1}{2} \ne 0$, the sequence $\{f_n\}$ is not uniformly convergent on $[0, 1]$, by Theorem 13.3.2.

4. For each natural number n, let $f_n(x) = 1 - \frac{x^n}{n}, x \in [0, 1]$. Show that the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$.

For $0 \le x \le 1$, $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} (1 - \frac{x^n}{n}) = 1$.

Hence the sequence $\{f_n\}$ converges pointwise on $[0, 1]$ to the function f where $f(x) = 1, x \in [0, 1]$.

Let $M_n = \sup\limits_{x\in[0,1]} |f_n(x) - f(x)|$.

Then $M_n = \sup\limits_{x\in[0,1]} \frac{|x|^n}{n} = \frac{1}{n}$ and $\lim\limits_{n\to\infty} M_n = 0$.

Hence the sequence $\{f_n\}$ converges uniformly on $[0, 1]$.

5. For each natural number n let $f_n(x) = \frac{x}{1+nx^2}, x \in [0, 1]$. Show that the sequence $\{f_n\}$ converges uniformly on $[0, 1]$.

For $x = 0$, the sequence is $\{0, 0, 0, \ldots\}$. This converges to 0.
For $0 < x \le 1$, $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{x}{1+nx^2} = 0$.

Therefore the sequence $\{f_n\}$ converges pointwise on $[0, 1]$ to the function f where $f(x) = 0, x \in [0, 1]$.

Let $M_n = \sup\limits_{x\in[0,1]} \mid f_n(x) - f(x) \mid$. Then $M_n = \sup\limits_{x\in[0,1]} \frac{x}{1+nx^2}$.

For $x > 0$, $\frac{\frac{1}{x}+nx}{2} \geq \sqrt{\frac{1}{x}.nx}$, the equality occurs when $x = \frac{1}{\sqrt{n}}$.

That is, for $x > 0$, $\frac{x}{1+nx^2} \leq \frac{1}{2\sqrt{n}}$ and $\frac{x}{1+nx^2} = \frac{1}{2\sqrt{n}}$ at $x = \frac{1}{\sqrt{n}}$.

For $x = 0$, $\frac{x}{1+nx^2} = 0$.

Clearly, $\sup\limits_{x\in[0,1]} \frac{x}{1+nx^2} = \frac{1}{2\sqrt{n}}$. Therefore $M_n = \frac{1}{2\sqrt{n}}$ and $\lim M_n = 0$.

Hence the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$.

6. Let $f_n(x) = x^n, x \in [0, 1]$. Show that the sequence of functions $\{f_n\}$ is not uniformly convergent on $[0, 1]$.

For all $x \in [0, 1)$, $\lim\limits_{n\to\infty} f_n(x) = 0$.

For $x = 1$, the sequence is $\{1, 1, 1, \ldots\}$. This converges to 1.

Therefore the sequence $\{f_n\}$ converges to the function f where

$$\begin{aligned} f(x) &= 0, 0 \leq x < 1 \\ &= 1, x = 1. \end{aligned}$$

Let $M_n = \sup\limits_{x\in[0,1]} \mid f_n(x) - f(x) \mid$.

Then $M_n = 1$ for all $n \in \mathbb{N}$ and therefore $\lim\limits_{n\to\infty} M_n = 1$.

Hence the convergence of the sequence $\{f_n\}$ is not uniform on $[0, 1]$.

7. Let $f_n(x) = nx(1 - x)^n, x \in [0, 1]$. Show that the sequence $\{f_n\}$ is not uniformly convergent on $[0, 1]$.

At $x = 0$, the sequence is $\{0, 0, 0, \ldots\}$. This converges to 0.

At $x = 1$, the sequence is $\{0, 0, 0, \ldots\}$. This converges to θ.

When $0 < x < 1$, $0 < 1 - x < 1$.

Let $1 - x = \frac{1}{y}, y > 1$. Then $y = 1 + a, a > 0$.

$f_n(x) = \frac{nx}{(1+a)^n} < \frac{2nx}{n(n-1)a^2}$ since $(1 + a)^n > \frac{n(n-1)}{2}a^2$.

Therefore $0 < f_n(x) < \frac{2x}{(n-1)a^2}$ for all $x \in (0, 1)$.

By Sandwich theorem 5.5.5, $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in (0, 1)$.

Thus the sequence $\{f_n\}$ converges to the function f on $[0, 1]$ where $f(x) = 0, x \in [0, 1]$.

Let $M_n = \sup\limits_{x\in[0,1]} \mid f_n(x) - f(x) \mid$.

$$\begin{aligned} \text{Then } M_n &= \sup_{x\in[0,1]} nx(1 - x)^n \\ &\geq n.\tfrac{1}{n}(1 - \tfrac{1}{n})^n, \text{ since } \tfrac{1}{n} \in [0, 1]. \end{aligned}$$

$\lim\limits_{n\to\infty} M_n \geq \lim\limits_{n\to\infty} (1 - \frac{1}{n})^n = e^{-1}$.

As $\lim\limits_{n\to\infty} M_n \neq 0$, the convergence of the sequence is not uniform on $[0, 1]$.

8. Let $f_n(x) = x^2e^{-nx}, x \in [0,\infty)$. Show that the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$.

The sequence $\{f_n\}$ converges to the function f where $f(x) = 0, x \in [0, 1]$. [worked Ex.10, 13.2.]

Let $M_n = \sup\limits_{x\in[0,\infty)} \mid f_n(x) - f(x) \mid$. Then $M_n = \sup\limits_{x\in[0,\infty)} x^2e^{-nx}$.

Let $u(x) = x^2e^{-nx}, x \geq 0$. Then $u'(x) = \frac{x(2-nx)}{e^{nx}}$.
$u'(x) = 0$ at $x = \frac{2}{n}$. $u'(x) > 0$ for $0 < x < \frac{2}{n}$. $u'(x) < 0$ for $x > \frac{2}{n}$.

Therefore u is an increasing function for $0 < x < \frac{2}{n}$, u is a maximum at $x = \frac{2}{n}$, u is a decreasing function for $x > \frac{2}{n}$ and $\lim\limits_{x\to\infty} u(x) = 0$.

Therefore $M_n = \sup\limits_{x\in[0,\infty)} u(x) = u(\frac{2}{n}) = \frac{4}{e^2n^2}$. $\lim\limits_{n\to\infty} M_n = 0$.

Hence the sequence $\{f_n\}$ is uniformly convergent on $[0,\infty)$.

9. Let $f_n(x) = \frac{x}{n+x^2}, x \in [0, 1]$. Show that the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$.

The sequence $\{f_n\}$ converges to the function f where $f(x) = 0, x \in [0, 1]$.

Let $M_n = \sup\limits_{x\in[0,1]} \mid f_n(x) - f(x) \mid$. Then $M_n = \sup\limits_{x\in[0,1]} \frac{x}{n+x^2}$.

Let $u_n(x) = \frac{x}{n+x^2}, x \in [0, 1]$. Then $u_n'(x) = \frac{n-x^2}{(n+x^2)^2} > 0$ for all $x \in [0, 1]$ and for all $n > 1$.

Therefore for all $n > 1, u_n$ is a strictly increasing function of x on $[0, 1]$ and therefore $\sup\limits_{x\in[0,1]} u_n(x) = \frac{1}{n+1}$.

That is, $M_n = \frac{1}{n+1}$ for all $n > 1$ and therefore $\lim M_n = 0$.

Hence the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$.

13.4. Consequences of uniform convergence.

Theorem 13.4.1. Let D be a subset of $\mathbb{R}$ and a sequence of functions $\{f_n\}$ be uniformly convergent on D to a function f. Let $x_0 \in D'$ (the derived set of D) and $\lim\limits_{x\to x_0} f_n(x) = a_n$. Then

(i) the sequence $\{a_n\}$ is convergent, and

(ii) $\lim\limits_{x\to x_0} f(x)$ exists and equals $\lim\limits_{n\to\infty} a_n$.

Proof. Let us choose $\epsilon > 0$. Since the sequence $\{f_n\}$ is uniformly convergent, there exists a natural number k such that

for all $x \in D$, $| f_m(x) - f_n(x) | < \frac{\epsilon}{2}$ for all $m, n \geq k$ ... ... (i)

As $\lim_{x \to x_0} f_n(x) = a_n$ and $\lim_{x \to x_0} f_m(x) = a_m$, it follows that $\lim_{x \to x_0} \{f_m(x) - f_n(x)\} = a_m - a_n$ and therefore $\lim_{x \to x_0} |f_m(x) - f_n(x)| = |a_m - a_n|$.

It follows from (i) that $| a_m - a_n | \leq \frac{\epsilon}{2} < \epsilon$ for all $m, n \geq k$.

This shows that $\{a_n\}$ is a Cauchy sequence in $\mathbb{R}$ and is therefore convergent.

Let $\lim a_n = l$. Let us choose $\epsilon > 0$.

Since the sequence $\{f_n\}$ converges uniformly on D, there exists a natural number k_1 such that

for all $x \in D$, $| f_n(x) - f(x) | < \frac{\epsilon}{3}$ for all $n \geq k_1$.

Since $\lim a_n = l$, there exists a natural number k_2 such that

$| a_n - l | < \frac{\epsilon}{3}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $| f_k(x) - f(x) | < \frac{\epsilon}{3}$ for all $x \in D$ and $| a_k - l | < \frac{\epsilon}{3}$.

Since $\lim_{x \to x_0} f_k(x) = a_k$, there exists a positive δ such that

$| f_k(x) - a_k | < \frac{\epsilon}{3}$ for all $x \in N'(x_0, \delta) \cap D$.

By triangle inequality,

$$\begin{aligned} | f(x) - l | &\leq | f(x) - f_k(x) | + | f_k(x) - a_k | + | a_k - l | \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} (= \epsilon) \text{ for all } x \in N'(x_0, \delta) \cap D. \end{aligned}$$

This proves $\lim_{x \to x_0} f(x) = l$. Therefore $\lim_{x \to x_0} f(x) = \lim_{n \to \infty} a_n$.

Note. In consequence of uniform convergence of the sequence $\{f_n\}$, $\lim_{x \to x_0} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to x_0} f_n(x)$. This indicates that the interchange of limits is permissible.

Corollary. Let I be an interval and a sequence of functions $\{f_n\}$ be uniformly convergent on I to a function f. Let $c \in I$ and each f_n be continuous at c. Then f is continuous at c.

Proof. Since each f_n is continuous at c, $\lim_{x \to c} f_n(x) = f_n(c)$, for all $n \in \mathbb{N}$. Since the sequence $\{f_n\}$ converges on I to the function f, the sequence $\{f_n(c)\}$ converges to $f(c)$.

By the theorem, $\lim_{x \to c} f(x)$ exists and equals $\lim_{n \to \infty} f_n(c)$, i.e., $\lim_{x \to c} f(x) = f(c)$.

This proves that f is continuous at c.

Example 1. Let $f_n(x) = x^n, x \in [0, 1)$.

The sequence $\{f_n\}$ is pointwise convergent on $[0, 1)$ and the limit function f is given by $f(x) = 0, x \in [0, 1)$. 1 is a limit point of $[0, 1)$.

Let $a_n = \lim\limits_{x \to 1} f_n(x)$. Then $a_n = \lim\limits_{x \to 1} x^n = 1$.

Since $a_n = 1$ for all $n \in \mathbb{N}$, $\lim\limits_{n \to \infty} a_n = 1$. $\lim\limits_{x \to 1} f(x) = 0 \neq \lim a_n$.

This proves that $\{f_n\}$ is not uniformly convergent on $[0, 1)$.

Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}, f_n : D \to \mathbb{R}$ is bounded on D. If $\{f_n\}$ be pointwise convergent on D then the limit function f may not be bounded on D.

For example, let $f_n(x) = 1 + x + x^2 + \cdots + x^{n-1}, x \in [0, 1)$.

Then $\lim\limits_{n \to \infty} f_n(x) = \frac{1}{1-x}, x \in [0, 1)$.

The sequence $\{f_n\}$ converges on $[0, 1)$ to the function f is given by $f(x) = \frac{1}{1-x}, x \in [0, 1)$.

$$| f_n(x) | = | 1 + x + x^2 + \cdots + x^{n-1} | \leq 1 + | x | + | x^2 | + \cdots + |x^{n-1}|$$
$$< n \text{ for all } x \in [0, 1).$$

Each f_n is bounded on $[0, 1)$. But f is unbounded on $[0, 1)$.

Theorem 13.4.2. Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}, f_n : D \to \mathbb{R}$ is bounded on D. If the sequence $\{f_n\}$ be uniformly convergent on D, then the limit function f is bounded on D.

Proof. Let us choose $\epsilon > 0$. Since $\{f_n\}$ is uniformly convergent on D to f, there exists a natural number k such that

for all $x \in D$, $| f_n(x) - f(x) | < \epsilon$ for all $n \geq k$.

Let $\epsilon = 1$. There exists a natural number m such that for all $x \in D$, $| f_n(x) - f(x) | < 1$ for all $n \geq m$.

Therefore for all $x \in D$, $| f_m(x) - f(x) | < 1$.

Since $| \, | f(x) | - | f_m(x) | \, | \leq | f(x) - f_m(x) |$, it follows that

$| f(x) | \leq | f_m(x) | + | f(x) - f_m(x) | < | f_m(x) | + 1$.

Since f_m is bounded on D, there exists a positive number B such that $| f_m(x) | < B$ for all $x \in D$.

Therefore for all $x \in D$, $|f(x)| < B + 1$ and this proves that f is bounded on D.

Note. If each f_n be bounded on D, the uniform convergence of the sequence $\{f_n\}$ on D is a sufficient but not a necessary condition for boundedness of the limit function f on D.

For example, let $f_n(x) = \frac{nx}{1+n^2x^2}, x \in [0, 1]$. Then the limit function f is defined by $f(x) = 0, x \in [0, 1]$.

Since $\sup\limits_{x\in[0,1]} | f_n(x) |= \frac{1}{2}$, each f_n is bounded on $[0,1]$. Also the limit function f is bounded in $[0,1]$. But the convergence of the sequence $\{f_n\}$ is not uniform on $[0,1]$.

Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}, f_n : D \to \mathbb{R}$ is continuous on D. If the sequence $\{f_n\}$ be pointwise convergent on D then the limit function f may not be continuous on D.

For example, let $f_n(x) = x^{n-1}, x \in [0,1]$.

Then each f_n is continuous on $[0,1]$.

The sequence $\{f_n\}$ converges on $[0,1]$ to the function f where

$f(x) = 0, 0 \le x < 1$

$\quad = 1, x = 1.$

The limit function f is not continuous on $[0,1]$.

Theorem 13.4.3. Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}, f_n : D \to \mathbb{R}$ is continuous on D. If the sequence $\{f_n\}$ be uniformly convergent on D to a function f, then f is continuous on D.

Proof. Let $c \in D$. Let us choose $\epsilon > 0$.

Since $\{f_n\}$ is uniformly convergent on D to the function f, there exists a natural number k such that

for all $x \in D, | f_n(x) - f(x) |< \frac{\epsilon}{3}$ for all $n \ge k$.

Therefore $| f_k(x) - f(x) |< \frac{\epsilon}{3}$ for all $x \in D$ and $| f_k(c) - f(c) |< \frac{\epsilon}{3}$.

Since f_k is continuous on at c, there exists a positive δ such that $| f_k(x) - f_k(c) |< \frac{\epsilon}{3}$ for all $x \in N(c,\delta) \cap D$.

By triangle inequality,

$$\begin{aligned} | f(x) - f(c) | &\le | f(x) - f_k(x) | + | f_k(x) - f_k(c) | + \\ &\quad | f_k(c) - f(c) | \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} \text{ for all } x \in N(c,\delta) \cap D. \end{aligned}$$

That is, $| f(x) - f(c) |< \epsilon$ for all $x \in N(c,\delta) \cap D$.

This proves that f is continuous at c. Since c is arbitrary, f is continuous on D.

Note 1. If each f_n be continuous on D, the uniform convergence of the sequence $\{f_n\}$ on D is a sufficient but not a necessary condition for continuity of the limit function f on D.

For example, let $f_n(x) = \frac{nx}{1+n^2x^2}, x \in [0,1]$.

Each f_n is continuous on $[0,1]$. The sequence $\{f_n\}$ converges on $[0,1]$ to the function f where $f(x) = 0, x \in [0,1]$.

The limit function f is continuous on $[0,1]$.

But the convergence of the sequence $\{f_n\}$ is not uniform on $[0, 1]$, as established in worked Example 3, page 543.

Note 2. If each f_n be continuous on D and the sequence $\{f_n\}$ converges pointwise on D to a function f not continuous on D, then it follows from the theorem that the convergence is not uniform on D.

Worked Examples .

1. Let $f_n(x) = \tan^{-1} nx, x \in [0, 1]$. Prove that the sequence $\{f_n\}$ is not uniformly convergent on $[0, 1]$.

$$\begin{aligned}\lim_{n\to\infty} f_n(x) &= \tfrac{\pi}{2}, \text{ if } x \in (0, 1]\\ &= 0, \text{ if } x = 0.\end{aligned}$$

The sequence $\{f_n\}$ is convergent on $[0, 1]$ to the function f where

$$\begin{aligned}f(x) &= 0, x = 0\\ &= \tfrac{\pi}{2}, 0 < x \leq 1.\end{aligned}$$

Each f_n is continuous on $[0, 1]$ but the limit function f is not continuous on $[0, 1]$.

This proves that the convergence of the sequence is not uniform on $[0, 1]$.

2. Prove that the sequence $\{f_n\}$ where $f_n(x) = \frac{x^n}{1+x^n}, x \in [0, 2]$ is not uniformly convergent on $[0, 2]$.

When $0 \leq x < 1$, $\lim\limits_{n\to\infty} f_n(x) = 0$.

When $x = 1$, $\lim\limits_{n\to\infty} f_n(x) = \frac{1}{2}$.

When $1 < x \leq 2$, $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{1}{(\frac{1}{x})^n + 1} = 1$.

The sequence $\{f_n\}$ converges pointwise to the function f where

$$\begin{aligned}f(x) &= 0, 0 \leq x < 1\\ &= \tfrac{1}{2}, x = 1\\ &= 1, 1 < x \leq 2.\end{aligned}$$

Each f_n is continuous on $[0, 2]$. The function f is not continuous on $[0, 2]$. Therefore the convergence of the sequence is not uniform on $[0, 2]$.

3. For each $n \in \mathbb{N}$, let
$$\begin{aligned}f_n(x) &= 1 - nx, 0 \leq x \leq \tfrac{1}{n}\\ &= 0, \quad \tfrac{1}{n} < x \leq 1.\end{aligned}$$

Show that the sequence $\{f_n\}$ converges on $[0, 1]$ to a function f but the convergence of the sequence is not uniform on $[0, 1]$.

At $x = 0$, the sequence is $\{1, 1, 1, \ldots\}$. This converges to 1.

At $x = 1$, the sequence is $\{0, 0, 0, \ldots\}$. This converges to 0.

Let $c \in (0,1)$. By Archimedean property of $\mathbb{R}$ there exists a natural number m such that $0 < \frac{1}{m} < c$ and therefore $0 < \frac{1}{n} < c$ for all $n \geq m$. $f_m(c) = 0$ and $f_n(c) = 0$ for all $n \geq m$. This proves $\lim_{n\to\infty} f_n(c) = 0$.

Therefore the sequence $\{f_n\}$ converges to the function f on $[0,1]$ given by
$$\begin{aligned} f(x) &= 1, x = 0 \\ &= 0, 0 < x \leq 1. \end{aligned}$$

Each f_n is continuous on $[0,1]$. The limit function f is not continuous on $[0,1]$.

Therefore the convergence of the sequence $\{f_n\}$ is not uniform on $[0,1]$, since uniform convergence of the sequence $\{f_n\}$ of continuous functions on $[0,1]$ implies continuity of f on $[0,1]$.

4. Prove that the sequence $\{f_n\}$ defined by $f_n(x) = \frac{nx}{1+nx}, x \geq 0$ is not uniformly convergent on $[0,\infty)$, but the convergence is uniform on $[a,\infty)$ if $a > 0$.

Each f_n is continuous on $[0,\infty)$ but the sequence $\{f_n\}$ converges to the function f which is not continuous on $[0,\infty)$. [Ex.5, 13.2]

Therefore the convergence of the sequence $\{f_n\}$ is not uniform on $[0,\infty)$.

For all $x > 0$, $| f_n(x) - f(x) | = \frac{1}{1+nx} < \frac{1}{nx}$.

Then for all $x \geq a$, $| f_n(x) - f(x) | < \frac{1}{na}$.

Let $\epsilon > 0$. Then for all $x \geq a$, $| f_n(x) - f(x) | < \epsilon$ holds if $n > \frac{1}{a\epsilon}$.

Let $k = [1/(a\epsilon)] + 1$. Then k is a natural number and for all $x \geq a$, $| f_n(x) - f(x) | < \epsilon$ for all $n \geq k$.

This proves that $\{f_n\}$ is uniformly convergent on $[a,\infty)$.

The following theorem due to U.Dini gives a partial converse of the theorem 13.4.3.

Theorem 13.4.4. (Dini)

Let D be a compact subset of $\mathbb{R}$ and $f_n : D \to \mathbb{R}$ be a sequence of continuous functions on D that converges pointwise to a continuous function f. If the sequence $\{f_n\}$ be a monotone sequence on D, i.e., either $f_{n+1}(x) \geq f_n(x)$ for each $n \in \mathbb{N}$ and each $x \in D$, or $f_{n+1}(x) \leq f_n(x)$ for each $n \in \mathbb{N}$ and each $x \in D$, then the convergence of the sequence $\{f_n\}$ is uniform on D.

Proof. If $\{f_n\}$ be monotone increasing, let $g_n = f - f_n$. If $\{f_n\}$ be monotone decreasing, let $g_n = f_n - f$.

Then $g_{n+1} - g_n \leq 0$ for all $n \in \mathbb{N}$ and for all $x \in D$.

So $\{g_n\}$ is a monotone decreasing sequence of continuous functions with $\lim g_n(x) = 0$ for every $x \in D$. Also $g_n(x) \geq 0$ for all $n \in \mathbb{N}$ and all $x \in D$.

Let $M_n = \sup\limits_{x \in D} g_n(x)$. Then $M_{n+1} \leq M_n$ for all $n \in \mathbb{N}$.

Since g_n is continuous on D, g_n attains the supremum M_n at a point, say $x_n, \in D$, i.e., $g_n(x_n) = M_n$ for all $n \in \mathbb{N}$.

The sequence $\{x_n\}$ is a sequence in a compact set D. Therefore there exists a subsequence $\{x_{r_n}\}$ of $\{x_n\}$ such that $\{x_{r_n}\}$ converges to a point x^* in D.

Since $\lim g_n(x^*) = 0$, for a pre-assigned positive ϵ, there exists a natural number m such that $g_n(x^*) < \frac{\epsilon}{2}$ for all $n \geq m$.

Since g_m is continuous at x^*, there exists a neighbourhood U of x^* such that $| g_m(x) - g_m(x^*) | < \frac{\epsilon}{2}$ for all $x \in U \cap D$.

It follows that $g_m(x) < \epsilon$ for all $x \in U \cap D$.

Since $\lim x_{r_n} = x^*$, there exists a natural number k_1 such that $x_{r_n} \in U \cap D$ (neighbourhood of x^*) for all $n \geq k_1$.

Also there exists a natural number k_2 such that $r_n > m$ for all $n \geq k_2$.

Let $k = max\{k_1, k_2\}$. Then $g_{r_n}(x_{r_n}) < g_m(x_{r_n}) < \epsilon$ for all $n \geq k$.

This proves $M_{r_n} < \epsilon$ for all $n \geq k$, i.e., $\lim M_{r_n} = 0$.

Since $\{M_n\}$ is a monotone decreasing sequence having a convergent subsequence $\{M_{r_n}\}$ with limit 0, the sequence $\{M_n\}$ converges to 0.

Therefore the sequence $\{g_n\}$ converges uniformly to 0 on D.

Consequently, the sequence $\{f_n\}$ converges uniformly to f on D.

This completes the proof.

Another proof.

Without loss of generality, let us assume that the sequence $\{f_n\}$ is monotone increasing.

Let $a \in D$. Since the sequence $\{f_n(a)\}$ is a monotone increasing sequence converging to $f(a)$, for a pre-assigned positive ϵ there exists a natural number k such that

$$f(a) - \epsilon < f_k(a) \leq f(a), \text{ i.e., } 0 \leq f(a) - f_k(a) < \epsilon.$$

Since f and f_k are continuous at a, there exists a positive δ such that $|f(x) - f(a)| < \epsilon$ and $|f_k(x) - f_k(a)| < \epsilon$ for all $x \in N(a, \delta) \cap D$.

Therefore
$$\begin{aligned} |f(x)-f_k(x)| &\leq |f(x)-f(a)|+|f(a)-f_k(a)|+|f_k(a)-f_k(x)| \\ &< 3\epsilon \text{ for all } x \in N(a, \delta) \cap D, \end{aligned}$$
$$\text{i.e., } 0 \leq f(x) - f_k(x) < 3\epsilon \text{ for all } x \in N(a, \delta) \cap D.$$

Thus corresponding to a chosen positive ϵ, for each point $a \in D$ there exist a positive number δ_a and a natural number k_a such that

$$0 \le f(x) - f_{k_a}(x) < 3\epsilon \text{ for all } x \in N(a, \delta_a) \cap D.$$

The set of neighbourhoods $\{N(a, \delta_a) : a \in D\}$ form an open cover of D and since D is compact, there exist a finite number of points $a_1, a_2, \ldots, a_m$ in D such that the union of the corresponding neighbourhoods of $a_1, a_2, \ldots, a_m$ determined by ϵ form an open cover of D.

Let the neighbourhood $N(a_i, \delta_i)$ and the natural number k_i correspond to the point a_i, for $i = 1, 2, \ldots, m$.

Then $0 \le f(x) - f_{k_i}(x) < 3\epsilon$ for all $x \in N(a_i, \delta_i) \cap D$.

Let $k_0 = \max\{k_1, k_2, \ldots, k_m\}$. Then

$$0 \le f(x) - f_{k_0}(x) < 3\epsilon \text{ for all } x \in D.$$

Since $\{f_n\}$ is a monotone increasing sequence converging to f, for all $x \in D$, $|f(x) - f_n(x)| < 3\epsilon$ for all $n \ge k_0$.

This proves that convergence of the sequence $\{f_n\}$ is uniform on D.

This completes the proof.

Note. The compactness of D in the theorem is essential.

Let us consider the following examples.

Let $I = [0, 1)$ and let $f_n : I \to \mathbb{R}$ be defined by $f_n(x) = x^n, x \in I$. The sequence $\{f_n\}$ is a monotone decreasing sequence of continuous functions on I. It converges to the 0-function on I, which is a continuous function. But the convergence is not uniform on I. [worked Example 1, 13.4]

This does not violate Dini's theorem, since I is not a compact subset of $\mathbb{R}$.

Let $I = [0, \infty)$ and let $f_n : I \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{n}, x \in I$. The sequence is a monotone decreasing sequence of continuous functions on I and it converges to the 0-function on I, which is a continuous function. But the convergence is not uniform on I. [worked Example 2,13.3.]

This does not violate Dini's theorem, since I is not a compact subset of $\mathbb{R}$.

Worked Examples (continued).

5. For each $n \in \mathbb{N}$, let $f_n(x) = x^{n-1} - x^n, x \in [0, 1]$. Use Dini's theorem to prove that the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$.

$\lim\limits_{n \to \infty} f_n(x) = \lim\limits_{n \to \infty} x^{n-1}(1 - x) = 0$ for $x \in [0, 1]$.

The sequence $\{f_n\}$ is convergent on $[0, 1]$ to the function f where $f(x) = 0, x \in [0, 1]$.

Each f_n is continuous on $[0, 1]$; f is also continuous on $[0, 1]$.

$$\begin{aligned}\text{For each } x \in [0,1], f_{n+1}(x) - f_n(x) &= (x^n - x^{n+1}) - (x^{n-1} - x^n) \\ &= -x^{n-1}(x-1)^2 \leq 0.\end{aligned}$$

Therefore the sequence $\{f_n\}$ is a monotone decreasing sequence on $[0, 1]$, a compact subset of $\mathbb{R}$.

By Dini's theorem, the convergence of the sequence is uniform on $[0, 1]$.

6. A sequence of functions $\{f_n\}$ is defined by $f_1(x) = \sqrt{x}, f_{n+1}(x) = \sqrt{x f_n(x)}$ for all $n \geq 1$.

Use Dini's theorem to prove that the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$.

$f_1(x) = x^{\frac{1}{2}}, f_2(x) = x^{\frac{1}{2}+\frac{1}{2^2}}, \ldots, f_n(x) = x^{\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}}, \ldots$

At $x = 0$, the sequence converges to 0.

When $x > 0$, $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} x^{1-\frac{1}{2^n}} = x$.

Therefore the sequence $\{f_n\}$ converges to f on $[0, 1]$ where $f(x) = x, x \in [0, 1]$.

Each f_n is continuous on $[0, 1]$ and the limit function f is also continuous on $[0, 1]$.

$f_{n+1}(x) - f_n(x) = x^{\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}}[x^{\frac{1}{2^{n+1}}} - 1] \leq 0$ for each $x \in [0, 1]$.

Therefore the sequence $\{f_n\}$ is a monotone decreasing sequence on $[0, 1]$, a compact subset of $\mathbb{R}$.

Thus the sequence $\{f_n\}$ is a sequence of continuous functions on the compact set $[0, 1]$ and converges to a function f continuous on $[0, 1]$. Also $\{f_n\}$ is a monotone decreasing sequence on $[0, 1]$.

By Dini's theorem the convergence of the sequence is uniform on $[0, 1]$.

Let $I = [a, b]$ be closed and bounded interval and for each natural number n, let $f_n : I \to \mathbb{R}$ be $\mathcal{R}$-integrable on I. If the sequence $\{f_n\}$ be pointwise convergent on I to a function f then f may not be $\mathcal{R}$-integrable on I.

For Example, let $I = [0, 1]$. Let $r_1, r_2, r_3, \ldots$ be an enumeration of the set of all rational points in I. For each $n \in \mathbb{N}$, let $f_n : I \to \mathbb{R}$ be defined by

$$\begin{aligned} f_n(x) &= 0, \ x = r_1, r_2, \ldots, r_n \\ &= 1, \ x \in [0,1] - \{r_1, r_2, \ldots, r_n\}. \end{aligned}$$

Then each f_n is $\mathcal{R}$-integrable on $[0, 1]$, since f_n is continuous on $[0, 1]$ except only at n points.

$$\begin{aligned} \lim_{n\to\infty} f_n(x) &= 0, \ \text{if } x \in [0,1] \cap \mathbb{Q} \\ &= 1, \ \text{if } x \in [0,1] - \mathbb{Q}. \end{aligned}$$

Therefore the sequence $\{f_n\}$ converges to f on $[0,1]$ where

$$f(x) = 0, \text{ if } x \in [0,1]\cap\mathbb{Q}$$
$$= 1, \text{ if } x \in [0,1]-\mathbb{Q}.$$

f is discontinuous at every point in $[0,1]$. So f is not $\mathcal{R}$- integrable on $[0,1]$.

Let $I=[a,b]$ and for each $n\in\mathbb{N}$, let $f_n : I \to \mathbb{R}$ be integrable on I and the sequence $\{f_n\}$ converges pointwise to a function f which is also integrable on I.

We now ask if it is true that the sequence $\{\int_a^b f_n\}$ converges to $\int_a^b f$.

That is, if it is true that $\lim\limits_{n\to\infty} (\int_a^b f_n) = \int_a^b(\lim\limits_{n\to\infty} f_n)$.

The answer is 'no'.

For example, let $f_n(x) = nxe^{-nx^2}, x\in[0,1]$.

When $x=0$, the sequence is $\{0,0,0,\ldots\}$. This converges to 0.

When $0<x\leq 1, e^{nx^2} > \frac{n^2x^4}{2}$.

For all $x\in(0,1]$, we have $0 < nxe^{-nx^2} < \frac{2}{nx^3}$.

By Sandwich theorem, $\lim\limits_{n\to\infty} nxe^{-nx^2} = 0$, for $x\in(0,1]$.

Therefore the sequence $\{f_n\}$ converges on $[0,1]$ to the function f where $f(x)=0, x\in[0,1]$.

Each f_n is integrable on $[0,1]$ and f is also integrable on $[0,1]$.

$\int_0^1 f_n(x)dx = [-\frac{1}{2}e^{-nx^2}]_0^1 = \frac{1}{2}(1-e^{-n})$.

$\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx = \lim\limits_{n\to\infty} \frac{1}{2}(1-e^{-n}) = \frac{1}{2}$.

Hence the sequence $\{\int_0^1 f_n\}$ converges to $\frac{1}{2}$ but $\int_0^1 f(x)dx = 0$.

Therefore $\lim\limits_{n\to\infty} (\int_0^1 f_n) \neq \int_0^1(\lim\limits_{n\to\infty} f_n)$.

Theorem 13.4.5. Let $I=[a,b]$ be a closed and bounded interval and for each $n\in\mathbb{N}, f_n : I\to\mathbb{R}$ be $\mathcal{R}$-integrable on I. If the sequence $\{f_n\}$ converges uniformly to a function f on I then f is $\mathcal{R}$-integrable on I and moreover, the sequence $\{\int_a^b f_n\}$ converges to $\int_a^b f$.

Proof. Let us choose $\epsilon>0$. Since $\{f_n\}$ is uniformly convergent on $[a,b]$ to the function f there exists a natural number k such that

for all $x\in[a,b], \mid f_n(x)-f(x)\mid < \frac{\epsilon}{4(b-a)}$ for all $n\geq k$.

Therefore for all $x\in[a,b], \mid f_k(x)-f(x)\mid < \frac{\epsilon}{4(b-a)}$.

or, $f_k(x) - \frac{\epsilon}{4(b-a)} < f(x) < f_k(x) + \frac{\epsilon}{4(b-a)}$ for all $x\in[a,b]$ ... (i)

Since f_k is integrable on $[a,b]$, there exists a partition $P = (x_0, x_1, x_2, \ldots, x_n)$ of $[a,b]$ such that $U(P,f_k) - L(P,f_k) < \frac{\epsilon}{2}$ ... (ii)

Let $M_r = \sup\limits_{x\in[x_{r-1},x_r]} f(x), m_r = \inf\limits_{x\in[x_{r-1},x_r]} f(x)$

$$M_r' = \sup_{x\in[x_{r-1},x_r]} f_k(x), m_r' = \inf_{x\in[x_{r-1},x_r]} f_k(x),\ r = 1,2,\ldots,n.$$

From (i) it follows that $m_r \geq m_r' - \frac{\epsilon}{4(b-a)}$; $M_r \leq M_r' + \frac{\epsilon}{4(b-a)}$.

$$\begin{aligned} U(P,f) &= M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1}) \\ &\leq M_1'(x_1 - x_0) + M_2'(x_2 - x_1) + \cdots + M_n'(x_n - x_{n-1}) + \tfrac{\epsilon}{4}. \end{aligned}$$

$$\begin{aligned} L(P,f) &= m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1}) \\ &\geq m_1'(x_1 - x_0) + m_2'(x_2 - x_1) + \cdots + m_n'(x_n - x_{n-1}) - \tfrac{\epsilon}{4}. \end{aligned}$$

Therefore $U(P,f) - L(P,f) \leq U(P,f_k) - L(P,f_k) + \frac{\epsilon}{2}$
$< \epsilon$, by using (ii).

This proves that f is $\mathcal{R}$-integrable on $[a,b]$.

Second part.

Let us choose $\epsilon > 0$.

Since the sequence $\{f_n\}$ converges uniformly to f on $[a,b]$, there exists a natural number k such that for all $x \in [a,b]$,
$|f_n(x) - f(x)| < \frac{\epsilon}{2(b-a)}$ for all $n \geq k$.

We have $\left| \int_a^b [f_n(x) - f(x)]dx \right| \leq \int_a^b |f_n(x) - f(x)|\, dx \leq \frac{\epsilon}{2(b-a)} \cdot (b-a)$, i.e., $< \epsilon$ for all $n \geq k$

or $\left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| < \epsilon$ for all $n \geq k$.

This implies $\lim\limits_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

In other words, the sequence $\{\int_a^b f_n\}$ converges to $\int_a^b f$.

This completes the proof.

Remarks. In symbols, $\lim\limits_{n\to\infty} (\int_a^b f_n) = \int_a^b (\lim\limits_{n\to\infty} f_n)$.

This says that if the convergence of the sequence $\{f_n\}$ be uniform on $[a,b]$, it is permissible to interchange $\lim\limits_{n\to\infty}$ and $\int_a^b$.

Corollary. For each $x \in [a,b]$, the sequence $\{\int_a^x f_n\}$ converges to $\int_a^x f$.

