EC5.203 Communication Theory (3-1-0-4):

Lecture 13:
Digital Modulation - 2

06 March 2025

INTERNATIONAL INSTITUTE OF
INFORMATION TECHNOLOGY

HYDERABAD

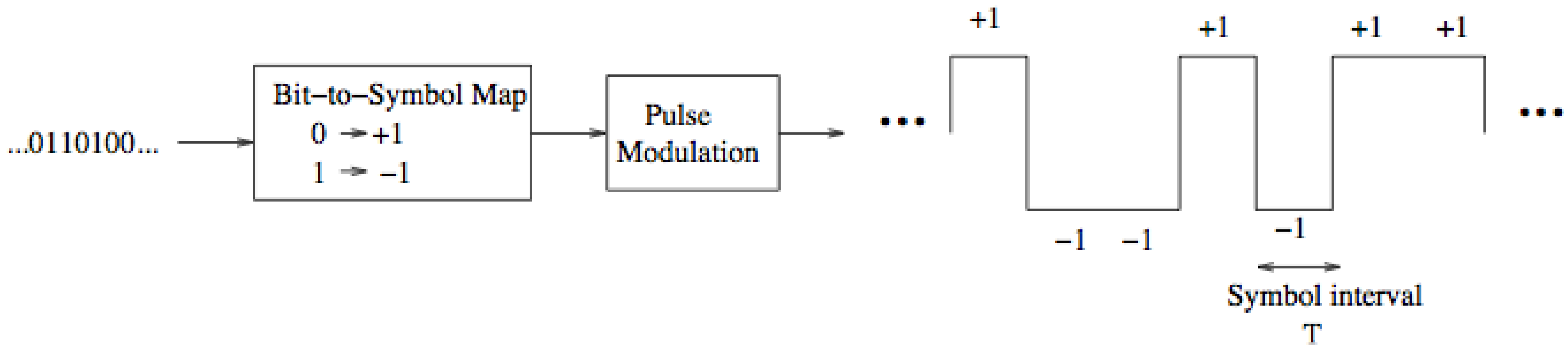
References

- Chap. 4 (Madhow)

Digital modulation

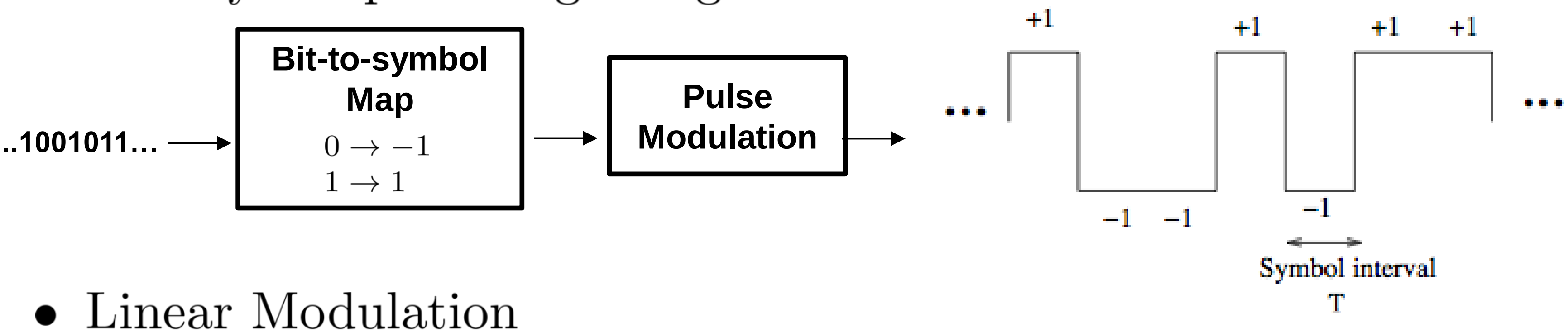
- Digital modulation is the process of translating bits to analog waveforms that can be sent over a physical channel.

- Baseband example: Binary antipodal Signaling



Digital modulation: baseband example

- Binary antipodal Signaling



- Linear Modulation

$$u(t) = \sum_n b[n]p(t - nT)$$

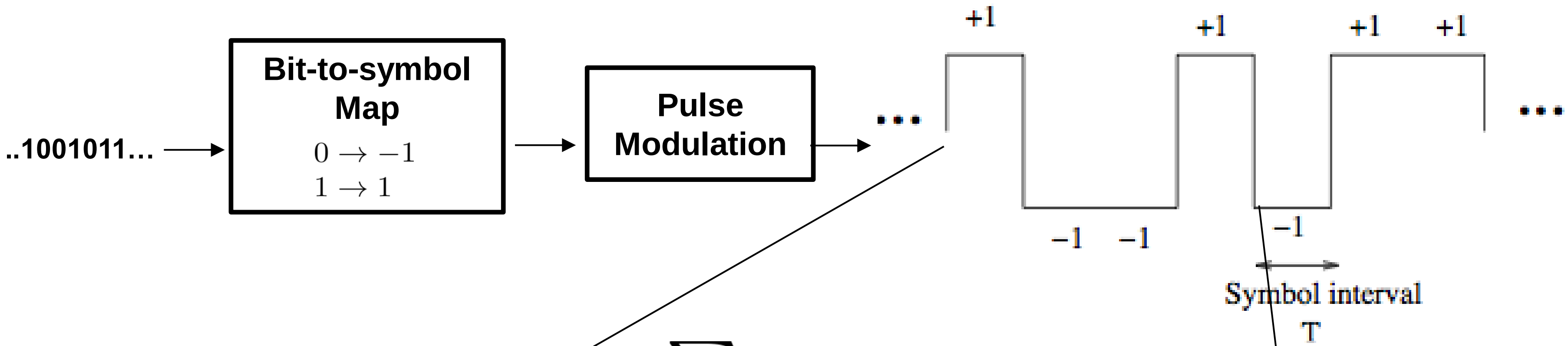
where $\{b[n]\}$ is sequence of symbols and $p(t)$ is modulating pulse for T seconds. For this example

$$p(t) = I_{[0,T]}(t)$$

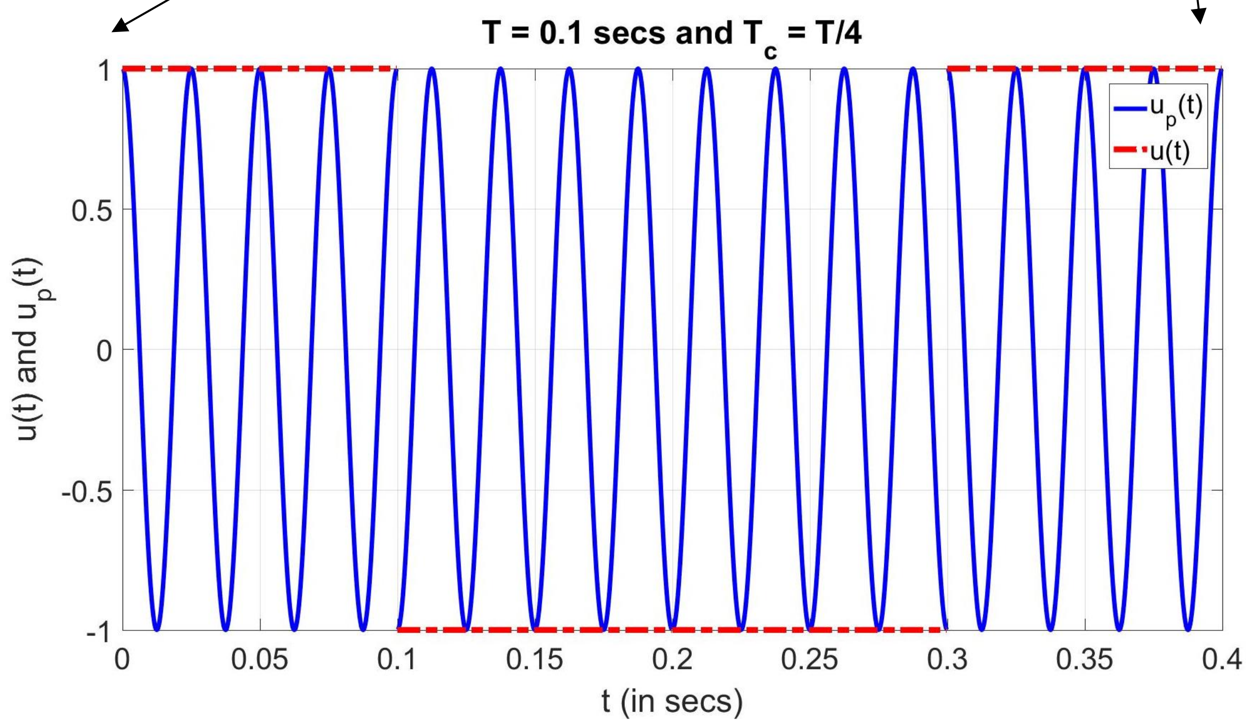
is rectangular window in time domain.

- Baseband signal sent directly over the physical baseband channel.

Digital modulation: passband example

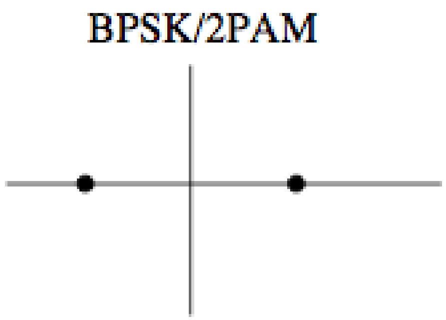


$$u_p(t) = u(t)\cos(2\pi f_c t) = \sum_n b[n]p(t - nT)\cos(2\pi f_c t)$$

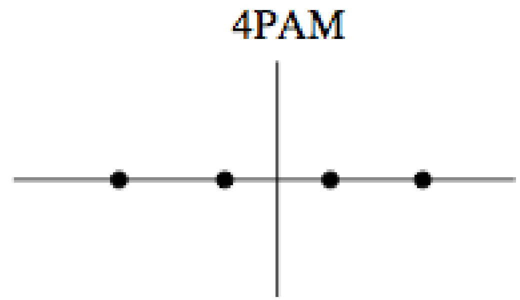


Can we load more bits per symbol?