Note 1. If each f_n be integrable on $[a,b]$ and the sequence $\{f_n\}$ converges pointwise to a function f which is also integrable on $[a,b]$, the uniform convergence of the sequence $\{f_n\}$ is a sufficient but not a necessary condition for the convergence of the sequence $\{\int_a^b f_n\}$ to $\int_a^b f$.

For example, let $f_n(x) = \frac{nx}{1+n^2x^2}, x \in [0,1]$.

This sequence $\{f_n\}$ converges on $[0,1]$ to the function f where $f(x) = 0, x \in [0,1]$.

Each f_n is integrable on $[0,1]$ and also f is integrable on $[0,1]$.

$\int_0^1 f_n(x)dx = [\frac{1}{2n}\log(1+n^2x^2)]_0^1 = \frac{1}{2n}\log(1+n^2)$.

$\lim\limits_{x\to\infty} \frac{\log(1+x^2)}{2x} = 0$. By sequential criterion for limits, $\lim\limits_{n\to\infty} \frac{\log(1+n^2)}{2n} = 0$.

$\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx = 0$ and $\int_0^1 f(x)dx = 0$.

Thus the sequence $\{\int_0^1 f_n\}$ converges to $\int_0^1 f$ but the convergence of the sequence $\{f_n\}$ is not uniform on $[0,1]$.

Note 2. If each f_n be integrable on $[a,b]$ and the sequence $\{f_n\}$ converges pointwise to a function f which is not integrable on $[a,b]$, then it follows from the theorem that the convergence of the sequence $\{f_n\}$ is not uniform on $[a,b]$.

If each f_n be integrable on $[a,b]$ and the sequence $\{f_n\}$ converges pointwise to a function f which is also integrable on $[a,b]$ but the sequence $\{\int_a^b f_n\}$ does not converge to $\int_a^b f$, then it follows from the theorem that the convergence of the sequence $\{f_n\}$ is not uniform on $[a,b]$.

Worked Examples (continued).

7. For each $n \in \mathbb{N}$, let $f_n(x) = nxe^{-nx^2}, x \in [0,1]$. Show that the sequence $\{f_n\}$ is not uniformly convergent on $[0,1]$.

For all $x \in (0,1], nxe^{-nx^2} > 0$. Let $u_n = nxe^{-nx^2}, x \in (0,1]$.

Then $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} (1+\frac{1}{n})e^{-x^2} = e^{-x^2} > 0$.

By theorem 5.8.1, $\lim\limits_{n\to\infty} u_n = 0$, i.e., $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in (0,1]$.

Also for $x = 0$, the sequence converges to 0.

Thus the sequence $\{f_n\}$ converges pointwise on $[0,1]$ to the function f where $f(x) = 0, x \in [0,1]$.

Each f_n is integrable on $[0,1]$. $\int_0^1 f_n(x)dx = [-\frac{1}{2}e^{-nx^2}]_0^1 = \frac{1}{2}(1-e^{-n})$.

f is integrable on $[0,1]$ and $\int_0^1 f(x)dx = 0$.

$\lim\limits_{n\to\infty} \int_0^n f_n(x)dx = \lim\limits_{n\to\infty} \frac{1}{2}(1-e^{-n}) = \frac{1}{2}$.

Hence the sequence $\{\int_0^1 f_n\}$ converges to $\frac{1}{2}$ which is not equal to $\int_0^1 f$.

This proves that the convergence of the sequence $\{f_n\}$ is not uniform on $[0,1]$.

8. Let $f_n(x) = nx(1-x^2)^n, x \in [0,1]$. Show that the sequence of functions $\{f_n\}$ converges to a function f integrable on $[0,1]$ but the convergence is not uniform on $[0,1]$.

When $x=0$, the sequence is $\{0,0,0,\ldots\}$. This converges to 0.
When $x=1$, the sequence is $\{0,0,0,\ldots\}$. This converges to 0.
$0 < x <$ implies $0 < 1-x^2 < 1$.

Let $1-x^2 = \frac{1}{y}, y > 1$. Then $y = 1+a, a > 0$.
Then $f_n(x) = \frac{nx}{(1+a)^n} < \frac{2nx}{n(n-1)a^2}$, since $(1+a)^n > \frac{n(n-1)}{2}a^2$.
We have, $0 < f_n(x) < \frac{2x}{(n-1)a^2}$, when $0 < x < 1$.

By Sandwich theorem 5.5.5, $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in (0,1)$.

Therefore the sequence $\{f_n\}$ is convergent on $[0,1]$ to the function f where $f(x) = 0, x \in [0,1]$.

Each f_n is integrable on $[0,1]$ and $\int_0^1 f_n(x)dx = \frac{n}{2(n+1)}$.
f is integrable on $[0,1]$ and $\int_0^1 f(x)dx = 0$. $\lim\limits_{n\to\infty} \int_0^1 f_n = \frac{1}{2} \neq \int_0^1 f$.

This proves that the sequence $\{f_n\}$ is not uniformly convergent on $[0,1]$.

Let $\{f_n\}$ be a sequence of functions on $[a,b]$ such that for each $n \in \mathbb{N}$, $f_n'(x)$ exists for all $x \in [a,b]$. Let $\{f_n\}$ be pointwise convergent on $[a,b]$ to a function f. We ask if $f'(x)$ exists for all $x \in [a,b]$. If it exists, we ask again if the sequence $\{f_n'\}$ converges to f' on $[a,b]$.

The answer to both the questions is 'no'.

For example, let $f_n(x) = x^{n-1}, x \in [0,1]$.
Then the limit function f is given by
$$\begin{aligned} f(x) &= 0, x \in [0,1) \\ &= 1, x = 1. \end{aligned}$$
For each $n \in \mathbb{N}$, $f_n'(x)$ exists for all $x \in [0,1]$.
But $f'(x) = 0, x \in [0,1]$ and $f'(1)$ does not exist.

To discuss the second question, let us consider the sequence $\{f_n\}$ where $f_n(x) = x - \frac{x^n}{n}, x \in [0,1]$. $\lim\limits_{n\to\infty} f_n(x) = x, x \in [0,1]$.

Therefore the sequence converges to the function f on $[0,1]$ where $f(x) = x, x \in [0,1]$.

$f_n'(x) = 1 - x^{n-1}$ and $f'(x) = 1, x \in [0,1]$.

For each $n \in \mathbb{N}$, $f_n'(x)$ exists for all $x \in [0,1]$. Also $f'(x)$ exists for all $x \in [0,1]$.
$$\begin{aligned} \lim_{n\to\infty} f_n'(x) &= 1, x \in [0,1) \\ &= 0, x = 1. \end{aligned}$$

This shows that the sequence $\{f'_n\}$ does not converge to f' on $[0,1]$.

Let $\{f_n\}$ be a sequence of functions on $[a,b]$ such that for each $n \in \mathbb{N}$, $f'_n(x)$ exists for all $x \in [a,b]$. Let the sequence $\{f_n\}$ converge uniformly to a function f on $[a,b]$.

We ask if the sequence $\{f'_n\}$ converges to f' on $[a,b]$.

The answer is 'no'.

For example, let $f_n(x) = \frac{\sin nx}{n}, x \in [0,1]$.

The sequence $\{f_n\}$ converges uniformly to the function f where $f(x) = 0, x \in [0,1]$. [Example 5, 13.3.]

$f'(x) = 0, x \in [0,1]$. $f'_n(x) = \cos nx$, $x \in [0,1]$.

The sequence $\{f'_n\}$ converges at $x = 0$ but does not converge for $x \in (0,1]$.

Theorem 13.4.6. Let $\{f_n\}$ be a sequence of functions on $[a,b]$ such that for each $n \in \mathbb{N}$, $f'_n(x)$ exists for all $x \in [a,b]$. If the sequence of derivatives $\{f'_n\}$ converges uniformly on $[a,b]$ to a function g and the sqeuence $\{f_n\}$ converges *at least at one point* $x_0 \in [a,b]$, then the sequence $\{f_n\}$ is uniformly convergent on $[a,b]$ and if the limit function be f then $f'(x) = g(x)$ for all $x \in [a,b]$.

Proof. Let us choose $\epsilon > 0$.

Since the sequence $\{f'_n\}$ is uniformly convergent on $[a,b]$, there exists a natural number k_1 such that

for all $x \in [a,b]$, $| f'_{n+p}(x) - f'_n(x) | < \frac{\epsilon}{2(b-a)}$ for all $n \geq k_1$ and $p = 1,2,3,\ldots$

Also since $\{f_n(x_0)\}$ is convergent, there exists a natural number k_2 such that $| f_{n+p}(x_0) - f_n(x_0) | < \frac{\epsilon}{2}$ for all $n \geq k_2$ and $p = 1,2,3,\ldots$

Let $k = \max\{k_1, k_2\}$.

Then for all $x \in [a,b]$, $| f'_{n+p}(x) - f'_n(x) | < \frac{\epsilon}{2(b-a)}$ for all $n \geq k$ and $p = 1,2,3,\ldots$ and $| f_{n+p}(x_0) - f_n(x_0) | < \frac{\epsilon}{2}$ for all $n \geq k$ and $p = 1,2,3,\ldots$

Applying Lagrange's Mean value theorem to the function $f_{n+p} - f_n$ on $[x_0, x]$ or $[x, x_0]$ where $x \in [a,b]$,

$| f_{n+p}(x) - f_n(x) - f_{n+p}(x_0) + f_n(x_0) | = | x - x_0 || f'_{n+p}(\xi) - f'_n(\xi) |$ where $x_0 < \xi < x$ or $x < \xi < x_0$, as the case may be.

Now $| f'_{n+p}(\xi) - f'_n(\xi) | < \frac{\epsilon}{2(b-a)}$ for all $n \geq k$ and $p = 1,2,3,\ldots$ and $| x - x_0 | < b - a$ for $x \in [a,b]$.

It follows that for all $x \in [a,b]$,

$| f_{n+p}(x) - f_n(x) - f_{n+p}(x_0) + f_n(x_0) | < \frac{\epsilon}{2}$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

Using triangle inequality,
for all $x \in [a, b]$, $| f_{n+p}(x) - f_n(x) | \leq | f_{n+p}(x) - f_n(x) - f_{n+p}(x_0) + f_n(x_0) | + | f_{n+p}(x_0) - f_n(x_0) | < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

This proves that the sequence $\{f_n\}$ is uniformly convergent on $[a, b]$.

Second part.

Let f be the uniform limit of the sequence $\{f_n\}$ on $[a, b]$.

We now prove that $f'(x) = g(x), x \in [a, b]$.

Let $c \in [a, b]$.

Let us define a sequence of functions $\{\phi_n\}$ on $D = [a, b] - \{c\}$ by

$$\phi_n(x) = \frac{f_n(x) - f_n(c)}{x - c}, \ x \in [a, b] - \{c\}.$$

Then for $x \in [a, b] - \{c\}$, $\lim\limits_{n \to \infty} \phi_n(x) = \lim\limits_{n \to \infty} \frac{f_n(x) - f_n(c)}{x - c} = \frac{f(x) - f(c)}{x - c}$.

Applying Mean value theorem to the function $f_{n+p} - f_n$ on $[c, x]$ or $[x, c]$, we have $| f_{n+p}(x) - f_n(x) - f_{n+p}(c) + f_n(c) | = | x - c || f'_{n+p}(\eta) - f'_n(\eta) |$ where $c < \eta < x$ or $x < \eta < c$.

Let us choose $\epsilon > 0$.

Since the sequence $\{f'_n\}$ converges uniformly to g on $[a, b]$, there exists a natural number k_1 such that

for all $x \in [a, b]$, $| f'_{n+p}(x) - f'_n(x) | < \epsilon$ for all $n \geq k$ and $p = 1, 2, \ldots$

Therefore $| f'_{n+p}(\eta) - f'_n(\eta) | < \epsilon$ for all $n \geq k$ and $p = 1, 2, \ldots$

For all $x \in [a, b] - \{c\}$, $|\phi_{n+p}(x) - \phi_n(x)|$

$$= \left| \frac{f_{n+p}(x) - f_n(x) - f_{n+p}(c) + f_n(c)}{x - c} \right|$$

$< \epsilon$ for all $n \geq k$ and $p = 1, 2, \ldots$

This proves that the sequence $\{\phi_n\}$ is uniformly convergent on D.

Now c is a limit point of D.

Since the the sequence $\{\phi_n\}$ is uniformly convergent on D, it follows that $\lim\limits_{n \to \infty} \lim\limits_{x \to c} \phi_n(x) = \lim\limits_{x \to c} \lim\limits_{n \to \infty} \phi_n(x)$, by Theorem 12.4.1.

$\lim\limits_{n \to \infty} \lim\limits_{x \to c} \phi_n(x) = \lim\limits_{n \to \infty} \lim\limits_{x \to c} \frac{f_n(x) - f_n(c)}{x - c} = \lim\limits_{n \to \infty} f'_n(c) = g(c)$ and
$\lim\limits_{x \to c} \lim\limits_{n \to \infty} \phi_n(x) = \lim\limits_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$.
Therefore $g(c) = f'(c)$.

Since c is arbitrary, $f'(x) = g(x)$ for all $x \in [a, b]$.

This completes the proof.

Note 1. For a sequence of functions $\{f_n\}$ where each f_n is differentiable on $[a, b]$, mere uniform convergence of the sequence $\{f'_n\}$ on $[a, b]$ is not enough to ensure uniform convergence of the sequence $\{f_n\}$ on $[a, b]$.

For example, let $f_n(x) = \log(n + x^2), x \in [0, 1]$.

Then $f'_n(x) = \frac{2x}{n+x^2}, x \in [0, 1]$. The sequence $\{f'_n\}$ converges to the function g where $g(x) = 0, x \in [0, 1]$.

Let $M_n = \sup\limits_{x\in[0,1]} \mid f'_n(x) - g(x) \mid$. Then $M_n = \sup\limits_{x\in[0,1]} \frac{2x}{n+x^2}$.

Let $u_n(x) = \frac{2x}{n+x^2}, x \in [0, 1]$. Then $u'_n(x) = \frac{2n-2x^2}{(n+x^2)^2} > 0$, for all $x \in [0, 1]$ and for all $n > 1$.

Therefore for all $n > 1, u_n$ is a strictly increasing function of x on $[0, 1]$ and therefore $\sup\limits_{x\in[0,1]} u_n(x) = \frac{2}{n+1}$.

That is, $M_n = \frac{2}{n+1}$ for all $n > 1$ and therefore $\lim M_n = 0$.

Hence the sequence $\{f'_n\}$ is uniformly convergent on $[0, 1]$.

But the sequence $\{f_n\}$ is not even pointwise convergent on $[0, 1]$.

Note 2. For a sequence of functions $\{f_n\}$ where each f_n is differentiable on $[a, b]$ and the sequence $\{f_n\}$ is pointwise convergent on $[a, b]$, the uniform convergence of the sequence $\{f'_n\}$ on $[a, b]$ is only a sufficient but not a necessary condition for the uniform convergence of $\{f_n\}$ on $[a, b]$.

For example, let $f_n(x) = x - \frac{x^n}{n}, x \in [0, 1]$. $\lim\limits_{n\to\infty} f_n(x) = x, x \in [0, 1]$.

The sequence $\{f_n\}$ converges to the function f where $f(x) = x, x \in [0, 1]$.

Let $M_n = \sup\limits_{x\in[0,1]} \mid f_n(x) - f(x) \mid$. Then $M_n = \frac{1}{n}$ and $\lim\limits_{n\to\infty} M_n = 0$.

This establishes uniform convergence of the sequence $\{f_n\}$ on $[0, 1]$.

$$
\begin{aligned}
f'_n(x) = 1 - x^{n-1}. \lim_{n\to\infty} f'_n(x) &= 1, 0 \le x < 1 \\
&= 0, x = 1.
\end{aligned}
$$

The limit function of the sequence $\{f'_n\}$ is not continuous on $[0, 1]$.

As each f'_n is continuous on $[0, 1]$ and the limit function is not continuous on $[0, 1]$, the convergence of the sequence $\{f'_n\}$ is not uniform on $[0, 1]$.

Thus the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$ inspite of non-uniform convergence of the sequence $\{f'_n\}$ on $[0, 1]$ and our assertion is established.

Worked Examples (continued).

9. Show that the sequence $\{f_n\}$ where $f_n(x) = \frac{x}{1+nx^2}, 0 \leq x \leq 1$ converges uniformly to a function f but $\lim\limits_{n\to\infty} f'_n(x) = f'(x)$ is true if $x \neq 0$.

When $x = 0$, the sequence converges to 0.
When $0 < x \leq 1$, $\lim\limits_{n\to\infty} f_n(x) = 0$.

Therefore the sequence $\{f_n\}$ converges to the function f on $[0,1]$ where $f(x) = 0, 0 \leq x \leq 1$.

Let $M_n = \sup\limits_{x\in[0,1]} \mid f_n(x) - f(x) \mid$. Then $M_n = \sup\limits_{x\in[0,1]} \dfrac{x}{1+nx^2}$.
$M_n = \frac{1}{2\sqrt{n}}$ [worked Ex.5, 13.3.] and therefore $\lim\limits_{n\to\infty} M_n = 0$.
This proves that the sequence $\{f_n\}$ is uniformly convergent on $[0,1]$.

$f'_n(x) = \frac{1-nx^2}{(1+nx^2)^2}$.
When $x = 0$, $\lim\limits_{n\to\infty} f'_n(x) = 1$. When $0 < x \leq 1$, $\lim\limits_{n\to\infty} f'_n(x) = 0$.
But $f'(x) = 0, 0 \leq x \leq 1$.
Therefore $\lim\limits_{n\to\infty} f'_n(x) = 0 = f'(x)$, when $0 < x \leq 1$
and $\lim\limits_{n\to\infty} f'_n(x) = 1 \neq f'(x)$ when $x = 0$.

Miscellaneous Examples.

1. Let a sequence of functions $\{f_n\}$ be uniformly convergent on an interval I and each f_n be bounded on I. Prove that the sequence f_n is uniformly bounded on I.

[A sequence of functions $f_n(x)$ is said to be uniformly bounded on an interval I if there exists a constant B such $|f_n(x)| < B$ for all $x \in I$ and for all $n \in \mathbb{N}$.]

Let the sequence $\{f_n\}$ be uniformly convergent on I to the function f. Since each f_n is bounded on I and the the sequence $\{f_n\}$ is uniformly convergent on I to the function f, f is bounded on I. Therefore there exists positive real number k_1 such that $|f(x)| < k_1$ for all $x \in I$.

Let $M_n = \sup\limits_{x\in I} \mid f_n(x) - f(x) \mid$.

Since $\{f_n\}$ is uniformly convergent on I to the function f, $\lim M_n = 0$. Consequently, the sequence $\{M_n\}$ is a bounded sequence and therefore there exists positive real number k_2 such that $|M_n| < k_2$ for all $n \in \mathbb{N}$.
That is, $\sup\limits_{x\in I} |f_n(x) - f(x)| < k_2$ for all $n \in \mathbb{N}$.
Therefore for all $x \in I$, $|f_n(x)| < k_1 + k_2$ for all $n \in \mathbb{N}$.

This proves that the sequence f_n is uniformly bounded on I.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $\mathbb{R}$. For each natural number n, let $f_n(x) = f(x + \frac{1}{n}), x \in \mathbb{R}$.

Prove that the sequence $\{f_n\}$ is uniformly convergent on $\mathbb{R}$.

For all $x \in \mathbb{R}$, $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} f(x + \frac{1}{n})$

$= f(x)$, since f is continuous at x.

Therefore the sequence $\{f_n\}$ converges to the function f on $\mathbb{R}$.

Let $\epsilon > 0$. Since f is uniformly continuous on $\mathbb{R}$ there exists a positive δ such that

for all $x, u \in \mathbb{R}, | x - u |< \delta \Rightarrow | f(x) - f(u) |< \epsilon$ ... ... (i)

There exists a natural number k such that $0 < \frac{1}{n} < \delta$ for all $n \geq k$.

It follows from (i) that

for all $x \in \mathbb{R}, | f(x + \frac{1}{n}) - f(x) |< \epsilon$ for all $n \geq k$.

That is, for all $x \in \mathbb{R}, | f_n(x) - f(x) |< \epsilon$ for all $n > k$.

This proves that the sequence $\{f_n\}$ is uniformly convergent on $\mathbb{R}$ to the function f.

3. Let $\{f_n\}$ be a sequence of functions on an interval I that converges uniformly on I to a continuous function f. Let $c \in I$ and $\{x_n\}$ is any sequence in I converging to c. Prove that $\lim\limits_{n\to\infty} f_n(x_n) = f(c)$.

Let $\epsilon > 0$. Since the sequence $\{f_n\}$ is uniformly convergent on I, there exists a natural number k_1 such that

for all $x \in I$, $| f_n(x) - f(x) |< \frac{\epsilon}{2}$ for all $n \geq k_1$.

Since $x_n \in I$, $| f_n(x_n) - f(x_n) |< \frac{\epsilon}{2}$ for all $n \geq k_1$.

Since f is continuous at c and $\lim x_n = c$, $\lim f(x_n) = f(c)$. Therefore there exists a natural number k_2 such that

$| f(x_n) - f(c) |< \frac{\epsilon}{2}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $| f_n(x_n) - f(x_n) |< \frac{\epsilon}{2}$ and $| f(x_n) - f(c) |< \frac{\epsilon}{2}$ for all $n \geq k$.

By triangle inequality,

$| f_n(x_n) - f(c) |\leq| f_n(x_n) - f(x_n) | + | f(x_n) - f(c) |< \epsilon$ for all $n \geq k$.

This implies $\lim\limits_{n\to\infty} f_n(x_n) = f(c)$.

4. Let $\{f_n\}$ be a sequence of differentiable functions on $[a, b]$ that converges on $[a, b]$ to f. If $\{f'_n\}$ be a sequence of continuous functions on $[a, b]$ that converges uniformly to g on $[a, b]$ then show that $g(x) = f'(x)$ for all $x \in [a, b]$.

Since $\{f'_n\}$ is a uniformly convergent sequence of continuous functions on $[a, b]$, the limit function g is continuous on $[a, b]$.

Since each f_n' is continuous on $[a,b]$, each f_n' is integrable on $[a,b]$.

Since $\{f_n'\}$ is uniformly convergent on $[a,b]$ to the function g, by the corollary of the theorem 13.4.5,

for all $x \in [a,b]$, $\lim\limits_{n\to\infty} \int_a^x f_n'(t)dt = \int_a^x g(t)dt$.

But $\int_a^x f_n'(t)dt = f_n(x) - f_n(a)$, by the fundamental theorem.

$\lim\limits_{n\to\infty} \int_a^x f_n'(t)dt = \lim\limits_{n\to\infty} \{f_n(x) - f_n(a)\} = f(x) - f(a)$.

Therefore for all $x \in [a,b]$, $f(x) - f(a) = \int_a^x g(t)dt$.

Since g is continuous on $[a,b]$, by the corollary of the theorem 11.8.2, $f'(x) = g(x)$ for all $x \in [a,b]$.

Exercises 24

1. Let $\{u_n\}$ and $\{v_n\}$ be sequences of functions uniformly convergent on $[a,b]$ to the limit functions u and v respectively. Prove that the sequence $\{u_n + v_n\}$ converges uniformly on $[a,b]$ to the limit function $u+v$.

2. For each $n \in \mathbb{N}$, let $f_n(x) = x - \frac{1}{n}, g_n(x) = x + \frac{2}{n}$ on $[0,\infty)$.

Show that the sequences $\{f_n\}$ and $\{g_n\}$ are uniformly convergent on $[0,\infty)$ but the sequence $\{f_n g_n\}$ is not so.

3. Define uniform convergence of a sequence of functions $\{f_n\}$ on an interval I. Use the definition to examine uniform convergence of the sequence $\{f_n\}$ on $[0,\infty)$, where

(i) $f_n(x) = \frac{x}{x+n}, x \in [0,\infty)$;

(ii) $f_n(x) = xe^{-nx}, x \in [0,\infty)$;

(iii) $f_n(x) = x^2e^{-nx}, x \in [0,\infty)$;

(iv) $f_n(x) = n^2x^2e^{-nx}, x \in [0,\infty)$.

4. Let $a < c < b$. Let $\{f_n\}$ be a sequence of functions converging uniformly on $[a,c]$ and $[c,b]$. Prove that $\{f_n\}$ converges uniformly on $[a,b]$.

5. Prove that a sequence of functions $\{f_n\}$ is uniformly convergent on $[a,b]$ to a function f if and only if $\lim\limits_{n\to\infty} M_n = 0$, where

$$M_n = \sup_{x\in[a,b]} | f_n(x) - f(x) | .$$

Utilise this to examine uniform convergence of the sequence $\{f_n\}$ on $[0,1]$:

(i) for each $n \in \mathbb{N}, f_n(x) = \frac{x}{1+nx^2}, x \in [0,1]$;

(ii) for each $n \in \mathbb{N}, f_n(x) = \frac{nx}{1+n^3x^2}, x \in [0,1]$;

(iii) for each $n \in \mathbb{N}$, $f_n(x) = \frac{n^2 x}{1+n^4 x^2}$, $x \in [0,1]$;

(iv) for each $n \in \mathbb{N}$, $f_n(x) = nx(1-x)^n$, $x \in [0,1]$;

(v) For each $n \in \mathbb{N}$, $f_n(x) = \frac{x}{1+nx}$, $x \in [0,1]$;

(vi) For each $n \in \mathbb{N}$, $f_n(x) = xe^{-nx}$, $x \in [0,\infty)$.

6. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and $f_n(x) = e^{-nx}$, $x \in D$.

(i) Show that the sequence $\{f_n\}$ converges to a function f on D.

(ii) Show that f is not continuous on D. Deduce that the convergence of the sequence $\{f_n\}$ is not uniform on D.

(iii) Show that the convergence of the sequence $\{f_n\}$ is uniform on $[a,\infty)$, if $a > 0$.

7. Let $f_n(x) = \frac{nx}{1+nx}$, $x \in [0,1]$.

(i) Show that the sequence $\{f_n\}$ converges to a function f on $[0,1]$.

(ii) Show that f is not continuous on $[0,1]$. Deduce that the convergence of the sequence is not uniform on $[0,1]$.

8. For each $n \in \mathbb{N}$, let
$$f_n(x) = nx, 0 \leq x \leq 1/n.$$
$$= 1, \tfrac{1}{n} < x \leq 1.$$

(i) Show that the sequence $\{f_n\}$ converges to a function f on $[0,1]$.

(ii) Show that f is not continuous on $[0,1]$. Deduce that the convergence of the sequence $\{f_n\}$ is not uniform on $[0,1]$.

9. For each $n \in \mathbb{N}$, let
$$f_n(x) = nx^2, 0 \leq x \leq 1/n$$
$$= x, \tfrac{1}{n} < x \leq 1.$$

(i) Show that the sequence $\{f_n\}$ converges to a function f on $[0,1]$.

(ii) Find M_n, where $M_n = \sup\limits_{x\in[0,1]} \mid f_n(x) - f(x) \mid$. Show that the sequence $\{f_n\}$ is uniformly convergent on $[0,1]$.

10. For each $n \geq 2$, let
$$f_n(x) = n^2 x, \ 0 \leq x \leq \tfrac{1}{n}$$
$$= -n^2 x + 2n, \ \tfrac{1}{n} < x < \tfrac{2}{n}$$
$$= 0, \ \tfrac{2}{n} \leq x \leq 1.$$

(i) Show that the sequence $\{f_n\}_2^\infty$ converges to a function f on $[0,1]$;

(ii) Show that the convergence of the sequence is not uniform on $[0,1]$ by establishing that $\lim\limits_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$.

[**Hint.** If $c \in (0,1)$ there exists a natural number $p > 2$ such that $0 < \frac{2}{p} < c$ and therefore $\frac{2}{n} < c < 1$ for all $n \geq p$. $f_p(c) = 0$ and $f_n(c) = 0$ for all $n \geq p$. Therefore $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in (0,1)$ and for all $n > 2$.]

11. Show that the sequence of functions f_n defined on $[0,1]$ by

$$f_n(x) = n(1 - nx),\ 0 \le x < \tfrac{1}{n}$$
$$= 0,\ \tfrac{1}{n} \le x \le 1$$

converges to the function f given by $f(x) = 0$, $x \in [0, 1]$.

Show that $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 f(x)dx$. Is the convergence of the sequence uniform ?

[**Hint.** If $c \in (0, 1)$ there exists a natural number m such that $0 < \frac{1}{m} < c$ and therefore $\frac{1}{n} < c < 1$ for all $n \ge m$. $f_m(c) = 0$ and $f_n(c) = 0$ for all $n \ge m$. Therefore $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in (0, 1)$.]

12. For each $n \in \mathbb{N}$, let $f_n(x) = \lim\limits_{m\to\infty} (\cos n!\pi x)^{2m}$.

(i) Show that the sequence $\{f_n\}$ converges on $\mathbb{R}$ to the function f defined by $f(x) = 1, x \in \mathbb{Q}$
$= 0, x \in \mathbb{R} - \mathbb{Q}$.

(ii) If $[a, b]$ be a closed and bounded interval, show that each f_n is integrable on $[a, b]$.

Deduce that the sequence $\{f_n\}$ is not uniformly convergent on $[a, b]$.

[**Hint.** If $x \in \mathbb{R} - \mathbb{Q}, 0 < \cos^2 n!\pi x < 1$ and $\lim\limits_{m\to\infty} (\cos n!\pi x)^{2m} = 0$.

If $x = \frac{p}{q}$, where p, q are integers and $q \ge 1$ then $(\cos n!\pi x)^{2m} = 1$ if $n \ge q$.]

13. Let $f_n(x) = n^2 x(1 - x^2)^n, 0 \le x \le 1$. Show that

(i) the sequence $\{f_n\}$ converges to a function f on $[0, 1]$;

(ii) the sequence is not uniformly convergent on $[0, 1]$ by establishing that $\lim\limits_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$.

14. Let $f_n(x) = \frac{nx}{1+nx}, x \in [0, 1]$. Show that

(i) the sequence $\{f_n\}$ converges to a function f on $[0, 1]$;

(ii) f is integrable on $[0, 1]$ and $\lim\limits_{n\to\infty} \int_0^1 f_n = \int_0^1 f$, but still the convergence of the sequence is not uniform on $[0, 1]$.

15. Let $f_n(x) = nx(1 - x)^n, x \in [0, 1]$. Show that

(i) the sequence $\{f_n\}$ converges to a function f on $[0, 1]$;

(ii) f is integrable on $[0, 1]$ and $\lim\limits_{n\to\infty} \int_0^1 f_n = \int_0^1 f$, but still the convergence of the sequence is not uniform on $[0, 1]$.

16. Let $\{f_n\}$ be a sequence of continuous functions on $[a, b]$ and the sequence is uniformly convergent on $[a, b]$. Let $g_n(x) = \int_a^x f_n(x)dx, a \le x \le b$.

Prove that the sequence $\{g_n\}$ is uniformly convergent on $[a, b]$.

[**Hint.** $g_n'(x) = f_n(x), x \in [a, b]$. $\{g_n'\}$ is uniformly convergent on $[a, b]$ and the sequence $\{g_n\}$ is convergent at a.]

17. Let $f_n(x) = \frac{x}{1+nx}, 0 \le x \le 1$. Show that

(i) the sequence $\{f_n\}$ converges uniformly to a function on $[0, 1]$;

(ii) the sequence $\{f_n'\}$ converges to a function g on $[0, 1]$ and $f'(0) \neq g(0)$.

18. Let $f_n(x) = \frac{x^n}{n}, 0 \le x \le 1$. Show that

(i) the sequence $\{f_n\}$ converges uniformly to a function f on $[0, 1]$;

(ii) the sequence $\{f_n'\}$ converges to a function g on $[0, 1]$ and $f'(x) = g(x), x \in (0, 1)$, $f'(1) \neq g(1)$.

19. Show that the sequence of functions f_n defined on $[-1, 1]$ by $f_n(x) = |x|^{1+\frac{1}{n}}$, $x \in [-1, 1]$ converges uniformly to the function given by $f(x) = |x|$, $x \in [-1, 1]$ but the convergence of the sequence $\{f_n'\}$ is not uniform on $[-1, 1]$.

20. Let $f_n(x) = \log(n^2 + x^2), x \in \mathbb{R}$. Show that

(i) the sequence $\{f_n'\}$ is uniformly convergent on $\mathbb{R}$;

(ii) the sequence $\{f_n\}$ is not uniformly convergent on $\mathbb{R}$.

21. Let $f_n(x) = \frac{\log(1+n^2x^2)}{n^2}, x \in [0, 1]$. Show that

(i) the sequence $\{f_n'\}$ is uniformly convergent on $[0, 1]$;

(ii) the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$.

22. Let $f_n(x) = n + \frac{x}{n}, x \in \mathbb{R}$. Show that

(i) the sequence $\{f_n'\}$ is uniformly convergent on $\mathbb{R}$;

(ii) the sequence $\{f_n\}$ is not uniformly convergent on $\mathbb{R}$.

Explain why the sequence $\{f_n\}$ is not uniformly convergent on $\mathbb{R}$ although (i) is satisfied.

14. SERIES OF FUNCTIONS

14.1. Uniform convergence.

Let $D \subset \mathbb{R}$. Let $\{f_n\}$ be a sequence of functions on D to $\mathbb{R}$. Then $f_1 + f_2 + f_3 + \cdots$ is said to be *a series of functions* on D. The infinite series is denoted by Σf_n (or by $\sum\limits_1^\infty f_n$).

Let the sequence of functions $\{s_n\}$ be defined for $x \in D$ by

$$\begin{aligned} s_1(x) &= f_1(x), \\ s_2(x) &= f_1(x) + f_2(x), \\ s_3(x) &= f_1(x) + f_2(x) + f_3(x), \\ &\cdots \quad \cdots \\ s_n(x) &= f_1(x) + f_2(x) + \cdots + f_n(x), \\ &\cdots \quad \cdots \end{aligned}$$

The sequence $\{s_n\}$ is said to be the sequence of *partial sums* of the infinite series Σf_n.

If the sequence $\{s_n\}$ be pointwise convergent on D to a function s then the series Σf_n is said to be *pointwise convergent* on D and s is said to be the *sum function* of the series Σf_n on D.

If the sequence $\{s_n\}$ be uniformly convergent on D to a function s then the series Σf_n is said to be *uniformly convergent* on D to the sum function s.

If the series $\Sigma \mid f_n(x) \mid$ converges for each $x \in D$, then the series Σf_n is said to be *absolutely convergent* on D.

Note. We shall use the symbol Σf_n (or $\sum\limits_1^\infty f_n$) to denote either the series of functions $f_1 + f_2 + f_3 + \cdots \cdots$, or the sum of the series, when it exists.

Worked Examples.

1. Prove that the series of functions $1 + x + x^2 + \cdots, 0 \le x < 1$ is convergent on $0 \le x < 1$, but the convergence is not uniform on $[0, 1)$.

Let $s_n(x) = 1 + x + x^2 + \cdots + x^{n-1}$, $0 \le x < 1$.
Then $s_n(x) = \frac{1-x^n}{1-x}, x \in [0, 1)$. $\lim\limits_{n \to \infty} s_n(x) = \frac{1}{1-x}, x \in [0, 1)$.

The sequence $\{s_n\}$ converges on $[0,1)$ to the function s where $s(x) = \frac{1}{1-x}, x \in [0,1)$.

The series Σf_n is, therefore, pointwise convergent on $[0,1)$ to the sum function s.

Each s_n is bounded on $[0,1)$ but the limit function s is not bounded on $[0,1)$.

Hence the convergence of the sequence $\{s_n\}$ is not uniform on $[0,1)$ and, by definition, Σf_n is not uniformly convergent on $[0,1)$.

2. Prove that the series of functions
$\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + \cdots, x \geq 0$
is convergent on $[0,\infty)$ but the convergence is not uniform on $[0,\infty)$.

Let $s_n(x) = \frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \cdots + \frac{x}{[(n-1)x+1](nx+1)]}$.

$$\begin{aligned}\text{Then } s_n(x) &= (1 - \tfrac{1}{x+1}) + (\tfrac{1}{x+1} - \tfrac{1}{2x+1}) + \cdots \\ &\quad + (\tfrac{1}{(n-1)x+1} - \tfrac{1}{nx+1}) \\ &= 1 - \tfrac{1}{nx+1} = \tfrac{nx}{nx+1}, x \in [0,\infty).\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} s_n(x) &= 0, x = 0 \\ &= 1, x > 0.\end{aligned}$$

Therefore the sequence $\{s_n\}$ converges on $[0,\infty)$ to the function s where $s(x) = 0, x = 0,$
$= 1, x > 0.$

The function s is not continuous on $[0,\infty)$ but each s_n is continuous on $[0,\infty)$. This implies that the convergence of the sequence $\{s_n\}$ is not uniform on $[0,\infty)$.

By definition, the convergence of the given series is not uniform on $[0,\infty)$.

3. Prove that the series $x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \cdots$, $x \in [0,1]$ is not uniformly convergent on $[0,1]$.

Let the series be $f_1(x) + f_2(x) + f_3(x) + \cdots$

Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$.

Then $s_n(x) = x^4[1 + \frac{1}{(1+x^4)} + \cdots + \frac{1}{(1+x^4)^{n-1}}]$.

When $x = 0, s_n(x) = 0$.

When $0 < x \leq 1, s_n(x) = (1+x^4).[1 - \frac{1}{(1+x^4)^n}]$.

$$\begin{aligned}\lim_{n\to\infty} s_n(x) &= 0,\ x = 0 \\ &= 1 + x^4,\ 0 < x \leq 1.\end{aligned}$$

The sequence $\{s_n\}$ converges to the function s where

$$\begin{aligned}s(x) &= 0, \quad x = 0 \\ &= 1 + x^4, 0 < x \leq 1.\end{aligned}$$

s is not continuous on $[0, 1]$, the point of discontinuity being 0. Each s_n is continuous on $[0, 1]$. Therefore the convergence of the sequence $\{s_n\}$ is not uniform on $[0, 1]$.

By definition, the convergence of the series is not uniform on $[0, 1]$.

Theorem 14.1.1. (Cauchy's principle of convergence)

Let $D \subset \mathbb{R}$ and Σf_n be a series of functions on D to $\mathbb{R}$. The series Σf_n is uniformly convergent on D if and only if for a pre-assigned positive ϵ there exists a natural number k such that for all $x \in D$,
$|f_{n+1}(x)+f_{n+2}(x)+\cdots+f_{n+p}(x)| < \epsilon$ for all $n \geq k$ and for $p = 1, 2, 3, \ldots$

Proof. Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x), x \in D$.

Let the series Σf_n be uniformly convergent on D. Then the sequence of functions $\{s_n\}$ is uniformly convergent on D.

By Cauchy's principle for the sequence, for a pre-assigned positive ϵ there exists a natural number k such that for all $x \in D$,
$| s_{n+p}(x) - s_n(x) | < \epsilon$ for all $n \geq k$ and for $p = 1, 2, 3, \ldots$

That is, for all $x \in D$, $| f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x) | < \epsilon$ for all $n \geq k$ and for $p = 1, 2, 3, \ldots$ showing that the condition is necessary.

Conversely, let the condition hold.

Then for all $x \in D$, $| s_{n+p}(x) - s_n(x) | < \epsilon$ for all $n \geq k$ and for $p = 1, 2, 3, \ldots$.

By Cauchy's principle of convergence for the sequence, the sequence $\{s_n\}$ is uniformly convergent on D and by definition, the series of functions Σf_n is uniformly convergent on D.

Worked Examples (continued).

4. Let a series of functions Σf_n be uniformly convergent on the intervals $[a, c]$ and $[c, b]$. Show that the series is uniformly convergent on $[a, b]$.

Let us choose $\epsilon > 0$.

Since Σf_n is uniformly convergent on $[a, c]$, there exists a natural number k_1 such that for all $x \in [a, c]$,
$| f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x) | < \epsilon$ for all $n \geq k_1, p = 1, 2, 3, \ldots$.

Since Σf_n is uniformly convergent on $[c, b]$, there exists a natural number k_2 such that for all $x \in [c, b]$,
$| f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x) | < \epsilon$ for all $n \geq k_2, p = 1, 2, 3, \ldots$

Let $k = \max\{k_1, k_2\}$. Then for all $x \in [a, b]$,
$| f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x) | < \epsilon$ for all $n \geq k, p = 1, 2, 3, \ldots$

This proves that Σf_n is uniformly convergent on $[a, b]$.

Theorem 14.1.2. (Weierstrass' M-test)

Let $D \subset \mathbb{R}$ and Σf_n be a series of functions on D to $\mathbb{R}$.

Let $\{M_n\}$ be a sequence of positive real numbers such that for all $x \in D, | f_n(x) | \leq M_n$ for all $n \in \mathbb{N}$. If the series ΣM_n be convergent then the series Σf_n is uniformly and absolutely convergent on D.

Proof. Let us choose $\epsilon > 0$.

Since ΣM_n is convergent, there exists a natural number k (by Cauchy's principle) such that

$| M_{n+1} + M_{n+2} + \cdots + M_{n+p} | < \epsilon$ for all $n \geq k, p = 1, 2, 3, \ldots$

For all $x \in D$,
$$\begin{aligned} & | f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x) | \\ \leq & | f_{n+1}(x) | + | f_{n+2}(x) | + \cdots + | f_{n+p}(x) | \\ \leq & M_{n+1} + M_{n+2} + \cdots + M_{n+p} \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Hence for all $x \in D, | f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x) | < \epsilon$ for all $n \geq k, p = 1, 2, 3, \ldots$.

By Cauchy's principle, the series Σf_n is uniformly convergent on D.

Again, for all $x \in D$,
$$\begin{aligned} & \big| \, |f_{n+1}(x)| + |f_{n+2}(x)| + \cdots + |f_{n+p}(x)| \, \big| \\ = & | f_{n+1}(x) | + | f_{n+2}(x) | + \cdots + | f_{n+p}(x) | \\ \leq & M_{n+1} + M_{n+2} + \cdots + M_{n+p}. \end{aligned}$$

Therefore for all $x \in D$, $\big| \, |f_{n+1}(x)| + \cdots + |f_{n+p}(x)| \, \big|$
$< \epsilon$ for all $n \geq k, p = 1, 2, 3, \ldots$

By Cauchy's principle, $\Sigma | f_n |$ is convergent on D.

This implies that the series Σf_n is absolutely convergent on D.

Worked Examples (continued).

5. Prove that the series $\Sigma \frac{1}{n^3 + n^4 x^2}$ is uniformly convergent for all real x.

Let $f_n(x) = \frac{1}{n^3 + n^4 x^2}, x \in \mathbb{R}$.

For all $x \in \mathbb{R}, f_n(x) \leq \frac{1}{n^3}$. This holds for all $n \in \mathbb{N}$.

Let $M_n = \frac{1}{n^3}$. Then for all $x \in \mathbb{R}, | f_n(x) | \leq M_n$ for all $n \in \mathbb{N}$.

The series ΣM_n is a convergent series of positive real numbers.

By Weierstrass' M-test, the series Σf_n is uniformly convergent on $\mathbb{R}$.

6. Prove that the series $\Sigma \frac{x}{n + n^2 x^2}$ is uniformly convergent for all real x.

Let $f_n(x) = \frac{x}{n + n^2 x^2}$.

When $x = 0, f_n(x) = 0$.

When $x \neq 0, \frac{n}{|x|} + n^2|x| \geq 2n^{3/2}$, the equality occurs when $| x | = \frac{1}{\sqrt{n}}$

or, $| f_n(x) | \leq \frac{1}{2n^{3/2}}$, the equality occurs when $| x | = \frac{1}{\sqrt{n}}$.

It follows that $| f_n(x) | \leq \frac{1}{2n^{3/2}}$ for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$.

Let $M_n = \frac{1}{2n^{3/2}}$. Then for all real x, $| f_n(x) | \leq M_n$ for all $n \in \mathbb{N}$. The series ΣM_n is a convergent series of positive real numbers.

By Weierstrass' M-test, Σf_n is uniformly convergent on $\mathbb{R}$.

7. Show that the series
$1 - \frac{e^{-2x}}{2^2-1} + \frac{e^{-4x}}{4^2-1} - \frac{e^{-6x}}{6^2-1} + \cdots$ converges uniformly for all $x \geq 0$.

Let $\sum\limits_{n=0}^{\infty} f_n$ be the given series.

Then $f_0(x) = 1, f_n(x) = (-1)^n \frac{e^{-2nx}}{4n^2-1}, n \geq 1$.

$$\begin{aligned} \text{For all } x \geq 0, | f_n(x) | &= \frac{e^{-2nx}}{4n^2-1}, \text{ for all } n \geq 1 \\ &\leq \frac{1}{4n^2-1}, \text{since } e^{2nx} \geq 1 \text{ for all } x \geq 0. \end{aligned}$$

Let $M_n = \frac{1}{4n^2-1}$. Then ΣM_n is a convergent series of positive real numbers and for all $x \geq 0$, $| f_n(x) | \leq M_n$ for all $n \geq 1$.

By Weierstrass' M-test, Σf_n is uniformly convergent for all $x \geq 0$.

14.2. Consequences of uniform convergence.

Theorem 14.2.1. Let D be a subset of $\mathbb{R}$ and a series of functions Σf_n be uniformly convergent on D to a function f. Let $x_0 \in D'$ (the derived set of D) and $\lim\limits_{x \to x_0} f_n(x) = a_n$. Then

(i) the series Σa_n is convergent, and

(ii) $\lim\limits_{x \to x_0} f(x)$ exists and equals Σa_n.

Proof. (i) Let us choose $\epsilon > 0$.

Since the series Σf_n is uniformly convergent on D, there exists a natural number k such that

for all $x \in D$, $|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)| < \frac{\epsilon}{2}$ for all $n \geq k$ and for $p = 1, 2, \ldots \ldots \ldots$ (i)

As $\lim\limits_{x \to x_0} f_n(x) = a_n$, it follows that

$\lim\limits_{x \to x_0} \{f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)\} = a_{n+1} + a_{n+2} + \cdots + a_{n+p}$
and therefore

$\lim\limits_{x \to x_0} |f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)| = |a_{n+1} + a_{n+2} + \cdots + a_{n+p}|$.

It follows from (i) that $|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| \leq \frac{\epsilon}{2} < \epsilon$ for all $n \geq k$ and for $p = 1, 2, \ldots$.

This shows that the series Σa_n is convergent, by Cauchy's principle.

(ii) Let $\Sigma a_n = s$. Let $b_n = a_1 + a_2 + \cdots + a_n$. Then $\lim b_n = s$.

Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$, $x \in D$.

Let us choose $\epsilon > 0$.

Since the series Σf_n converges uniformly on D to the function f, the sequence $\{s_n\}$ is uniformly convergent on D to the function f and therefore there exists a natural number k_1 such that for all $x \in D$, $|s_n(x) - f(x)| < \frac{\epsilon}{3}$ for all $n \geq k_1$.

Since $\lim b_n = s$, there exists a natural number k_2 such that $|b_n - s| < \frac{\epsilon}{3}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $|s_k(x) - f(x)| < \frac{\epsilon}{3}$ for all $x \in D$ and $|b_k - s| < \frac{\epsilon}{3}$.

Since $\lim\limits_{x \to x_0} f_n(x) = a_n$ for $n = 1, 2, \ldots$, $\lim\limits_{x \to x_0} s_k(x) = b_k$ and therefore there exists a positive δ such that

$| s_k(x) - b_k | < \frac{\epsilon}{3}$ for all $x \in N'(x_0, \delta) \cap D$.

By triangle inequality,

$$\begin{aligned} |f(x) - s| &\leq |f(x) - s_k(x)| + |s_k(x) - b_k| + |b_k - s| \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} (= \epsilon) \text{ for all } x \in N'(x_0, \delta) \cap D. \end{aligned}$$

This proves $\lim\limits_{x \to x_0} f(x) = s$. That is, $\lim\limits_{x \to x_0} f(x) = \Sigma a_n$.

Note. In consequence of uniform convergence of the series Σf_n, $\lim\limits_{x \to x_0} \Sigma f_n(x) = \Sigma \lim\limits_{x \to x_0} f_n(x)$. This means that the interchange of the symbols Σ and $\lim\limits_{x \to x_0}$ is permissible.

Corollary. Let I be an interval and a series of functions Σf_n be uniformly convergent on I to a function f. Let $c \in I$ and each f_n be continuous at c. Then f is continuous at c.

Worked Example.

1. Find $\lim\limits_{x \to 0} \sum\limits_{n=1}^{\infty} \frac{\cos nx}{n(n+1)}$.

Let $f_n(x) = \frac{\cos nx}{n(n+1)}$ for $n = 1, 2, \ldots$. Then $|f_n(x)| \leq \frac{1}{n(n+1)}$ for $n = 1, 2, \ldots$ and for all $x \in \mathbb{R}$.

Let $M_n = \frac{1}{n(n+1)}$ for $n = 1, 2, \ldots$. Then $|f_n(x)| \leq M_n$ for $n = 1, 2, \ldots$.

ΣM_n is a convergent series of positive real numbers and therefore by Weierstrass' M-test, $\sum\limits_{n=1}^{\infty} f_n(x)$ is uniformly convergent for all real x.

Since the series is uniformly convergent for all real x, $\lim\limits_{x \to 0} \sum\limits_{n=1}^{\infty} f_n(x) = \sum\limits_{n=1}^{\infty} \lim\limits_{x \to 0} f_n(x)$. That is, $\lim\limits_{x \to 0} f(x) = \sum\limits_{n=1}^{\infty} \lim\limits_{x \to 0} \frac{\cos nx}{n(n+1)} = \sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Let $t_n = \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)}$. Then $t_n = 1 - \frac{1}{n+1}$ and $\lim\limits_{n \to \infty} t_n = 1$.

This implies $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ and therefore $\lim\limits_{x \to 0} \sum\limits_{n=1}^{\infty} \frac{\cos nx}{n(n+1)} = 1$.

Theorem 14.2.2. Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}, f_n : D \to \mathbb{R}$ is a continuous function on D. If the series Σf_n be uniformly convergent on D then sum function s is continuous on D.

Proof. Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x), x \in D$.

Then each s_n is continuous on D.

Since the series Σf_n is uniformly convergent on D, the sequence $\{s_n\}$ is uniformly convergent on D to the function s.

Let us choose $\epsilon > 0$. Then there exists a natural number k such that for all $x \in D, |s_n(x) - s(x)| < \frac{\epsilon}{3}$ for all $n \geq k$.

It follows that for all $x \in D, |s_k(x) - s(x)| < \frac{\epsilon}{3}$ ... ... (i)

Let $c \in D$. Then (i) gives $| s_k(c) - s(c) |< \frac{\epsilon}{3}$.

Since s_k is continuous at c, there exists a positive δ such that $| s_k(x) - s_k(c) |< \frac{\epsilon}{3}$ for all $x \in \mathbb{N}(c, \delta) \cap D$.

By triangle inequality,

$$\begin{aligned} | s(x) - s(c) | &\leq |s(x) - s_k(x)| + |s_k(x) - s_k(c)| + |s_k(c) - s(c)| \\ &< \epsilon \text{ for all } x \in N(c, \delta) \cap D. \end{aligned}$$

This proves that s is continuous at c.

Since c is arbitrary, s is continuous on D.

Note 1. If for each $n \in \mathbb{N}, f_n$ is continuous on D and the sum function s of the series Σf_n is not continuous on D then it follows from the theorem that the convergence of the series Σf_n is not uniform on D.

Note 2. If each f_n be continuous on D, the uniform convergence of the series Σf_n on D is a sufficient but not a necessary condition for continuity of the sum function s on D.

For example, let $f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}, x \in \mathbb{R}$.

Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x), x \in \mathbb{R}$.

Then $s_n(x) = \frac{nx}{1+n^2x^2}$. $\lim\limits_{n \to \infty} s_n(x) = 0$ for all $x \in \mathbb{R}$.

The sequence $\{s_n\}$ converges to the function s where $s(x) = 0$, for all $x \in \mathbb{R}$.

The convergence of the sequence $\{s_n\}$ is not uniform on $\mathbb{R}$ by the Example 5 of 13.3 and therefore the convergence of the series Σf_n is not uniform on $\mathbb{R}$.

But each f_n is continuous on $\mathbb{R}$ and also the sum function s is continuous on $\mathbb{R}$.

This proves that for a convergent series of continuous functions, the uniform convergence of the series is not necessary for continuity of the sum function.

Worked Example (continued).

2. Show that the series $(1-x)+x(1-x)+x^2(1-x)+\cdots$ is not uniformly convergent on $[0,1]$.

Let the series be $\sum\limits_1^\infty f_n(x)$.

Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x), x \in [0,1]$.

Then $s_n(x) = (1-x)[1+x+x^2+\cdots x^{n-1}]$.

$$\begin{aligned} \lim s_n(x) &= 1, \text{ for } 0 \le x < 1 \\ &= 0, \text{ for } x = 1. \end{aligned}$$

The sequence $\{s_n\}$ converges to the function s

$$\begin{aligned} \text{where } s(x) &= 1, 0 \le x < 1 \\ &= 0, x = 1. \end{aligned}$$

Therefore the series Σf_n converges to the function s on $[0,1]$.

Each f_n is continuous on $[0,1]$ but s is not continuous on $[0,1]$.

This proves that the convergence of the series Σf_n is not uniform on $[0,1]$.

Theorem 14.2.3. Let $I = [a,b]$ be a closed and bounded interval and for each $n \in \mathbb{N}, f_n : I \to \mathbb{R}$ be intergrable on I. If the series Σf_n be uniformly convergent on I to the function s then

(i) s is integrable on I,

(ii) $\Sigma \int_a^b f_n(x)dx = \int_a^b s(x)dx$.

Proof. Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x), x \in [a,b]$.

Then each s_n is integrable on $[a,b]$.

Since the series Σf_n is uniformly convergent on $[a,b]$ to the function s, the sequence $\{s_n\}$ is uniformly convergent on $[a,b]$ to s.

By the Theorem 13.4.5, s is integrable on $[a,b]$ and $\lim\limits_{n\to\infty} \int_a^b s_n(x)dx = \int_a^b s(x)dx$.

But $\int_a^b s_n(x)dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx + \cdots + \int_a^b f_n(x)dx$.

Since $\lim\limits_{n\to\infty} \int_a^b s_n(x)dx = \int_a^b s(x)dx$, the series $\Sigma \int_a^b f_n(x)dx$ converges to $\int_a^b s(x)dx$. This completes the proof.

Note 1. The equality $\Sigma \int_a^b f_n(x) = \int_a^b s(x)dx$ can be expressed as

$$\sum_{k=1}^\infty \int_a^b f_k(x)dx = \int_a^b \sum_{k=1}^\infty f_k(x)dx.$$

Thus it is permissible to interchange the symbols $\sum\limits_{k=1}^\infty$ and $\int_a^b$ if each f_n be integrable on $[a,b]$ and the convergence of the series Σf_n be uniform on $[a,b]$.

Equivalently, this can be expressed as
$\int_a^b [f_1(x) + f_2(x) + \cdots]dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx + \cdots$.
That is, the series of functions can be integrated term-by-term on $[a, b]$ if the convergence of the series is uniform on $[a, b]$.

Note 2. If each f_n be integrable on $[a, b]$ and the series Σf_n converges to a function s which is not integrable on $[a, b]$, then it follows from the theorem that the convergence of the series is not uniform on $[a, b]$.

If each f_n be integrable on $[a, b]$ and the series Σf_n converges to a function s which is also integrable on $[a, b]$ but the series $\Sigma \int_a^b f_n(x)dx$ does not converge to $\int_a^b s(x)dx$, then it follows from the theorem that the convergence of the series is not uniform on $[a, b]$.

Note 3. If each f_n be integrable on $[a, b]$, the uniform convergence of the series is only a sufficient but not a necessary condition for the integrability of the sum function.

For example, let $f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}, x \in [0, 1]$.
Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x), x \in [0, 1]$.
Then $s_n(x) = \frac{nx}{1+n^2x^2}, x \in [0, 1]$.
The sequence $\{s_n\}$ converges to the function s where $s(x) = 0$.
But the convergence is not uniform on $[0, 1]$ by Example 5, 13.3.

Thus the series Σf_n is such that each f_n is integrable on $[0, 1]$ and it converges to the function s which is also integrable on $[0, 1]$ but the convergence of the series is not uniform on $[0, 1]$.

This proves that for a series of integrable functions on $[0, 1]$ the uniform convergence of the series is not necessary for integrability of the sum function.

Note 4. If each f_n be integrable on $[a, b]$, and the series Σf_n converges to a function s which is also integrable on $[a, b]$, the uniform convergence of the series is only a sufficient but not a necessary condition for term-by-term integration of the series on $[a, b]$.

For example, let $f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}, x \in [0, 1]$.
Each f_n is integrable on $[0, 1]$.
The series converges to the function s on $[0, 1]$ where $s(x) = 0, x \in [0, 1]$.
$\int_0^1 s(x)dx = 0$.
$\int_0^1 f_1(x)dx = \int_0^1 \frac{x}{1+x^2}dx = \frac{1}{2}\log 2$, and
for $n \geq 2$, $\int_0^1 f_n(x)dx = \frac{1}{2n}\log(1 + n^2) - \frac{1}{2(n-1)}\log(1 + (n-1)^2)$.

Let $t_n = \int_0^1 f_1(x)dx + \int_0^1 f_2(x)dx + \cdots + \int_0^1 f_n(x)dx$. Then $t_n = \frac{\log(1+n^2)}{2n}$ and $\lim\limits_{n\to\infty} t_n = 0$. Therefore $\int_0^1 s(x)dx = \lim\limits_{n\to\infty} t_n$.

or, $\int_0^1 \{f_1(x) + f_2(x) + \cdots\}dx = \int_0^1 f_1(x)dx + \int_0^1 f_2(x)dx + \cdots$

But the convergence of the sequence $\{s_n\}$ and hence the convergence of the series Σf_n is not uniform on $[0, 1]$.

Thus the series can be integrated term-by-term on $[0, 1]$, although the convergence of the series is not uniform on $[0, 1]$.

This establishes that uniform convergence of a series on $[a, b]$ is a sufficient but not a necessary condition for term -by-term integration of the series on $[a, b]$.

Worked Examples (continued).

3. For the series $\sum\limits_1^\infty f_n(x)$ where

$f_n(x) = n^2xe^{-n^2x^2} - (n-1)^2xe^{-(n-1)^2x^2}, x \in [0, 1]$ show that

$\sum\limits_1^\infty \int_0^1 f_n(x)dx \neq \int_0^1 (\sum\limits_1^\infty f_n(x))dx$.

Is the series $\sum\limits_1^\infty f_n(x)$ uniformly convergent on $[0, 1]$?

Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$. Then $s_n(x) = n^2xe^{-n^2x^2}$.

For all $x \in (0, 1]$, $e^{n^2x^2} > \frac{n^4x^4}{2} > 0$.

Therefore $0 < s_n(x) < \frac{2}{n^2x^3}$ for all $x \in (0, 1]$.

By Sandwich theorem, $\lim\limits_{n\to\infty} s_n(x) = 0$, for all $x \in (0, 1]$.

And for $x = 0$, the sequence $\{s_n\}$ converges to 0.

Hence the series $\sum\limits_1^\infty f_n(x)$ is convergent on $[0, 1]$ and the sum function f is given by $f(x) = 0, x \in [0, 1]$.

Therefore $\int_0^1 (\sum\limits_1^\infty f_n(x))dx = 0$.

$\int_0^1 f_n(x)dx = \frac{1}{2}[-e^{-n^2x^2} + e^{-(n-1)^2x^2}]_0^1 = \frac{1}{2}[e^{-(n-1)^2} - e^{-n^2}]$.

Let $t_n = \int_0^1 f_1(x)dx + \int_0^1 f_2(x)dx + \cdots + \int_0^1 f_n(x)dx$.

Then $t_n = \frac{1}{2}[1 - e^{-n^2}]$ and $\lim\limits_{n\to\infty} t_n = \frac{1}{2}$.

Therefore $\sum\limits_1^\infty \int_0^1 f_n(x)dx = \frac{1}{2} \neq \int_0^1 (\sum\limits_1^\infty f_n(x))dx$.

Note. It follows that the series $\sum\limits_1^\infty f_n(x)$ is not uniformly convergent on $[0, 1]$, since uniform convergence of the series implies the equality of $\sum\limits_1^\infty \int_0^1 f_n(x)dx$ and $\int_0^1 (\sum\limits_1^\infty f_n(x))dx$.

4. If $f(x)$ be the sum of the series $e^{-x} + 2e^{-2x} + 3e^{-3x} + \cdots, x > 0$ show that f is continuous for all $x > 0$. Evaluate $\int_{\log 2}^{\log 3} f(x)dx$.

Let the series be $\sum\limits_{n=1}^{\infty} f_n(x)$. Then $f_n(x) = ne^{-nx}$.

$| f_n(x) | = \frac{n}{e^{nx}} < \frac{6n}{n^3 x^3}$ for all $x > 0$.

Let $[a, b]$ be a closed and bounded interval $\subset (0, \infty)$.

For all $x \in [a, b]$, $| f_n(x) | < \frac{6}{a^3 n^3}$ for all $n \in \mathbb{N}$.

Let $M_n = \frac{6}{a^3 n^3}$. Then ΣM_n is a convergent series of positive real numbers.

By Weierstrass' M-test, Σf_n is uniformly convergent on $[a, b]$.

Let $c > 0$. Let us choose a positive δ such that $c - \delta > 0$.

Then Σf_n is uniformly convergent on $[c - \delta, c + \delta]$.

Since each f_n is continuous on $[c - \delta, c + \delta]$, the sum function f is continuous on $[c - \delta, c + \delta]$. Hence f is continuous at c. It follows that f is continuous for all $x > 0$.

Let $a = \log 2, b = \log 3$. Then $[a, b]$ is a closed and bounded interval $\subset (0, \infty)$.

Each f_n is integrable on $[a, b]$. Since the series is uniformly convergent on $[a, b]$,

$\int_a^b f(x)dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx + \int_a^b f_3(x)dx + \cdots$

$$\begin{aligned} \text{That is, } \int_{\log 2}^{\log 3} f(x)dx &= \int_{\log 2}^{\log 3} e^{-x}dx + \int_{\log 2}^{\log 3} 2e^{-2x}dx + \cdots \\ &= (\tfrac{1}{2} - \tfrac{1}{3}) + (\tfrac{1}{2^2} - \tfrac{1}{3^2}) + (\tfrac{1}{2^3} - \tfrac{1}{3^3}) + \cdots \\ &= (1 + \tfrac{1}{2} + \tfrac{1}{2^2} + \cdots) - (1 + \tfrac{1}{3} + \tfrac{1}{3^2} + \cdots) \\ &= 2 - \tfrac{3}{2} = \tfrac{1}{2}. \end{aligned}$$

Theorem 14.2.4. Let $[a, b]$ be a closed and bounded interval and for each $n \in \mathbb{N}$, let f_n be differentiable on $[a, b]$. If the series of functions $f_1' + f_2' + f_3' + \cdots$ converges uniformly on $[a, b]$ to a function g and the series $f_1 + f_2 + f_3 + \cdots$ converges *at least at one point* $x_0 \in [a, b]$, then the series $f_1 + f_2 + f_3 + \cdots$ converges uniformly on $[a, b]$ to a function s such that $s'(x) = g(x)$ for all $x \in [a, b]$.

Proof. Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x), x \in [a, b]$.

Each s_n is differentiable on $[a, b]$ and $s_n'(x) = f_1'(x) + f_2'(x) + \cdots + f_n'(x)$.

Since $\Sigma f_n'$ converges uniformly on $[a, b]$ to g, the sequence $\{s_n'\}$ converges uniformly to g on $[a, b]$.

Since Σf_n converges at x_0, the sequence $\{s_n\}$ converges at x_0.

Hence by the Theorem 13.4.6, the sequence $\{s_n\}$ is uniformly convergent on $[a, b]$ to a function s such that $s'(x) = g(x)$ for all $x \in [a, b]$.

This implies that the series Σf_n is uniformly convergent on $[a, b]$ to a function s such that $s'(x) = g(x)$ for all $x \in [a, b]$.

Note 1. The theorem says that under the conditions stated, $f_1(x) + f_2(x) + f_3(x + \cdots = s(x)$ for all $x \in [a, b]$ and

$\frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x) + \frac{d}{dx} f_3(x) + \cdots = \frac{d}{dx} s(x)$ for all $x \in [a, b]$.

That is, $\frac{d}{dx}[f_1(x)+f_2(x)+f_3(x)+\cdots] = \frac{d}{dx} f_1(x)+\frac{d}{dx} f_2(x)+\frac{d}{dx} f_3(x)+\cdots$ for all $x \in [a, b]$.

In other words, term-by-term differentiation of the series of functions is valid under the conditions.

Note 2. Only the uniform convergence of the series of functions $f_1(x) + f_2(x) + f_3(x) + \cdots$ on $[a, b]$ is not sufficient to ensure validity of term-by-term differentiation of the series on $[a, b]$.

For example, let the series be $f_1(x) + f_2(x) + f_3(x) + \cdots, x \in [0, 1]$ such that $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x) = \frac{x}{1+nx^2}, x \in [0, 1]$.

$\lim\limits_{n \to \infty} s_n(x) = \lim\limits_{n \to \infty} \frac{x}{1+nx^2}$
$= 0$ for all $x \in [0, 1]$.

The sequence $\{s_n\}$ converges to the function s where $s(x) = 0$.

Let $M_n = \sup\limits_{x \in [0,1]} | s_n(x) - s(x) |$. Then $M_n = \sup\limits_{x \in [0,1]} \frac{x}{1+nx^2}$.

$M_n = \frac{1}{2\sqrt{n}}$ [by Example 5, 13.3.2.] and therefore $\lim\limits_{n \to \infty} M_n = 0$.

This implies that the convergence of the sequence $\{s_n\}$ is uniform on $[0, 1]$. Hence the series $f_1(x) + f_2(x) + \cdots$ converges uniformly to the function s on $[0, 1]$.

$\frac{d}{dx} s_n(x) = \frac{1-nx^2}{(1+nx^2)^2}$. $\lim\limits_{n \to \infty} s'_n(x) = 0, 0 < x \leq 1$
$= 1, x = 0$.

Therefore the series $f'_1(x) + f'_2(x) + \cdots$ converges to the function g where $g(x) = 0, 0 < x \leq 1$
$= 1, x = 0$.

Hence $\frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x) + \cdots = 0 = \frac{d}{dx}[f_1(x) + f_2(x) + \cdots]$ for $0 < x \leq 1$; and $\frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x) + \cdots = 1 \neq \frac{d}{dx}[f_1(x) + f_2(x) + \cdots]$ for $x = 0$.

Note 3. If the series $f_1(x) + f_2(x) + f_3(x) + \cdots$ be convergent, then the uniform convergence of the series $f'_1(x) + f'_2(x) + f'_3(x) + \cdots$ is only a sufficient but not a necessary condition for the validity of term-by-term differentiation of the series $f_1(x) + f_2(x) + f_3(x) + \cdots$

For example, let the series be

$f_1(x) + f_2(x) + f_3(x) + \cdots, x \in [0, 1]$ such that
$s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x) = \frac{\log(1+n^2x^2)}{2n}, x \in [0, 1]$.
$\lim\limits_{n\to\infty} s_n(x) = \lim\limits_{n\to\infty} \frac{\log(1+n^2x^2)}{2n} = 0$, for all $x \in [0, 1]$.

The sequence $\{s_n\}$ converges to the function s where $s(x) = 0, x \in [0, 1]$. Hence the series $f_1(x) + f_2(x) + \cdots$ converges to the function s on $[0, 1]$.

$s_n'(x) = f_1'(x) + f_2'(x) + \cdots + f_n'(x) = \frac{nx}{1+n^2x^2}, x \in [0, 1]$.
$\lim\limits_{n\to\infty} s_n'(x) = 0$, for all $x \in [0, 1]$.

The sequence $\{s_n'\}$ converges to the function g where $g(x) = 0, x \in [0, 1]$. Hence the series $f_1'(x) + f_2'(x) + \cdots$ converges to the function $g(x)$ on $[0, 1]$.

Now $\frac{d}{dx}s(x) = 0, x \in [0, 1]$. Therefore $\frac{d}{dx}s(x) = g(x), x \in [0, 1]$.
Hence $\frac{d}{dx}f_1(x) + \frac{d}{dx}f_2(x) + \cdots = \frac{d}{dx}[f_1(x) + f_2(x) + \cdots]$.

This shows that term-by-term differentiation of the series Σf_n is valid.

But the convergence of the series $\Sigma f_n'$ is not uniform on $[0, 1]$ since the convergence of the sequence $\{s_n'\}$ is not uniform on $[0, 1]$ by worked Example 3, 13.3.2.

Theorem 14.2.5. Let $[a, b]$ be a closed and bounded interval and for each $n \in \mathbb{N}$, let f_n be differentiable on $[a, b]$. If each f_n' be continuous on $[a, b]$ and the series of functions $f_1' + f_2' + f_3' + \cdots$ converges uniformly on $[a, b]$ to a function g and the series $f_1 + f_2 + f_3 + \cdots$ converges to s on $[a, b]$, then $s'(x) = g(x)$ for all $x \in [a, b]$.

Proof. Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x), x \in [a, b]$.

Each s_n is differentiable on $[a, b]$ and $s_n'(x) = f_1'(x) + f_2'(x) + \cdots + f_n'(x)$.

Since each $f_n'(x)$ is continuous on $[a, b]$, each s_n' is continuous on $[a, b]$.

Since the series $\Sigma f_n'$ is uniformly convergent to g on $[a, b]$, the sequence $\{s_n'\}$ converges uniformly to g on $[a, b]$.

Since s_n' is continuous on $[a, b]$, g is continuous on $[a, b]$.

Therefore each s_n' is integrable on $[a, b]$ and g is also integrable on $[a, b]$.

By the corollary of the Theorem 13.4.5, for each $x \in [a, b]$
$\lim\limits_{n\to\infty} \int_a^x s_n'(x)dx = \int_a^x g(x)dx \quad \ldots \ldots \quad$ (i)
But $\int_a^x s_n'(x)dx = s_n(x) - s_n(a)$ by the fundamental theorem.
Therefore $\lim\limits_{n\to\infty} \int_a^x s_n'(x)dx = s(x) - s(a)$.
From (i) $s(x) - s(a) = \int_a^x g(x)dx$.
Since g is continuous on $[a, b]$, $s'(x) = g(x)$ for all $x \in [a, b]$.

Worked Examples (continued).

5. Let $f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}, x \in [0,1]$.

Show that at $x = 0$, $\frac{d}{dx}\Sigma f_n(x) \neq \Sigma \frac{d}{dx} f_n(x)$.

Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$.

Then $s_n(x) = \frac{nx}{1+n^2x^2}$. $\lim\limits_{n\to\infty} s_n(x) = 0$ for all $x \in [0,1]$.

The sequence $\{s_n\}$ converges to the function s where $s(x) = 0, x \in [0,1]$. Therefore the series $\Sigma f_n(x)$ converges to $s(x)$ for all $x \in [0,1]$.

$\frac{d}{dx}(\Sigma f_n(x)) = \frac{d}{dx}s(x) = 0$ for all $x \in [0,1]$.

$\frac{d}{dx} f_n(x) = \frac{n-n^2x^2}{(1+n^2x^2)} - \frac{(n-1)-(n-1)^2x^2}{[1+(n-1)^2x^2]^2}$.

At $x = 0$, $\frac{d}{dx} f_n(x) = n - (n-1) = 1$.

At $x = 0, \Sigma \frac{d}{dx} f_n(x) = 1 + 1 + 1 + \cdots$ This is divergent.

Hence $\frac{d}{dx}\Sigma f_n(x) \neq \Sigma \frac{d}{dx} f_n(x)$ at $x = 0$.

6. Show that the series $\Sigma \frac{1}{n^3+n^4x^2}$ is uniformly convergent for all real x. If $s(x)$ be the sum function verify that $s'(x)$ is obtained by term-by-term differentiation.

Let $f_n(x) = \frac{1}{n^3+n^4x^2}, x \in \mathbb{R}$.

For all $x \in \mathbb{R}, | f_n(x) | \leq \frac{1}{n^3}$ for all $n \in \mathbb{N}$.

Let $M_n = \frac{1}{n^3}$. Then ΣM_n is a convergent series of positive terms.

By Weierstrass' M-test, Σf_n is uniformly convergent for all real x.

$f_n'(x) = \frac{-2x}{n^2(1+nx^2)^2} = u(x)$, say. Then $u'(x) = \frac{2(3nx^2-1)}{n^2(1+nx^2)^3}$.

$u'(x) = 0$ at $x = \pm\frac{1}{\sqrt{3n}}, u'(x) < 0$ for $| x | < \frac{1}{\sqrt{3n}}, u'(x) > 0$ for $| x | > \frac{1}{\sqrt{3n}}$.

Therefore u is a minimum at $\frac{1}{\sqrt{3n}}$ and maximum at $= \frac{-1}{\sqrt{3n}}$.

$u_{\max} = \frac{9}{8\sqrt{3}} \cdot \frac{1}{n^{5/2}}, u_{\min} = \frac{-9}{8\sqrt{3}} \cdot \frac{1}{n^{5/2}}$

$u(0) = 0, u$ is decreasing on $(0, \frac{1}{\sqrt{3n}}), u$ is a minimum at $\frac{1}{\sqrt{3n}}, u$ is increasing for $x > \frac{1}{\sqrt{3n}}$ and $\lim\limits_{x\to\infty} u(x) = 0$.

Since u is an odd function, it follows that for all real x,

$$| f_n'(x) | \leq \frac{3\sqrt{3}}{8} \cdot \frac{1}{n^{5/2}}.$$

By Weierstrass' M-test, $\Sigma f_n'$ is uniformly convergent for all real x.

This ensures validity of term-by-term differentiation of the series $\Sigma f_n(x)$. Therefore $f_1'(x) + f_2'(x) + \cdots = s'(x)$.