1 bit per symbol of T secs

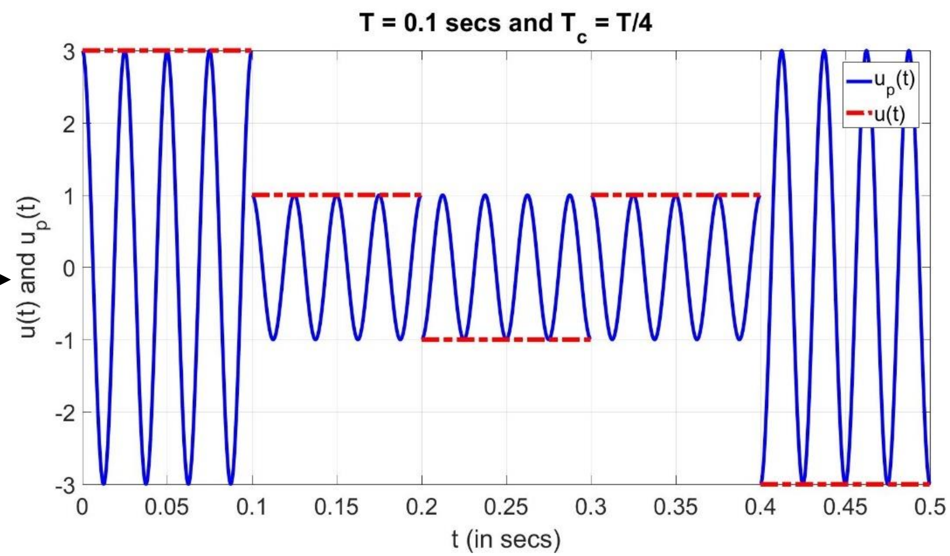


2 bits per symbol



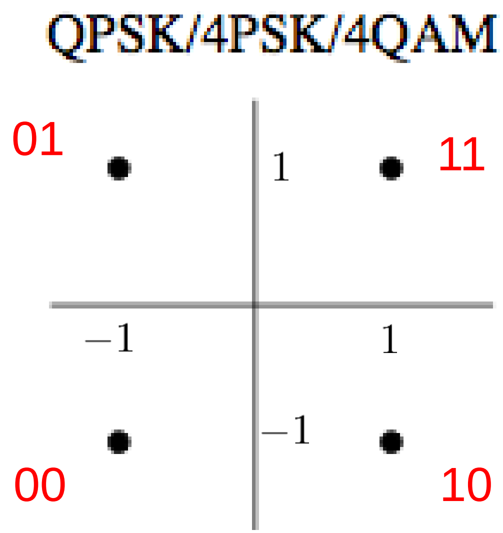
1 1 1 0 0 1 1 0 0 0

$$
\begin{aligned}
00 &\to -3 \\
01 &\to -1 \\
10 &\to +1 \\
11 &\to +3
\end{aligned}
$$

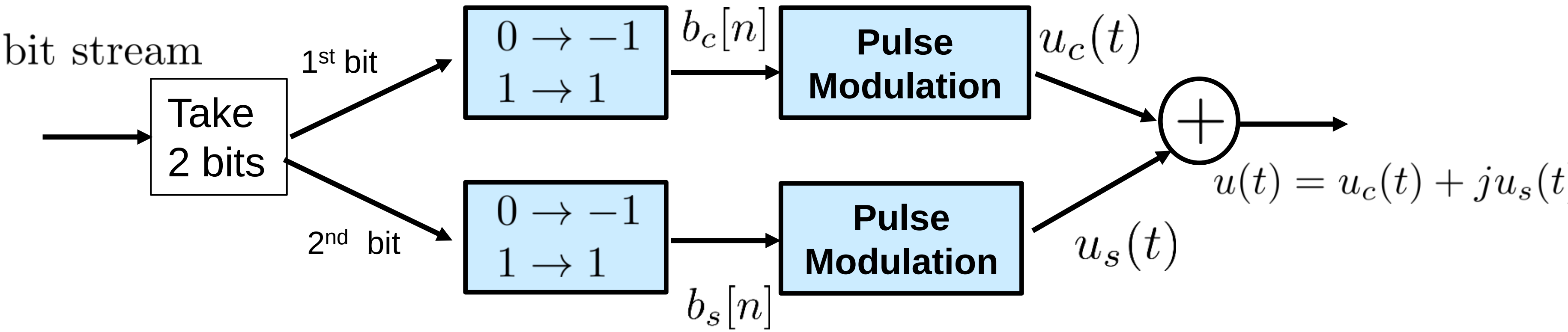


Another way for 2 bits/symbol: QPSK

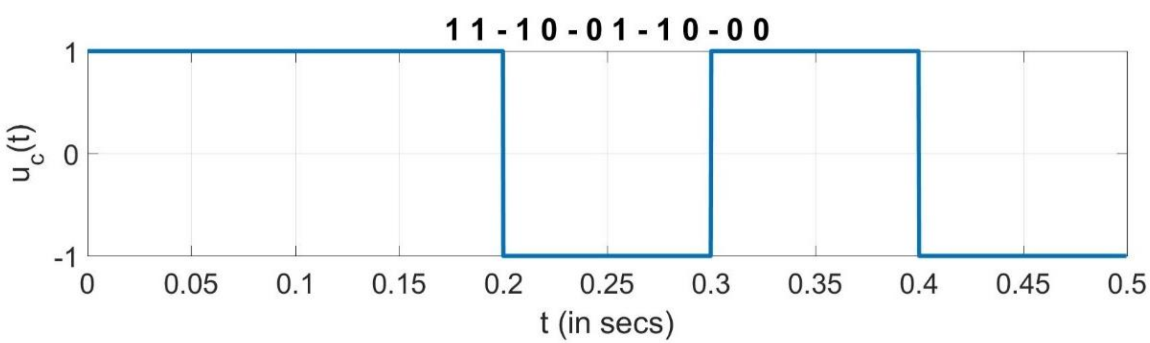
- Load Q component also giving rise to QAM and PSK modulation schemes



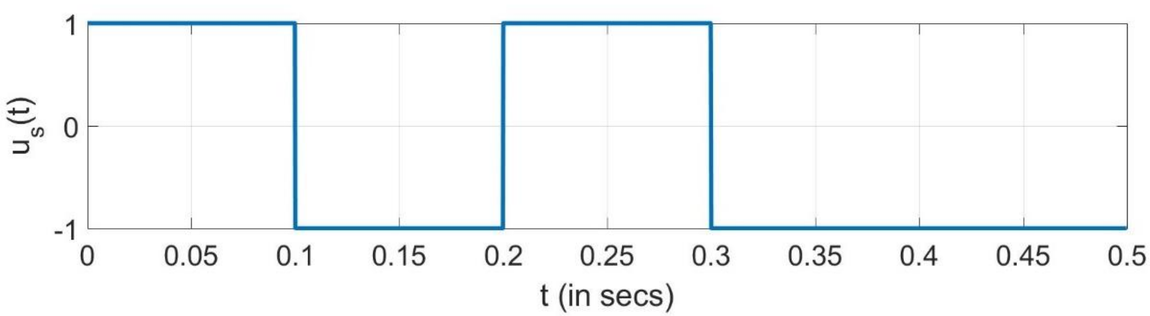
QPSK baseband



$$b[n] = b_c[n] + jb_s[n]$$



$$u_c(t) = \sum_n b_c[n]p(t - nT)$$

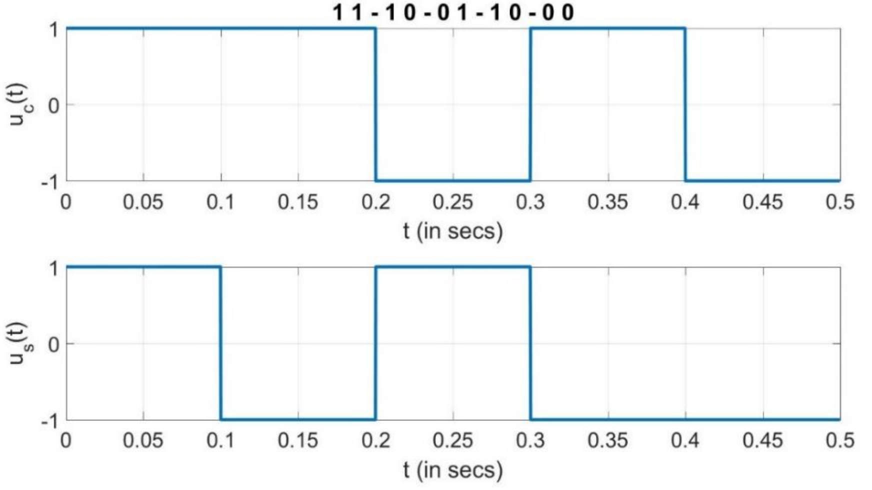


$$u_s(t) = \sum_n b_s[n]p(t - nT)$$

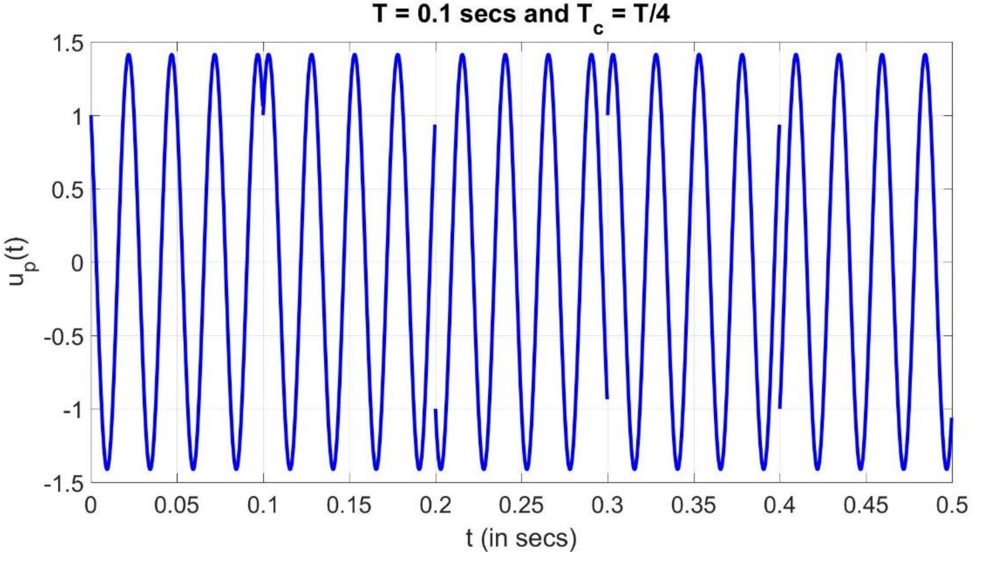
- For single physical baseband channel, QPSK is not possible; simply set $b_s[n] = 0$