14.3. Abel's and Dirichlet's tests.

Definition. A sequence of functions $\{f_n\}$ is said to be *uniformly bounded* on an interval I if there exists a positive real number k such that for all $x \in I$, $| f_n(x) | < k$ for all $n \in \mathbb{N}$.

For example, the sequence $\{f_n\}$ where $f_n(x) = \sin nx, x \in \mathbb{R}$ is uniformly bounded on $\mathbb{R}$.

The sequence $\{f_n\}$ where $f_n = \frac{nx}{1+x}, x \in [0, \infty)$ is not uniformly bounded on $[0, \infty)$ although each f_n is bounded on $[0, \infty)$.

Theorem 14.3.1. (Abel's test)

Let (i) the series of functions Σu_n be uniformly convergent on $[a, b]$

and (ii) the sequence of functions $\{v_n\}$ be monotonic for every $x \in [a, b]$ and uniformly bounded on $[a, b]$.

Then the series $u_1(x)v_1(x)+u_2(x)v_2(x)+u_3(x)v_3(x)+\cdots$ is uniformly convergent on $[a, b]$.

Proof. Since the sequence $\{v_n\}$ is uniformly bounded on $[a, b]$, there is a positive real number B such that $| v_n(x) | < B$ for all $[a, b]$ and for all $n \in \mathbb{N}$.

Let us choose $\epsilon > 0$. Since the series Σu_n is uniformly convergent on $[a, b]$, there exists a natural number k such that for all $x \in [a, b]$,

$| u_{n+1}(x) + u_{n+2}(x) + \cdots + u_{n+p}(x) | < \frac{\epsilon}{3B}$ for all $n \geq k, p = 1, 2, 3, \ldots$

Let $R_{n,p}(x) = u_{n+1}(x) + u_{n+2}(x) + \cdots + u_{n+p}(x)$.

Then $| u_{n+1}(x)v_{n+1}(x) + u_{n+2}(x)v_{n+2}(x) + \cdots + u_{n+p}(x)v_{n+p}(x) |$

$=| R_{n,1}(x)v_{n+1}(x) + \{R_{n,2}(x) - R_{n,1}(x)\}v_{n+2}(x) + \cdots + \{R_{n,p}(x) - R_{n,p-1}(x)\}v_{n+p}(x) |$

$=| R_{n,1}(x)\{v_{n+1}(x) - v_{n+2}(x)\} + R_{n,2}(x)\{v_{n+2}(x) - v_{n+3}(x)\} + \cdots + R_{n,p-1}(x)\{v_{n+p-1}(x) - v_{n+p}(x)\} + R_{n,p}(x).v_{n+p}(x) |$

$\leq| R_{n,1}(x) || v_{n+1}(x) - v_{n+2}(x) | + | R_{n,2}(x) || v_{n+2}(x) - v_{n+3}(x) | + \cdots + | R_{n,p-1}(x) || v_{n+p-1}(x) - v_{n+p}(x) | + | R_{n,p}(x) || v_{n+p}(x) |$

$< \frac{\epsilon}{3B}[| v_{n+1}(x)-v_{n+2}(x) | + | v_{n+2}(x)-v_{n+3}(x) | + \cdots | v_{n+p-1}(x) - v_{n+p}(x) | + | v_{n+p}(x) |]$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

Since $\{v_n\}$ is monotonic for every $x \in [a, b]$,

$| v_{n+1}(x) - v_{n+2}(x) | + \cdots + | v_{n+p-1}(x) - v_{n+p}(x) |$

$=| v_{n+1}(x) - v_{n+p}(x) | \leq| v_{n+1}(x) | + | v_{n+p}(x) |$.

Therefore $| u_{n+1}(x)v_{n+1}(x) + \cdots + u_{n+p}(x)v_{n+p}(x) | < \frac{\epsilon}{3B}.2B + \frac{\epsilon}{3B}.B$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

Hence for all $x \in [a, b]$, $| u_{n+1}(x)v_{n+1}(x) + \cdots + u_{n+p}(x)v_{n+p}(x) | < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

This proves that the series $u_1(x)v_1(x) + u_2(x)v_2(x) + \cdots$ is uniformly convergent on $[a, b]$.

Worked Examples.

1. If $a_0 + a_1 + a_2 + \cdots$ be a convergent series of real numbers prove that the series $a_0 + a_1 x + a_2 x^2 + \cdots$ is uniformly convergent on $[0, 1]$.

Let $v_n(x) = x^n, x \in [0, 1]$.
Then $| v_n(x) | \leq 1$ for all $x \in [0, 1]$ and all $n \in \mathbb{N}$.
The sequence $\{v_n\}$ is monotonic for every fixed $x \in [0, 1]$.
The series $a_0 + a_1 + a_2 \cdots$ being a convergent series of real numbers is uniformly convergent on $[0, 1]$.
By Abel's test, the series $a_0 + a_1 x + a_2 x^2 + \cdots$ is uniformly convergent on $[0, 1]$.

2. If $a_1 + a_2 + a_3 + \cdots$ be a convergent series of real numbers prove that the series $a_1 + \frac{a_2}{2^x} + \frac{a_3}{3^x} + \cdots$ is uniformly convergent on $[0, \infty)$.

Let $v_n(x) = \frac{1}{n^x}, x \geq 0$.
For a fixed $x \in [0, \infty)$ the sequence $\{v_n\}$ is monotonic and for all $x \geq 0, | v_n(x) | = \frac{1}{n^x} \leq 1$ for all $n \in \mathbb{N}$.
The series $a_1 + a_2 + a_3 + \cdots$ being a convergent series of real numbers is uniformly convergent on $[0, \infty)$.
By Abel's test, the series $a_1 + \frac{a_2}{2^x} + \frac{a_3}{3^x} + \cdots$ is uniformly convergent on $[0, \infty)$.

3. Prove that the series $e^{-x} - \frac{e^{-2x}}{2} + \frac{e^{-3x}}{3} - \frac{e^{-4x}}{4} + \cdots$ is uniformly convergent on $[0, 1]$.

Let $v_n(x) = e^{-nx}, x \in [0, 1]$.
For each $x \in [0, 1]$, the sequence $\{v_n\}$ is monotonic, and for all $x \in [0, 1], | v_n(x) | = \frac{1}{e^{nx}} \leq 1$ for all $n \in \mathbb{N}$.
The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is a convergent series of real numbers and therefore it is uniformly convergent on $[0, 1]$.
By Abel's test, the series $1 - \frac{1}{2}e^{-2x} + \frac{1}{3}e^{-3x} - \frac{1}{4}e^{-4x} + \cdots$ is uniformly convergent on $[0, 1]$.

4. Prove that the series $\Sigma \frac{(-1)^{n-1} x^n}{n(1+x^n)}$ is uniformly convergent on $[0, 1]$.
Let $v_n(x) = \frac{x^n}{1+x^n}, x \in [0, 1]$.
Then $v_{n+1} - v_n = \frac{x^n(x-1)}{(1+x^n)(1+x^{n+1})} \leq 0$ for all $x \in [0, 1]$.

For each $x \in [0,1]$, the sequence $\{v_n\}$ is monotonic and for all $x \in [0,1], \mid v_n(x) \mid < 1$ for all $n \in \mathbb{N}$.

The series $\Sigma \frac{(-)^{n-1}}{n}$ is a convergent series of real numbers and therefore it is uniformly convergent on $[0,1]$.

By Abel's test, the series $\Sigma \frac{(-1)^{n-1} x^n}{n(1+x^n)}$ is uniformly convergent on $[0,1]$.

Theorem 14.3.2. (Dirichlet's test)

Let (i) the sequence of partial sums $\{s_n\}$ of the series of functions $u_1(x) + u_2(x) + u_3(x) + \cdots$ be uniformly bounded on $[a,b]$

(ii) the sequence of functions $\{v_n\}$ be monotonic for every $x \in [a,b]$,

(iii) the sequence $\{v_n\}$ be uniformly convergent to 0 on $[a,b]$.

Then the series of functions $u_1(x)v_1(x) + u_2(x)v_2(x) + u_3(x)v_3(x) + \cdots$ is uniformly convergent on $[a,b]$.

Proof. Since the sequence $\{s_n\}$ is uniformly bounded on $[a,b]$, there exists a positive number B such that for all $x \in [a,b], \mid s_n(x) \mid < B$ for all $n \in \mathbb{N}$.

Let us choose $\epsilon > 0$.

Since the sequence $\{v_n\}$ converges uniformly to 0 on $[a,b]$, there exists a natural number k such that for all $x \in [a,b], \mid v_n(x) \mid < \frac{\epsilon}{4B}$ for all $n \geq k$.

Now $u_{n+1}(x)v_{n+1}(x) + u_{n+2}(x)v_{n+2}(x) + \cdots + u_{n+p}(x)v_{n+p}(x)$

$= [s_{n+1}(x) - s_n(x)]v_{n+1}(x) + [s_{n+2}(x) - s_{n+1}(x)]v_{n+2}(x) + \cdots + [s_{n+p}(x) - s_{n+p-1}(x)].v_{n+p}(x)$

$= s_{n+1}(x)[v_{n+1}(x) - v_{n+2}(x)] + \cdots + s_{n+p-1}(x)[v_{n+p-1}(x) - v_{n+p}(x)] + s_{n+p}(x)v_{n+p}(x) - s_n(x)v_{n+1}(x)$.

For all $x \in [a,b], \mid u_{n+1}(x)v_{n+1}(x) + \cdots + u_{n+p}(x)v_{n+p}(x) \mid$

$\leq \mid s_{n+1}(x) \mid \mid v_{n+1}(x) - v_{n+2}(x) \mid + \cdots + \mid s_{n+p-1}(x) \mid \mid v_{n+p-1}(x) - v_{n+p}(x) \mid + \mid s_{n+p}(n) \mid \mid v_{n+p}(x) \mid + \mid s_n(x) \mid v_{n+1}(x) \mid$

$< B[\mid v_{n+1}(x) - v_{n+2}(x) \mid + \cdots + \mid v_{n+p-1}(x) - v_{n+p}(x) \mid + \mid v_{n+p}(x) \mid + \mid v_{n+1}(x) \mid]$.

Since $\{v_n\}$ is monotonic for every $x \in [a,b]$,
$\mid v_{n+1}(x) - v_{n+2}(x) \mid + \cdots + \mid v_{n+p-1}(x) - v_{n+p}(x) \mid$
$= \mid v_{n+1}(x) - v_{n+p}(x) \mid < \frac{\epsilon}{2B}$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

Therefore for all $x \in [a,b]$,
$\mid u_{n+1}(x)v_{n+1}(x) + \cdots + u_{n+p}(x)v_{n+p}(x) \mid < B.\frac{\epsilon}{2B} + 2B.\frac{\epsilon}{4B} (= \epsilon)$
for all $n \geq k$ and $p = 1, 2, 3, \ldots$

This proves that the series $u_1(x)v_1(x) + u_2(x)v_2(x) + \cdots$ is uniformly convergent on $[a, b]$.

Worked Examples (continued).

5. Prove that the series $\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots$ is uniformly convergent on any closed interval $[a, b]$ contained in the open interval $(0, 2\pi)$.

Let $u_n(x) = \sin nx, x \in [a, b] \subset (0, 2\pi)$; and $v_n = \frac{1}{n}$.
Then $\{v_n\}$ is a monotone decreasing sequence converging to 0.
Let $s_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$.

Then $s_n(x) = \sin x + \sin 2x + \cdots + \sin nx$
$= \frac{\sin \frac{n x}{2} \sin \frac{(n+1)x}{2}}{\sin \frac{x}{2}}$.

For each $n \in \mathbb{N}$, $| s_n(x) | \leq | \frac{1}{\sin x/2} |$.

$\sin \frac{x}{2} \neq 0$ for all $x \in [a, b] \subset (0, 2\pi)$. Therefore the function f defined by $f(x) = \frac{1}{\sin \frac{x}{2}}, x \in [a, b]$ is continuous on $[a, b]$ and therefore it is bounded on $[a, b]$. Then there exists a positive real number k such that $| \frac{1}{\sin \frac{x}{2}} | \leq k$ for all $x \in [a, b] \subset (0, 2\pi)$.

This shows that the sequence $\{s_n\}$ is uniformly bounded on $[a, b]$.

By Dirichlet's test, the series $\Sigma u_n v_n$, i.e., the series $\Sigma \frac{\sin nx}{n}$ is uniformly convergent on $[a, b] \subset (0, 2\pi)$.

6. Prove that the series $\Sigma(-1)^n x^n(1 - x)$ converges uniformly on $[0, 1]$, but the series $\Sigma x^n(1 - x)$ is not uniformly convergent on $[0, 1]$.

Let $u_n = (-1)^n, v_n = x^n(1 - x), x \in [0, 1]$.
Let $s_n = u_1 + u_2 + \cdots + u_n$. Then the sequence $\{s_n\}$ is bounded.

$v_{n+1} - v_n = x^{n+1}(1 - x) - x^n(1 - x) = -x^n(1 - x)^2 \leq 0$ for all $x \in [0, 1]$.

This implies $v_{n+1}(x) \leq v_n(x)$ for each x in $[0, 1]$.

$\lim_{n \to \infty} v_n(x) = 0$ for all $x \in [0, 1]$.

Then the sequence of functions $\{v_n\}$ is such that each v_n is continuous on $[0, 1]$, the sequence converges to a continuous function on $[0, 1]$ and $\{v_n\}$ is a monotone decreasing sequence on $[0, 1]$.

By Dini's theorem, the sequence $\{v_n\}$ is uniformly convergent on $[0, 1]$.

Since (i) the sequence $\{s_n\}$ is bounded and (ii) the sequence $\{v_n\}$ is a monotone decreasing sequence on $[0, 1]$ converging uniformly to 0, the series $\Sigma(-1)^n v_n$ is uniformly convergent on $[0, 1]$, by Dirichlet's test.

Second part. Let $v_n(x) = x^n(1-x), x \in [0,1]$.

Then $v_1(x) + v_2(x) + \cdots + v_n(x) = (x + x^2 + \cdots + x^n)(1-x)$.

Let $s_n(x) = v_1(x) + \cdots + v_n(x)$. Then $s_n(x) = x(1-x^n)$ if $x \neq 1$
$= 0$ if $x = 1$.

$\lim\limits_{n\to\infty} s_n(x) = x$ if $x \neq 1$
$= 0$ if $x = 1$.

The sequence $\{s_n\}$ converges to a function discontinuous on $[0,1]$, while each s_n is continuous on $[0,1]$.

This implies that the sequence $\{s_n\}$ not uniformly convergent on $[0,1]$ and consequently the series Σv_n is not uniformly convergent on $[0,1]$.

Note. This example shows that the uniform convergence of a series $\Sigma f_n(x)$ does not necessarily imply uniform convergence of the series $\Sigma \mid f_n(x) \mid$.

7. Prove that the series $\Sigma(-1)^n \frac{x^2+n}{n^2}$ converges uniformly in any closed and bounded interval $[a,b]$, but does not converge absolutely for any real x.

Let $u_n = (-1)^n, v_n = \frac{x^2+n}{n^2}, x \in [a,b]$.

Let $s_n = u_1 + u_2 + \cdots + u_n$. Then the sequence $\{s_n\}$ is bounded.

$v_{n+1} - v_n = \frac{x^2+n+1}{(n+1)^2} - \frac{x^2+n}{n^2} = x^2[\frac{1}{(n+1)^2} - \frac{1}{n^2}] + [\frac{1}{n+1} - \frac{1}{n}] < 0$ for all $x \in [a,b]$.

This shows that $\{v_n\}$ is a monotone decreasing sequence for each x in $[a,b]$.

$\lim\limits_{n\to\infty} v_n(x) = 0$ for all $x \in [a,b]$.

Thus the sequence of functions $\{v_n\}$ is such that each v_n is continuous on $[a,b]$, the sequence converges to a continuous function on $[a,b]$ and $\{v_n\}$ is a monotone decreasing sequence on $[a,b]$.

By Dini's theorem, the sequence $\{v_n\}$ is uniformly convergent on $[a,b]$.

Since (i) the sequence $\{s_n\}$ is uniformly bounded on $[a,b]$ and (ii) the sequence $\{v_n\}$ is a monotone decreasing sequence on $[a,b]$ and converges uniformly to 0, the series $\Sigma(-1)^n v_n$ is uniformly convergent on $[a,b]$, by Dirichlet's test.

Second part. Let the series be $(-1)^n v_n(x)$.

For each real x, the series $\Sigma v_n(x)$ is a series of positive terms.

Let $w_n = \frac{1}{n}$. Then $\lim\limits_{n\to\infty} \frac{v_n}{u_n} = 1$.

By comparison test, the series $\Sigma v_n(x)$ is divergent for each real x.

Since $\{v_n\}$ is a monotone decreasing sequence for each $x \in \mathbb{R}$ and $\lim_{n\to\infty} v_n(x) = 0$ for each $x \in \mathbb{R}$, the series $\Sigma(-1)^{n-1}v_n(x)$ is convergent for each real x.

Therefore the series $\Sigma(-1)^{n-1}\frac{x^2+n}{n^2}$ does not converge absolutely for any real x.

Exercises 25

1. If $\Sigma u_n(x)$ be uniformly convergent on $[a,b]$ and $v : [a,b] \to \mathbb{R}$ be a bounded function on $[a,b]$, prove that $\Sigma u_n(x)v(x)$ is uniformly convergent on $[a,b]$.

2. Prove that the series $\sum\limits_1^\infty f_n(x)$ where $f_n(x) = n^2x^2e^{-n^2x^2} - (n-1)^2x^2$ $e^{-(n-1)^2x^2}, x \in [0,1]$ is not uniformly convergent on $[0,1]$.

3. Prove that the series $(1-x)^2 + x(1-x)^2 + x^2(1-x)^2 + \cdots$ is uniformly convergent on $[0,1]$.

4. Show that the series $x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots$ is not uniformly convergent on $[0,1]$.

5. Show that the following series are uniformly convergent for all real x.

 (i) $\sum\limits_1^\infty \dfrac{1}{(n+x^2)^2}$, (ii) $\sum\limits_1^\infty r^n \cos nx, 0 < r < 1$,

 (iii) $\sum\limits_1^\infty r^n \sin nx, 0 < r < 1$, (iv) $\sum\limits_1^\infty \dfrac{(-1)^{n-1}x^{2n}}{n^2(1+x^{2n})}$,

 (v) $\sum\limits_1^\infty \dfrac{x}{n(1+nx^2)}$.

6. If $p \neq 0, \neq \pm 1, \neq \pm 2, \cdots$ prove that the series $\frac{1}{p^2} - \frac{\cos x}{p^2-1^2} + \frac{\cos 2x}{p^2-2^2} - \frac{\cos 3x}{p^2-3^2} + \cdots$ is uniformly convergent on any closed and bounded interval $[a,b]$.

[**Hint.** For a fixed p, there exists a natural number k such that $n^2 > 2p^2$ for all $n \geq k$. Then for all $n \geq k$, $|\frac{(-1)^n \cos nx}{p^2-n^2}| \leq \frac{1}{n^2-p^2} < \frac{2}{n^2}$.]

7. Show that the series $\sum\limits_1^\infty \dfrac{1}{n^2+[f(x)]^2}$ is uniformly convergent on any set $D \subset \mathbb{R}$ on which f is defined.

8. Show that the series $\sum\limits_1^\infty \dfrac{nx^2}{n^3+x^3}$ is uniformly convergent on $[0,k]$ for any $k > 0$.

9. Show that both the series $\sum\limits_0^\infty \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $\sum\limits_1^\infty \dfrac{(-1)^n x^{2n}}{(2n)!}$ are uniformly convergent on any closed and bounded interval $[a,b]$.

10. Show that the series $\sum_1^\infty \frac{n^5+1}{n^7+3}(\frac{x}{3})^n$ is absolutely and uniformly convergent on $[-3, 3]$.

11. If Σa_n be an absolutely convergent series of real numbers prove that the series

(i) $\Sigma \frac{a_n x^n}{1+x^{2n}}$ is absolutely and uniformly convergent for all real x;

(ii) $\Sigma a_n \sin nx$ is absolutely and uniformly convergent for all real x;

(iii) $\Sigma a_n x^n$ is absolutely and uniformly convergent on $[-1, 1]$.

12. Let $\sum_1^\infty f_n(x)$ be uniformly convergent to $f(x)$ on $[a, b]$ where each f_n is continuous on $[a, b]$. If $g : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, prove that

$$\int_a^b f(x)g(x)dx = \sum_1^\infty \int_a^b f_n(x)g(x)dx.$$

13. If $f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}, x \in [0, 1]$, show that $\int_0^1 (\sum_1^\infty f_n(x))dx = \sum_1^\infty (\int_0^1 f_n(x)dx)$, although the series $\sum_1^\infty f_n(x)$ is not uniformly convergent on $[0, 1]$.

14. Prove that the series is uniformly convergent for all real x. Show that the derivative of the sum function can be obtained by differentiating the series term-by-term.

(i) $\sum_1^\infty \frac{1}{n^2+nx^2}$, (ii) $\sum_1^\infty \frac{1}{n^2+n^3x^2}$.

15. Show that the series $\sum_1^\infty \frac{\sin nx}{n^3}$ is uniformly convergent for all real x and the derivative of the sum function $s(x)$ can be obtained by differentiating the series term-by-term, i.e., $\sum_1^\infty \frac{\cos nx}{n^2} = s'(x)$.

16. Prove that the series $\sum_{n=1}^\infty \frac{\sin nx}{n^2}$ is uniformly convergent for every real x. If $f(x)$ be the sum of the series prove that f is continuous on $\mathbb{R}$.

Also prove that $\int_0^\pi f(x)dx = 2(1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \cdots\cdots)$.

17. Prove that the series $\sum_1^\infty \frac{x}{n(x+n)}$ is uniformly convergent on $[0, 1]$. If $f(x)$ be the sum of the series prove that f is continuous on $[0, 1]$.

Also prove that $\int_0^1 f(x)dx = \gamma$, where $\gamma = \lim_{n\to\infty} (1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n)$.

18. If Σa_n is a convergent series of real numbers prove that the series

(i) $\Sigma a_n e^{-nx}$ is uniformly convergent on $[0, \infty)$;

(ii) $\Sigma \frac{a_n}{n^x}$ is uniformly convergent on $[0, 1]$.

19. Prove that the series $\sum_{1}^{\infty} \frac{\cos nx}{n^p}$ and $\sum_{1}^{\infty} \frac{\sin nx}{n^p}$ both converge uniformly for all p satisfying $0 < p \leq 1$ on any closed interval $[a, b]$ contained in $[0, 2\pi)$.

[**Hint.** Use Dirichlet's test.]

20. Prove that the series $\sum_{1}^{\infty} \frac{(-1)^{n-1} x^n}{n^p (1+x^n)}$ converges uniformly for all p satisfying $p > 0$ on $[0, 1]$.

[**Hint.** If $p > 1$, use Weierstrass' M-test. If $0 < p \leq 1$, use Abel's test.]

21. Prove that the series $\sum_{1}^{\infty} \frac{\cos nx}{\{\log(n+1)\}^{x}}$ is uniformly convergent on any closed interval $[a, b]$ lying within $(0, 2\pi)$.

15. POWER SERIES

15.1. Introduction.

We now study an important class of series of functions, called a power series.

A series of the form $a_0 + a_1x + a_2x^2 + \cdots$ where $a_0, a_1, a_2, \ldots$ are real numbers, is called a *power series*. The general form of a power series is

$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$ where $a_0, a_1, a_2, \ldots, x_0 \in \mathbb{R}$.

This is called a power series about the point x_0.

The general form reduces to the form $a_0 + a_1x + a_2x^2 + \cdots$ (which is a power series about 0) by the substitution $x' = x - x_0$.

To study the nature and properties of a power series we shall consider the power series about 0, i.e., a series of the form

$$a_0 + a_1x + a_2x^2 + \cdots$$

This is denoted by $\sum\limits_{n=0}^{\infty} a_nx^n$. It is a series of functions $\sum\limits_{n=0}^{\infty} f_n(x)$ where, for $n = 0, 1, 2, \ldots, f_n(x) = a_nx^n, x \in \mathbb{R}$.

Although each f_n is defined for all real x, it is not expected that the series $\sum\limits_{n=0}^{\infty} a_nx^n$ will converge for all real x.

For example,

the series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ converges for all real x; while
the series $1 + x + x^2 + \cdots$ converges only for all $x \in (-1, 1)$; and
the series $1 + x + 2!x^2 + 3!x^3 + \cdots$ converges only for $x = 0$.

It appears that some power series converge for all $x \in \mathbb{R}$. They are called *everywhere convergent* power series. Some power series converge *only* for $x = 0$. They are called *nowhere convergent* power series. Some power series converge for *some real* x and diverge for the others.

We shall see, however, that an arbitrary subset of $\mathbb{R}$ cannot be the precise set of points of convergence of a power series.

Note. We shall use the symbol $\sum\limits_{n=0}^{\infty} a_nx^n$ to denote the power series $a_0 + a_1x + a_2x^2 + \cdots$ and also to denote the sum of the series, when the sum exists.

Theorem 15.1.1. If a power series $a_0 + a_1x + a_2x^2 + \cdots$ converges for $x = x_1$, then the series converges absolutely for all real x satisfying $|x| < |x_1|$.

Proof. Since the series converges for $x = x_1$, $\sum\limits_{n=0}^{\infty} a_n x_1^n$ is convergent. It follows that $\lim a_n x_1^n = 0$. Again the convergence of the sequence $\{a_n x_1^n\}$ implies that the sequence $\{a_n x_1^n\}$ is bounded.

Therefore there exists a positive real number k such that $| a_n x_1^n | \leq k$ for all $n \in \mathbb{N}$.

$|a_n x^n| = |a_n x_1^n|.|\frac{x}{x_1}|^n \leq k|\frac{x}{x_1}|^n$.

For all x satisfying $|\frac{x}{x_1}| < 1$, $\sum\limits_{n=0}^{\infty} |\frac{x}{x_1}|^n$ is a convergent series of positive real numbers.

By Comparison test, $\sum\limits_{n=0}^{\infty} | a_n x^n |$ is convergent if $|x| < |x_1|$.

Therefore $\sum\limits_{n=0}^{\infty} a_n x^n$ is absolutely convergent if $|x| < |x_1|$.

This completes the proof.

Theorem 15.1.2. If a power series $a_0 + a_1x + a_2x^2 + \cdots$ diverges for $x = x_1$, then the series diverges for all real x satisfying $|x| > |x_1|$.

Proof. Let the power series be convergent for $x = c$ such that $|c| > |x_1|$. Since the series converges for $x = c$ and $|x_1| < |c|$, by the previous theorem, the series would be convergent for $x = x_1$, a contradiction to the hypothesis.

This proves that the series is divergent for all real x satisfying $|x| > |x_1|$.

Theorem 15.1.3. If a power series $\sum\limits_{n=0}^{\infty} a_n x^n$ be neither nowhere convergent nor everywhere convergent, then there exists a positive real number R such that the series converges absolutely for all real x satisfying $| x | < R$ and diverges for all x satisfying $| x | > R$.

Proof. Since the series is neither nowhere convergent nor everywhere convergent, there exists at least one non-zero point of convergence, say $x = c$ and there exists at least one point of divergence, say $x = d$.

Let $c_1 > 0$ be such that $c_1 <| c |$ and $d_1 > 0$ be such that $d_1 >| d |$. Then c_1 is a point of convergence and d_1 is a point of divergence of the series, by Theorems 15.1.1 and 15.1.2.

We assert that $c_1 < d_1$. Because if $c_1 > d_1$ then c_1 being a point of convergence of the series, d_1 will also be a point of convergence by the Theorem 15.1.1 and a contradiction will arise.

Let I_1 be the closed and bounded interval $[c_1, d_1]$. Then the series converges at c_1 and diverges at d_1.

Let $c_1' = \frac{1}{2}(c_1 + d_1)$. If c_1' be a point of convergence of the series we select the closed subinterval $[c_1', d_1]$ and call it $[c_2, d_2]$.

If c_1' be a point of divergence of the series we select the closed subinterval $[c_1, c_1']$ and call it $[c_2, d_2]$.

Thus the closed interval $I_2 = [c_2, d_2]$ is such that

(i) c_2 is a point of convergence and d_2 is a point of divergence of the series,

(ii) $I_2 \subset I_1$, and

(iii) $| I_2 |= \frac{1}{2}(d_1 - c_1)$.

Let $c_2' = \frac{1}{2}(c_2 + d_2)$. If c_2' be a point of convergence of the series we select the closed subinterval $[c_2', d_2]$ and call it $[c_3, d_3]$.

If c_2' be a point of divergence of the series we select the closed subinterval $[c_2, c_2']$ and call it $[c_3, d_3]$.

Thus the closed interval $I_3 = [c_3, d_3]$ is such that

(i) c_3 is a point of convergence and d_3 is a point of divergence of the series,

(ii) $I_3 \subset I_2$, and

(iii) $| I_3 |= \frac{1}{2^2}(d_1 - c_1)$.

Let $c_3' = \frac{1}{2}(c_3 + d_3)$. Proceeding in a similar manner we obtain a sequence of closed and bounded intervals $\{I_n\}$ such that for every $n \in \mathbb{N}$,

(i) c_n is a point of convergence and d_n is a point of divergence of the series,

(ii) $I_{n+1} \subset I_n$, and

(iii) $| I_n |= \frac{1}{2^{n-1}}(d_1 - c_1)$.

The sequence $\{I_n\}$ is a sequence of nested intervals and $\lim | I_n |= 0$.

By Cantor's theorem, there exists one and only one point α such that $c_n \leq \alpha \leq d_n$ for all $n \in \mathbb{N}$ and $\sup\{c_n\} = \alpha = \inf\{d_n\}$.

Let x_0 be such that $0 < x_0 < \alpha$.

Since $\alpha = \sup\{c_n\}$, there exists a natural number m such that

$x_0 < c_m \leq \alpha$.

Since the power series converges at c_m and $0 < x_0 < c_m$, the power series converges for $x = x_0$.

By Theorem 15.1.1, the power series converges absolutely for all x such that $| x |< x_0$. Since x_0 is arbitrary, the power series converges for all x satisfying $| x |< \alpha$.

Let x_1 be such that $x_1 > \alpha$.

Since $\alpha = \inf\{d_n\}$, there exists a natural number k such that $\alpha \leq d_k < x_1$.

Since the power series is divergent for $x = d_k$ and $0 < d_k < x_1$, the power series diverges for $x = x_1$.

By Theorem 15.1.2, the power series diverges for all x satisfying $|x| > x_1$. Since x_1 is arbitrary, the power series diverges for all x satisfying $|x| > \alpha$.

Hence $\alpha = R$ and the theorem is proved.

Definition. R is called the *radius of convergence* of the power series $\sum_{n=0}^{\infty} a_n x^n$. The open interval $(-R, R)$ is called the *interval of convergence* of the series.

Note 1. We define $R = 0$ for a power series which is nowhere convergent; and $R = \infty$ for a power series which is everywhere convergent.

Note 2. The convergence of the power series at $x = R, x = -R$ depends on the nature of the sequence $\{a_n\}$. There are power series for which both R and $-R$ are points of convergence, or both R and $-R$ are points of divergence, or one of R and $-R$ is a point of convergence and the other is a point of divergence.

15.2. Determination of the radius of convergence.

Theorem 15.2.1. (Cauchy-Hadamard)

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and let $\overline{\lim} |a_n|^{1/n} = \mu$. Then

(i) if $\mu = 0$, the series is everywhere convergent;

(ii) if $0 < \mu < \infty$, the series is absolutely convergent for all x satisfying $|x| < 1/\mu$ and is divergent for all x satisfying $|x| > 1/\mu$;

(iii) if $\mu = \infty$, the series is nowhere convergent.

Proof. (i) Let $x_0 \neq 0$ and $\epsilon = \frac{1}{2|x_0|}$.

Since $\overline{\lim} |a_n|^{1/n} = 0$, there exists a natural number k such that $|a_n|^{1/n} < \epsilon$ for all $n \geq k$. or, $|a_n x_0^n| < \frac{1}{2^n}$ for all $n \geq k$.

Since $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is a convergent series of positive terms, $\sum_{n=0}^{\infty} |a_n x_0^n|$ is a convergent series, by Comparison test.

It follows that $\sum_{n=0}^{\infty} a_n x_0^n$ is absolutely convergent and is therefore convergent.

As x_0 is arbitrary, the series $\sum_{n=0}^{\infty} a_n x^n$ is everywhere convergent.

(ii) $\overline{\lim} \sqrt[n]{| a_n x^n |} = \overline{\lim}(\sqrt[n]{| a_n |} \mid x \mid) = \mid x \mid \mu$.

By Cauchy's root test, the series $\sum_{n=0}^{\infty} \mid a_n x^n \mid$ is convergent if $|x|\mu < 1$.

Therefore if $\mid x \mid < 1/\mu$, the series $\sum_{n=0}^{\infty} |a_n x^n|$ is convergent i.e., the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent.

If $\mid x \mid \mu > 1, \overline{\lim} \sqrt[n]{| a_n x^n |} = \overline{\lim} \sqrt[n]{| a_n |}.|x| > 1$.

Let $u_n = a_n x^n$. Then $\overline{\lim} \mid u_n \mid > 1$ and this implies $\lim \mid u_n \mid \neq 0$. Consequently, $\lim u_n \neq 0$ and it follows that $\sum_{n=0}^{\infty} u_n$ is divergent.

(iii) If possible, let the series $\sum_{n=0}^{\infty} a_n x^n$ is convergent for $x = x_0, (x_0 \neq 0)$. Then $\lim a_n x_0^n = 0$.

The sequence $\{a_n x_0^n\}$ being a bounded sequence, there exists a positive real number B such that $\mid a_n x_0^n \mid < B$ for all $n \in \mathbb{N}$.

This shows that the sequence $\{| a_n |^{1/n}\}$ is a bounded sequence and this contradicts that $\overline{\lim} \mid a_n \mid^{1/n} = \infty$.

Thus the series $\Sigma a_n x^n$ is not convergent for $x = x_0$. As x_0 is an arbitrary non-zero real number, the series $\Sigma a_n x^n$ is nowhere convergent.

This completes the proof.

Note. The radius of convergence of the series is $\frac{1}{\overline{\lim}|a_n|^{1/n}}$.

When $0 < \mu < \infty, R = \frac{1}{\mu}$; when $\mu = 0, R = \infty$; when $\mu = \infty, R = 0$.

Theorem 15.2.2. (Ratio test)

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and let $\lim \mid \frac{a_{n+1}}{a_n} \mid = \mu$. Then

(i) if $\mu = 0$ the series is everywhere convergent;

(ii) if $0 < \mu < \infty$ the series is absolutely convergent for all x satisfying $\mid x \mid < \frac{1}{\mu}$ and the series is divergent for all x satisfying $\mid x \mid > \frac{1}{\mu}$;

(iii) if $\mu = \infty$, the series is nowhere converegnt.

Proof. (i) Let $x \neq 0$ and let $u_n = a_n x^n$.

Then $\lim\limits_{n \to \infty} \mid \frac{u_{n+1}}{u_n} \mid = \lim\limits_{n \to \infty} \mid \frac{a_{n+1}}{a_n} \mid\mid x \mid = 0 < 1$.

By D'Alembert's Ratio test, the series $\Sigma |u_n|$ is convergent.

Therefore $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for all non-zero real x.

Consequently, $\sum_{n=0}^{\infty} a_n x^n$ is convergent for all non-zero real x, i.e., the series $\sum_{n=0}^{\infty} a_n x^n$ is everywhere convergent.

(ii) Let $x \neq 0$. Then $\lim \mid \frac{a_{n+1}x^{n+1}}{a_n x^n} \mid = \mu|x|$.

By D'Alembert's ratio test, the series $\sum_{n=0}^{\infty} \mid a_n x^n \mid$ is convergent if $|x| < \frac{1}{\mu}$.

The series is convergent for $x = 0$ also.

Hence the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for all real x satisfying $\mid x \mid < \frac{1}{\mu}$.

When $\mid x \mid > \frac{1}{\mu}$, $\lim \mid \frac{a_{n+1}x^{n+1}}{a_n x^n} \mid > 1$. Let $u_n = a_n x^n$.
Then $\lim \mid \frac{u_{n+1}}{u_n} \mid > 1$. Let $\lim \mid \frac{u_{n+1}}{u_n} \mid = l$. Then $l > 1$.
Let us choose $\epsilon > 0$ such that $l - \epsilon > 1$. There exists a natural number m such that $l - \epsilon < \mid \frac{u_{n+1}}{u_n} \mid < l + \epsilon$ for all $n \geq m$.

Therefore $\mid \frac{u_{n+1}}{u_n} \mid > 1$ for all $n \geq m$
or, $\mid u_{n+1} \mid > \mid u_n \mid$ for all $n \geq m$.

This shows that the sequence $\{\mid u_n \mid\}$ is ultimately a monotone increasing sequence of positive real numbers and therefore $\lim \mid u_n \mid$ cannot be 0.