QPSK: Passband

1 1 1 0 0 1 1 0 0 0

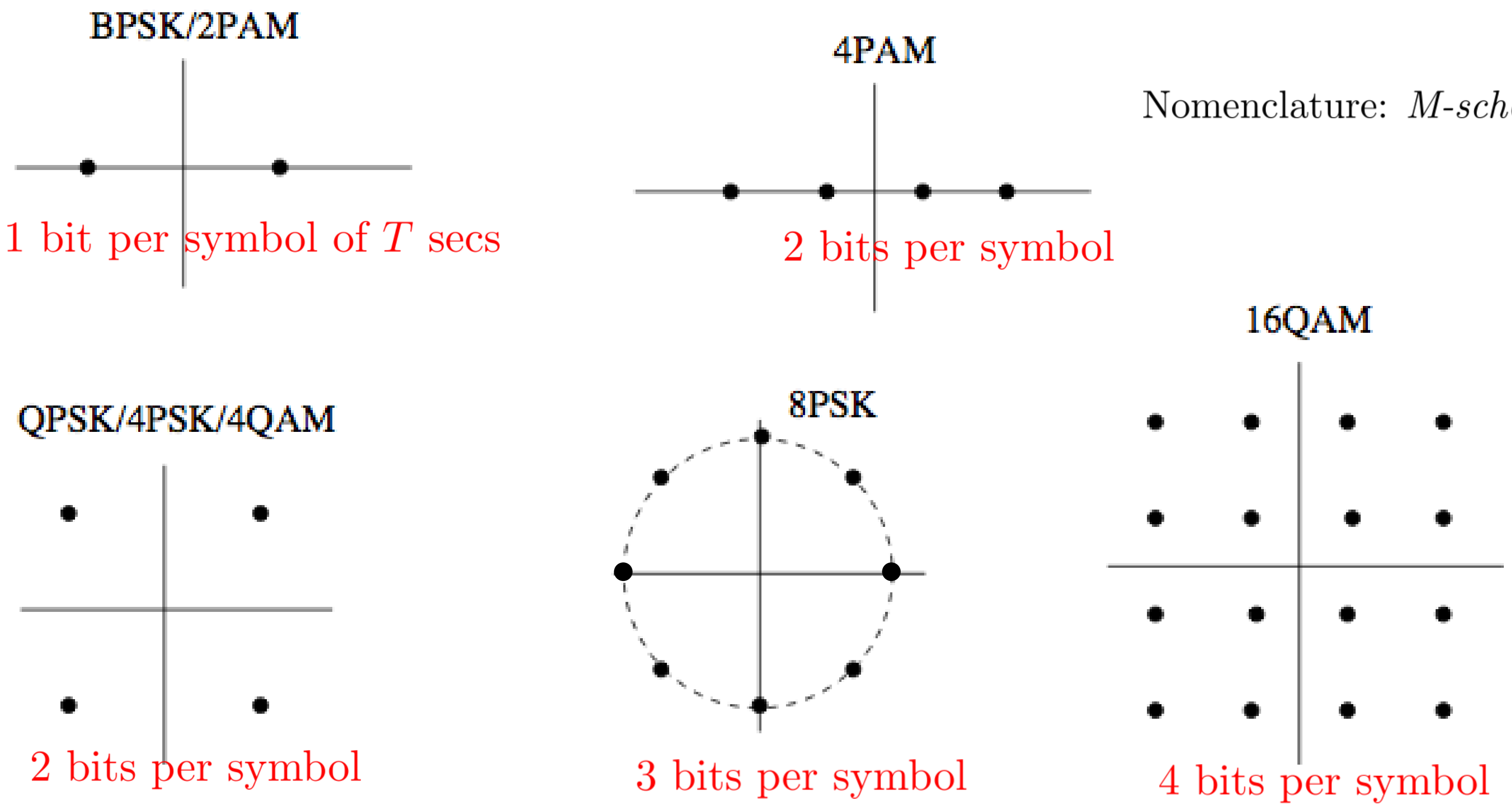


$$u_p(t) = \Re\{u(t)e^{j2\pi f_c t}\} = u_c(t)\cos(2\pi f_c t) - u_s(t)\sin(2\pi f_c t)$$



Can we load even more bits?

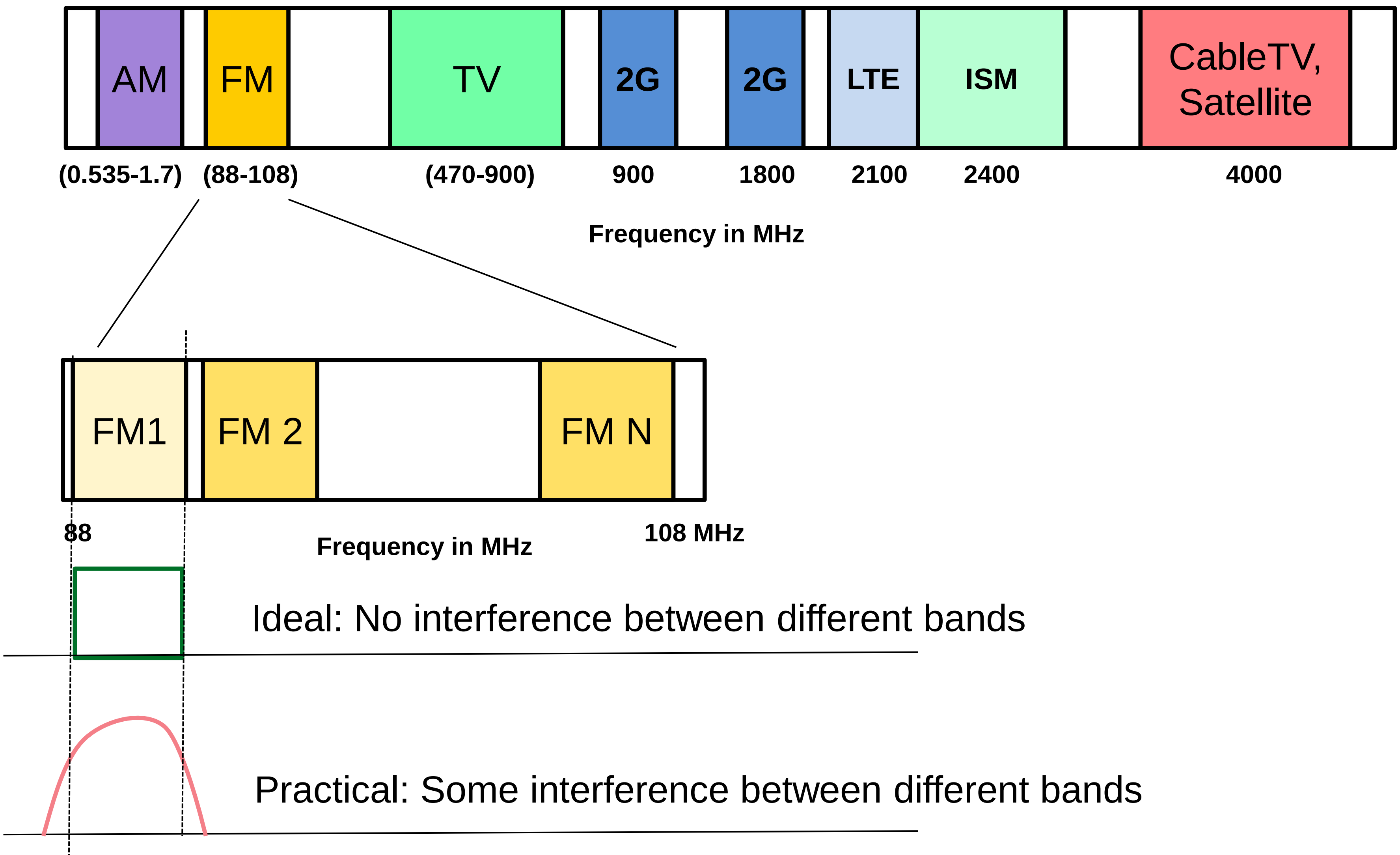
Nomenclature: *M-scheme*



COMMON CONSTELLATIONS

- In general, M-ary modulation scheme can transmit $\log_2 M$ bits per symbol.
- Information rate $= \frac{\log_2 M}{T}$ bits/sec.

Motivation for Bandwidth Occupancy



Modeling bandwidth occupancy

- Consider the complex envelope of a linearly modulated signal

$$u(t) = \sum_n b[n]p(t - nT)$$

where $\{b[n]\}$ is sequence of symbols and $p(t)$ is modulating pulse for T seconds.

- $\{b[n]\}$ is modeled as random at the transmitter as well as receiver.
- However for characterizing the bandwidth occupancy of digitally modulated signal u, we define the quantities of interest in terms of average across time.
- We treat $u(t)$ as a finite power signal that can be modeled as a deterministic sequences once $\{b[n]\}$ is fixed.
- Bandwidth is then defined in terms of power spectral density.

Power Spectral Density (PSD)

- Power spectral density $S_x(f)$ for signal $x(t)$ specifies how the power in a signal is distributed in different frequency bands.
- PSD is defined as power per unit frequency.
- Units of watts/hertz or joules (since power/frequency = energy)
- Total power in $x(t)$ is given by

$$P_x = \int_{-\infty}^{\infty} S_x(f)\ df = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{T_0}^{T_0} |x(t)|^2\ dt$$

- PSD is generally used for random or periodic signals.
  - PSD for random process: We are interested in modeling the complete random process instead of a particular realization of it in a given time.

Periodogram-based PSD estimation

- Limit the signal $x(t)$ to a finite observation interval

$$x_{T_0} = x(t) I_{[-T_0/2, T_0/2]}(t)$$

  where T_0 is the length of the observation interval. Since T_0 is finite and $x(t)$ has finite power, $x_{T_0}(t)$ has finite energy. So its Fourier transform is given by

$$X_{T_0} = \mathcal{F}(x_{T_0}(t))$$

- The energy spectral density of x_{T_0} is given by $|X_{T_0}(f)|^2$.
- Therefore the estimated PSD is given by

$$\hat{S}_x(f) = \frac{|X_{T_0}(f)|^2}{T_0}$$

- Formally, the PSD is in the limit of large time windows as follows

$$\hat{S}_x(f) = \lim_{T_0 \to \infty} \frac{|X_{T_0}(f)|^2}{T_0}$$

PSD of Linearly Modulated Signal

- Theorem 4.2.1: Consider a linearly modulated signal where the symbol sequence $\{b[n]\}$ is zero mean and uncorrelated with average symbol energy

$$\sigma_b^2 = \overline{|b[n]|^2} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |b[n]|^2,$$

then the PSD is given by

$$S_u(f) = \frac{|P(f)|^2}{T} \sigma_b^2$$

and the power of the modulated signal is

$$P_u = \frac{\sigma_b^2 ||p||^2}{T}$$

where $||p||^2$ denotes the energy of the modulating pulse. Proof.

- Assumptions:
  - The symbols have zero DC value: $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} b[n] = 0$.
  - The symbols are uncorrelated: $\lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} b[n] b^*[n-k] = 0 \quad \text{for } k \neq 0$.

* Intuitive interpretation: Every T time units, we send a pulse of form $b[n]p(t-nT)$ with average energy spectral density $\sigma_b^2 |P(f)|^2$ so that the PSD is obtained as ESD$/T$

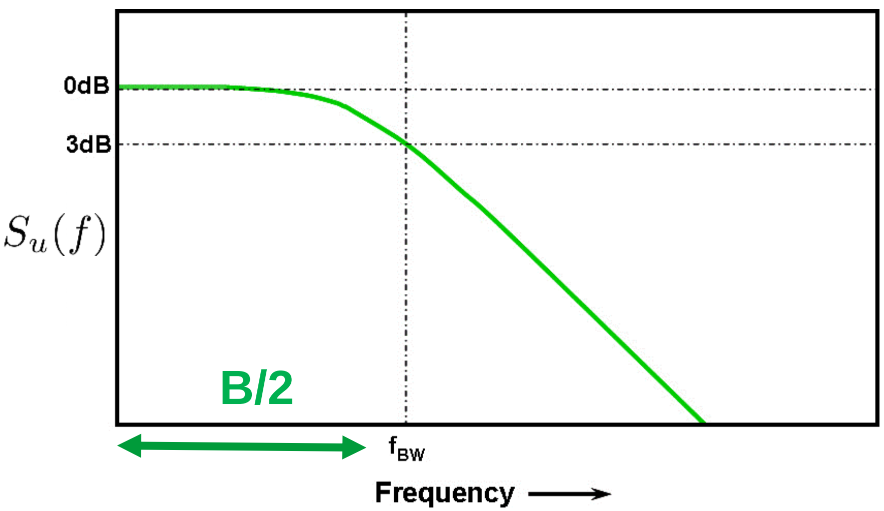
* Same reasoning applies for power in time domain. Every T time units, we send a pulse $b[n]p(t-nT)$ with average $\sigma_b^2 \|p\|^2$ so that the PSD is $\frac{\sigma_b^2 \|p\|^2}{T}$

Today’s Class

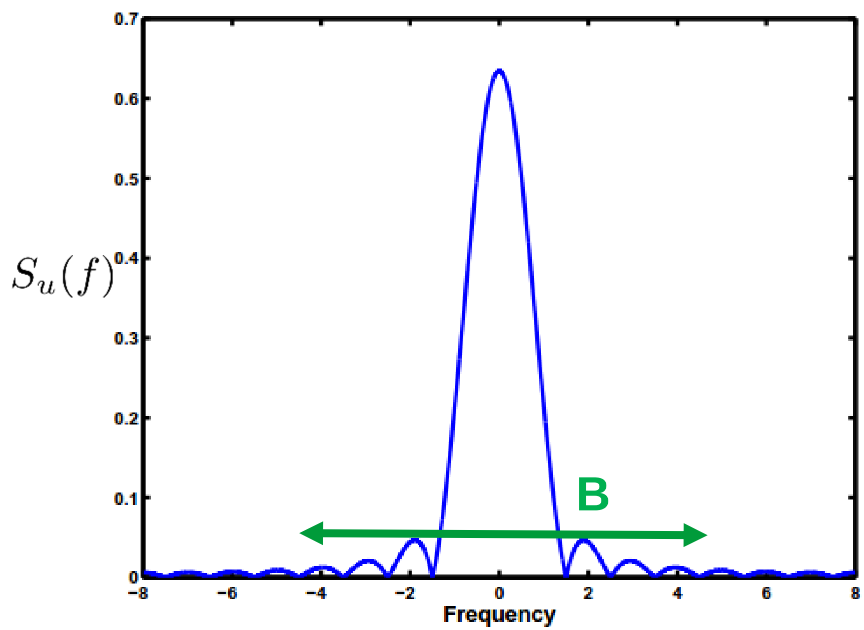
Bandwidth based on PSD

- 3-dB bandwidth
- Fractional power-containment bandwidth: This is the smallest interval that contains a given fraction of the power

$$\int_{-B/2}^{B/2} S_u(f)df = \gamma P_u = \gamma \int_{-\infty}^{\infty} S_u(f)df$$



$$S_u(B_{3dB}/2) = S_u(-B_{3dB}/2) = S_u(0)/2$$



We will focus mostly on this!

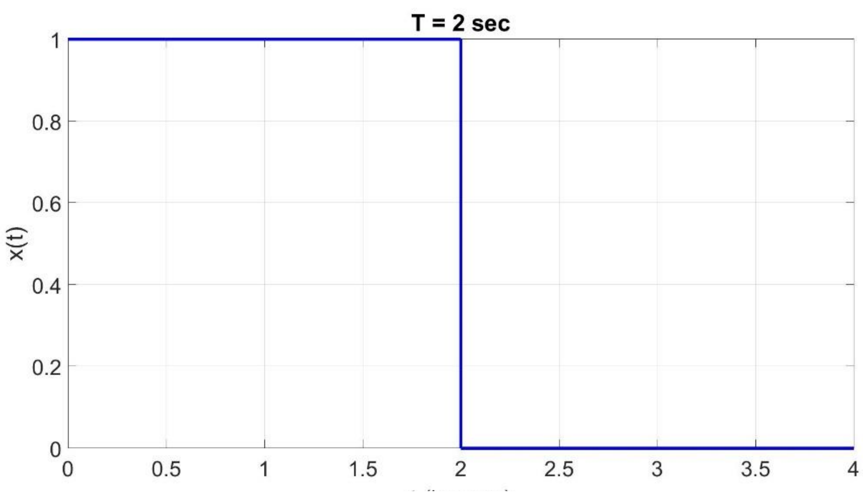
Time Frequency Normalization: *Time Domain*

- If we are sending one symbol every T time units, then the symbol rate is $1/T$ in units of symbols/time unit.

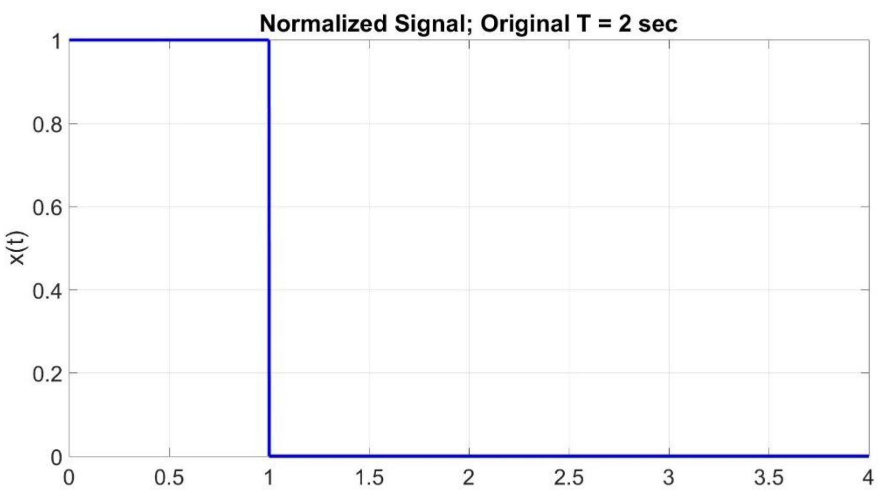
- If we normalize the system for the symbol rate of 1, where the unit of time is T. This implies unit of frequency is $1/T$. In terms of new unit, the linearly modulated signals can be written as

$$u_1(t) = \sum_n b[n] p_1(t-n)$$

where $p_1(t)$ is the modulation pulse for the normalized system.