It follows that $\lim u_n \neq 0$ and consequently $\sum_{n=0}^{\infty} a_n x^n$ is divergent for all real x satisfying $\mid x \mid > \frac{1}{\mu}$.

(iii) Let $u_n = a_n x^n$.
$\lim \mid \frac{u_{n+1}}{u_n} \mid = \lim \mid \frac{a_{n+1}}{a_n} \mid \mid x \mid = \infty$. (not considering the case, $x = 0$.)
Let us choose $G > 1$. There exists a natural number m such that $\mid \frac{u_{n+1}}{u_n} \mid > G$ for all $n \geq m$.
Therefore $\mid u_{n+1} \mid > \mid u_n \mid$ for all $n \geq m$.
Proceeding with similar arguments as in the last part of case (ii) it can be proved that $\sum_{n=0}^{\infty} a_n x^n$ is divergent for all real $x (\neq 0)$.

Thus the series $\sum_{n=0}^{\infty} a_n x^n$ is nowhere convergent.

This completes the proof.

Note 1. The radius of convergence of the power series is $\frac{1}{\lim \mid \frac{a_{n+1}}{a_n} \mid}$.

Note 2. We have $\underline{\lim} \mid \frac{a_{n+1}}{a_n} \mid \leq \underline{\lim} \mid a_n \mid^{1/n}$
$\leq \overline{\lim} \mid a_n \mid^{1/n} \leq \overline{\lim} \mid \frac{a_{n+1}}{a_n} \mid$. [Theorem 5.16.4.]

Therefore if $\lim |\frac{a_{n+1}}{a_n}|$ exists then $\lim |a_n|^{1/n}$ exists, but the converse is not true.

It follows that if the ratio test be applicable to a series then Cauchy-Hadamard test is also applicable to the series. But there are cases where Cauchy-Hadamard test is applicable while the ratio test fails to be applicable. Therefore Cauchy-Hadamard test is more powerful than the ratio test for the determination of the nature of a power series.

Worked Examples.

1. Determine the radius of convergence of the power series
$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \cdots$
Let $\sum\limits_{n=0}^{\infty} a_nx^n$ be the given series. Then $a_0 = 0, a_n = \frac{n^n}{n!}$ for all $n \in \mathbb{N}$.
$|\frac{a_{n+1}}{a_n}| = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = (1 + \frac{1}{n})^n$ for $n \geq 1$. $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = e$.
The radius of convergence of the power series is $1/e$.

2. Determine the radius of convergence of the power series
$\frac{1}{3} - x + \frac{x^2}{3^2} - x^3 + \frac{x^4}{3^4} - x^5 + \cdots$
Let $\sum\limits_{n=0}^{\infty} a_nx^n$ be the given series.
Then $a_0 = \frac{1}{3}, a_1 = -1, a_2 = \frac{1}{3^2}, a_3 = -1, a_4 = \frac{1}{3^4}, \cdots$
$\overline{\lim} |a_n|^{1/n} = 1$.
The radius of convergence of the series is 1.

3. Find the radius of convergence of the power series
$x + \frac{(2!)^2}{4!}x^2 + \frac{(3!)^2}{6!}x^3 + \cdots + \frac{(n!)^2}{(2n)!}x^n + \cdots$
Let $\sum\limits_{n=0}^{\infty} a_nx^n$ be the given series.
Then $a_0 = 0, a_1 = 1, a_n = \frac{(n!)^2}{(2n)!}$ for all $n \geq 2$.
$\lim |\frac{a_{n+1}}{a_n}| = \lim \frac{n+1}{2(2n+1)} = \frac{1}{4}$.
The radius of convergence of the series is 4.

15.3. Properties of a power series.

Theorem 15.3.1. Let $\sum\limits_{n=0}^{\infty} a_nx^n$ be a power series with radius of convergence $R(> 0)$. Then the series is uniformly convergent on $[-s, s]$, where $0 < s < R$.

Proof. Let $f_n(x) = a_nx^n, n \geq 0$.

Since R is the radius of convergence of the power series, the series is absolutely convergent for all real x satisfying $|x| < R$.

Since $0 < s < R$, the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for all x satisfying $|x| \leq s < R$.

Therefore the series $\sum_{n=0}^{\infty} |a_n s^n|$ is convergent.

Now $|f_n(x)| = |a_n x^n| \leq |a_n| s^n$ for all real x satisfying $|x| \leq s$.

Let $M_n = |a_n| s^n$ for all $n \in \mathbb{N}$.

Then $\sum_{n=1}^{\infty} M_n$ is a convergent series of positive real numbers and for all $n \in \mathbb{N}$, $|f_n(x)| \leq M_n$ for all $x \in [-s, s]$.

By Weierstrass' M-test, the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on $[-s, s]$. Consequently, the series $\sum_{n=0}^{\infty} f_n(x)$, i.e., the power series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[-s, s]$.

Corollary 1. Let $R(> 0)$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$. Then the series is uniformly convergent on $[-R+\epsilon, R-\epsilon]$ where ϵ is an arbitrarily small positive number satisfying $R - \epsilon > 0$.

Proof. $R - \epsilon > 0$. Let $s = R - \epsilon$. Then $0 < s < R$ and therefore the power series is uniformly convergent on $[-s, s]$, i.e., on $[-R + \epsilon, R - \epsilon]$.

Corollary 2. Let $R(> 0)$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$. If $[a, b]$ be any closed interval contained in $(-R, R)$, then the series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[a, b]$.

Proof. Let us choose a positive ϵ such that $R - \epsilon > 0$ and $-R < -R + \epsilon < a < b < R - \epsilon < R$.

Let $R - \epsilon = s$. Then $0 < s < R$ and $-R < -s < a < b < s < R$.

Since the power series is uniformly convergent on $[-s, s]$ and $[a, b] \subset [-s, s]$, the power series is uniformly convergent on $[a, b]$.

Theorem 15.3.2. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R(> 0)$. Let $f(x)$ be the sum of the series on $(-R, R)$. Then f is continuous on $(-R, R)$.

Proof. Since R is the radius of convergence of the power series, the series is uniformly convergent on $[-R + \delta, R - \delta]$ where δ is an arbitrarily small positive number satisfying $R - \delta > 0$.

Let $f_n(x) = a_n x^n, n \geq 0$.

Let $s_n(x) = f_0(x) + f_1(x) + \cdots + f_n(x), n \geq 1$.

Since the series is uniformly convergent on $[-R+\delta, R-\delta]$ to the function f, the sequence $\{s_n\}$ is uniformly convergent to f on $[-R+\delta, R-\delta]$. Let $c \in [-R+\delta, R-\delta]$.

Let us choose $\epsilon > 0$. There exists a natural number k such that for all $x \in [-R+\delta, R-\delta]$, $| s_n(x) - f(x) | < \frac{\epsilon}{3}$ for all $n \geq k$.
Hence for all $x \in [-R+\delta, R-\delta]$, $| s_k(x) - f(x) | < \frac{\epsilon}{3}$
Therefore $| s_k(c) - f(c) | < \frac{\epsilon}{3}$.
Since each f_n is continuous at c, $s_n(x)$ is continuous at c for all $n \geq 1$.
Therefore there exists a positive δ' such that
$| s_k(x) - s_k(c) | < \frac{\epsilon}{3}$ for all $x \in N(c, \delta') \cap [-R+\delta, R-\delta]$. We have

$$\begin{aligned} | f(x) - f(c) | &= | f(x) - s_k(x) + s_k(x) - s_k(c) + s_k(c) - f(c) | \\ &\leq | f(x) - s_k(x) | + | s_k(x) - s_k(c) | \\ &\quad + | s_k(c) - f(c) | \\ &< \epsilon \text{ for all } x \in N(c, \delta') \cap [-R+\delta, R-\delta] \end{aligned}$$

or, $| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta') \cap [-R+\delta, R-\delta]$.
This shows that f is continuous at c.

Since c is arbitrary, f is continuous on $[-R+\delta, R-\delta]$.

Since δ is arbitrary, f is continuous on $(-R, R)$ and this completes the proof.

Note. A power series with radius of convergence $R(> 0)$ has a continuous sum function on the interval of convergence $(-R, R)$.

Theorem 15.3.3. A power series can be integrated term-by-term on any closed and bounded interval contained within the interval of convergence.

Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R and let $f(x)$ be the sum of the series on $(-R, R)$.

The theorem states that for any closed interval $[a, b] \subset (-R, R)$,
$\int_a^b a_0 dx + \int_a^b a_1 x dx + \int_a^b a_2 x^2 dx + \cdots = \int_a^b f(x) dx$.

Proof. Since R is the radius of convergence of the series and the closed and bounded interval $[a, b] \subset (-R, R)$, the series is uniformly convergent on $[a, b]$ to the sum function f.

Since each term of the series is integrable on $[a, b]$, f is also intergrable on $[a, b]$ and
$\int_a^b a_0 dx + \int_a^b a_1 x dx + \int_a^b a_2 x^2 dx + \cdots = \int_a^b f(x) dx$.

Note. For any x satisfying $| x | < R$, the series is uniformly convergent on $[0, x]$ or $[x, 0]$ and
$\int_0^x a_0 dx + \int_0^x a_1 x dx + \int_0^x a_2 x^2 dx + \cdots = \int_0^x f(x) dx$
or, $a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \cdots + \frac{a_n x^{n+1}}{n+1} + \cdots = \int_0^x f(x) dx$.

The convergence of the left hand series (obtained by term-by-term integration) to $\int_0^x f(x)dx$ is established by the theorem.

Since the left hand series is a power series we now determine the radius of convergence of the series.

Lemma 15.3.4. Let $\{u_n\}$ be a bounded sequence where $u_n \geq 0$ for all $n \in \mathbb{N}$ and let $v_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim v_n = v$. Then $\overline{\lim}\ (u_n v_n) = v.\overline{\lim}\ u_n$.

Proof. **Case 1.** $v \neq 0$.

Since the sequence $\{v_n\}$ is convergent, it is a bounded sequence. So the sequence $\{u_n v_n\}$ is bounded and therefore $\overline{\lim}\ (u_n v_n)$ exists.

Let $\overline{\lim}\ (u_n v_n) = p$. Then there exists a subsequence $\{u_{r_n} v_{r_n}\}$ of $\{u_n v_n\}$ such that $\lim u_{r_n} v_{r_n} = p$. Again $\lim v_n = v \Rightarrow \lim v_{r_n} = v$.

Since $\lim u_{r_n} v_{r_n} = p$ and $\lim v_{r_n} = v \neq 0$, $\lim u_{r_n} = \frac{p}{v}$.

Clearly, $\frac{p}{v}$ is a subsequential limit of the sequence $\{u_n\}$ and so $\frac{p}{v} \leq \overline{\lim}\ u_n (= u$, say). Therefore $p \leq uv$ since $v > 0$.

Since $\overline{\lim}\ u_n = u$, there exists a subsequence $\{u_{k_n}\}$ of $\{u_n\}$ such that $\lim u_{k_n} = u$. Also $\lim v_n = v \Rightarrow \lim v_{k_n} = v$.

Therefore $\lim u_{k_n} v_{k_n} = uv$.

Clearly, uv is a subsequential limit of $\{u_n v_n\}$ and therefore $uv \leq p$.

It follows that $p = uv$. That is, $\overline{\lim}(u_n v_n) = v.\overline{\lim}\ u_n$.

Case 2. $v = 0$.

Let $\overline{\lim}\ u_n = u$. Then there exists a natural number k_1 such that $u_n < u + 1$ for all $n \geq k_1$. Clearly $u + 1 > 0$.

Let $\epsilon > 0$. Since $\lim v_n = 0$, there exists a natural number k_2 such that $v_n < \frac{\epsilon}{u+1}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $u_n v_n < \epsilon$ for all $n \geq k$. This proves that $\lim u_n v_n = 0$. Therefore $\overline{\lim}(u_n v_n) = \lim u_n v_n = 0 = v\overline{\lim}\ u_n$.

This completes the proof.

Lemma 15.3.5. Let $u_n > 0$ for all $n \in \mathbb{N}$ and $\{u_n\}$ be a bounded sequence such that $\overline{\lim}\ u_n > 0$. Let $v_n > 0$ for all $n \in \mathbb{N}$ and $\lim v_n = v > 0$. Then $\overline{\lim}\ (u_n)^{v_n} = (\overline{\lim}\ u_n)^v$.

Proof. Since $\{u_n\}$, $\{v_n\}$ are bounded sequences and $u_n > 0$ for all $n \in \mathbb{N}$, the sequence $\{(u_n)^{v_n}\}$ is a bounded sequence. So $\overline{\lim}\ (u_n)^{v_n}$ exists.

Let $\overline{\lim}\ (u_n)^{v_n} = p$. Then $p > 0$.

Let $\overline{\lim}\ u_n = u$. Since $\overline{\lim}\ (u_n)^{v_n} = p$, there exists a subsequence $\{(u_{r_n})^{v_{r_n}}\}$ of the sequence $\{(u_n)^{v_n}\}$ such that $\lim (u_{r_n})^{v_{r_n}} = p$. Also

$\lim v_n = v \Rightarrow \lim v_{r_n} = v$.

$$\begin{aligned}\log p &= \log \lim (u_{r_n})^{v_{r_n}} \\ &= \lim [v_{r_n} \log(u_{r_n})].\end{aligned}$$

$$\begin{aligned}\text{Since } \lim v_{r_n} = v \neq 0, \lim \log(u_{r_n}) &= \tfrac{\log p}{v}. \\ \text{or, } \log \lim u_{r_n} &= \tfrac{1}{v} \log p \\ \text{or, } \lim u_{r_n} &= e^{\frac{1}{v}\log p} = p^{\frac{1}{v}}.\end{aligned}$$

Clearly, $p^{\frac{1}{v}}$ is a subsequential limit of the sequence $\{u_n\}$ and so $p^{\frac{1}{v}} \leq u$. Therefore $p \leq u^v$, since $p > 0, v > 0$.

Since $\overline{\lim}\ u_n = u$, there exists a subsequence $\{u_{k_n}\}$ of the sequence $\{u_n\}$ such that $\lim u_{k_n} = u$. Also $\lim v_n = v \Rightarrow \lim v_{k_n} = v$.

Since $\lim u_{k_n} = u > 0$ and $\lim v_{k_n} = v$, $\lim (u_{k_n})^{v_{k_n}} = u^v$.

Clearly, u^v is a subsequential limit of the sequence $\{(u_n)^{v_n}\}$. Therefore $u^v \leq p$. It follows that $p = u^v$. That is, $\overline{\lim}(u_n)^{v_n} = (\overline{\lim} u_n)^v$.

This completes the proof.

Theorem 15.3.6. Let $R(> 0)$ be the radius of convergence of the power series $a_0 + a_1 x + a_2 x^2 + \cdots$ Then the radius of convergence of the power series$a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots + \frac{a_{n-1}}{n} x^n + \cdots$, obtained by term-by-term intergration, is also R.

Proof. $\frac{1}{R} = \overline{\lim}\ \sqrt[n]{\mid a_n \mid}$. Let R' be the radius of convergence of the power series $a_0 x + \frac{a_1}{2} x^2 + \cdots + \frac{a_{n-1}}{n} x^n + \cdots$ Then $\frac{1}{R'} = \overline{\lim}\ \sqrt[n]{\frac{|a_{n-1}|}{n}}$.

$\frac{1}{R'} = \overline{\lim}\ \sqrt[n]{\frac{|a_{n-1}|}{n}} = \overline{\lim} \frac{\{|a_{n-1}|^{\frac{1}{n-1}}\}^{\frac{n-1}{n}}}{\sqrt[n]{n}} = \overline{\lim}\ (u_n . v_n)$, where $u_n = \frac{1}{\sqrt[n]{n}}$ and $v_n = \{\mid a_{n-1} \mid^{\frac{1}{n-1}}\}^{\frac{n-1}{n}}$.

As $\overline{\lim}\ \sqrt[n]{\mid a_n \mid} = \frac{1}{R}$, we have $\overline{\lim} \mid a_{n-1} \mid^{\frac{1}{n-1}} = \frac{1}{R}$.

Since $\overline{\lim} \mid a_{n-1} \mid^{\frac{1}{n-1}} = \frac{1}{R}$ and $\lim \frac{n-1}{n} = 1$, it follows that $\overline{\lim}\ v_n = \frac{1}{R}$.

As $\lim \sqrt[n]{n} = 1$, we have $\lim u_n = 1$.

Since $\lim u_n = 1$ and $\overline{\lim}\ v_n = \frac{1}{R}$, we have $\overline{\lim}\ (u_n v_n) = \lim u_n . \overline{\lim}\ v_n = \overline{\lim}\ v_n = \frac{1}{R}$.

Therefore $\frac{1}{R'} = \frac{1}{R}$, i.e., $R' = R$.

Note. It follows that the series obtained by integrating the power series $\Sigma a_n x^n$ term-by-term is also uniformly convergent on any closed and bounded sub-interval contained in the interval of convergence.

Worked Example.

1. A function f is defined on $(-\frac{1}{3}, \frac{1}{3})$ by
$f(x) = 1 + 2.3x + 3.3^2 x^2 + \cdots + n.3^{n-1} x^{n-1} + \cdots$
Show that f is continuous on $(-\frac{1}{3}, \frac{1}{3})$. Evaluate $\int_0^{\frac{1}{4}} f$.

Let $\sum_{n=0}^{\infty} a_n x^n$ be the given series.

Then $a_0 = 1, a_n = (n+1)3^n$ for $n \geq 1$.

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{3(n+2)}{n+1} = 3$.

The radius of convergence of the power series is $\frac{1}{3}$.

Therefore f is continuous on $(-\frac{1}{3}, \frac{1}{3})$.

The series can be integrated term-by-term on any closed interval contained within $(-\frac{1}{3}, \frac{1}{3})$. $[0, \frac{1}{4}] \subset (-\frac{1}{3}, \frac{1}{3})$.

Therefore $\int_0^{1/4} f(x)dx$

$= \int_0^{1/4} dx + \int_0^{1/4} 2.3x dx + \cdots + \int_0^{1/4} n.3^{n-1}x^{n-1}dx + \cdots$

$= \frac{1}{4} + \frac{1}{4}(\frac{3}{4}) + \frac{1}{4}(\frac{3}{4})^2 + \cdots$

$= \frac{1}{4} \cdot \frac{1}{1-3/4} = 1$.

Theorem 15.3.7. Let $R(>0)$ be the radius of convergence of the power series $a_0 + a_1x + a_2x^2 + \cdots$ Then the radius of convergence of the power series $a_1 + 2a_2x + 3a_3x^2 + \cdots + (n+1)a_{n+1}x^n + \cdots$ obtained by term-by-term differentiaton, is also R.

Proof. $\frac{1}{R} = \overline{\lim} \sqrt[n]{|a_n|}$. Let R' be the radius of convergence of the series $a_1 + 2a_2 + 3a_3x^2 + \cdots$ Then $\frac{1}{R'} = \overline{\lim} \sqrt[n]{(n+1)|a_{n+1}|}$.

$\frac{1}{R'} = \overline{\lim} \sqrt[n]{(n+1)|a_{n+1}|}$

$= \overline{\lim}[\{(n+1)^{\frac{1}{n+1}}\}^{\frac{n+1}{n}}.\{|a_{n+1}|^{\frac{1}{n+1}}\}^{\frac{n+1}{n}}] = \overline{\lim}(u_n v_n)$ where $u_n = \{(n+1)^{\frac{1}{n+1}}\}^{\frac{n+1}{n}}$, and $v_n = \{|a_{n+1}|^{\frac{1}{n+1}}\}^{\frac{n+1}{n}}$.

As $\overline{\lim} \sqrt[n]{|a_n|} = \frac{1}{R}$, we have $\overline{\lim} |a_{n+1}|^{\frac{1}{n+1}} = \frac{1}{R}$.

Since $\overline{\lim} |a_{n+1}|^{\frac{1}{n+1}} = \frac{1}{R}$ and $\lim \frac{n+1}{n} = 1$, it follows that $\overline{\lim}\, v_n = \frac{1}{R}$.

As $\lim \sqrt[n]{n} = 1$, we have $\lim(n+1)^{\frac{1}{n+1}} = 1$.

Since $\lim(n+1)^{\frac{1}{n+1}} = 1$ and $\lim \frac{n+1}{n} = 1$, it follows that $\lim u_n = 1$.

Since $\lim u_n = 1$ and $\overline{\lim} v_n = \frac{1}{R}$, we have $\overline{\lim}(u_n v_n) = \lim u_n . \overline{\lim} v_n = \overline{\lim}\, v_n = \frac{1}{R}$.

Therefore $\frac{1}{R'} = \frac{1}{R}$, i.e., $R' = R$.

Theorem 15.3.8. A power series can be differentiated term-by-term within the interval of convergence.

Proof. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R(>0)$. Let $f(x)$ be the sum of the series on $(-R, R)$. The theorem states that $\frac{d}{dx}(a_0) + \frac{d}{dx}(a_1x) + \frac{d}{dx}(a_2x^2) + \cdots = \frac{d}{dx}f(x)$ on $(-R, R)$.

Differentiating the series $\sum_{n=0}^{\infty} a_n x^n$ term-by-term, we obtain the series $a_1 + 2a_2x + 3a_3x^2 + \cdots$

Let R' be the radius of convergence of this power series.

Then $R' = R$, by Theorem 15.3.6.

Since R is the radius of convergence of both the series, both of these are uniformly convergent on $[-R+\epsilon, R-\epsilon]$ for any positive ϵ satisfying $R - \epsilon > 0$.

Let $f(x)$ be the sum of the series $a_0 + a_1x + a_2x^2 + \cdots$ on $[-R+\epsilon, R-\epsilon]$. Then

$\frac{d}{dx}(a_0) + \frac{d}{dx}(a_1x) + \frac{d}{dx}(a_2x^2) + \cdots = \frac{d}{dx}f(x)$ on $[-R+\epsilon, R-\epsilon]$, by Theorem 14.2.3.

Since ϵ is arbitrary, it follows that f is differentiable at each point of $(-R, R)$ and $\frac{d}{dx}(a_0) + \frac{d}{dx}(a_1x) + \frac{d}{dx}(a_2x^2) + \cdots = \frac{d}{dx}f(x)$ on $(-R, R)$.

Theorem 15.3.9. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R(>0)$ and $f(x)$ be the sum of the series on $(-R, R)$. Then $f^k(0) = k!a_k$ $(k = 0, 1, 2, \ldots)$.

Proof. $a_0 + a_1x + a_2x^2 + \cdots = f(x)$ on $(-R, R)$ ... ... (i)

Therefore $a_0 = f(0)$.

Differentiating the series (i) term-by-term, we have

$a_1 + 2a_2x + 3a_3x^2 + \cdots = f'(x)$ on $(-R, R)$ ... ... (ii)

Therefore $a_1 = f'(0)$.

Differentiating the series (ii) term-by-term, we have

$1.2a_2 + 2.a_3x + 3.4a_4x^2 + \cdots = f''(x)$ on $(-R, R)$ ... ... (iii)

Therefore $2!a_2 = f''(0)$.

Differentiating the series (iii) term-by-term, we have

$1.2.3a_3 + 2.3.4a_4x + \cdots = f'''(x)$ on $(-R, R)$.

Therefore $3!a_3 = f'''(0)$.

Proceeding similarly, $k!a_k = f^k(0)$ for $k = 0, 1, 2, \ldots$

Note. The power series takes the form $\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$, the coefficients depending on the values at the origin of the sum function f and its successive derivatives.

Definition. If a function f defined on some neighbourhood $N(0)$ of 0, has derivatives of all orders on $N(0)$, then the series

$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$ is called the *Taylor's series* of f about the point 0.

The Theorem 15.3.9 shows that every power series $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $R(>0)$ is the Taylor's series about 0 of its sum function f.

Now it is natural to ask if a function f, having derivatives of all orders on some neighbourhood $N(0)$ of 0, be chosen first and the Taylor's series $\sum\limits_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n$ be constructed, does this power series will have f as its sum function on $N(0)$?

The answer is 'no'.

Let us consider the function f (due to Cauchy) defined on some neighbourhood $N(0)$ of 0 by $f(x) = e^{-1/x^2}$, $x \neq 0$
$= 0$, $x = 0$.

We have $f^n(0) = 0$ for $n = 0, 1, 2, \ldots$

The Taylor's series of f about 0 is $0 + 0 + 0 + \cdots$ and this converges obviously to 0, and not to f, on $N(0)$.

Theorem 15.3.10. (Abel)

Let $\sum\limits_{0}^{\infty} a_n x^n$ be a power series with radius of convergence $R(>0)$. If the series converges at the end point R of the interval of convergence $(-R, R)$, then the series is uniformly convergent on the closed interval $[0, R]$.

[i.e., the range of uniform convergence extends upto and includes R].

If the series converges at the end point $-R$ of the interval of convergence $(-R, R)$, then the series is uniformly convergent on the closed interval $[-R, 0]$.

[i.e., the range of uniform convergence of the series extends upto and includes $-R$].

Proof. First part.

The series $\sum\limits_{n=0}^{\infty} a_n R^n$ is convergent.

Let us choose $\epsilon > 0$. Then there exists a natural number k such that $|a_{n+1}R^{n+1} + a_{n+2}R^{n+2} + \cdots + a_{n+p}R^{n+p}| < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

$$\begin{aligned}
\text{Let } s_{n,1} &= a_{n+1}R^{n+1}, \\
s_{n,2} &= a_{n+1}R^{n+1} + a_{n+2}R^{n+2}, \\
&\quad \cdots \quad \cdots \\
s_{n,p} &= a_{n+1}R^{n+1} + a_{n+2}R^{n+2} + \cdots + a_{n+p}R^{n+p}, \\
&\quad \cdots \quad \cdots
\end{aligned}$$

Then $|s_{n,p}| < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

$| a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots + a_{n+p}x^{n+p} |$

$= | a_{n+1}R^{n+1}(\frac{x}{R})^{n+1} + a_{n+2}R^{n+2}(\frac{x}{R})^{n+2} + \cdots + a_{n+p}R^{n+p}(\frac{x}{R})^{n+p} |$

$=| s_{n,1}(\frac{x}{R})^{n+1} + (s_{n,2} - s_{n,1})(\frac{x}{R})^{n+2} + (s_{n,3} - s_{n,2})(\frac{x}{R})^{n+3} + \cdots + (s_{n,p} - s_{n,p-1})(\frac{x}{R})^{n+p} |$

$=| s_{n,1}\{(\frac{x}{R})^{n+1} - (\frac{x}{R})^{n+2}\} + s_{n,2}\{(\frac{x}{R})^{n+2} - (\frac{x}{R})^{n+3}\} + \cdots + s_{n,p-1}\{(\frac{x}{R})^{n+p-1} - (\frac{x}{R})^{n+p}\} + s_{n,p}(\frac{x}{R})^{n+p} |$

$\leq | s_{n,1} || (\frac{x}{R})^{n+1} - (\frac{x}{R})^{n+2} | + | s_{n,2} || (\frac{x}{R})^{n+2} - (\frac{x}{R})^{n+3} | + \cdots + | s_{n,p-1} || (\frac{x}{R})^{n+p-1} - (\frac{x}{R})^{n+p} | + | s_{n,p} || \frac{x}{R} |^{n+p}$

$=| s_{n,1} | \{(\frac{x}{R})^{n+1} - (\frac{x}{R})^{n+2}\} + | s_{n,2} | \{(\frac{x}{R})^{n+2} - (\frac{x}{R})^{n+3}\} + \cdots + | s_{n,p} | (\frac{x}{R})^{n+p}$,

since for all $x \in [0, R], 0 \leq (\frac{x}{R})^{n+p} \leq (\frac{x}{R})^{n+p-1} \leq \cdots \leq (\frac{x}{R})^{n+1} \leq 1$

$< \epsilon[\{(\frac{x}{R})^{n+1} - (\frac{x}{R})^{n+2}\} + \{(\frac{x}{R})^{n+2} - (\frac{x}{R})^{n+3}\} + \cdots + \{(\frac{x}{r})^{n+p-1} - (\frac{x}{R})^{n+p}\} + (\frac{x}{R})^{n+p}]$ for all $n \geq k, p = 1, 2, 3, \ldots$

$= \epsilon.(\frac{x}{R})^{n+1}$.

Therefore for all $x \in [0, R]$, $| a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots + a_{n+p}x^{n+p} | < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

This proves that $\sum\limits_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[0, R]$.

Second part. Similar proof.

Theorem 15.3.11. Abel's theorem (Another form)

Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence 1. If $\sum\limits_{n=0}^{\infty} a_n$ be convergent then the series $\sum\limits_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[0, 1]$.

Proof. The series $\sum\limits_{n=0}^{\infty} a_n$ is convergent.

Let us choose $\epsilon > 0$. Then there exists a natural number k such that $| a_{n+1} + a_{n+2} + \cdots + a_{n+p} | < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

$$\begin{aligned} \text{Let } s_{n,1} &= a_{n+1}, \\ s_{n,2} &= a_{n+1} + a_{n+2}, \\ &\cdots \quad \cdots \\ s_{n,p} &= a_{n+1} + a_{n+2} + \cdots + a_{n+p}, \\ &\cdots \quad \cdots \end{aligned}$$

Then $| s_{n,p} | < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

$| a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots + a_{n+p}x^{n+p} |$

$=| s_{n,1}x^{n+1} + (s_{n,2} - s_{n,1})x^{n+2} + \cdots + (s_{n,p} - s_{n,p-1})x^{n+p} |$

$=| s_{n,1}\{x^{n+1} - x^{n+2}\} + s_{n,2}\{x^{n+2} - x^{n+3}\} + \cdots + s_{n,p-1}\{x^{n+p-1} - x^{n+p}\} + s_{n,p}x^{n+p} |$

$\leq | s_{n,1} || x^{n+1} - x^{n+2} | + | s_{n,2} || x^{n+2} - x^{n+3} | + \cdots + | s_{n,p-1} || x^{n+p-1} - x^{n+p} | + | s_{n,p} || x^{n+p} |$

$= | s_{n,1} | \{x^{n+1} - x^{n+2}\} + | s_{n,2} | \{x^{n+2} - x^{n+3}\} + \cdots + | s_{n,p-1} | \{x^{n+p-1} - x^{n+p}\} + | s_{n,p} | x^{n+p}$, for all $x \in [0,1]$

$< \epsilon . x^{n+1}$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

Therefore for all $x \in [0,1]$, $| a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots + a_{n+p}x^{n+p} | < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

This proves that $\sum\limits_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[0, 1]$.

Theorem 15.3.12. Abel's theorem (Limit form)

Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R and let the sum of the series be $f(x)$ on $(-R, R)$. If the series $\sum\limits_{n=0}^{\infty} a_n R^n$ be convergent then $\sum\limits_{n=0}^{\infty} a_n R^n = \lim\limits_{x \to R-} f(x)$.

Proof. Since R is the radius of convergence of the series and $\sum\limits_{n=0}^{\infty} a_n R^n$ is convergent, the series is uniformly convergent on $[0, R]$. Let $\phi(x)$ be the sum of the series on $[0, R]$.

Since each term of the series is continuous on $[0, R]$, the sum function ϕ is also continuous on $[0, R]$. Also $\phi(x) = f(x)$ on $[0, R)$.

Since ϕ is continuous at R, $\phi(R) = \lim\limits_{x \to R-} \phi(x) = \lim\limits_{x \to R-} f(x)$.

Therefore $\Sigma a_n R^n = \lim\limits_{x \to R-} f(x)$.

Corollary. Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence 1 and let the sum of the series be $f(x)$ on $(-1, 1)$. Then

(i) if the series Σa_n be convergent, then $\Sigma a_n = \lim\limits_{x \to 1-} f(x)$;

(ii) if the series $\Sigma(-1)^n a_n$ be convergent, then $\Sigma(-1)^n a_n = \lim\limits_{x \to -1+0} f(x)$.

Note. The converse of Abel's theorem is not true. For a power series $\sum\limits_{n=0}^{\infty} a_n x^n$ with radius of convergence R, $\lim\limits_{x \to R-} f(x)$ may exist, yet the series $\sum\limits_{n=0}^{\infty} a_n x^n$ may not converge at R.

For example, the sum of the series $1 - x + x^2 - x^3 + \cdots$ is $\frac{1}{1+x}$ on $(-1, 1)$, 1 being the radius of convergence of the series. $\lim\limits_{x \to 1-} \dfrac{1}{1+x} = 2$, but the series $1 - x + x^2 - x^3 + \cdots$ is not convergent at $x = 1$.

Theorem 15.3.13. (Uniqueness theorem)

If two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge on the same interval $(-R, R), R > 0$, to the same function f, then $a_n = b_n$ for $n = 0, 1, 2, \ldots$

Proof. By the given condition,

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ on $(-R, R)$

and $f(x) = b_0 + b_1 x + b_2 x^2 + \cdots$ on $(-R, R)$.

At $x = 0$, $f(0) = a_0 = b_0$.

Differentiating both the series term-by-term, we have

$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$ on $(-R, R)$

and $f'(x) = b_1 + 2b_2 x + 3b_3 x^2 + \cdots$ on $(-R, R)$.

At $x = 0$, $f'(0) = a_1 = b_1$.

Differentiating again, we have by similar arguments

$f''(0) = a_2 = b_2$.

Proceeding similarly, we have $a_n = b_n$ for $n = 0, 1, 2, \ldots$

Theorem 15.3.14. If R_1, R_2 be the radii of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ respectively and $\sum_{n=0}^{\infty} a_n x^n = f(x)$ for $|x| < R_1$, $\sum_{n=0}^{\infty} b_n x^n = g(x)$ for $|x| < R_2$, then the radius of convergence of the series $\sum_{n=0}^{\infty} (a_n + b_n) x^n$ is $R = \min\{R_1, R_2\}$ and the sum of the series is $f(x) + g(x)$ on $(-R, R)$.

Proof left to the reader.

An alternative proof of Theorem 6.6.4

Abel's theorem.

If the series $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, $\sum_{n=0}^{\infty} c_n$ converge to A, B, C respectively and if $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$, then $C = AB$.

Proof. Let $a_0 + a_1 x + a_2 x^2 + \cdots$, $b_0 + b_1 x + b_2 x^2 + \cdots$, $c_0 + c_1 x + c_2 x^2 + \cdots$ be three power series with radius of convergence 1 having the sums $f(x), g(x), h(x)$ respectively.

Each of the series is absolutely convergent for $0 \le x < 1$. Therefore $f(x).g(x) = h(x)$ for all x satisfying $0 \le x < 1$, by theorem 6.6.2.

Since the series $a_0 + a_1 x + a_2 x^2 + \cdots$ is convergent for $x = 1$, f is continuous at 1 and $\lim_{x \to 1-} f(x) = f(1) = A$.

Similarly, $\lim_{x \to 1-} g(x) = g(1) = B$ and $\lim_{x \to 1-} h(x) = h(1) = C$.

Since $f(x).g(x) = h(x)$ for $0 \le x < 1$ and the functions f, g, h are continuous at 1, $\lim\limits_{x \to 1-} [f(x).g(x)] = \lim\limits_{x \to 1-} h(x)$, i.e., $AB = C$.

Worked Examples (continued).

2. Let $f(x)$ be the sum of the power series $\sum\limits_{n=0}^{\infty} a_n x^n$ on $(-R, R)$ for some $R > 0$. If $f(x) = f(-x)$ for all $x \in (-R, R)$, show that $a_n = 0$ for all odd n.

$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = f(x)$ for all $x \in (-R, R)$
Therefore $a_0 - a_1 x + a_2 x^2 - a_3 x^3 + \cdots = f(-x)$ for all $x \in (-R, R)$.

As $f(x) = f(-x)$, both the power series
$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ and $a_0 - a_1 x + a_2 x^2 - a_3 x^3 + \cdots$
have the same sum $f(x)$ on $(-R, R)$.

By uniqueness theorem, $a_1 = -a_1, a_3 = -a_3, a_5 = -a_5, \cdots$
Hence $a_n = 0$ for all odd n.

3. Assuming the power series expansion for $\frac{1}{\sqrt{1-x^2}}$ as
$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}.x^4 + \frac{1.3.5}{2.4.6}x^6 + \cdots$
obtain the power series expansion for $\sin^{-1} x$.
Deduce that $1 + \frac{1}{2.3} + \frac{1.3}{2.4.5} + \frac{1.3.5}{2.4.6.7} + \cdots = \frac{\pi}{2}$.

Let $x^2 = y$. The series becomes $1 + \frac{1}{2}y + \frac{1.3}{2.4}y^2 + \frac{1.3.5}{2.4.6}y^3 + \cdots$

Let $\sum\limits_{n=0}^{\infty} a_n y^n$ be the series. Then $a_0 = 1, a_n = \frac{1.3.5\cdots(2n-1)}{2.4.6\cdots 2n}$ for $n \ge 1$.

$\lim\limits_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1.$

Hence the interval of convergence of the series is $\{y \in \mathbb{R} : -1 < y < 1\}$. It follows that the interval of convergence of the given series is $\{x \in \mathbb{R} : -1 < x < 1\}$.

Integrating term-by-term on $[0, x]$ where $|x| < 1$, we have
$x + \frac{1}{2}.\frac{x^3}{3} + \frac{1.3}{2.4}.\frac{x^5}{5} + \frac{1.3.5}{2.4.6}.\frac{x^7}{7} + \cdots = \sin^{-1} x$ for $-1 < x < 1$.

At $x = 1$, the series becomes $1 + \frac{1}{2}.\frac{1}{3} + \frac{1.3}{2.4}.\frac{1}{5} + \cdots$
The series is convergent by Raabe's test. (Ex. 1, Theorem 6.3.10)

By Abel's theorem, the sum of the series at $x = 1$ is $\sin^{-1} 1$.

At $x = -1$, the series becomes $-1 - \frac{1}{2}.\frac{1}{3} - \frac{1.3}{2.4}.\frac{1}{5} - \cdots$
This is also convergent.
By Abel's theorem the sum of the series at $x = -1$ is $\sin^{-1}(-1)$.

Hence $\sin^{-1} x = x + \frac{1}{2}.\frac{x^3}{3} + \frac{1.3}{2.4}.\frac{x^5}{5} + \frac{1.3.5}{2.4.6}.\frac{x^7}{7} + \cdots$ for $-1 \le x \le 1$

and $1 + \frac{1}{2.3} + \frac{1.3}{2.4.5} + \frac{1.3.5}{2.4.6.7} + \cdots = \sin^{-1} 1 = \frac{\pi}{2}$.

4. Assuming the expansion

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ for $-1 < x \le 1$

prove that $\int_0^1 \frac{\log(1+x)}{x} dx = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

Let us consider the series $1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots$ ... ... (i)

The radius of convergence of the series is 1. Let $\phi(x)$ be the sum of the series on $-1 < x < 1$.

$$\begin{aligned} \text{Then } \phi(x) &= \frac{\log(1+x)}{x}, \text{ for } 0 < |x| < 1 \\ &= 1, \text{ for } x = 0. \end{aligned}$$

At $x = 1$, the series is convergent. By Abel's theorem, the sum of the series at $x = 1$ is $\lim\limits_{x \to 1-} \phi(x) = \log 2$.

At $x = -1$, the series is divergent.

Hence $1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots = f(x)$ for $-1 < x \le 1$,

$$\begin{aligned} \text{where } f(x) &= \frac{\log(1+x)}{x}, \text{ for } -1 < x \le 1, x \ne 0 \\ &= 1, x = 0. \end{aligned}$$

The series (i) is uniformly convergent on $[0, 1]$. Integrating term-by-term on $[0, 1]$, we have

$\int_0^1 f(x)dx = \int_0^1 dx - \frac{1}{2}\int_0^1 xdx + \frac{1}{3}\int_0^1 x^2dx - \frac{1}{4}\int_0^1 x^3dx + \cdots$

or, $\int_0^1 \frac{\log(1+x)}{x} dx = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

5. Assuming the expansion

$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$ for $-1 \le x < 1$

prove that $\int_0^1 \log(1-x)dx = -1$.

The radius of convergence of the series $-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$ is 1.

Integrating term-by-term on $[0, x]$ where $|x| < 1$, we have

$-\frac{x^2}{2} - \frac{x^3}{2.3} - \frac{x^4}{3.4} - \cdots = \int_0^x \log(1-x)dx$ for $-1 < x < 1$.

At $x = 1$, the series becomes $-\frac{1}{1.2} - \frac{1}{2.3} - \frac{1}{3.4} - \cdots$

Let $s_n = -\frac{1}{1.2} - \frac{1}{2.3} - \frac{1}{3.4} - \cdots - \frac{1}{n(n+1)}$.

Then $s_n = -[1 - \frac{1}{n+1}]$ and $\lim s_n = -1$.

Therefore the series converges to -1 at $x = 1$.

By Abel's theorem, the sum of the series at $x = 1$ is $\lim\limits_{x \to 1-} \int_0^x \log(1-x)dx$.

Therefore $\lim\limits_{x \to 1-} \int_0^x \log(1-x)dx = -1$.

Since $\lim\limits_{x \to 1-} \int_0^x \log(1-x)dx$ exists, this limit is $\int_0^1 \log(1-x)dx$. Therefore $\int_0^1 \log(1-x)dx = -1$.

6. Find the sum of the series $\sum_{n=0}^{\infty}(2^n + 3^n)x^n$, indicating the range of validity.

Let the series be $\sum_{n=0}^{\infty} a_n x^n$. Then $a_n = 2^n + 3^n$.

$\sum_{n=0}^{\infty} 2^n x^n$ is a power series whose radius of convergence is $\frac{1}{2}$ and the sum of the series is $\frac{1}{1-2x}$ for $|x| < \frac{1}{2}$.

$\sum_{n=0}^{\infty} 3^n x^n$ is a power series whose radius of convergence is $\frac{1}{3}$ and the sum of the series is $\frac{1}{1-3x}$ for $|x| < \frac{1}{3}$.

Hence the radius of convergence of the series $\sum_{n=0}^{\infty} a_n x^n$ is $\frac{1}{3}$ and $\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1-2x} + \frac{1}{1-3x}$ for $|x| < \frac{1}{3}$.

At $x = \frac{1}{3}$ the series becomes $\sum_{n=0}^{\infty} \frac{2^n+3^n}{3^n}$.

As $\lim[(\frac{2}{3})^n + 1] \neq 0$, the series is divergent at $x = \frac{1}{3}$.
By similar arguments, the series is divergent at $x = -\frac{1}{3}$.

Hence the sum of the series is $\frac{1}{1-2x} + \frac{1}{1-3x}$ for $-\frac{1}{3} < x < \frac{1}{3}$.

7. Assuming that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ for $-1 \leq x \leq 1$ and $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$ for $-1 < x < 1$, deduce that

$\frac{1}{2}(\tan^{-1} x)^2 = \frac{1}{2}x^2 - \frac{1}{4}(1+\frac{1}{3})x^4 + \frac{1}{6}(1+\frac{1}{3}+\frac{1}{5})x^6 - \cdots$ for $-1 \leq x \leq 1$.

The radius of convergence of each of the series $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ and $1 - x^2 + x^4 - x^6 + \cdots$ is 1 and therefore both the series are absolutely convergent for $-1 < x < 1$.

So their Cauchy product will converge absolutely to the product of their sums for $-1 < x < 1$.

Let the Cauchy product be $c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$
Then $c_0 = 0, c_1 = 1, c_2 = 0, c_3 = -(1+\frac{1}{3}), c_4 = 0, c_5 = (1+\frac{1}{3}+\frac{1}{5}), \ldots$

Therefore $\frac{\tan^{-1} x}{1+x^2} = x - (1+\frac{1}{3})x^3 + (1+\frac{1}{3}+\frac{1}{5})x^5 + \cdots$ for $-1 < x < 1$.

Integrating the series term-by-term on $[0, x]$ where $|x| < 1$, we have $\frac{1}{2}(\tan^{-1} x)^2 = \frac{1}{2}x^2 - \frac{1}{4}(1+\frac{1}{3})x^4 + \frac{1}{6}(1+\frac{1}{3}+\frac{1}{5})x^6 - \cdots$ for $-1 < x < 1$.

At $x = \pm 1$ the series becomes $\frac{1}{2} - \frac{1}{4}(1+\frac{1}{3}) + \frac{1}{6}(1+\frac{1}{3}+\frac{1}{5}) - \cdots$ This is an alternating series and it is convergent by Leibnitz's test.

By Abel's theorem, $\frac{1}{2}(\tan^{-1} x)^2 = \frac{1}{2}x^2 - \frac{1}{4}(1+\frac{1}{3})x^4 + \frac{1}{6}(1+\frac{1}{3}+\frac{1}{5})x^6 - \cdots$ for $-1 \leq x \leq 1$.

Exercises 26

1. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ where

(i) $a_n = \frac{(-1)^n n^n}{n! 2^n}, n = 1, 2, \ldots, a_0 = 0$; (ii) $a_n = \frac{2^n}{n^2}, n = 1, 2, \ldots, a_0 = 0$;

(iii) $a_n = (1/3)^n$ if n be odd
$= (1/2)^n$ if n be even ;

(iv) $a_n = 2^n + 3^n,\ n \geq 1$;

(v) $a_0 = 1, a_n = (\sqrt[n]{n} + 1)^n, n \geq 1$; (vi) $a_0 = 1, 2 \leq |a_n| \leq 3$ for $n \geq 1$.

2. Find the radius of convergence of the power series

(i) $1 + 3x + \frac{3^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \cdots$

(ii) $1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \cdots$

(iii) $1 - \frac{2^2}{3^2}x + \frac{2^2.4^2}{3^2.5^2}x^2 - \frac{2^2.4^2.6^2}{3^2.5^2.7^2}x^3 + \cdots$

(iv) $1 - \frac{x}{1.2} + x^2 - \frac{x^3}{2.4} + x^4 - \frac{x^5}{4.8} + \cdots$

(v) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}(x+1)^n$

(vi) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)(n+2)}(x-2)^n$.

3. $\sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence $R(>0)$. Construct a power series $\sum_{n=0}^{\infty} b_n x^n$, other than $\sum_{n=0}^{\infty} x^n$, such that the radius of convergence of the series $\sum_{n=0}^{\infty} a_n b_n x^n$ is also R.

[**Hint.** Take $b_n = \frac{1}{n+1}$.]

4. $\sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence $R(>0)$. Construct a power series $\sum_{n=0}^{\infty} b_n x^n$, other than $\sum_{n=0}^{\infty} (x/2)^n$, such that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n b_n x^n$ is $2R$.

[**Hint.** Take $b_n = \frac{1}{2^n}$ if n be even
$= \frac{1}{3^n}$ if n be odd .]

5. Find the sum of the power series $1 + x + x^2 + \cdots$ on its interval of convergence.

Deduce the power series expansion of $\log(1-x)$ and use Abel's theorem to prove that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2$.

6. Prove that $\frac{1}{2}[\log(1-x)]^2 = \frac{1}{2}x^2 + (1 + \frac{1}{2})\frac{x^3}{3} + (1 + \frac{1}{2} + \frac{1}{3})\frac{x^4}{4} + \cdots$ for $-1 < x \leq 1$.

7. Find the sum of the power series $1+x+x^2+\cdots$ on its interval of convergence. By repeated differentiation prove that

$$(1-x)^{-3} = 1 + 3x + \tfrac{3.4}{1.2}x^2 + \tfrac{3.4.5}{1.2.3}x^3 + \cdots \text{ on } (-1,1).$$

8. Assuming the power series expansion for $(1+x)^{-1}$ as

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

obtain the power series expansion for $\log(1+x)$. Deduce that

(i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2$.

(ii) $\frac{1}{2}[\log(1+x)]^2 = \frac{1}{2}x^2 - \frac{1}{3}(1+\frac{1}{2})x^3 + \frac{1}{4}(1+\frac{1}{2}+\frac{1}{3})x^4 - \cdots$ for $-1 < x \le 1$.

9. Assuming the power series expansion for $(1+x^2)^{-1}$ as

$$(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots$$

obtain the power series expansion for $\tan^{-1} x$.

Deduce that (i) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$;

(ii) $\int_0^1 \frac{\tan^{-1} x}{x}\,dx = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots$

10. Assuming that the sum of the power series $x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ on its interval of convergence is $\log(1+x)$, deduce that

$$\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \cdots = 2\log 2 - 1.$$

11. Assuming that $\sin^{-1} x = x + \frac{1}{2}.\frac{x^3}{3} + \frac{1.3}{2.4}.\frac{x^5}{5} + \cdots$ for $-1 \le x \le 1$ prove that

$$\int_0^1 \frac{\sin^{-1} x}{x}\,dx = 1 + \frac{1}{2}.\frac{1}{3^2} + \frac{1.3}{2.4}.\frac{1}{5^2} + \cdots$$

12. Assuming the expansion $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ for $-1 < x < 1$ prove that

(i) $\int_0^1 \frac{x}{1+x}\,dx = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$

(ii) $\int_0^1 \frac{x^2}{1+x}\,dx = \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

13. Find the sum of the series indicating the range of validity.

(i) $\sum\limits_{n=0}^{\infty}(1+2^n)x^n$, (ii) $\sum\limits_{n=0}^{\infty}(n+3)x^n$.

14. Let $f(x)$ be the sum of the power series $\sum\limits_{n=0}^{\infty} a_n x^n$ on $(-R,R)$ for $R > 0$. If $f(x) + f(-x) = 0$ for all $x \in (-R,R)$ prove that $a_n = 0$ for all even n.

15. Let $f(x)$ be the sum of the power series $a_0 + a_1x + a_2x^2 + \cdots$ on R. If $f'(x) = f(x)$ for all $x \in R$ and $f(0) = 1$, prove that $a_n = \frac{1}{n!}$ for all $n \in \mathbb{N}$.

16. If a function f be defined for $|x| < R$ and if there exists a constant k such that for all $x \in (-R,R)$, $|f^n(x)| \le k$ for all $n \in \mathbb{N}$, prove that the Taylor's series $\sum\limits_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n$ converges to $f(x)$ for all $x \in (-R,R)$.

A1.1. Introduction.

$\mathbb{R}^2$ is the set of all ordered pairs $\{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$. An ordered pair $\{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$ is also called a *point* in $\mathbb{R}^2$, denoted by $\mathbf{x}$.

Definition. Let $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. We define

(i) $\mathbf{x} = \mathbf{y}$ if and only if $x_1 = x_2, y_1 = y_2$;

(ii) $\mathbf{x} + \mathbf{y} = (x_1 + x_2, y_1 + y_2)$;

(iii) $c\mathbf{x} = (cx_1, cy_1) \quad (c \in \mathbb{R})$.

Norm. Let $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. The norm of $\mathbf{x}$ is denoted by $\| \mathbf{x} \|$ and is defined by $\| \mathbf{x} \| = \sqrt{x_1^2 + x_2^2}$.

Note. Norm is a mapping from $\mathbb{R}^2$ to $\mathbb{R}$.

Properties of the norm. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. Then

(i) $\| \mathbf{x} \| \geq 0$, and $\| \mathbf{x} \| = 0$ if and only if $\mathbf{x} = \mathbf{0}$;

(ii) $\| c\mathbf{x} \| = | c | \,.\, \| \mathbf{x} \|$ for all $c \in \mathbb{R}$;

(iii) $\| \mathbf{x} + \mathbf{y} \| \leq \| \mathbf{x} \| + \| \mathbf{y} \|$ (Triangle inequality).

Note. The triangle inequality is also expressed in the form –

$\| \mathbf{x} - \mathbf{y} \| + \| \mathbf{y} - \mathbf{z} \| \leq \| \mathbf{x} - \mathbf{z} \|$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$.

A1.2. Cell in $\mathbb{R}^2$. Disc in $\mathbb{R}^2$.

Let $a_1, b_1 \in \mathbb{R}$ and $a_1 < b_1 : a_2, b_2 \in \mathbb{R}$ and $a_2 < b_2$.

Let $I_1 = \{x_1 \in \mathbb{R} : a_1 < x_1 < b_1\}$ and $I_2 = \{x_2 \in \mathbb{R} : a_2 < x_2 < b_2\}$ be open intervals in $\mathbb{R}$. Then the Cartesian product $I_1 \times I_2 = \{(x_1, x_2) \in \mathbb{R}^2 : a_1 < x_1 < b_1, a_2 < x_2 < b_2\}$ is said to be an *open cell* in $\mathbb{R}^2$.

Let $J_1 = \{x_1 \in \mathbb{R} : a_1 \leq x_1 \leq b_1\}$ and $J_2 = \{x_2 \in \mathbb{R} : a_2 \leq x_2 \leq b_2\}$ be closed intervals in $\mathbb{R}$. Then the Cartesian product $J_1 \times J_2 = \{(x_1, x_2) \in \mathbb{R}^2 : a_1 \leq x_1 \leq b_1, a_2 \leq x_2 \leq b_2\}$ is said to be a *closed cell* in $\mathbb{R}^2$.

If I_1 and I_2 be open (closed) bounded intervals in $\mathbb{R}$ then the Cartesian product $I_1 \times I_2$ is said to be an *open (closed) bounded cell* in $\mathbb{R}^2$.

An open cell in $\mathbb{R}^2$ is also called an *open rectangle* in $\mathbb{R}^2$ and a closed cell in $\mathbb{R}^2$ is also called a *closed rectangle* in $\mathbb{R}^2$.

Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$ and $\delta_1 > 0, \delta_2 > 0$. The set $S = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - a_1| < \delta_1, |x_2 - a_2| < \delta_2\}$ is said to be an *open cell about* $\mathbf{a}$ (or an *open rectangle about* $\mathbf{a}$).

This is a rectangular region in $\mathbb{R}^2$ with the centre at $\mathbf{a}$.

Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$ and $\delta > 0$. The set of all points $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ satisfying the condition $\| \mathbf{x} - \mathbf{a} \| < \delta$, i.e., $\sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} < \delta$ is said to be an *open δ-disc* about $\mathbf{a}$.

This is a circular region in $\mathbb{R}^2$ with the centre at $\mathbf{a}$ and radius δ.

It is a matter of simple verification that an open cell about $\mathbf{a}$ contains an open disc about $\mathbf{a}$ and vice-versa.

Note. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and $\delta, \delta_1, \delta_2, \ldots, \delta_n > 0$.

An open n-cell about $\mathbf{a}$ is the set of all points $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ satisfying the condition $|x_1 - a_1| < \delta_1, |x_2 - a_2| < \delta_2, \ldots, |x_n - a_n| < \delta_n$.

An open n-ball about $\mathbf{a}$ is the set of all points $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ satisfying the condition $\| \mathbf{x} - \mathbf{a} \| < \delta$, i.e., $\sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2} < \delta$ and it is denoted by $B(\mathbf{a}, \delta)$.

In particular, if $n = 1$, the open 1-ball is the open interval $(a - \delta, a + \delta)$; if $n = 2$, the open 2-ball is the open disc about $\mathbf{a}$.

A1.3. Neighbourhood of a point in $\mathbb{R}^2$.

Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$.

A set $S \subset \mathbb{R}^2$ is said to be a neighbourhood of $\mathbf{a}$ if an open cell about $\mathbf{a}$ (or an open disc about $\mathbf{a}$) is contained in S. A neighbourhood of $\mathbf{a}$ is denoted by $N(\mathbf{a})$.

Clearly, an open disc about $\mathbf{a}$ is also a neighbourhood of the point $\mathbf{a}$. This is a *circular neighbourhood* of the point $\mathbf{a}$. It is also denoted by $N(\mathbf{a}, \delta)$, if δ be the radius.

An open cell about $\mathbf{a}$ is also a neighbourhood of the point $\mathbf{a}$. This is a *rectangular neighbourhood* of the point $\mathbf{a}$ (and also a *square neighbourhood*, in particular) of the point $\mathbf{a}$.

It can be observed that a rectangular neighbourhood of a point $\mathbf{a}$ contains a circular neighbourhood of the point and vice-versa.

Any type of neighbourhood of the point $\mathbf{a}$ is denoted by $N(\mathbf{a})$.

A1.4. Interior point.

Let S be a subset of $\mathbb{R}^2$. A point $\mathbf{x}$ in S is said to be an *interior point* of S if there exists a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x})$ is contained in S.

The set of all interior points of a set $S \subset \mathbb{R}^2$ is called the *interior* of S and is denoted by int S (or by S^o).

Examples.

1. Let $S = \{(x, y) \in \mathbb{R}^2 : 1 < x < 3, 1 < y < 5\}$. The set S is an open cell.

Let $(p, q) \in S$. Then $1 < p < 3, 1 < q < 5$. There exists a positive δ_1 such that $(p - \delta_1, p + \delta_1) \subset (1, 3)$ and there exists a positive δ_2 such that $(q - \delta_2, q + \delta_2) \subset (1, 5)$. $N = \{(x, y) \in \mathbb{R}^2 : p - \delta_1 < x < p + \delta_1, q - \delta_2 < y < q + \delta_2\}$ is a neighbourhood of the point (p, q) and $N \subset S$. Therefore (p, q) is an interior point of S. Thus each point of S is an interior point of S.

2. Let $S = \{(x, y) \in \mathbb{R}^2 : x + y < 1\}$. The set S is the open half plane containing the origin bounded by the line $x + y = 1$.

Let $(a, b) \in S$. Then $a + b < 1$. Let p be the length of perpendicular from the point (a, b) upon the line $x + y = 1$. Then $p > 0$. The set $N = \{(x, y) \in \mathbb{R}^2 : \sqrt{(x - a)^2 + (y - b)^2} < p$ is a neighbourhood of (a, b) and $N \subset S$. Therefore (a, b) is an interior point of S. Thus each point of S is an interior point of S.

3. Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. The set S is the interior of the circle of unit radius with centre at $(0, 0)$.

Let $(a, b) \in S$. Then $a^2 + b^2 < 1$. Let $a^2 + b^2 = r^2$. Then $0 \leq r < 1$. Let $p = \frac{1}{2}(1 - r)$. The set $N = \{(x, y) \in \mathbb{R}^2 : \sqrt{(x - a)^2 + (y - b)^2} < p$ is a neighbourhood of (a, b) and $N \subset S$. Therefore (a, b) is an interior point of S. Thus each point of S is an interior point of S.

4. Let $S = \{(x, y) \in \mathbb{R}^2 : x \leq 1\}$. The set S is the half plane containing the origin bounded by the line $x = 1$.

No point on the line $x = 1$ is an interior point of S, because every neighbourhood of a point on the line $x = 1$ contains points not in S. Every point in S not on the line $x = 1$ is an interior point of S.

5. Let $S = \mathbb{R}^2$. Each point of S is an interior point of S.

6. Let $S = \phi$. Here int $S = \phi$, i.e., int $S = S$.

A1.5. Open set.

Let S be a subset of $\mathbb{R}^2$. S is said to be an *open set* in $\mathbb{R}^2$ if each point of S is an interior point of S.

Clearly, S is an open set in $\mathbb{R}^2$ if int $S = S$.

Examples.

1. Let $S = \{(x, y) \in \mathbb{R}^2 : 1 < x < 3, 1 < y < 5\}$. Since each point of S is an interior point of S, S is an open set.

2. Let $S = \{(x, y) \in \mathbb{R}^2 : x + y < 1\}$. Since each point of S is an interior point of S, S is an open set.

3. Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Since each point of S is an interior point of S, S is an open set.

4. Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. No point of S is an interior point of S. S is not an open set.

5. Let $S = \mathbb{R}^2$. Each point of S is an interior point of S. Therefore S is an open set.

6. Let $S = \phi$. Here int $S = S$. Therefore S is an open set.

Theorem A1.5.1. The union of a finite number of open sets in $\mathbb{R}^2$ is an open set in $\mathbb{R}^2$.

Proof. Let $G_1, G_2 \ldots, G_m$ be m open sets in $\mathbb{R}^2$.

Let $G = G_1 \cup G_2 \cup \cdots \cup G_m$.

Let $\mathbf{x} \in G$. Then $\mathbf{x}$ belongs to at least one of the sets $G_i, i = 1, 2, \ldots, m$.

Let $\mathbf{x} \in G_k$. Since G_k is an open set in $\mathbb{R}^2$, $\mathbf{x}$ is an interior point of G_k. Therefore there exists a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subset G_k$. It follows that $N(\mathbf{x}) \subset G$ and this shows that $\mathbf{x}$ is an interior point of G.

Thus every point of G is an interior point of G. Hence G is an open set. This completes the proof.

Theorem A1.5.2. The intersection of a finite number of open sets in $\mathbb{R}^2$ is an open set in $\mathbb{R}^2$.

Proof. Let $G_1, G_2 \ldots, G_m$ be m open sets in $\mathbb{R}^2$.

Let $G = G_1 \cap G_2 \cap \cdots \cap G_m$.

Case 1. Let $G = \phi$. Then G is an open set in $\mathbb{R}^2$, since ϕ is an open set in $\mathbb{R}^2$.

Case 2. Let $G \neq \phi$.

Let $\mathbf{x} \in G$. Then $\mathbf{x} \in G_i$ for $i = 1, 2, \cdots, m$.

Since G_1 is an open set in $\mathbb{R}^2$, $\mathbf{x}$ is an interior point of G_1. Therefore there exists a neighbourhood $N(\mathbf{x}, \delta_1)$ of $\mathbf{x}$ such that $N(\mathbf{x}, \delta_1) \subset G_1$.

Since G_2 is an open set in $\mathbb{R}^2$, $\mathbf{x}$ is an interior point of G_2. Therefore there exists a neighbourhood $N(\mathbf{x}, \delta_2)$ of $\mathbf{x}$ such that $N(\mathbf{x}, \delta_2) \subset G_2$.

... ...

Since G_m is an open set in $\mathbb{R}^2$, $\mathbf{x}$ is an interior point of G_m. Therefore there exists a neighbourhood $N(\mathbf{x}, \delta_m)$ of $\mathbf{x}$ such that $N(\mathbf{x}, \delta_m) \subset G_m$.

Let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_m\}$. Then $\delta > 0$ and the neighbourhood $N(\mathbf{x}, \delta)$ of $\mathbf{x}$ is such that $N(\mathbf{x}, \delta) \subset G$. This shows that $\mathbf{x}$ is an interior point of G.

Thus every point of G is an interior point of G. Hence G is an open set. This completes the proof.

Theorem A1.5.3. The union of an arbitrary collection of open sets in $\mathbb{R}^2$ is an open set in $\mathbb{R}^2$.

Proof. Let $\{G_\lambda : \lambda \in \Lambda\}$, Λ being the index set, be a collection of open sets in $\mathbb{R}^2$.

Let $G = \cup G_\lambda$. Let $\mathbf{x} \in G$. Then $\mathbf{x}$ belongs to at least one set G_α of the collection, where $\alpha \in \Lambda$.

Since G_α is an open set in $\mathbb{R}^2$ and $\mathbf{x} \in G_\alpha$, $\mathbf{x}$ is an interior point of G_α. Therefore there exists a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subset G_\alpha$. This implies $N(\mathbf{x}) \subset G$. This shows that $\mathbf{x}$ is an interior point of G.

Thus every point of G is an interior point of G hence G is an open set. This completes the proof.

Note. The intersection of an infinite collection of open sets in $\mathbb{R}^2$ is not necessarily an open set in $\mathbb{R}^2$.

Let us consider the sets
$$\begin{aligned} G_1 &= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \\ G_2 &= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < \tfrac{1}{2}\} \\ &\cdots \qquad \cdots \qquad \cdots \\ G_n &= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < \tfrac{1}{n}\} \\ &\cdots \qquad \cdots \qquad \cdots \end{aligned}$$

Each G_i is an open set in $\mathbb{R}^2$. $\bigcap\limits_{i=1}^{\infty} G_i = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 0\}$, i.e., $\{(0, 0)\}$. This is not an open set in $\mathbb{R}^2$.

Let us consider the sets
$$\begin{aligned} G_1 &= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \\ G_2 &= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 2\} \\ &\cdots \qquad \cdots \qquad \cdots \\ G_n &= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < n\} \\ &\cdots \qquad \cdots \qquad \cdots \end{aligned}$$

Each G_i is an open set in $\mathbb{R}^2$. $\bigcap\limits_{i=1}^{\infty} G_i = G_1$ and it is an open set in $\mathbb{R}^2$.

These two examples establish that the intersection of an infinite number of open sets in $\mathbb{R}^2$ is not necessarily an open set in $\mathbb{R}^2$.

Worked Examples.

1. Let G_1, G_2 be open sets in $\mathbb{R}$. Prove that $G_1 \times G_2$ is an open set in $\mathbb{R}^2$.

If one or both of G_1 and G_2 be the null set then $G_1 \times G_2 = \phi$, an open set in $\mathbb{R}^2$.

Let $(c, d) \in G_1 \times G_2$. Then $c \in G_1$ and $d \in G_2$.

Since G_1 is an open set in $\mathbb{R}$ and $c \in G_1, c$ is an interior point of G_1. Therefore there exists a positive δ_1 such that the set $N_1 = \{x \in \mathbb{R} : c - \delta_1 < x < c + \delta_1\}$ is entirely contained in G_1.

Since G_2 is an open set in $\mathbb{R}$ and $d \in G_2, d$ is an interior point of G_2. Therefore there exists a positive δ_2 such that the set $N_2 = \{y \in \mathbb{R} : d - \delta_2 < y < d + \delta_2\}$ is entirely contained in G_2.

Clearly, the set $N_1 \times N_2$ is a neighbourhood of (c, d) and $N_1 \times N_2$ is entirely contained in $G_1 \times G_2$. This shows that (c, d) is an interior point of $G_1 \times G_2$. Hence $G_1 \times G_2$ is an open set in $\mathbb{R}^2$.

Corollary. An open cell in $\mathbb{R}^2$, being the Cartesian product of two open intervals in $\mathbb{R}$, is an open set in $\mathbb{R}^2$.

2. Prove that the set $S = \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}$ is an open set in $\mathbb{R}^2$.

The set S can be expressed as $S = A \cap B \cap C \cap D$ where $A = \{(x, y) \in \mathbb{R}^2 : x + y < 1\}$, $B = \{(x, y) \in \mathbb{R}^2 : x - y < 1\}$, $C = \{(x, y) \in \mathbb{R}^2 : -x + y < 1\}$ and $D = \{(x, y) \in \mathbb{R}^2 : -x - y < 1\}$.

Each of A, B, C, D is an open set in $\mathbb{R}^2$. S being the intersection of a finite number of open sets in $\mathbb{R}^2$, is an open set in $\mathbb{R}^2$.

3. Prove that an open bounded interval in $\mathbb{R}$ is not an open set in $\mathbb{R}^2$.

Let $a, b \in \mathbb{R}$ and $a < b$. Then $S = \{x \in \mathbb{R} : a < x < b\}$ is an open bounded interval in $\mathbb{R}$. In $\mathbb{R}^2$, S can be considered as the set $T = \{(x, y) \in \mathbb{R}^2 : a < x < b, y = 0\}$.

Let $(c, 0) \in T$. Then $a < c < b$. Any neighbourhood of $(c, 0)$ contains points of T and also points not in T. Therefore $(c, 0)$ is not an interior point of T. Hence T is not an open set in $\mathbb{R}^2$.

A1.6. Closed set.

Let S be a subset of $\mathbb{R}^2$.S is said to be a *closed set* in $\mathbb{R}^2$ if the complement of S in $\mathbb{R}^2$ is an open set in $\mathbb{R}^2$.

Examples.

1. Let $S = \{(x, y) \in \mathbb{R}^2 : x + y \leq 1\}$. S is the complement of the set A in $\mathbb{R}^2$ where $A = \{(x, y) \in \mathbb{R}^2 : x + y > 1\}$.

A is an open sets in $\mathbb{R}^2$. S being the complement of an open set in $\mathbb{R}^2$, is a closed set in $\mathbb{R}^2$.

2. Let $S = \{(x, y) \in \mathbb{R}^2 : x+y = 1\}$. S is the complement of the set $A \cup B$ in $\mathbb{R}^2$ where $A = \{(x, y) \in \mathbb{R}^2 : x+y < 1\}$, $B = \{(x, y) \in \mathbb{R}^2 : x+y > 1\}$.

Since A and B are open sets in $\mathbb{R}^2$, $A \cup B$ is an open set in $\mathbb{R}^2$.S being the complement of an open set in $\mathbb{R}^2$, is an closed set in $\mathbb{R}^2$.

3. Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. S is the complement of the set $A \cup B$ in $\mathbb{R}^2$ where $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$.

Since A and B are open sets in $\mathbb{R}^2$, $A \cup B$ is an open set in $\mathbb{R}^2$.S being the complement of an open set in $\mathbb{R}^2$, is a closed set in $\mathbb{R}^2$.

4. Let S be the closed cell $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

S is the complement of the set $A \cup B \cup C \cup D$, in $\mathbb{R}^2$ where $A = \{(x, y) \in \mathbb{R}^2 : x < 0\}$, $B = \{(x, y) \in \mathbb{R}^2 : x > 1\}$, $C = \{(x, y) \in \mathbb{R}^2 : y < 0\}$, $D = \{(x, y) \in \mathbb{R}^2 : y > 1\}$.

A, B, C, D being open sets in $\mathbb{R}^2$, $A \cup B \cup C \cup D$ is an open set in $\mathbb{R}^2$. S being the complement of an open set in $\mathbb{R}^2$, is a closed set in $\mathbb{R}^2$.

Note. A closed cell in $\mathbb{R}^2$ is a closed set in $\mathbb{R}^2$.

5. Let $S = \mathbb{R}^2$. The complement of S in $\mathbb{R}^2$ is ϕ and it is an open set in $\mathbb{R}^2$. Therefore S is a closed set.

6. Let $S = \phi$. The complement of S in $\mathbb{R}^2$ is $\mathbb{R}^2$ and it is an open set. Therefore S is a closed set.

The following theorems are immediate consequences of the definition of a closed set.

Theorem A1.6.1. The union of a finite number of closed sets in $\mathbb{R}^2$ is a closed set in $\mathbb{R}^2$.

Theorem A1.6.2. The intersection of a finite number of closed sets in $\mathbb{R}^2$ is a closed set in $\mathbb{R}^2$.

Theorem A1.6.3. The intersection of an arbitrary collection of closed sets in $\mathbb{R}^2$ is a closed set in $\mathbb{R}^2$.

Note. The union of an infinite collection of closed sets in $\mathbb{R}^2$ is not necessarily a closed set in $\mathbb{R}^2$.

Let us consider the sets
$$\begin{aligned} F_1 &= \{(x,y) \in \mathbb{R}^2 : -1 \le x \le 1\} \\ F_2 &= \{(x,y) \in \mathbb{R}^2 : -2 + \tfrac{1}{2} \le x \le 2 - \tfrac{1}{2}\} \\ &\cdots \quad \cdots \quad \cdots \\ F_n &= \{(x,y) \in \mathbb{R}^2 : -2 + \tfrac{1}{n} \le x \le 2 - \tfrac{1}{n}\} \\ &\cdots \quad \cdots \quad \cdots \end{aligned}$$

Each F_i is a closed set in $\mathbb{R}^2$.

$\bigcup\limits_{i=1}^{\infty} F_i = \{(x,y) \in \mathbb{R}^2 : -2 < x < 2\}$. This is not a closed set in $\mathbb{R}^2$.

Let us consider the sets
$$\begin{aligned} F_1 &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\} \\ F_2 &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le \tfrac{1}{2}\} \\ &\cdots \quad \cdots \quad \cdots \\ F_n &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le \tfrac{1}{n}\} \\ &\cdots \quad \cdots \quad \cdots \end{aligned}$$

Each F_i is a closed set in $\mathbb{R}^2$. $\bigcup\limits_{i=1}^{\infty} F_i = F_1$ and it is a closed set in $\mathbb{R}^2$.

These two examples establish that the union of an infinite number of closed sets in $\mathbb{R}^2$ is not necessarily a closed set in $\mathbb{R}^2$.

Worked Examples.

1. Prove that a closed and bounded interval in $\mathbb{R}$ is a closed set in $\mathbb{R}^2$.

Let $a, b \in \mathbb{R}$ and $a < b$. Then $S = \{x \in \mathbb{R} : a \le x \le b\}$ is a closed and bounded interval in $\mathbb{R}$.

In $\mathbb{R}^2$, S can be considered as the set $T = \{(x,y) \in \mathbb{R}^2 : a \le x \le b, y = 0\}$.

The set T^c (the complement of T in $\mathbb{R}^2$) can be expressed as $T^c = A \cup B \cup C \cup D$ where $A = \{(x,y) \in \mathbb{R}^2 : x < a\}$, $B = \{(x,y) \in \mathbb{R}^2 : x > b\}$, $C = \{(x,y) \in \mathbb{R}^2 : y < 0\}$, $D = \{(x,y) \in \mathbb{R}^2 : y > 0\}$.

Since A, B, C, D are open sets in $\mathbb{R}^2$, $A \cup B \cup C \cup D$ is an open set in $\mathbb{R}^2$. T being the complement of an open set in $\mathbb{R}^2$, is a closed set in $\mathbb{R}^2$.

Note. An infinite closed interval in $\mathbb{R}$ is a closed set in $\mathbb{R}^2$.

2. Prove that a one-element set $\{(a,b)\}$ $[a \in \mathbb{R}, b \in \mathbb{R}]$ is a closed set in $\mathbb{R}^2$.

Let $S = \{(a,b)\}$. S can be considered as the complement of the union $A \cup B \cup C \cup D$ where $A = \{(x,y) \in \mathbb{R}^2 : x < a\}$, $B = \{(x,y) \in \mathbb{R}^2 : x > a\}$, $C = \{(x,y) \in \mathbb{R}^2 : y < b\}$, $D = \{(x,y) \in \mathbb{R}^2 : y > b\}$.

A, B, C, D are open sets in $\mathbb{R}^2$. Therefore $A \cup B \cup C \cup D$ is an open set in $\mathbb{R}^2$. S being the complement of an open set in $\mathbb{R}^2$, is a closed set in $\mathbb{R}^2$.