t in secs

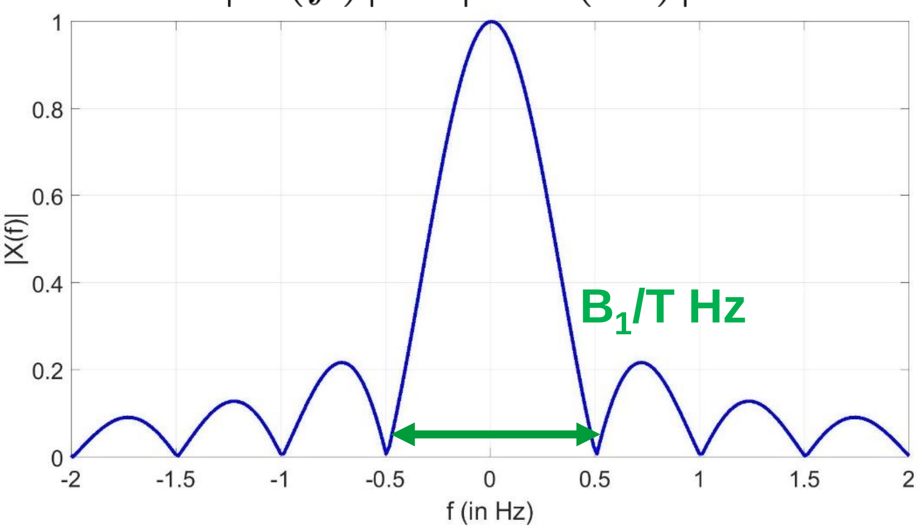


$t_0 = t/T$ Time Unit = T secs

Time Frequency Normalization: *Freq. Domain*

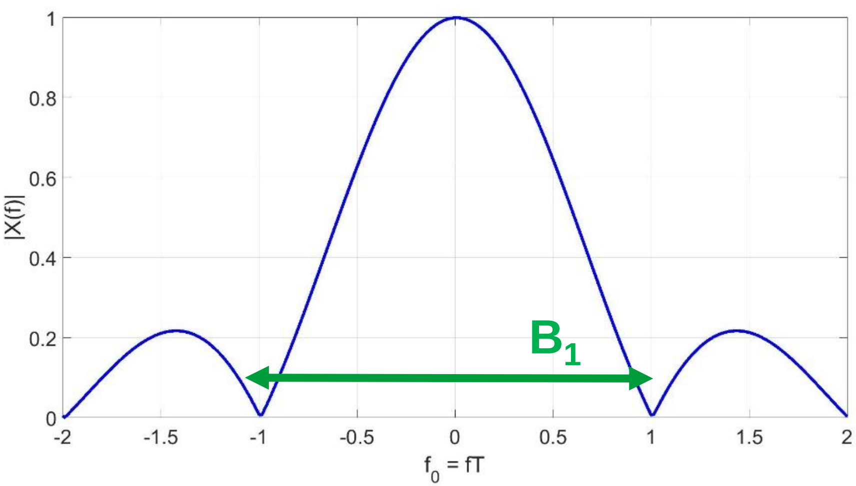
- Let us denote B_1 to be the bandwidth of the normalized system, then the bandwidth of the original system is B_1/T.
- In terms of determining the bandwidth occupancy, we can work, without loss of generality, with the normalized system.
- In essesence, we are working in the original system with the normalized time domain t/T and normalized frequency fT.

$$|P(f)| = |\mathrm{sinc}(\mathrm{fT})|$$



f   Frequency unit = 1 cycle/sec

$$|P(f)| = |\mathrm{sinc}(\mathrm{f})|$$ **normalized**



f_0 = fT   Frequency unit = 1/T

Bandwidth computation for Rectangular Pulse

- Consider normalized system with $p_1(t) = I_{[0,1]}(t)$ for which

$$P_1(f) = \mathrm{sinc}(f) e^{j\pi f}$$

- For $\{b[n]\}$ iid, taking values between ± 1 with equal probability, $\sigma_b^2 = 1$, we get $S_{u_1}(f) = \mathrm{sinc}^2(f)$. Real and Positive

- For a fractional power-containment bandwidth with fraction γ

$$\int_{-B_1/2}^{B_1/2} S_{u_1}(f)\, df = \int_{-B_1/2}^{B_1/2} \mathrm{sinc}^2(f)\, df = \gamma \int_{-\infty}^{\infty} \mathrm{sinc}^2(f)\, df$$

$$= \gamma \int_0^1 1^2(t)\, dt = \gamma$$ Parseval

$$\int_0^{B_1/2} S_{u_1}(f)\, df = \gamma/2$$ Symmetry of PSD

- For $\gamma = 0.99$, we obtain $B_1 = 10.2$ while for For $\gamma = 0.9$, we obtain $B_1 = 0.85$.

Bandwidth Computation: *Example*

- Consider a passband system operating at a carrier frequency of 2.4 GHz at a bit-rate of a 20 Mbps. A rectangular modulation pulse timelimited to the symbol interval is employed.
  - Find the 99% and 90% power-containment bandwidths if the constellation used is 16 QAM.
  - Find the 99% and 90% power-containment bandwidths if the constellation used is QPSK.

Poll!

- Consider the two statements:

  1. Transmission of $M_1 > 2$-PSK baseband signal is possible over a single physical channel
  2. Transmission of $M_2 > 4$-QAM passband signal is possible over a single physical channel

  Which of the statement is true?
  (a) Both are true (b) Both are false (c) First is false but second is true (d) First is true but second is false

Bandwidth computation for Sine Pulse

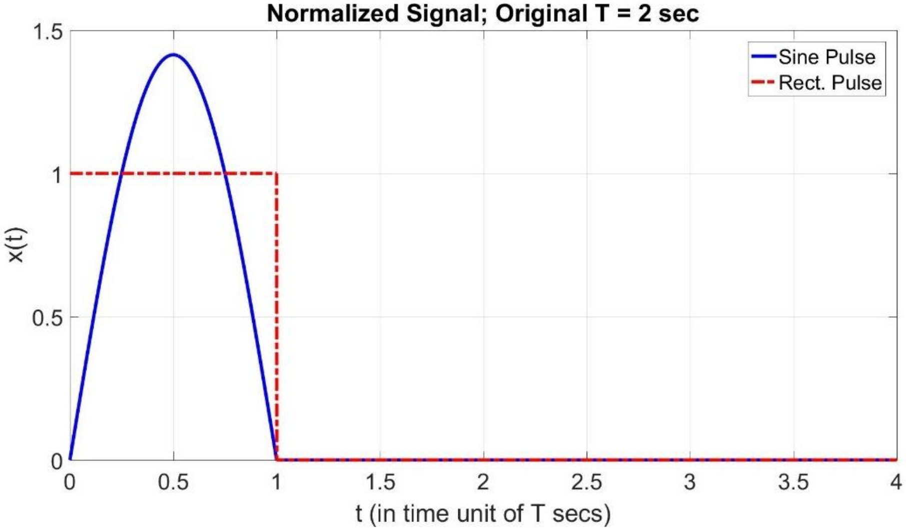
- Consider normalized sine pulse with $p_1(t) = \sqrt{2}\sin(\pi t) I_{[0,1]}(t)$ for which Fourier transform is given by

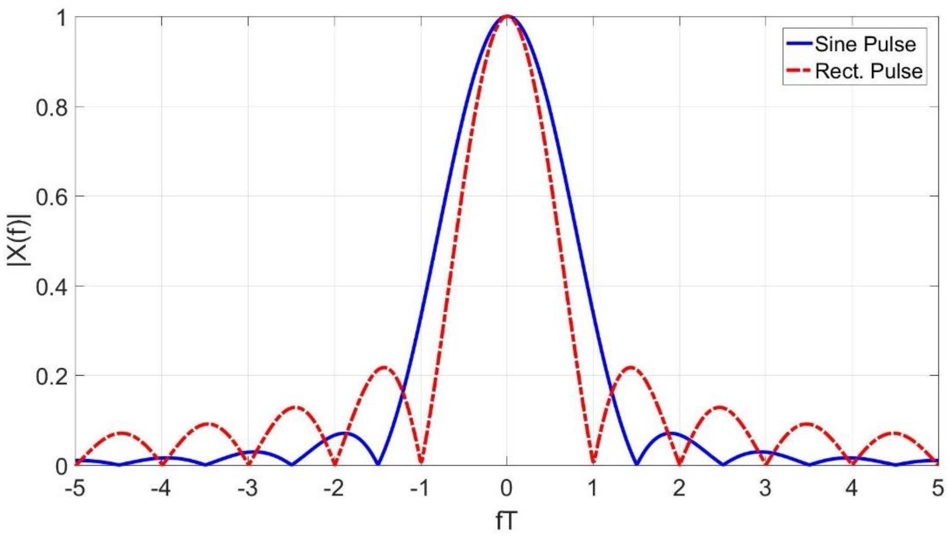
$$P_1(f) = \frac{2\sqrt{2}}{\pi} \frac{\cos(\pi f)}{1 - 4f^2}$$

- Corresponding PSD

$$S_{u_1}(f) = \frac{8}{\pi^2} \frac{\cos^2(\pi f)}{(1 - 4f^2)^2}$$

- Corresponding $\gamma = 0.99$, we obtain $B_1 = 1.2$.