Note. A finite subset in $\mathbb{R}^2$ is a closed set in $\mathbb{R}^2$.

A1.7. Limit point.

Let S be a subset of $\mathbb{R}^2$. A point $\mathbf{x}$ in $\mathbb{R}^2$ is said to be a *limit point* (or, an *accumulation point*) of S if every neighbourhood of $\mathbf{x}$ contains a point of S other than $\mathbf{x}$. That is, every deleted neighbourhood of $\mathbf{x}$ contains a point of S.

If $N(\mathbf{x})$ be a neighbourhood $\mathbf{x}$ then $N'(\mathbf{x}) = N(\mathbf{x}) - \{\mathbf{x}\}$ is called a *deleted neighbourhood* of the point $\mathbf{x}$.

This is to note that a limit point of a set S may not belong to S.

The set of all limit points of a set $S \subset \mathbb{R}^2$ is called the *derived set* of S. The derived set of S is denoted by S'.

Theorem A1.7.1. If $\mathbf{x}$ be a limit point of a set $S \subset \mathbb{R}^2$ then every neighbourhood of $\mathbf{x}$ contains infinitely many points of S.

Proof. Let us assume the contrary. Let a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ contains only a finite number of points of S distinct from $\mathbf{x}$, say, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$.

Let $r = \min\{\| \mathbf{x} - \mathbf{x}_1 \|, \| \mathbf{x} - \mathbf{x}_2 \|, \cdots, \| \mathbf{x} - \mathbf{x}_m \|\}$. Then the neighbourhood $N(\mathbf{x}, \frac{r}{2})$ of $\mathbf{x}$ contains no point of S, contradicting that $\mathbf{x}$ is a limit point of S. This proves the theorem.

Corollary. A finite subset of $\mathbb{R}^2$ has no limit point.

Theorem A1.7.2. Let $S \subset \mathbb{R}^2$. S is a closed set if and only if $S' \subset S$.

Proof. Let S be a closed set in $\mathbb{R}^2$. Then S^c, the complement of S in $\mathbb{R}^2$, is an open set in $\mathbb{R}^2$.

If $S^c = \phi$, then $S = \mathbb{R}^2$ and $S' \subset S$.

If $S^c \neq \phi$, let $\mathbf{x} \in S^c$. Then there exists a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subset S^c$.

So $N(\mathbf{x}) \cap S = \phi$. This shows that $\mathbf{x}$ is not a limit point of S. That is, $\mathbf{x} \notin S'$.

$\mathbf{x} \in S^c \Rightarrow \mathbf{x} \notin S'$. Contrapositively, $\mathbf{x} \in S' \Rightarrow \mathbf{x} \notin S^c$, i.e., $\mathbf{x} \in S' \Rightarrow \mathbf{x} \in S$. Therefore $S' \subset S$.

Conversely, let S be a subset of $\mathbb{R}^2$ such that $S' \subset S$.

Let $\mathbf{x} \in S^c$. Then $\mathbf{x} \notin S$ and therefore $\mathbf{x} \notin S'$. Since $\mathbf{x}$ is not a limit point of S, there exists a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \cap S = \phi$, i.e., $N(\mathbf{x}) \subset S^c$.

$\mathbf{x} \in S^c \Rightarrow N(\mathbf{x}) \subset S^c$. Therefore $\mathbf{x}$ is an interior point of S^c.

Thus every point of S^c is an interior point of S^c. Therefore S^c is an open set in $\mathbb{R}^2$. S being the complement of an open set in $\mathbb{R}^2$, is a closed set in $\mathbb{R}^2$. This completes the proof.

A1.8. Isolated point.

Let S be a subset of $\mathbb{R}^2$. A point $\mathbf{x}$ in S is said to be an *isolated point* of S if there exists a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \cap \{\mathbf{x}\} = \{\mathbf{x}\}$.

Clearly, an isolated point of S is not a limit point of S.

Worked Example.

1. Let $S = \{(1,1), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, \frac{1}{3}), \ldots\}$.

Prove that (i) each point of S is an isolated point of S,

(ii) $(0,0)$ is a limit point of S.

(i) Let $p \in \mathbb{N}$. Then $(\frac{1}{p}, \frac{1}{p}) \in S$. Let $\mathbf{a} = (\frac{1}{p}, \frac{1}{p})$ and $\epsilon = \frac{1}{p(p+1)}$.

Then the neighbourhood $N(\mathbf{a}, \epsilon)$ defined by $\{(x,y) \in \mathbb{R}^2 : | x - \frac{1}{p} | < \epsilon, | y - \frac{1}{p} | < \epsilon\}$ contains no point of S. This proves that $(\frac{1}{p}, \frac{1}{p})$ is an isolated point of S.

Therefore every point of S is an isolated point of S.

(ii) Let $\mathbf{0} = (0,0)$.

Let $\epsilon > 0$. by Archimedean property of $\mathbb{R}$, there exists a natural number p such that $0 < \frac{1}{p} < \frac{\epsilon}{\sqrt{2}}$. Therefore $0 < \sqrt{\frac{1}{p^2} + \frac{1}{p^2}} < \epsilon$.

Then the neighbourhood $N(\mathbf{0}, \epsilon)$ defined by $\{(x,y) \in \mathbb{R}^2 : \sqrt{(x-0)^2 + (y-0)^2} < \epsilon\}$ contains a point $(\frac{1}{p}, \frac{1}{p})$ of S other than $(0,0)$.

This proves that $(0,0)$ is a limit point of S.

Note. Since the point $(0,0)$ does not belong to S, it follows that S is not a closed set in $\mathbb{R}^2$.

A1.9. Adherent point.

Let S be a subset of $\mathbb{R}^2$. A point $\mathbf{x}$ in $\mathbb{R}^2$ is said to be an *adherent point* of S if every neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ contains a point of S. [That is, if $N(\mathbf{x}) \cap S \neq \phi$.]

The set of all adherent points of a set $S \subset \mathbb{R}^2$ is called the *closure* of S. The closure of S is denoted by $\bar{S}$.

Theorem A1.9.1. $\bar{S} = S \cup S'$.

From the definition it follows that (i) if $\mathbf{x} \in S$ then $\mathbf{x} \in \bar{S}$ and (ii) if $\mathbf{x} \in S'$ then $\mathbf{x} \in \bar{S}$. Therefore $S \cup S' \subset \bar{S}$ ... (i)

Let $\mathbf{x} \notin S \cup S'$. Then $\mathbf{x} \notin S$ and $\mathbf{x} \notin S'$.

Therefore there exists a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \cap S = \phi$. This shows that $\mathbf{x}$ is not an adherent point of S, i.e., $\mathbf{x} \notin \bar{S}$.

$\mathbf{x} \notin S \cup S' \Rightarrow \mathbf{x} \notin \bar{S}$. Contrapositively, $\mathbf{x} \in \bar{S} \Rightarrow \mathbf{x} \in S \cup S'$ and therefore $\bar{S} \subset S \cup S'$ ... (ii)

From (i) and (ii) $S \cup S' = \bar{S}$.

Theorem A1.9.2. A set S in $\mathbb{R}^2$ is a closed set in $\mathbb{R}^2$ if and only if $S = \bar{S}$.

Proof. Let S be a closed set in $\mathbb{R}^2$. Then $\mathbb{R}^2 - S$ is an open set in $\mathbb{R}^2$.

Let $\mathbf{x} \in \mathbb{R}^2 - S$. Then there exists a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subset \mathbb{R}^2 - S$.

Therefore $N(\mathbf{x}) \cap S = \phi$. This implies $\mathbf{x} \notin \bar{S}$.

$\mathbf{x} \in \mathbb{R}^2 - S \Rightarrow \mathbf{x} \notin \bar{S}$. Contrapositively, $\mathbf{x} \in \bar{S} \Rightarrow \mathbf{x} \notin \mathbb{R}^2 - S$, i.e., $\mathbf{x} \in S$.

Therefore $\bar{S} \subset S$. Also by definition, $S \subset \bar{S}$. Combining, we have $S = \bar{S}$.

Conversely, let $S = \bar{S}$.

Let $\mathbf{x} \in \mathbb{R}^2 - S$. Then $\mathbf{x} \notin S$ and therefore $\mathbf{x} \notin \bar{S}$, since $S = \bar{S}$.

Therefore there exists a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \cap S = \phi$. That is, $N(\mathbf{x}) \subset \mathbb{R}^2 - S$.

Thus $\mathbf{x} \in \mathbb{R}^2 - S \Rightarrow N(\mathbf{x}) \subset \mathbb{R}^2 - S$, showing that $\mathbf{x}$ is an interior point of $\mathbb{R}^2 - S$. This proves that $\mathbb{R}^2 - S$ is an open set. Therefore S is a closed set in $\mathbb{R}^2$.

This completes the proof.

Theorem A1.9.3. Let $S \subset \mathbb{R}^2$. Then $\bar{S}$ is a closed set in $\mathbb{R}^2$.

Proof left to the reader.

A1.10. Nested cells in $\mathbb{R}^2$.

Let $\{I_n : n \in \mathbb{N}\}$ be a family of cells in $\mathbb{R}^2$ such that $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$, then the family $\{I_n : n \in \mathbb{N}\}$ is said to be a family of *nested cells* in $\mathbb{R}^2$.

Theorem A1.10.1. (Theorem on nested cells in $\mathbb{R}^2$)

Let $\{I_1, I_2, I_3 \ldots\}$ be a family of non-empty closed and bounded cells such that $I_1 \supset I_2 \supset I_3 \supset \cdots$ Then $\bigcap_{n=1}^{\infty} I_n$ is non-empty.

Proof. Let $I_1 = \{(x_1, x_2) \in \mathbb{R}^2 : a_{11} \le x_1 \le b_{11}, a_{21} \le x_2 \le b_{21}\}$,

$$I_2 = \{(x_1, x_2) \in \mathbb{R}^2 : a_{12} \le x_1 \le b_{12}, a_{22} \le x_2 \le b_{22}\},$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$I_n = \{(x_1, x_2) \in \mathbb{R}^2 : a_{1n} \le x_1 \le b_{1n}, a_{2n} \le x_2 \le b_{2n}\},$$

$$\cdots \qquad \cdots \qquad \cdots$$

Then $I_1 = [a_{11}, b_{11}] \times [a_{21}, b_{21}], I_2 = [a_{12}, b_{12}] \times [a_{22}, b_{22}], \quad \cdots, I_n = [a_{1n}, b_{1n}] \times [a_{2n}, b_{2n}], \cdots$

Since $I_1 \supset I_2 \supset I_3 \supset \cdots$ it follows that $[a_{11}, b_{11}] \supset [a_{12}, b_{12}] \supset [a_{13}, b_{13}] \supset \cdots$ and $[a_{21}, b_{21}] \supset [a_{22}, b_{22}] \supset [a_{23}, b_{23}] \supset \cdots$

$\{[a_{1n}, b_{1n}] : n \in \mathbb{N}\}$ is a family of nested closed and bounded intervals in $\mathbb{R}$. By the theorem on nested intervals, there exists a point $x_1 \in \mathbb{R}$ such that $x_1 \in \bigcap_{i=1}^{\infty} [a_{1i}, b_{1i}]$.

$\{[a_{2n}, b_{2n}] : n \in \mathbb{N}\}$ is a family of nested closed and bounded intervals in $\mathbb{R}$. By the theorem on nested intervals, there exists a point $x_2 \in \mathbb{R}$ such that $x_2 \in \bigcap_{i=1}^{\infty} [a_{2i}, b_{2i}]$.

Therefore $(x_1, x_2) \in \bigcap_{n=1}^{\infty} I_n$, showing that $\bigcap_{n=1}^{\infty} I_n$ is non-empty.

This completes the proof.

A1.11. Bounded set in $\mathbb{R}^2$.

Definition. Let S be a subset of $\mathbb{R}^2$. S is said to be a *bounded set* in $\mathbb{R}^2$ if there exists a positive real number b such that $\| \mathbf{x} \| \le b$ for all $\mathbf{x} \in S$.

If $\mathbf{x} = (x_1, x_2)$ then $| x_1 | \le \| \mathbf{x} \|$ and $| x_2 | \le \| \mathbf{x} \|$. It follows that if S is a bounded set then S is contained in the closed cell $[-b, b] \times [-b, b]$.

Diameter of a set. Let S be a subset of $\mathbb{R}^2$. The *diameter* of S is the supremum of the set $\{\| \mathbf{x} - \mathbf{y} \| : \mathbf{x} \in S, \mathbf{y} \in S\}$. It is denoted by $d(S)$.

Definition. Let S be a subset of $\mathbb{R}^2$. S is said to be a *bounded set* if $d(S)$ is finite.

Note. These two definitions are equivalent.

Theorem A1.11.1. Bolzano-Weierstrass theorem.

Every bounded infinite subset of $\mathbb{R}^2$ has at least one limit point.

Proof. Let S be a bounded subset of $\mathbb{R}^2$ containing infinite number of points. Since S is a bounded set, there exists a closed cell I_1 in $\mathbb{R}^2$ such that $S \subset I_1$.

Let $I_1 = \{(x_1, x_2) \in \mathbb{R}^2 : a_{11} \le x_1 \le b_{11}, a_{21} \le x_2 \le b_{21}\}$.

Let $l(I_1) = \max\{b_{11} - a_{11}, b_{21} - a_{21}\}$. Then $d(I_1)=$ the diameter of $I_1 \le \sqrt{2}l(I_1)$.

Let us divide I_1 into four closed subcells by bisecting each side of the rectangle. At least one of these subcells must contain infinitely many elements of S. We call one such I_2.

Let $I_2 = \{(x_1, x_2) \in \mathbb{R}^2 : a_{12} \le x_1 \le b_{12}, a_{22} \le x_2 \le b_{22}\}$.

Then $I_2 \subset I_1$ and $d(I_2) = \frac{1}{2}d(I_1)$. I_2 contains infinite number of points of S.

Let us divide I_2 into four closed subcells by bisecting each side of the rectangle. At least one of these subcells must contain infinitely many elements of S. We call one such I_3.

Then $I_3 \subset I_2 \subset I_1$ and $d(I_3) = \frac{1}{4}d(I_1)$.$I_3$ contains infinite number of points of S.

Continuing thus we obtain a family of closed cells $\{I_1, I_2, I_3, \ldots\}$ in $\mathbb{R}^2$ such that for all $n \in \mathbb{N}$,
(i) $I_{n+1} \subset I_n$, (ii) I_n contains infinite number of points of S and (iii) $d(I_n) = \frac{1}{2^{n-1}}d(I_1)$.

Since each I_n is a closed and bounded cell, by the theorem on nested cells in $\mathbb{R}^2$ there exists a point $\mathbf{x} = (x_1, x_2)$ in $\mathbb{R}^2$ such that $\mathbf{x} \in \bigcap\limits_{n=1}^{\infty} I_n$ (i)

We prove that $\mathbf{x}$ is a limit point of S.

Since $d(I_1) \in \mathbb{R}$ and $d(I_n) = \frac{1}{2^{n-1}}d(I_1)$ for all $n \in \mathbb{N}$, the sequence $\{d(I_n)\}$ is a null sequence in $\mathbb{R}$.

Let us choose a positive ϵ. Then there exists a natural number m such that $d(I_n) < \epsilon$ for all $n \ge m$ ... (ii)

By (i), $\mathbf{x} \in I_m$ and by (ii), $d(I_m) < \epsilon$. Therefore $I_m \subset N(\mathbf{x}, \epsilon)$.

Since I_m contains infinite number of points of S, the neighbourhood $N(\mathbf{x}, \epsilon)$ of $\mathbf{x}$ contains infinite number of points of S

Since ϵ is arbitrary, it follows that $\mathbf{x}$ is a limit point of S.

This proves the existence of a limit point of S.

A1.12. Cover of a set in $\mathbb{R}^2$.

Let S be a subset of $\mathbb{R}^2$ and C be a collection of sets in $\mathbb{R}^2$ given by $\{A_\alpha : \alpha \in \Lambda\}, \Lambda$ being the index set. C is said to be a *cover* of S if

$S \subset \bigcup_{\alpha \in \Lambda} A_\alpha$.

Let $\mathcal{G}$ be a collection of open sets in $\mathbb{R}^2$ given by $\{A_\alpha : \alpha \in \Lambda\}, \Lambda$ being the index set. $\mathcal{G}$ is said to be an *open cover* of S if $S \subset \bigcup_{\alpha \in \Lambda} A_\alpha$.

Let $\mathcal{G}$ be a collection of sets in $\mathbb{R}^2$ such that $\mathcal{G}$ covers S. If $\mathcal{G}'$ be a subcollection of $\mathcal{G}$ such that G' also covers S, then $\mathcal{G}'$ is said to be a *subcover* of the cover $\mathcal{G}$.

If the subcollection $\mathcal{G}'$ contains a finite number of sets of $\mathcal{G}$ and $\mathcal{G}'$ covers S, then $\mathcal{G}'$ is said to be a *finite subcover* of the cover $\mathcal{G}$.

Worked Example.

1. Let $I_n = \{(x,y) \in \mathbb{R}^2 : -n < x < n, -n < y < n\}, n = 1,2,3,\ldots$ and $\mathcal{G} = \{I_n : n \in \mathbb{N}\}$. Prove that the family $\mathcal{G}$ is an open cover of the set $\mathbb{R}^2$. Show that there is no finite subfamily $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers the set $\mathbb{R}^2$.

Let $(p,q) \in \mathbb{R}^2$. Then $p, q \in \mathbb{R}$ and $|p| \geq 0, |q| \geq 0$. There exist natural numbers u, v such that $u - 1 \leq |p| < u, v - 1 \leq |q| < v$. Let $w=$ max $\{u,v\}$. Then $w \in \mathbb{N}$ and $(p,q) \in I_w$.

$(p,q) \in \mathbb{R}^2 \Rightarrow (p,q) \in \bigcup_{n \in \mathbb{N}} I_n$ and this implies $\mathbb{R}^2 \in \bigcup_{n=1}^{\infty} I_n$. Therefore $\mathcal{G}$ is an open cover of the set $\mathbb{R}^2$.

Let $\mathcal{G}' = \{I_{r_1}, I_{r_2}, \ldots, I_{r_m}\}$ be a finite subcollection of $\mathcal{G}$ such that $\mathcal{G}'$ covers the set $\mathbb{R}^2$.

Then $\mathbb{R}^2 \subset I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m} \quad \ldots \quad$ (i)

Let $p =$ max $\{r_1, r_2, \ldots, r_m\}$. Then $I_{r_1} \subset I_p, I_{r_2} \subset I_p, \cdots, I_{r_m} \subset I_p$ and therefore $I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m} \subset I_p$.

From (i) it follows that $\mathbb{R}^2 \subset I_p$ but this cannot be, since $(p,p) \in \mathbb{R}^2$ but $(p,p) \notin I_p$.

Therefore $\mathcal{G}'$ cannot cover the set $\mathbb{R}^2$. So no finite subfamily of $\mathcal{G}$ can be a cover of the set $\mathbb{R}^2$.

Compact set. Let S be a subset of $\mathbb{R}^2$. S is said to be a *compact set* in $\mathbb{R}^2$ if every open cover $\mathcal{G}$ of S has a finite subcover. That is, if $\mathcal{G}$ be a collection of open sets in $\mathbb{R}^2$ that covers S then there exists a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers S.

Heine-Borel theorem. Let S be a closed and bounded set in $\mathbb{R}^2$. Then every open cover of S has a finite subcover.

Heine-Borel theorem states that a closed and bounded set in $\mathbb{R}^2$ is a compact set in $\mathbb{R}^2$.

Converse of Heine-Borel theorem. A compact set in $\mathbb{R}^2$ is a closed and bounded set in $\mathbb{R}^2$.

The Heine-Borel theorem and its converse characterise the compact sets in $\mathbb{R}^2$. The closed and bounded sets in $\mathbb{R}^2$ are the only compact sets in $\mathbb{R}^2$.

Worked Examples.

1. Let T be a finite subset of $\mathbb{R}^2$. Using the definition of a compact set, show that T is a compact set in $\mathbb{R}^2$.

Let $T = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m\}$ be a finite set in $\mathbb{R}^2$. Let $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$, Λ being the index set, be an open cover of T.

Each $\mathbf{x}_i$ is contained in some open set G_{α_i} of the collection $\mathcal{G}$. Then the union of the finite collection $\{G_{\alpha_1}, G_{\alpha_2}, \cdots, G_{\alpha_m}\}$ is a cover of T.

So $\mathcal{G}$ has a finite subcover. Since $\mathcal{G}$ is arbitrary, every open cover of T has a finite subcover. Consequently, T is compact in $\mathbb{R}^2$.

2. Show that the set $S = \{(1,1), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, \frac{1}{3}), \ldots\}$ is not a compact set in $\mathbb{R}^2$.

$(0,0)$ is a limit point of S and $(0,0)$ does not belong to S. Therefore the set S is not a closed set in $\mathbb{R}^2$. [Worked Ex.1, page 620.]

S is not a compact set in $\mathbb{R}^2$, since a compact set in $\mathbb{R}^2$ is a closed and bounded set in $\mathbb{R}^2$.

Exercises

1. Show that the set S is an open set in $\mathbb{R}^2$.
   (i) $S = \{(x,y) \in \mathbb{R}^2 : |x| < 1\}$,
   (ii) $S = \{(x,y) \in \mathbb{R}^2 : |x| > 1\}$,
   (iii) $S = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0, x + y < 1\}$,
   (iv) $S = \{(x,y) \in \mathbb{R}^2 : x > 0, x^2 + y^2 < 1\}$.
2. Show that the set S is a closed set in $\mathbb{R}^2$.
   (i) $S = \{(x,y) \in \mathbb{R}^2 : |x| \leq 1\}$,
   (ii) $S = \{(x,y) \in \mathbb{R}^2 : |x| = 1\}$,
   (iii) $S = \{(x,y) \in \mathbb{R}^2 : 2x + 3y = 1\}$,
   (iv) $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$,
3. Show that the set S is not a compact set in $\mathbb{R}^2$.
   (i) $S = \{(x,y) \in \mathbb{R}^2 : |x| \geq 1\}$,
   (ii) $S = \{(x,y) \in \mathbb{R}^2 : x = y\}$,

(iii) $S = \{(1,1), (1, \frac{1}{2}), (1, \frac{1}{3}), (1, \frac{1}{4}), \cdots\}$,

(iv) $S = \{(\frac{1}{m}, \frac{1}{n}) : m \in \mathbb{N}, n \in \mathbb{N}\}$,

(v) $S = \{(1,1), (2,2), (3,3), (4,4), \cdots\}$.

4. Use the definition of a compact set in $\mathbb{R}^2$ to prove that

(i) the union of two compact sets in $\mathbb{R}^2$ is a compact set in $\mathbb{R}^2$;

(ii) the intersection of two compact sets in $\mathbb{R}^2$ is a compact set in $\mathbb{R}^2$.

5. Define a compact set in $\mathbb{R}^2$. Use the definition to prove that

(i) the set $\mathbb{Z} \times \mathbb{Z}$ is not a compact set in $\mathbb{R}^2$;

(ii) the set $\mathbb{N} \times \mathbb{N}$ is not a compact set in $\mathbb{R}^2$.

[**Hint.** Let $C_n = \{(x, y) \in \mathbb{R}^2 : |x| < n, |y| < n\}$, $n = 1, 2, 3, \ldots$ and $\mathcal{G} = \{C_n : n \in \mathbb{N}\}$. Show that $\mathcal{G}$ is an open cover of the set having no finite subcover.]

6. Let $C_n = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < n\}, n = 1, 2, 3, \ldots$ and $\mathcal{G} = \{C_n : n \in \mathbb{N}\}$. Show that

(i) $\mathcal{G}$ is an open cover of the set $\mathbb{R}^2$;

(ii) $\mathcal{G}$ has no finite subcover.

7. Let A and B be subsets of $\mathbb{R}^2$ of which A is closed in $\mathbb{R}^2$ and B is compact in $\mathbb{R}^2$. Prove that $A \cap B$ is a compact set in $\mathbb{R}^2$.

A2.1. Introduction.

$\mathbb{R}^2$ is the set of all ordered pairs $\{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$.

A sequence in $\mathbb{R}^2$ is a mapping $X : \mathbb{N} \to \mathbb{R}^2$. To each natural number n the X-image, generally denoted by X_n, is an element in $\mathbb{R}^2$ and it is called the *nth element* of the sequence X. Let $X_n = (x_{1n}, x_{2n})$. Then the elements of the sequence are $(x_{11}, x_{21}), (x_{12}, x_{22}), (x_{13}, x_{23}), \quad \ldots \quad \ldots$

The sequence X is also denoted by the symbol $\{X_n\}$. The symbol $\{(x_{11}, x_{21}), (x_{12}, x_{22}), (x_{13}, x_{23}), \quad \ldots \quad \ldots\}$ is also used to describe the sequence X. The elements of the sequence have an order induced by the order of the natural numbers. The image set of the sequence is $\{X_n : n \in \mathbb{N}\}$.

Examples.

1. Let $X : \mathbb{N} \to \mathbb{R}^2$ be defined by $X_n = (n, \frac{1}{n}), n \in \mathbb{N}$. Then $X_1 = (1, 1), X_2 = (2, \frac{1}{2}), X_3 = (3, \frac{1}{3}), \quad \ldots \quad \ldots$ The sequence is $\{(1, 1), (2, \frac{1}{2}), (3, \frac{1}{3}), \quad \ldots \quad \ldots\}$.

2. Let $X : \mathbb{N} \to \mathbb{R}^2$ be defined by $X_n = ((-1)^n, (-1)^{n+1}), n \in \mathbb{N}$. The sequence is $\{(-1, 1), (1, -1), (-1, 1), \quad \ldots \quad \ldots\}$. The image set of the sequence is $\{(-1, 1), (1, -1)\}$, a set containing only two elements.

3. Let $X : \mathbb{N} \to \mathbb{R}^2$ be defined by $X_n = (\frac{1}{n}, 1 - \frac{1}{n}), n \in \mathbb{N}$. The sequence is $\{(1, 0), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, \frac{2}{3}), (\frac{1}{4}, \frac{3}{4}), \quad \ldots \quad \ldots\}$.

4. Let $X : \mathbb{N} \to \mathbb{R}^2$ be defined by $X_n = (1, 1)$ for all $n \in \mathbb{N}$. The sequence is $\{(1, 1), (1, 1), (1, 1), \quad \ldots \quad \ldots\}$. This is called a *constant sequence.*

A2.2. Bounded sequence.

A sequence $\{X_n\}$ in $\mathbb{R}^2$ is said to be a *bounded sequence* if there exists a positive real number b such that $\| X_n \| \leq b$ for all $n \in \mathbb{N}$, where $\| X_n \| = \sqrt{{x_{1n}}^2 + {x_{2n}}^2}$.

Examples.

1. Let $X_n = (\frac{1}{2^n}, \frac{1}{3^n})$. Then the sequence $\{X_n\}$ is a bounded sequence, because $\| X_n \| = \sqrt{(\frac{1}{2^n})^2 + (\frac{1}{3^n})^2} < \frac{1}{\sqrt{2}}$ for all $n \in \mathbb{N}$.

2. Let $X_n = ((-1)^n, (-1)^{n+1})$. Then the sequence $\{X_n\}$ is a bounded sequence because $\| X_n \| = \sqrt{(-1)^{2n} + (-1)^{2n+2}} = \sqrt{2}$ for all $n \in \mathbb{N}$.

3. Let $X_n = (n, \frac{1}{n})$. Then the sequence $\{X_n\}$ is not a bounded sequence.

Theorem A2.2.1. A sequence $\{X_n\}$ in $\mathbb{R}^2$ where $X_n = (x_{1n}, x_{2n})$ for all $n \in \mathbb{N}$, is a bounded sequence if and only if both the sequences $\{x_{1n}\}$ and $\{x_{2n}\}$ are bounded.

[The sequences $\{x_{1n}\}$ and $\{x_{2n}\}$ are called the *co-ordinate sequences* of the sequence $\{X_n\}$.]

Proof. Let $\{X_n\}$ be a bounded sequence. Then there exists a positive real number b such that $\| X_n \| \leq b$ for all $n \in \mathbb{N}$,

i.e., $\sqrt{{x_{1n}}^2 + {x_{2n}}^2} \leq b$ for all $n \in \mathbb{N}$.

But $\sqrt{{x_{1n}}^2 + {x_{2n}}^2} \geq | x_{1n} |$ and also $\sqrt{{x_{1n}}^2 + {x_{2n}}^2} \geq | x_{2n} |$ for all $n \in \mathbb{N}$.

Hence $| x_{1n} | \leq b$ as well as $| x_{2n} | \leq b$ for all $n \in \mathbb{N}$.

This implies that $\{x_{1n}\}$ and $\{x_{2n}\}$ are both bounded sequences.

Conversely, let $\{x_{1n}\}$ and $\{x_{2n}\}$ be both bounded sequences.

Then there exist positive real numbers b_1, b_2 such that

$$| x_{1n} | \leq b_1, | x_{2n} | \leq b_2 \text{ for all } n \in \mathbb{N}.$$

Let $b = \max \{b_1, b_2\}$.

Then $\| X_n \| = \sqrt{x_{1n}^2 + x_{2n}^2} \leq \sqrt{2}b$ for all $n \in \mathbb{N}$.

This proves that $\{X_n\}$ is a bounded sequence.

A2.3. Limit of a sequence in $\mathbb{R}^2$.

Let $\{X_n\}$ be a sequence in $\mathbb{R}^2$. An element $\mathbf{x}$ in $\mathbb{R}^2$ is said to be a *limit* of $\{X_n\}$ if for a pre-assigned positive ϵ there exists a natural number k such that

$$\| X_n - \mathbf{x} \| < \epsilon \text{ for all } n \geq k.$$

Theorem A2.3.1. A sequence in $\mathbb{R}^2$ can have at most one limit.

Proof. If possible, let a sequence $\{X_n\}$ in $\mathbb{R}^2$ have two distinct limits $\mathbf{x}'$ and $\mathbf{x}''$.

Let us choose $\epsilon = \frac{1}{2} \| \mathbf{x}' - \mathbf{x}'' \|$. Then the ϵ -balls $B(\mathbf{x}', \epsilon)$ and $B(\mathbf{x}'', \epsilon)$ are disjoint.

[Note that $B(\mathbf{x}', \epsilon) = \{\mathbf{x} \in \mathbb{R}^2 : \| \mathbf{x}' - \mathbf{x} \| < \epsilon\}, B(\mathbf{x}'', \epsilon) = \{\mathbf{x} \in \mathbb{R}^2 : \| \mathbf{x}'' - \mathbf{x} \| < \epsilon\}$.]

Since $\mathbf{x}'$ is a limit of the sequence, there exists a natural number k_1 such that $\| X_n - \mathbf{x}' \| < \epsilon$ for all $n \geq k_1$.

Since $\mathbf{x}''$ is a limit of the sequence, there exists a natural number k_2 such that $\| X_n - \mathbf{x}'' \| < \epsilon$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $\| X_k - \mathbf{x}' \| < \epsilon$ as well as $\| X_k - \mathbf{x}'' \| < \epsilon$.

Consequently, $X_k \in B(\mathbf{x}', \epsilon)$ and $X_k \in B(\mathbf{x}'', \epsilon)$ i.e., $X_k \in B(\mathbf{x}', \epsilon) \cap B(\mathbf{x}'', \epsilon)$.

But $B(\mathbf{x}', \epsilon) \cap B(\mathbf{x}'', \epsilon) = \phi$ and we arrive at a contradiction.

Therefore our assumption is wrong and this proves the theorem.

A2.4. Convergent sequence.

A sequence in $\mathbb{R}^2$ is said to be a *convergent sequence* if it has a limit $\mathbf{x}$ in $\mathbb{R}^2$. In this case we also say that the sequence $\{X_n\}$ *converges to* $\mathbf{x}$. We write $\lim X_n = \mathbf{x}$.

If a sequence $\{X_n\}$ has no limit then we say that the sequence $\{X_n\}$ is *divergent.*

The following theorem establishes a connection between the convergence of $\{X_n\}$ and the convergence of the co-ordinate sequences.

Theorem A2.4.1. A sequence $\{X_n\}$ in $\mathbb{R}^2$ where $X_n = (x_{1n}, x_{2n}), n \in \mathbb{N}$ converges to an element $\mathbf{x} = (x_1, x_2)$ in $\mathbb{R}^2$ if and only if the real sequences $\{x_{1n}\}$ and $\{x_{2n}\}$ converge to x_1 and x_2 respectively.

Proof. Since $\lim X_n = \mathbf{x}$, for a pre-assigned positive ϵ, there exists a natural number k such that

$$\| X_n - \mathbf{x} \| < \epsilon \text{ for all } n \geq k.$$

$$\begin{aligned} \text{But } \| X_n - \mathbf{x} \| &= \sqrt{(x_{1n} - x_1)^2 + (x_{2n} - x_2)^2} \\ &\geq | x_{1n} - x_1 |. \\ \text{Also } \| X_n - \mathbf{x} \| &\geq | x_{2n} - x_2 |. \end{aligned}$$

Therefore $| x_{1n} - x_1 | < \epsilon$ and $| x_{2n} - x_2 | < \epsilon$ for all $n \geq k$.

This shows that the real sequences $\{x_{1n}\}$ and $\{x_{2n}\}$ are convergent with the limits x_1 and x_2 respectively.

Conversely, let us suppose that the sequences $\{x_{1n}\}$ and $\{x_{2n}\}$ converge to x_1 and x_2 respectively.

Let $\epsilon > 0$. There exist natural numbers k_1 and k_2 such that

$| x_{1n} - x_1 | < \frac{\epsilon}{\sqrt{2}}$ for all $n \geq k_1$ and $| x_{2n} - x_2 | < \frac{\epsilon}{\sqrt{2}}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$.

Then $| x_{1n} - x_1 | < \frac{\epsilon}{\sqrt{2}}$ and $| x_{2n} - x_2 | < \frac{\epsilon}{\sqrt{2}}$ for all $n \geq k$.

Now $\| (x_{1n}, x_{2n}) - (x_1, x_2) \| = \sqrt{(x_{1n} - x_1)^2 + (x_{2n} - x_2)^2} < \epsilon$ for all $n \geq k$.

That is, $\| X_n - \mathbf{x} \| < \epsilon$ for all $n \geq k$. This proves that the sequence $\{X_n\}$ converges to $\mathbf{x}$.

Examples.

1. Let $X_n = (\frac{1}{n}, \frac{1}{n+1}), n \geq 1$. Prove that the sequence $\{X_n\}$ is convergent.

Let $X_n = (x_{1n}, x_{2n})$. Then $x_{1n} = \frac{1}{n}$ for $n \geq 1$; $x_{2n} = \frac{1}{n+1}$ for $n \geq 1$.
Now $\lim x_{1n} = 0, \lim x_{2n} = 0$. Therefore $\lim X_n = (0, 0)$.

2. Let $X_n = ((-1)^n, \frac{1}{n}), n \geq 1$. Prove that the sequence $\{X_n\}$ is divergent.

Let $X_n = (x_{1n}, x_{2n})$. Then $x_{1n} = (-1)^n$ and $x_{2n} = \frac{1}{n}$ for $n \geq 1$.
$\{x_{1n}\}$ is a divergent sequence.

Therefore $\{X_n\}$ is a divergent sequence, because the convergence of the sequence $\{X_n\}$ implies the convergence of both the real sequences $\{x_{1n}\}$ and $\{x_{2n}\}$.

Theorem A2.4.2. A convergent sequence in $\mathbb{R}^2$ is bounded.

Proof. Let $\{X_n\}$ be a convergent sequence in $\mathbb{R}^2$. Let $\lim X_n = \mathbf{x}$.

Let $\epsilon = 1$. Then there exists a natural number k such that

$$\| X_n - \mathbf{x} \| < 1 \text{ for all } n \geq k.$$

$\| X_n \| = \| X_n - \mathbf{x} + \mathbf{x} \| \leq \| X_n - \mathbf{x} \| + \| \mathbf{x} \|$.

Therefore $\| X_n \| \leq 1 + \| \mathbf{x} \|$ for all $n \geq k$.

Let $b = \max\{\| X_1 \|, \| X_2 \|, \ldots, \| X_{k-1} \|, \| \mathbf{x} \| + 1\}$. Then $\| X_n \| \leq b$ for all $n \in \mathbb{N}$. This proves that the sequence $\{X_n\}$ is bounded.

Note. The converse of the theorem is not true.

The sequence $\{((-1)^n, \frac{1}{n})\}$ is a bounded sequence but it not a convergent sequence.

Theorem A2.4.3. Let $\{X_n\}$ and $\{Y_n\}$ be two convergent sequences in $\mathbb{R}^2$ and $\lim X_n = \mathbf{x}, \lim Y_n = \mathbf{y}$. Then

(i) $\lim(X_n + Y_n) = \mathbf{x} + \mathbf{y}$,

(ii) $\lim cX_n = c\mathbf{x}$, c being a real number.

Proof. (i) To show that $\lim(X_n + Y_n) = \mathbf{x} + \mathbf{y}$, we need to establish that for a pre-assigned positive ϵ there exists a natural number k such that

$$\| (X_n + Y_n) - (\mathbf{x} + \mathbf{y}) \| < \epsilon \text{ for all } n \geq k.$$

$\| (X_n + Y_n) - (\mathbf{x} + \mathbf{y}) \| \leq \| X_n - \mathbf{x} \| + \| Y_n - \mathbf{y} \|$, by triangle inequality.

Let $\epsilon > 0$. Since $\lim X_n = \mathbf{x}$ and $\lim Y_n = \mathbf{y}$, there exist natural numbers k_1 and k_2 such that $\| X_n - \mathbf{x} \| < \frac{\epsilon}{2}$ for all $n \geq k_1$ and $\| Y_n - \mathbf{y} \| < \frac{\epsilon}{2}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$.

Then $\| X_n - \mathbf{x} \| < \frac{\epsilon}{2}$ and $\| Y_n - \mathbf{y} \| < \frac{\epsilon}{2}$ for all $n \geq k$.

Therefore $\| (X_n + Y_n) - (\mathbf{x} + \mathbf{y}) \| < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim(X_n + Y_n) = \mathbf{x} + \mathbf{y}$.

(ii) Proof left to the reader.

Theorem A2.4.4. Let $\{X_n\}$ be a convergent sequence in $\mathbb{R}^2$ and let $\{c_n\}$ be a sequence in $\mathbb{R}$ that converges to $c \in \mathbb{R}$. Then the sequence $\{c_n X_n\}$ in $\mathbb{R}^2$ converges to $c\mathbf{x}$.

Proof.
$$\begin{aligned} \| c_n X_n - c\mathbf{x} \| &= \| c_n X_n - cX_n + cX_n - c\mathbf{x} \| \\ &\leq \| c_n X_n - cX_n \| + | c | \| X_n - \mathbf{x} \| \\ &= | c_n - c | \| X_n \| + | c | \| X_n - \mathbf{x} \| . \end{aligned}$$

Since $\{X_n\}$ is a convergent sequence, it is bounded. Therefore there exists a positive number b_1 such that $\| X_n \| \leq b_1$ for all $n \in \mathbb{N}$.

Let $b = \max\{b_1, | c |\}$. Then $b > 0$ and
$$\| c_n X_n - c\mathbf{x} \| \leq b \, | c_n - c | + b \, \| X_n - \mathbf{x} \| .$$

Let $\epsilon > 0$. Since $\lim c_n = c$ and $\lim X_n = \mathbf{x}$ there exist natural numbers k_1, k_2 such that $| c_n - c | < \frac{\epsilon}{2b}$ for all $n \geq k_1$ and $\| X_n - \mathbf{x} \| < \frac{\epsilon}{2b}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$.

Then $\| c_n X_n - c\mathbf{x} \| < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim c_n X_n = c\mathbf{x}$.

Theorem A2.4.5. Let $\{X_n\}, \{Y_n\}$ be two convergent sequences in $\mathbb{R}^2$ and $\lim X_n = \mathbf{x}, \lim Y_n = \mathbf{y}$.

Let the sequence $\{X_n . Y_n\}$ be defined by
$$\begin{aligned} X_n . Y_n &= (x_{1n}, x_{2n}).(y_{1n}, y_{2n}) \\ &= x_{1n} y_{1n} + x_{2n} y_{2n} \quad \text{(the inner product of } X_n \text{ and } Y_n\text{)}. \end{aligned}$$

Then the real sequence $\{X_n . Y_n\}$ converges to $\mathbf{x}.\mathbf{y}$.

Proof. $| X_n . Y_n - \mathbf{x}.\mathbf{y} | = | X_n . Y_n - X_n . \mathbf{y} + X_n . \mathbf{y} - \mathbf{x}.\mathbf{y} |$
$\leq | X_n .(Y_n - \mathbf{y}) + (X_n - \mathbf{x}).\mathbf{y} |$
$\leq | X_n .(Y_n - \mathbf{y}) | + | (X_n - \mathbf{x}).\mathbf{y} |$.

By Schwarz's inequality,

$| X_n .(Y_n - \mathbf{y}) | \leq \| X_n \| \| Y_n - \mathbf{y} \|$, $\quad | (X_n - \mathbf{x}).\mathbf{y} | \leq \| X_n - \mathbf{x} \| \| \mathbf{y} \|$.

Therefore $| X_n.Y_n - \mathbf{x}.\mathbf{y} | \leq \| X_n \| \| Y_n - \mathbf{y} \| + \| X_n - \mathbf{x} \| \| \mathbf{y} \|$.

Since $\{X_n\}$ is a convergent sequence, it is bounded.

Therefore There exists a positive number b_1, such that $\| X_n \| \leq b_1$ for all $n \in \mathbb{N}$. Let $b = \max\{b_1, \| \mathbf{y} \|\}$.

Then $| X_n.Y_n - \mathbf{x}.\mathbf{y} | \leq b\{\| Y_n - \mathbf{y} \| + \| X_n - \mathbf{x} \|\}$.

Let $\epsilon > 0$. There exist natural numbers k_1 and k_2 such that

$\| X_n - \mathbf{x} \| < \frac{\epsilon}{2b}$ for all $n \geq k_1$ and $\| Y_n - \mathbf{y} \| < \frac{\epsilon}{2b}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$.

Then $\| X_n - \mathbf{x} \| < \frac{\epsilon}{2b}$ and $\| Y_n - \mathbf{y} \| < \frac{\epsilon}{2b}$ for all $n \geq k$.

Therefore $| X_n.Y_n - \mathbf{x}.\mathbf{y} | < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim X_n.Y_n = \mathbf{x}.\mathbf{y}$.

This completes the proof.

A2.5. Subsequence.

Let $\{X_n\}$ be a sequence in $\mathbb{R}^2$ and $\{r_n\}$ be a strictly increasing sequence of natural numbers. Then $\{X_{r_n}\}$ is said to be a *subsequence* of the sequence $\{X_n\}$.

Examples.

1. Let $X_n = (\frac{1}{n}, (-1)^n), n \geq 1$ and $r_n = 2n, n \geq 1$.

Then $\{X_{r_n}\} = \{X_{2n}\} = \{(\frac{1}{2}, 1), (\frac{1}{4}, 1), (\frac{1}{6}, 1), \quad \ldots\}$ is a subsequence of the sequence $\{X_n\}$.

2. Let $X_n = (\frac{1}{n}, (-1)^n), n \geq 1$ and $r_n = 2n - 1, n \geq 1$.

Then $\{X_{r_n}\} = \{X_{2n-1}\} = \{(1, -1), (\frac{1}{3}, -1), (\frac{1}{5}, -1), \quad \ldots \quad \ldots\}$ is a subsequence of the sequence $\{X_n\}$.

Theorem A2.5.1. If a sequence $\{X_n\}$ in $\mathbb{R}^2$ converges to a limit $\mathbf{x}$ in $\mathbb{R}^2$ then every subsequence of $\{X_n\}$ converges to $\mathbf{x}$.

Proof. Let $X_n = (u_n, v_n), n \geq 1$ and let $\mathbf{x} = (u, v)$.

$\lim X_n = \mathbf{x}$ implies $\lim u_n = u$ and $\lim v_n = v$.

Let $\{r_n\}$ be a strictly increasing sequence of natural numbers.

Then $\{X_{r_n}\}$ is a subsequence of $\{X_n\}$. $\quad X_{r_n} = (u_{r_n}, v_{r_n})$.

$\{u_{r_n}\}$ is a subsequence of the sequence $\{u_n\}$ and $\{v_{r_n}\}$ is a subsequence of the sequence $\{v_n\}$.

$\lim u_n = u$ implies $\lim u_{r_n} = u$ and $\lim v_n = v$ implies $\lim v_{r_n} = v$.

Hence $\lim X_{r_n} = (u, v) = \mathbf{x}$. This proves the theorem.

Note. If there exist two different subsequences of $\{X_n\}$ that converge to two distinct limits then the sequence $\{X_n\}$ is divergent.

Worked Examples.

1. Prove that the sequence $\{(-1,1),(1,\frac{1}{2}),(-1,\frac{1}{3}),(1,\frac{1}{4}),\ \ldots\ \}$ is not convergent.

Let $\{X_n\}$ be the given sequence. Then $X_n = ((-1)^n, \frac{1}{n}), n \geq 1$.
$X_{2n-1} = (-1, \frac{1}{2n-1})$. The subsequence $\{X_{2n-1}\}$ converges to $(-1,0)$.
$X_{2n} = (1, \frac{1}{2n})$. The subsequence $\{X_{2n}\}$ converges to $(1,0)$.

As the subsequences $\{X_{2n-1}\}$ and $\{X_{2n}\}$ converge to two different limits, the sequence $\{X_n\}$ is divergent.

2. Prove that the sequence $\{(1,1),(1,-1),(\frac{1}{2},1),(\frac{1}{2},-1),(\frac{1}{3},1),(\frac{1}{3},-1),\ \ldots\ \}$ is not convergent.

Let $\{X_n\}$ be the given sequence. Let $X_n = (x_n, y_n)$.

Then $x_n = \frac{2}{n+1}$ if n be odd $\qquad y_n = 1$ if n be odd
$\qquad\ \ = \frac{2}{n}$ if n be even ; $\qquad\quad = -1$ if n be even .

$X_{2n-1} = (\frac{1}{n}, 1)$. The sequence $\{X_{2n-1}\}$ converges to $(0,1)$.
$X_{2n} = (\frac{1}{n}, -1)$. The sequence $\{X_{2n}\}$ converges to $(0,-1)$.

As the subsequences $\{X_{2n-1}\}$ and $\{X_{2n}\}$ converge to two different limits, the sequence $\{X_n\}$ is divergent.

Theorem A2.5.2. Let $\{I_n\}$ be a sequence of non-empty closed and bounded cells in $\mathbb{R}^2$ such that $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$, then $\bigcap\limits_{n=1}^{\infty} I_n$ is non-empty.

Proof. Let $I_1 = [a_{11}, b_{11}] \times [a_{21}, b_{21}], I_2 = [a_{12}, b_{12}] \times [a_{22}, b_{22}],\ \cdots$
$\cdots\ , I_n = [a_{1n}, b_{1n}] \times [a_{2n}, b_{2n}],\ \cdots\ \cdots$

Since $I_{n+1} \subset I_n, [a_{1\,n+1}, b_{1\,n+1}] \times [a_{2\,n+1}, b_{2\,n+1}] \subset [a_{1n}, b_{1n}] \times [a_{2n}, b_{2n}]$.

It follows that $[a_{1\,n+1}, b_{1\,n+1}] \subset [a_{1n}, b_{1n}]$ and $[a_{2\,n+1}, b_{2\,n+1}] \subset [a_{2n}, b_{2n}]$ for all $n \in \mathbb{N}$.

Therefore the sequence $\{[a_{1n}, b_{1n}]\}$ is a sequence of closed and bounded intervals in $\mathbb{R}$ such that $[a_{1\,n+1}, b_{1\,n+1}] \subset [a_{1n}, b_{1n}]$ for all $n \in \mathbb{N}$.
By the nested intervals theorem in $\mathbb{R}$, there exists a real number ξ such that $\xi \in [a_{1n}, b_{1n}]$ for all $n \in \mathbb{N}$.

By similar arguments there exists a real number η such that
$\eta \in [a_{2n}, b_{2n}]$ for all $n \in \mathbb{N}$. Hence $(\xi, \eta) \in I_n$ for all $n \in \mathbb{N}$ and this

proves that $\bigcap\limits_{n=1}^{\infty} I_n$ is non-empty.

This completes the proof.

Theorem A2.5.3. (Bolzano)

Every bounded sequence in $\mathbb{R}^2$ has a convergent subsequence.

Proof. Let $\{X_n\}$ be a bounded sequence in $\mathbb{R}^2$. Then there exists a closed and bounded cell in $\mathbb{R}^2$, say $I = [a_1, b_1] \times [a_2, b_2]$ such that $X_n \in I$ for all $n \in \mathbb{N}$.

Let $l(I) = \max\{b_1 - a_1, b_2 - a_2\}$. Then $d(I) =$ the diameter of $I \leq \sqrt{2}l(I)$.

Let $c_1 = \frac{a_1+b_1}{2}, c_2 = \frac{a_2+b_2}{2}$. Then I is divided into 4 closed subcells $[a_1, c_1] \times [a_2, c_2], [a_1, c_1] \times [c_2, b_2], [c_1, b_1] \times [a_2, c_2], [c_1, b_1] \times [c_2, b_2]$ in $\mathbb{R}^2$.

At least one of these subcells contains infinite number of elements of $\{X_n\}$. We call it I_1 and let $I_1 = [a_{11}, b_{11}] \times [a_{21}, b_{21}]$.

Then $I_1 \subset I$ and $d(I_1) = \frac{1}{2}d(I)$.

Let $c_{11} = \frac{a_{11}+b_{11}}{2}, c_{21} = \frac{a_{21}+b_{21}}{2}$. Then I_1 is divided into 4 subcells

$[a_{11}, c_{11}] \times [a_{21}, c_{21}], [a_{11}, c_{11}] \times [c_{21}, b_{21}], [c_{11}, b_{11}] \times [a_{21}, c_{21}], [c_{11}, b_{11}] \times [c_{21}, b_{21}]$ in $\mathbb{R}^2$.

At least one of these subcells contains infinite number of elements of $\{X_n\}$. We call it I_2 and let $I_2 = [a_{12}, b_{12}] \times [a_{22}, b_{22}]$.

Then $I_2 \subset I_1$ and $d(I_2) = \frac{1}{2}d(I_1) = \frac{1}{2^2}d(I)$.

By induction, we obtain a sequence of closed cells $\{I_n\}$ in $\mathbb{R}^2$ such that each I_n contains infinite number of elements of $\{X_n\}$ and $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$. Therefore $\bigcap\limits_{n=1}^{\infty} I_n$ is non-empty.

Let $\mathbf{x} = (\xi, \eta) \in \bigcap\limits_{n=1}^{\infty} I_n$. I_1 contains infinitely many elements of the sequence $\{X_n\}$.

Therefore the set $S_1 = \{n : X_n \in I_1\}$ is an infinite subset of $\mathbb{N}$.

By the well ordering property of the set $\mathbb{N}, S_1$ has a least element, say r_1. Then $X_{r_1} \in I_1$.

Since I_2 contains infinitely many elements of the sequence $\{X_n\}$ the set $S_2 = \{n : X_n \in I_2\}$ is an infinite subset of $\mathbb{N}$. Therefore there exists a natural number $r_2 > r_1$ such that $X_{r_2} \in I_2$.

By induction, we obtain a strictly increasing sequence $\{r_1, r_2, r_3, \ldots\}$ of natural numbers such that $X_{r_k} \in I_k$ for all $k \in \mathbb{N}$.

We now prove that the subsequence $\{X_{r_n}\}$ converges to $\mathbf{x}$.

Let $\epsilon > 0$. There exists a natural number m such that $\frac{d(I)}{2^m} < \epsilon$.

Since $X_{r_m} \in I_m$ and $\mathbf{x} \in I_m$, $\| X_{r_m} - \mathbf{x} \| \leq d(I_m) = \frac{1}{2^m}d(I) < \epsilon$.

$n \geq m \Rightarrow X_{r_n} \in I_m$. Hence $\| X_{r_n} - X \| \leq d(I_m) < \epsilon$ for all $n \geq m$.

Since ϵ is arbitrary, $\lim X_{r_n} = \mathbf{x}$. This shows that the subsequence $\{X_{r_n}\}$ is a convergent subsequence of $\{X_n\}$.

A2.6. Cauchy sequence.

A sequence $\{X_n\}$ in $\mathbb{R}^2$ is said to be a *Cauchy sequence* if for a pre-assigned positive ϵ there exists a natural number k such that

$$\| X_m - X_n \| < \epsilon \text{ for all } m, n \geq k.$$

Replacing m by $n+p$, where $p = 1, 2, 3, \ldots$ the condition for a Cauchy sequence can be equivalently stated as –

$$\| X_{n+p} - X_n \| < \epsilon \text{ for all } n \geq k \text{ and } p = 1, 2, 3, \ldots$$

Theorem A2.6.1. A convergent sequence in $\mathbb{R}^2$ is a Cauchy sequence.

Proof. Let $\{X_n\}$ be a convergent sequence in $\mathbb{R}^2$ and let $\lim X_n = \mathbf{x}$.

Let $\epsilon > 0$. There exists a natural number k such that

$$\| X_n - \mathbf{x} \| < \tfrac{\epsilon}{2} \text{ for all } n \geq k.$$

Hence if $m, n \geq k$, $\| X_n - \mathbf{x} \| < \frac{\epsilon}{2}$ and $\| X_m - \mathbf{x} \| < \frac{\epsilon}{2}$.

$$\begin{aligned} \| X_m - X_n \| &= \| X_m - \mathbf{x} + \mathbf{x} - X_n \| \\ &\leq \| X_m - \mathbf{x} \| + \| X_n - \mathbf{x} \| \\ &< \epsilon \text{ for all } m, n \geq k. \end{aligned}$$

This proves that $\{X_n\}$ is a Cauchy sequence.

Theorem A2.6.2. A Cauchy sequence in $\mathbb{R}^2$ is a bounded sequence.

Proof left to the reader.

Theorem A2.6.3. If a subsequence $\{X_{r_n}\}$ of a Cauchy sequence $\{X_n\}$ converges to a limit $\mathbf{x}$ then the sequence $\{X_n\}$ also converges to $\mathbf{x}$.

Proof. Let $\epsilon > 0$. Since $\{X_n\}$ is a Cauchy sequence, there exists a natural number k such that

$$\| X_m - X_n \| < \tfrac{\epsilon}{2} \text{ for all } m, n \geq k \quad \cdots \quad \text{(i)}$$

Since $\{X_{r_n}\}$ converges to $\mathbf{x}$, there is a natural number $p > k$ belonging to the set $\{r_1, r_2, \ldots\}$ such that $\| X_p - \mathbf{x} \| < \frac{\epsilon}{2}$.

Also from (i) $\| X_n - X_p \| < \frac{\epsilon}{2}$ for all $n \geq k$.

$$\begin{aligned} \| X_n - \mathbf{x} \| &\leq \| X_n - X_p \| + \| X_p - \mathbf{x} \| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon \text{ for all } n \geq k. \end{aligned}$$

This proves that $\lim X_n = \mathbf{x}$.

Theorem A2.6.4. A Cauchy sequence in $\mathbb{R}^2$ is convergent.

Proof. Let $\{X_n\}$ be a Cauchy sequence in $\mathbb{R}^2$. Then $\{X_n\}$ is a bounded sequence.

By Bolzano's theorem, $\{X_n\}$ has a convergent subsequence, say $\{X_{r_n}\}$. Let $\lim X_{r_n} = \mathbf{x}$.

Then by the previous theorem, $\lim X_n = \mathbf{x}$ and therefore the sequence $\{X_n\}$ is convergent.

Exercises

1. Let $\{X_n\}$ be a sequence in $\mathbb{R}^2$ that converges to $\mathbf{x}$. Prove that the real sequence $\{\| X_n \|\}$ converges to $\| \mathbf{x} \|$.

2. Let $X_n = ((-1)^n, (-1)^{n+1})$ for $n \geq 1$. Show that $\{X_n\}$ is a divergent sequence but the real sequence $\{\| X_n \|\}$ is convergent.

3. Let $X_n = ((-1)^n, \frac{1}{n})$ and $Y_n = ((-1)^{n+1}, \frac{1}{n})$ for $n \geq 1$.
Show that the sequence $\{X_n + Y_n\}$ is convergent but none of the sequences $\{X_n\}$ and $\{Y_n\}$ is convergent.

4. Let $X_n = (\frac{1}{n}, n)$ and $Y_n = (n, \frac{1}{n})$ for $n \geq 1$.
Show that the sequence $\{X_n . Y_n\}$ (where $X_n . Y_n$ denotes the inner product of X_n and Y_n) is convergent but none of the sequences $\{X_n\}$ and $\{Y_n\}$ is convergent.

5. Let $u_n = \frac{n+1}{n}$ and $X_n = (\frac{1}{n}, \frac{n}{2n+1})$ for $n \geq 1$.
Verify that $\lim u_n X_n = \lim u_n . \lim X_n$.

6. If the subsequences $\{X_{2n-1}\}$ and $\{X_{2n}\}$ of a sequence $\{X_n\}$ converge to the same limit $\mathbf{x}$ prove that the sequence $\{X_n\}$ converges to $\mathbf{x}$.
Show that the sequence $\{(1,1), (1,-1), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}), \quad \cdots \quad \}$ is convergent.

7. Let $X_1 = (1,1)$ and $X_2 = (2,2)$ and $X_{n+2} = \frac{1}{2}(X_{n+1} + X_n)$ for $n \geq 1$. Prove that the sequence $\{X_n\}$ converges to $(\frac{5}{3}, \frac{5}{3})$.

8. If $\{X_n\}$ and $\{Y_n\}$ are Cauchy sequences in $\mathbb{R}^2$, prove directly that the sequence $\{X_n + Y_n\}$ is a Cauchy sequence in $\mathbb{R}^2$.

ANSWERS TO EXERCISES

Exercises 1. (Page 12)

2. (i) $(-1,1), \{0\}$; (ii) $(-1,2), [0,1]$.

5. (i) $g(x) = \cos x - \sin x, x \in [0, \frac{\pi}{4})$
$= 0, x = \frac{\pi}{4}$
$= \sin x - \cos x, x \in (\frac{\pi}{4}, \frac{\pi}{2}]$
Therefore $f \neq g$.

(ii) $2\tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}$ for $|x| < 1$
$= \pi + \tan^{-1} \frac{2x}{1-x^2}$ for $x > 1$
$= -\pi + \tan^{-1} \frac{2x}{1-x^2}$ for $x < -1$.
Therefore $f \neq g$.

Exercises 2. (Page 41)

8. (i) $(-\infty, \frac{1}{2}) \cup (1, \infty)$, (ii) $(-\infty, 1) \cup (2, \infty)$, (iii) $(-\frac{3}{2}, -\frac{3}{14}) \cup (\frac{3}{2}, \infty)$,
(iv) $(-\infty, \frac{9}{11}) \cup (\frac{5}{3}, \infty)$, (v) $(-\infty, \frac{23}{5}) \cup (13, \infty)$.

10. (i) $1, -1$; (ii) $\frac{1}{3}, -3$; (iii) $2, 0$; (iv) $\frac{3}{2}, 0$.

Exercises 3. (Page 69)

1. (iii) $\{\cos \frac{n\pi}{2} + \frac{1}{n} : n \in \mathbb{N}\}$ (iv) $\{\cos \frac{n\pi}{3} + \frac{1}{n} : n \in \mathbb{N}\}$. **2.** $\mathbb{N}$. **3.**(ii) $\{-1, 1\}$.
6. (iii) $0, \frac{1}{2}, \frac{1}{2^2}, \cdots$ **9.** (i) yes, (ii) – (v) no. **17.** (iv) no.

Exercises 4. (Page 83)

1. $I_n = \{x \in \mathbb{R} : x \geq n\}$. **2.** $I_n = \{x \in \mathbb{R} : 0 < x < \frac{1}{n}\}$.

Exercises 5. (Page 97)

4. $I_n = \{x \in \mathbb{R} : \frac{1}{n+1} < x < \frac{n+1}{n}\}$; $\mathcal{G} = \{I_n : n \in \mathbb{N}\}$.

5. $I_n = \{x \in \mathbb{R} : -\frac{1}{n} < x < n\}$; $\mathcal{G} = \{I_n : n \in \mathbb{N}\}$.

6. Let $K_n = \{x \in \mathbb{R} : \frac{1}{n+1} \leq x \leq 1\}$. Then $\bigcup\limits_{n=1}^{\infty} K_n$ is not compact.

Exercises 6. (Page 119)

1. (i) $(-2, 2)$, (ii) $[-\frac{1}{3}, 1]$, (iii) $[-1, 2]$, (iv) $[-2, 0]$,
(v) $\bigcup\limits_{m \in z} (2m\pi, \overline{2m+1}\pi)$, (vi) $(-\infty, -1) \cup [0, \infty)$.

3. (i) odd, (ii) odd, (iii) even, (iv) odd.

4. (i)$[\frac{1}{2}(\sqrt{1+x} + \sqrt{1-x})] + [\frac{1}{2}(\sqrt{1+x} - \sqrt{1-x})]$,
(ii) $[\frac{1}{2}(x + \sqrt{1+x^2}) + \frac{1}{2}(-x + \sqrt{1-x^2})] + [\frac{1}{2}(x + \sqrt{1+x^2}) - \frac{1}{2}(-x + \sqrt{1-x^2})]$.

5. (i) $\frac{2\pi}{3}$, (ii) π, (iii) π.

Exercises 7. (Page 145)

2. (i) $2, -1 - \frac{1}{\sqrt{2}}$; (ii) $1, -1 - \frac{1}{3}$. **12.** 3.

Exercises 8. (Page 169)

1. (i) $e^{\frac{1}{3}}$; (ii) e; (iii) $e^{\frac{1}{3}}$; (iv) e.

7. (i) $1, -1$; (ii) ∞, ∞; (iii) $\infty, 0$; (iv) $\sqrt{2}, -\sqrt{2}$.

Exercises 9. (Page 198)

8. (i) convergent if $p > 2$, divergent if $p \leq 2$; (ii) convergent, (iii) divergent, (iv) divergent, (v) divergent, (vi) convergent.

9. (i)–(vi) convergent, (vii)–(ix) divergent.

10. (i) conv, (ii) conv, (iii) div, (iv) conv, (v) conv, (vi) conv, (vii) conv, (viii) conv, (ix) div, (x) conv, (xi) div, (xii) conv, (xiii) div.

(xiv)–(xv) conv if $0 < x < 1$, div if $x \geq 1$; (xvi) conv if $0 < x < e$, div if $x \geq e$; (xvii) convergent if $0 < x \leq 1$, divergent if $x > 1$; (xviii) convergent if $0 < x < \frac{1}{e}$, divergent if $x \geq \frac{1}{e}$; (xix) convergent if $0 < x < 1$, divergent if $x \geq 1$; (xx) convergent if $0 < x < 4$, divergent if $x \geq 4$.

Exercises 10. (Page 215)

9. (i) abs, (ii) abs, (iii) cond, (iv) abs, (v) div, (vi) abs, (vii)-(viii) cond.

Exercises 11. (Page 243)

4. (i) 0, 1; (ii) 0, 1; (iii) 0, 0; (iv) 1, 1; (v) 0, 0; (vl) 1, 1.

5. (i) 1, (ii) 0, (iii) 1, (iv) 0, (v) e^2, (vi) 0, (vii) 0, (viii) 0.

Exercises 12. (Page 270)

1. $$\begin{aligned} f(x) &= 1, x \in \mathbb{Q} & g(x) &= -1, x \in \mathbb{Q} \\ &= -1, x \in \mathbb{R} - \mathbb{Q}, & &= 1, x \in \mathbb{R} - \mathbb{Q} \end{aligned}$$
Therefore $(f + g)(x) = 0, x \in \mathbb{R}$.

2. $$\begin{aligned} f(x) &= 1, x \in \mathbb{Q} & g(x) &= -1, x \in \mathbb{Q} \\ &= -1, x \in \mathbb{R} - \mathbb{Q}, & &= 1, x \in \mathbb{R} - \mathbb{Q}. \end{aligned}$$
Therefore $fg(x) = -1, x \in \mathbb{R}$.

12. (i) $n\pi, (4n + 1)\frac{\pi}{2}$ where n is an integer; (ii) $0, \pm 1, \pm 2, \cdots\cdots$
(iii) $0, \pm 1, \pm 2, \cdots\cdots$ (iv) $1, \frac{1}{2}, \frac{1}{3}, \cdots\cdots$ (v) $1, -1$.

13. (i) infinite, (ii) removable, (iii) infinite, (iv) removable, (v) oscillatory, (vi) infinite, (vii) jump, (viii) infinite.

Exercises 13. (Page 303)

1. (i) for $x \in [-1, 1]$, $f(x) = \sin \frac{1}{x}, x \neq 0$ (ii) $f(x) = \operatorname{sgn} x, x \in [-1, 1]$.
$= 0, x = 0$.

4. (i) $f(x) = x \sin \frac{1}{x}, x \neq 0$ (ii) $f(x) = \sin x, x \in \mathbb{R}$.
$= 0, x = 0$.

11. (i) $I = [0, \infty), f(x) = x^2, x \in I$. (ii) $I = [0, \infty), f(x) = \frac{1}{1+x^2}, x \in I$.

Exercises 14. (Page 317)

2. $Lf'(0) = 1$, $Rf'(0) = 0$. **4.** $(-\infty, 0) \cup (0, 1) \cup (1, 2) \cup (2, \infty)$.

5. (i) $f'(x) = \frac{2}{\sqrt{1-x^2}}, x \in (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
$= -\frac{2}{\sqrt{1-x^2}}, x \in (-1, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, 1)$

$f'(x)$ does not exist at $x = \pm 1, \pm\frac{1}{\sqrt{2}}$.

(ii) $f'(x) = \frac{3}{\sqrt{1-x^2}}, x \in (-\frac{1}{2}, \frac{1}{2})$
$= -\frac{3}{\sqrt{1-x^2}}, x \in (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)$

$f'(x)$ does not exist at $x = \pm 1, \pm\frac{1}{2}$.

(iii) $f'(x) = \frac{4}{\sqrt{1-x^2}}, x \in (-1, -\frac{1}{\sqrt{2}}) \cup (0, \frac{1}{\sqrt{2}})$
$= -\frac{4}{\sqrt{1-x^2}}, x \in (-\frac{1}{\sqrt{2}}, 0) \cup (\frac{1}{\sqrt{2}}, 1)$

$f'(x)$ does not exist at $x = \pm 1, \pm\frac{1}{\sqrt{2}}, 0$.

12. (ii) $\frac{1}{4}, \frac{1}{9}$.

Exercises 15. (Page 336)

6. (i) $\frac{\pi}{2}$, (ii) $\sqrt{3}$, (iii) $\frac{e}{e-1}$, (iv) $\frac{\sqrt{6}-1}{\sqrt{6}}$.

Exercises 18. (Page 364)

1. (i) a minimum, (ii) neither a maximum nor a minimum, (iii) neither a maximum nor a minimum, (iv) a maximum, (v) a minimum (vi) a maximum, (vii) a minimum, (viii) a minimum.

2. (i) max at 1, min at 2; (ii) max at -2, min at 2, global max 7 at -2, global min $\frac{1}{7}$ at 2; (iii) max at -1, min at 0; (iv) max at 1, min at -1, global max 3 at 1, global min $\frac{1}{3}$ at -1; (v) max at $\frac{1}{2}$, min at 2; (vi) max at 1, min at -1, global max $\frac{1}{2}$ at 1, global min $-\frac{1}{2}$ at -1; (vii) max at 1, max at $\frac{5}{3}$, min at 2; (viii) max at 2, min at $\frac{9}{14}$; (ix) max at $\frac{1}{\sqrt{2}}$, min at $-\frac{1}{\sqrt{2}}$; (x) max at $\frac{1}{2}$, min at $-\frac{1}{2}$.

3. (i) $\frac{3\sqrt{3}}{4}, \frac{-3\sqrt{3}}{4}$; (ii) max 2, 0; min $-\frac{9}{8}$;

(iii) max $\frac{4\sqrt{3}\pm 3}{6}$, min $\frac{\sqrt{3}}{4}$; (iv) max $\frac{11}{6}, -\frac{5}{12}$; min $-\frac{1}{2}, -\frac{5}{6}$.

4. (i) $(\frac{1}{e})^{1/e}$, (ii) $e^{1/e}$, (iii) $\frac{1}{e}$, (iv) $\frac{e\log 2}{2}$. 6. (i) 5, 5; (ii) 6, 6.

7. (i) $\frac{3s}{4}, \frac{3s}{4}, \frac{s}{2}$; (ii) $\frac{3s}{5}, \frac{3s}{5}, \frac{4s}{5}$. 8. (i) $\frac{2\sqrt{3}}{3}r$, (ii) $\frac{4r}{3}$.

Exercises 19 (Page 377)

2. (i) 0, (ii) 1, (iii) $\frac{1}{3}$, (iv) 1, (v) 0, (vi) 0, (vii) 2, (viii) $\frac{1}{2}$.

5. (i) $a = 2$, (ii) $a = 1, b = -1$; (iii) $a = -\frac{1}{4}, b = -\frac{3}{8}$; (iv) $a = 2, b = 1, c = -3$.

Exercises 20. (Page 400)

5. (i) $V(x) = \sin 2x,\ 0 \le x < \frac{\pi}{4}; V(x) = 2 - \sin 2x,\ \frac{\pi}{4} \le x \le \frac{\pi}{2}$.

(ii) $V(x) = \sin x + \cos x - 1,\ 0 \le x < \frac{\pi}{4}; V(x) = -\sin x - \cos x + 2\sqrt{2} - 1\ \frac{\pi}{4} \le x \le \frac{\pi}{2}$. (iii) $V(x) = 0,\ 0 \le x < \frac{\pi}{4}; V(x) = 1,\ x = \frac{\pi}{4}; V(x) = 2,\ \frac{\pi}{4} < x \le \frac{\pi}{2}$.

6. (i) $V(x) = -x^2 + 2x,\ 0 \le x < 1; V(x) = x^2 - 2x + 2,\ 1 \le x \le 2$.

(ii) $V(x) = x,\ 0 \le x < 1; V(x) = 1 + x,\ 1 \le x < 2; V(x) = 4,\ x = 2$.

(iii) $V(x) = x,\ 0 \le x \le 2$.

7. (i) $V_f[0,3] = 5, p_f[0,3] = 1, n_f[0,3] = 4$. (ii) $V_f[0,3] = 2, p_f[0,3] = 2, n_f[0,3] = 0$.
(iii) $V_f[0,3] = 3, p_f[0,3] = 1, n_f[0,3] = 2$.

8. (i) $p(x) = 0,\ 0 \le x < 1; p(x) = x^2 - 2x + 1,\ 1 \le x \le 2$.
$n(x) = -x^2 + 2x,\ 0 \le x < 1; n(x) = 1,\ 1 \le x \le 2$.

(ii) $p(x) = 0.\ 0 \le x < 1; p(x) = 1,\ x = 1; p(x) = 2,\ 1 < x \le 2$.
$n(x) = 0,\ 0 \le x \le 2$.

(iii) $p(x) = 0,\ 0 \le x < 1; p(x) = x - 1,\ 1 \le x \le 2$.
$n(x) = x,\ 0 \le x < 1; n(x) = 1,\ 1 \le x \le 2$.

Exercises 21. (Page 479)

5. $\frac{1}{4}, \frac{1}{3}$. **6.** $\frac{7}{12}, \frac{5}{6}$. **7.** $1, \frac{\pi^2}{8}$. **11.** $\frac{13}{12}$. **12.** $\frac{3}{2}$. **13.** 10.

21. $F(x) = 0,\quad 0 \le x \le 1$
$= x - 1, 1 < x \le 2$
$= 2x - 3, 2 < x \le 3$.

22. $F(x) = \frac{1}{2}x^2,\quad 0 \le x \le 1$
$= x - \frac{1}{2}, 1 < x \le 2$
$= \frac{1}{2}(x^2 - 2x + 3), 2 < x \le 3$.

25. (i) $e^x\sqrt{1 + e^{2x}} - \sqrt{1 + x^2}$, (ii) $2x \sin x - \sin\sqrt{x}$. **29.** (ii) 2. **30.** (ii) $\frac{13}{2}$.

33. (i) $2 \log 2$, (ii) $\frac{2}{\pi}$, (iii) $\tan^{-1} 2$, (iv) $\frac{4}{e}$, (v) $\frac{4}{e}$.

40. (i) $\frac{1}{3}(5^{3/2} - 1)$, (ii) $\frac{52}{9}$, (iii) $\frac{1}{2}(3e^4 - 1)$, (iv) $\frac{1}{4}$.

Exercises 22. (Page 525)

1. (i) convergent, (ii) convergent, (iii) convergent, (iv) convergent, (v) divergent, (vi) divergent, (vii) convergent if $n < 1$, (viii) convergent if $0 < p < 1$, (ix) convergent if $p > -1$, (x) convergent, (xi) convergent, (xii) divergent.

2. (i) convergent, (ii) convergent, (iii) convergent, (iv) convergent, (v) divergent, (vi) convergent.

Exercises 24. (Page 563)

3. (i) no, (ii) yes, (iii) yes, (iv) no.

5. (i)–(ii) uniform, (iii)–(iv) non-uniform, (v)–(vi) uniform.

Exercises 26. (Page 609)

1. (i) $\frac{2}{e}$, (ii) $\frac{1}{2}$, (iii) 2, (iv) 1, (v) $\frac{1}{2}$, (vi) 1. **2.** (i) $\frac{1}{3}$, (ii)–(vi) 1.

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