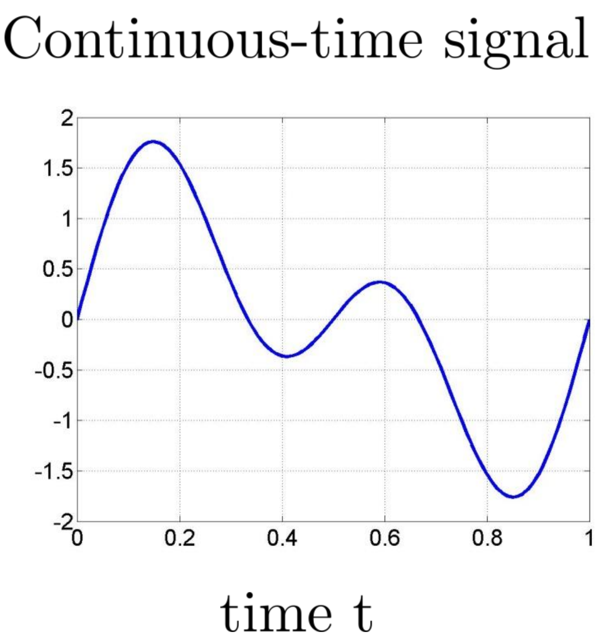


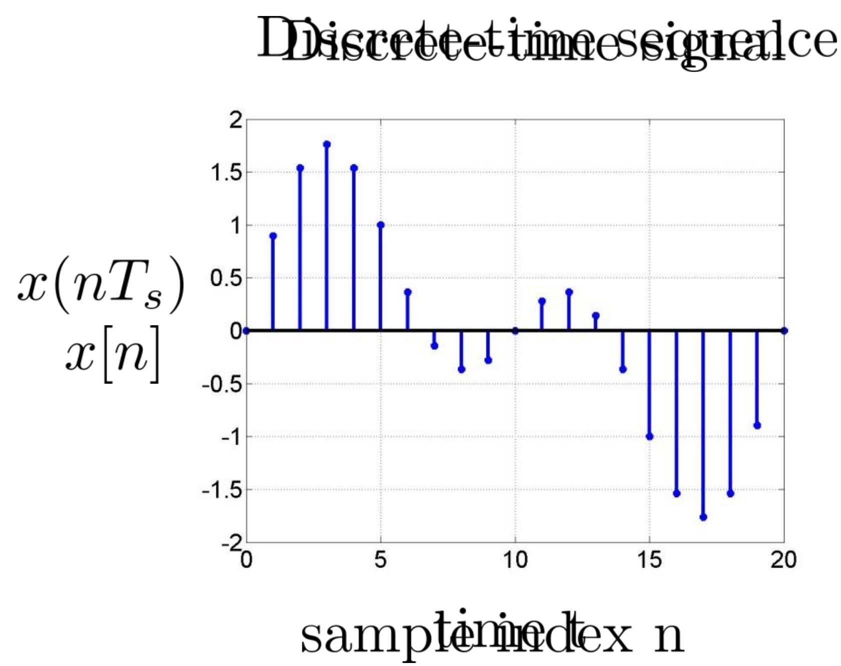
Design for Bandlimited Channels: Nyquist Sampling Criteria

Recap of S&S Oppenheim Chapter 7
[Not in syllabus]

Relation between CT and DT signals

$$x(t) \longrightarrow \boxed{\text{Sampling}} \longrightarrow x(nT_s) = x[n]$$



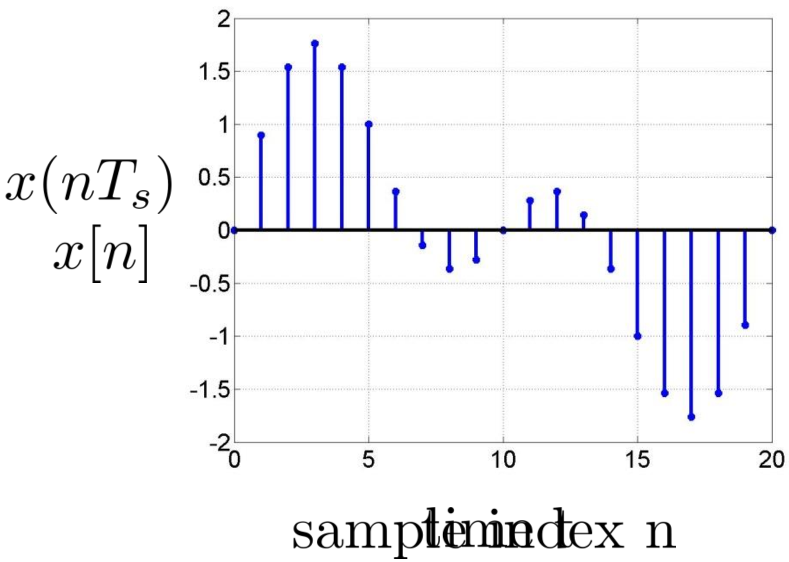


Relation between CT and DT signals

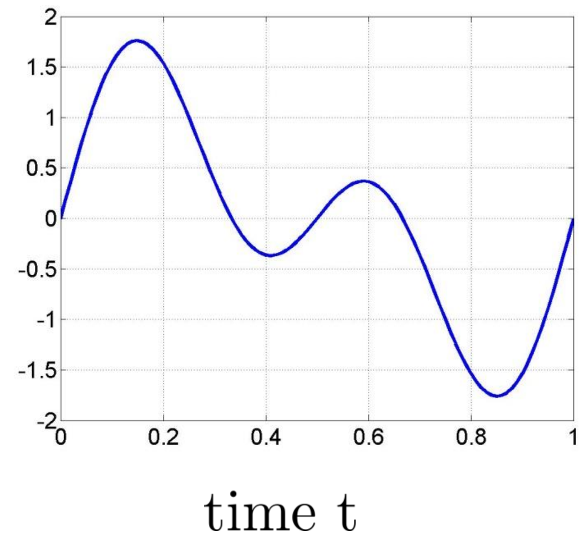
- Reconstruction

$$x(nT_s) = x[n] \longrightarrow \boxed{\text{Interpolation}} \longrightarrow x(t)$$

Discrete-time sequence
Discrete-time signal

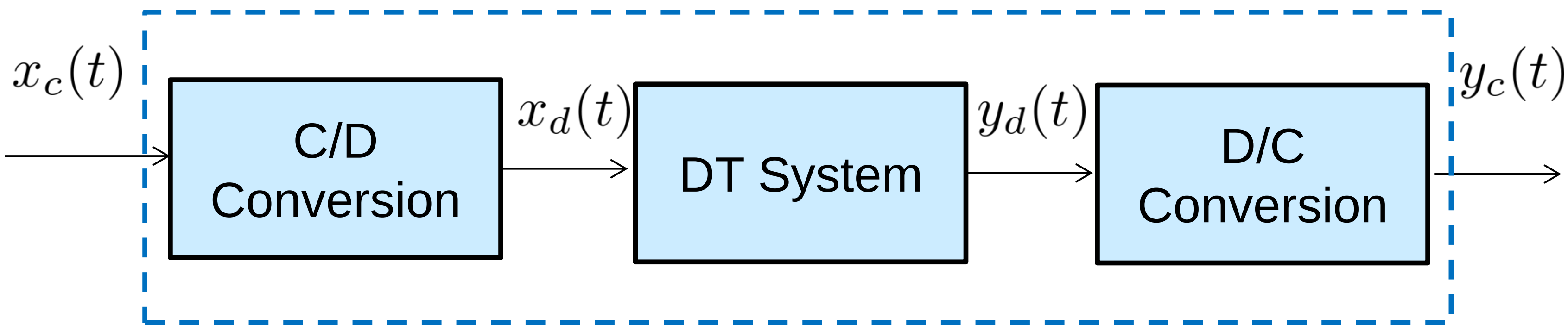


Continuous-time signal

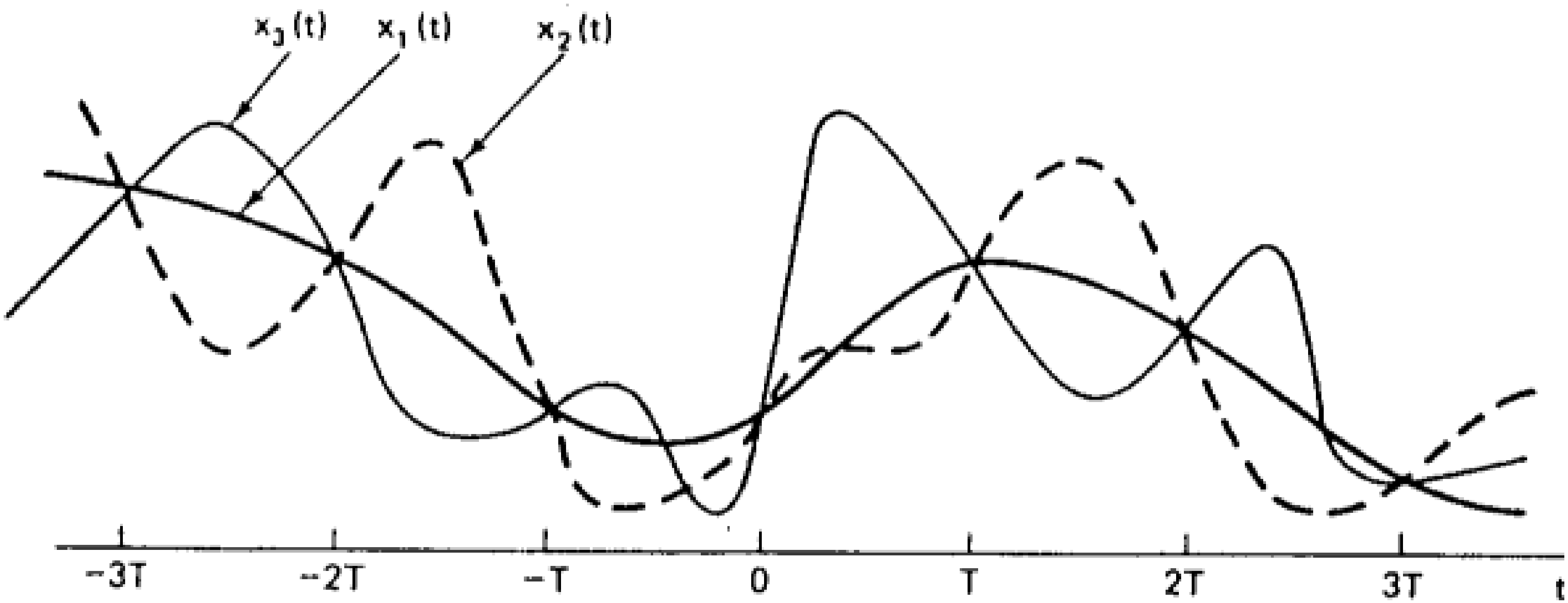


Importance of Sampling

- Bridge between CT and DT signals
- CT Signals can be represented by DT sequence (Sampling Theorem)
- DT signal processing of CT signals
  - DT systems are inexpensive, flexible, reliable, programmable, and efficient than CT systems



Issue with sampling



Sampling Theorem

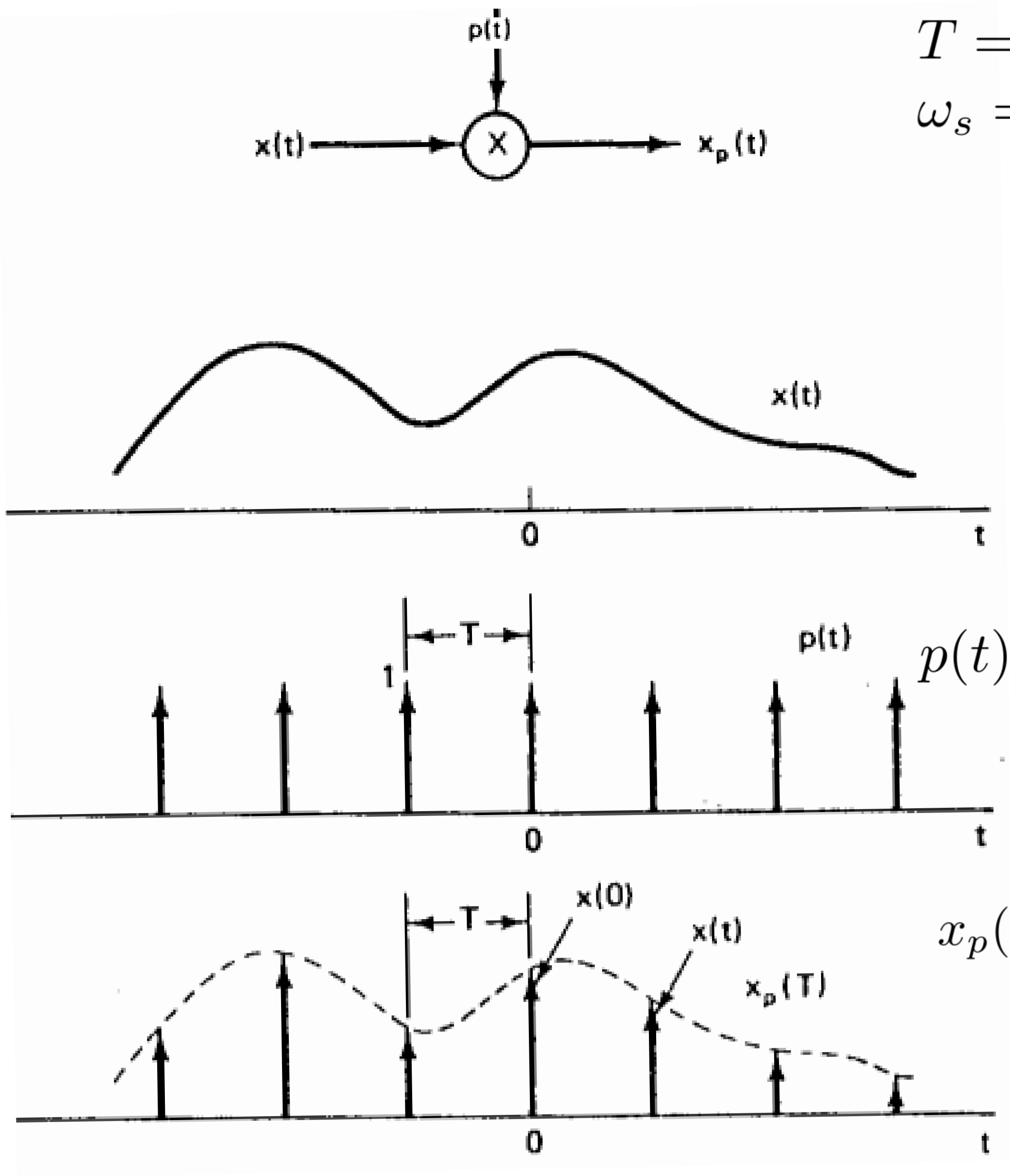
- If $x(t)$ be a bandlimited signal with $X(j\omega) = 0$ for $\omega > \omega_M$, and

$$w_s > 2w_M$$

then $x(t)$ is uniquely determined by its samples $x(nT)$, for $n = 0, \pm 1, \pm 2, \ldots,$. Here $w_s = 2\pi/T$.

- To develop sampling theorem, we will have a look at impulse train sampling and reconstruction.

S&S Recap: *Impulse Train Sampling (Time)*



$T =$ sampling time

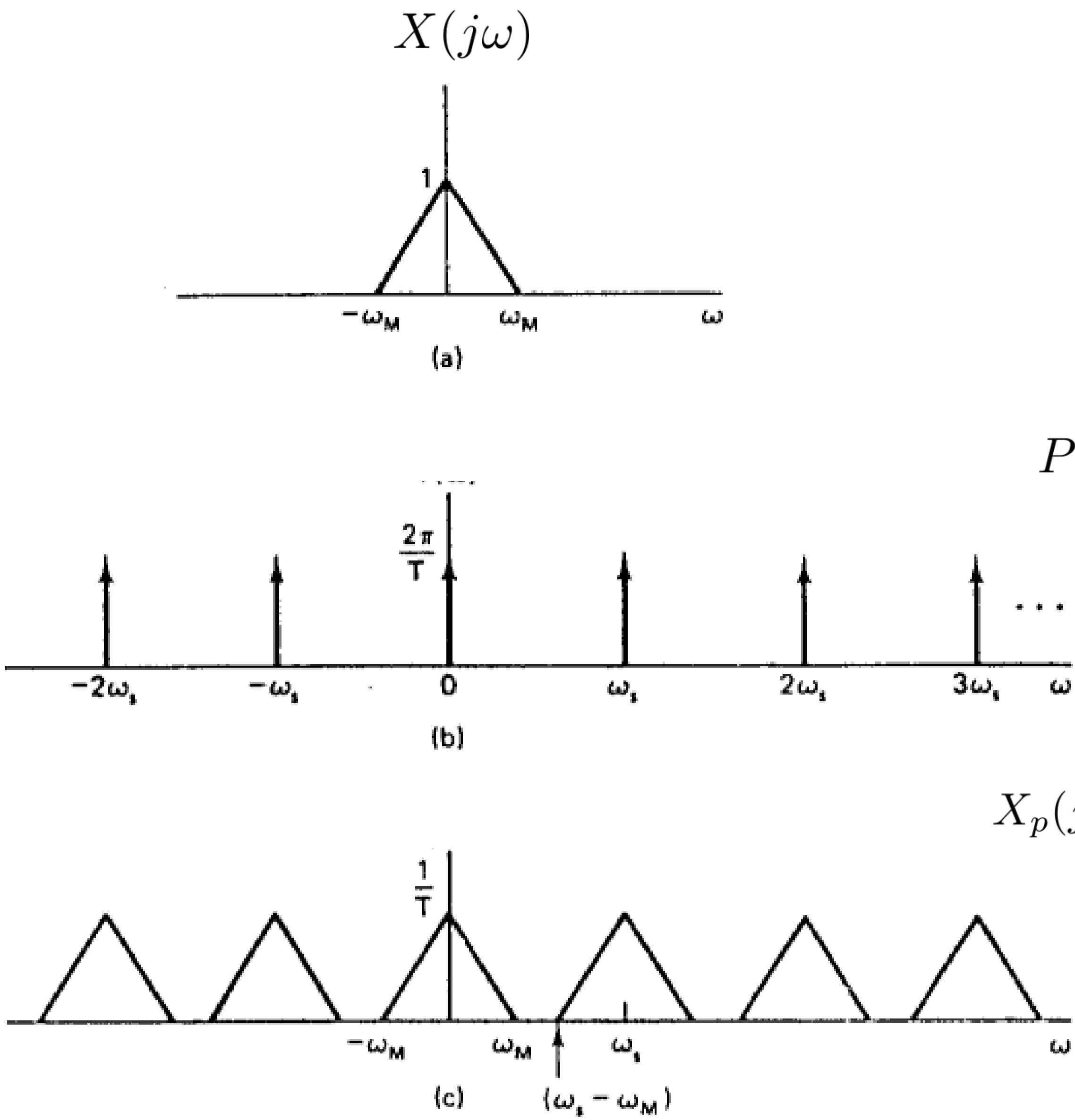
$\omega_s = 2\pi/T =$ sampling frequency

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

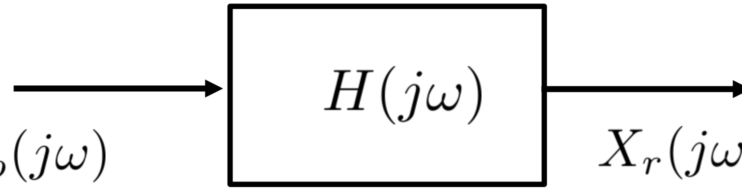
S&S Recap: *Impulse Train Sampling (Freq.)*



$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{k=\infty} \delta(\omega - k\omega_s)$$

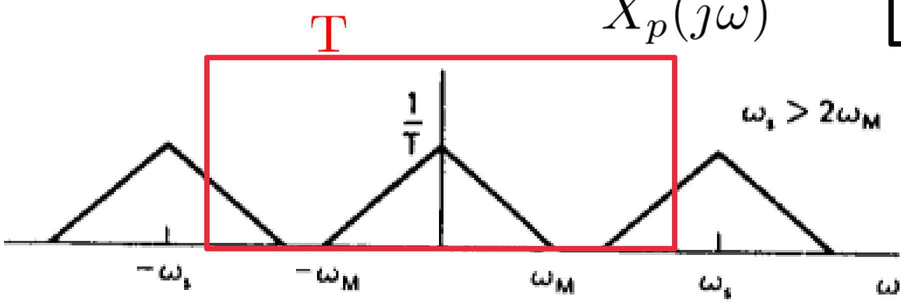
$$\begin{aligned} X_p(j\omega) &= \frac{1}{2\pi} X(j\omega) * P(j\omega) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} P(j\theta) X(j(w - \theta))\, d\theta \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \end{aligned}$$

Reconstruction of signal: Frequency Domain

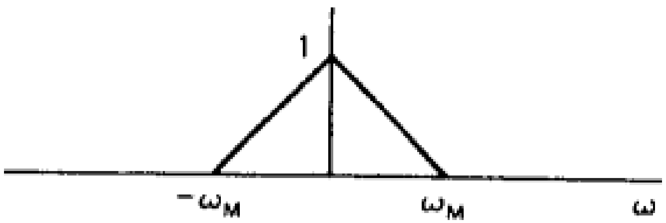


$X_p(j\omega)$

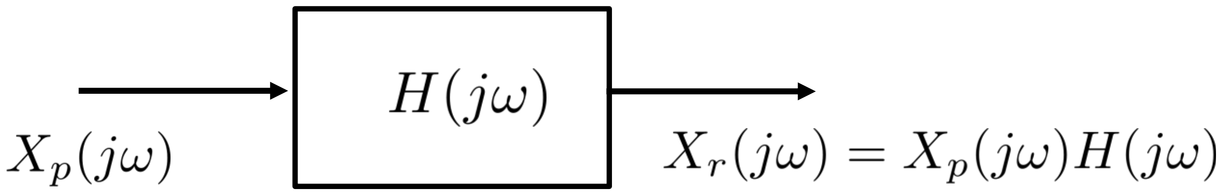
$X_r(j\omega) = X_p(j\omega)H(j\omega)$

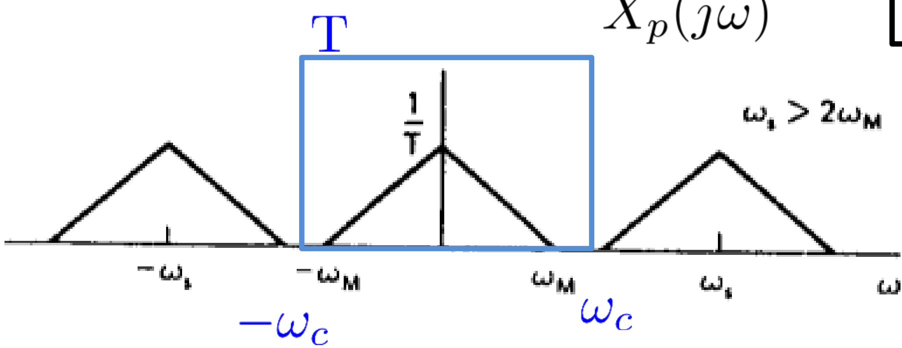


What should be $H(j\omega)$ if $X_r(j\omega) = X(j\omega)$?

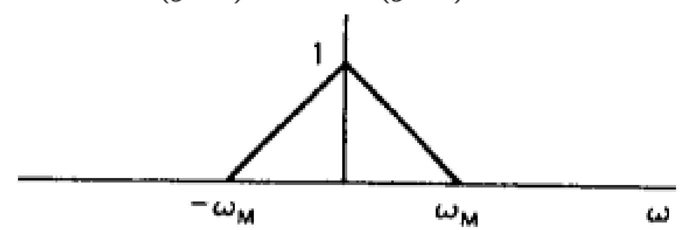


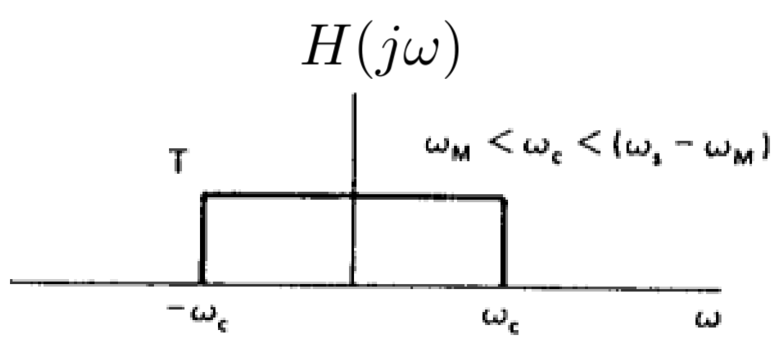
S&S Recap: *Frequency Domain*



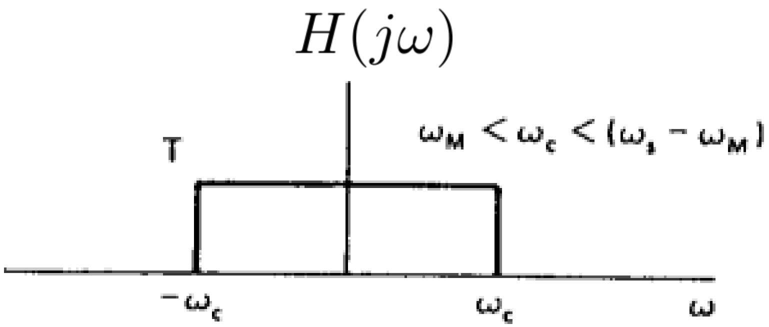


What should be $H(j\omega)$ if $X_r(j\omega) = X(j\omega)$?

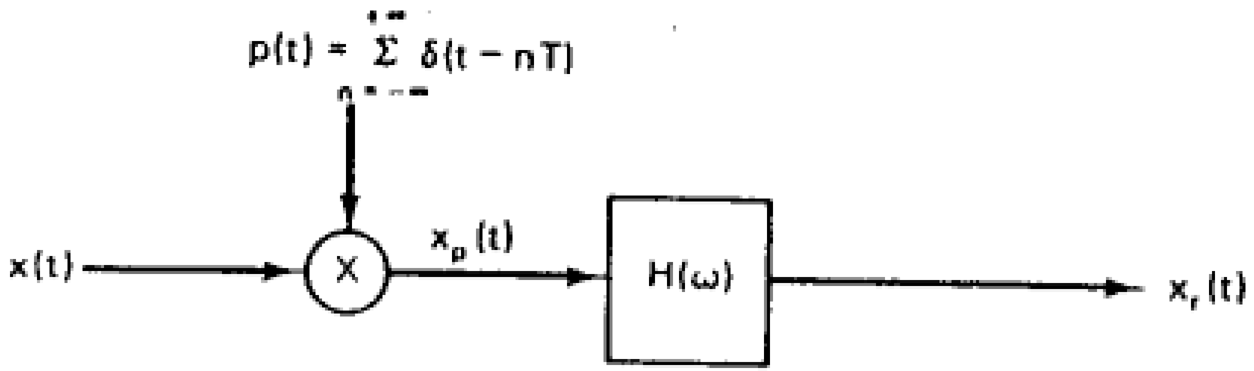




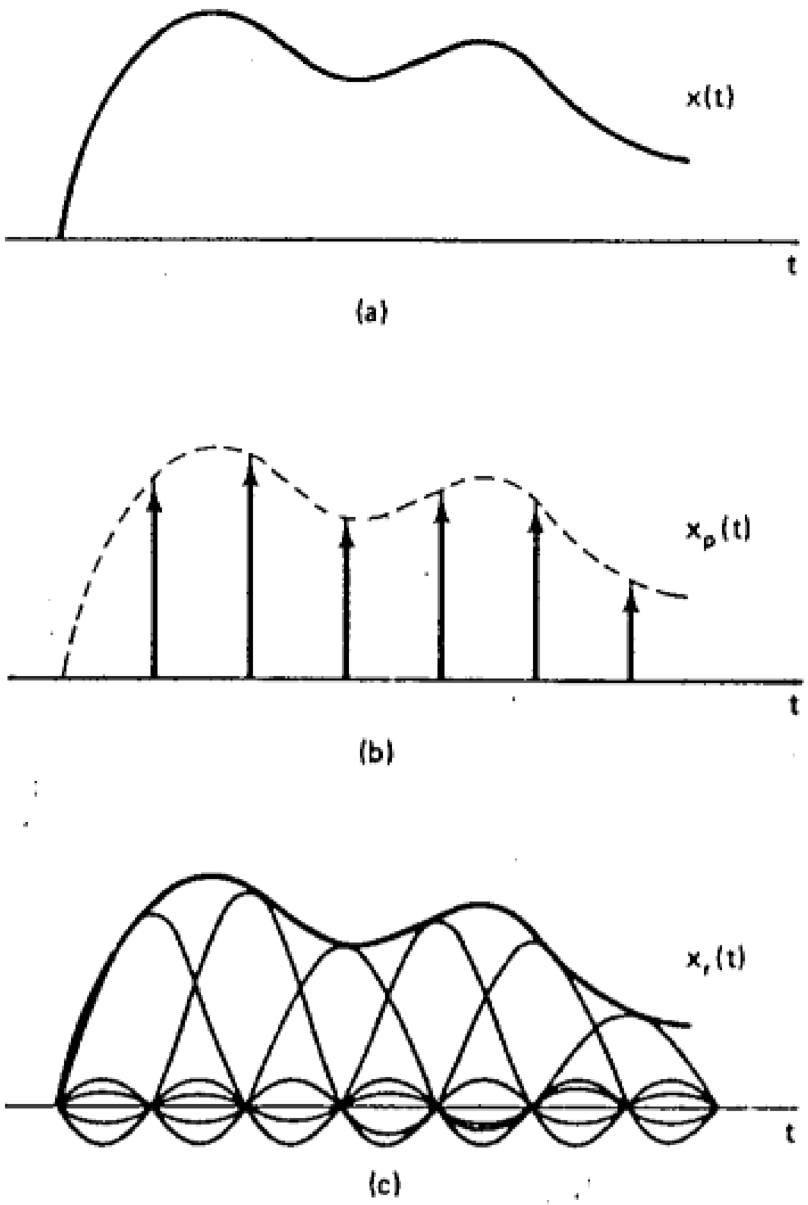
S&S Recap: *Reconst. of signal (Freq. Domain)*



$$h(t) = \frac{\omega_c T \sin(\omega_c t)}{\pi \omega_c t} = \frac{\omega_c T}{\pi} \mathrm{sinc}(\omega_c t)$$



S&S Recap: *Reconst. of signal (Time Domain)*



$$
\begin{aligned}
x_r(t) &= x_p(t) * h(t) \\
&= \left(\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) \right) * h(t) \\
&= \sum_{n=-\infty}^{\infty} x(nT)h(t-nT) \\
&= \sum_{n=-\infty}^{\infty} x(nT)\frac{\omega_c T}{\pi}\mathrm{sinc}(\omega_c(t-nT))
\end{aligned}
$$

Questions?

Design for Bandlimited Channels:

Nyquist Sampling Criteria

[Madhow]

Nyquist Sampling Theorem (Book Notations!!!)

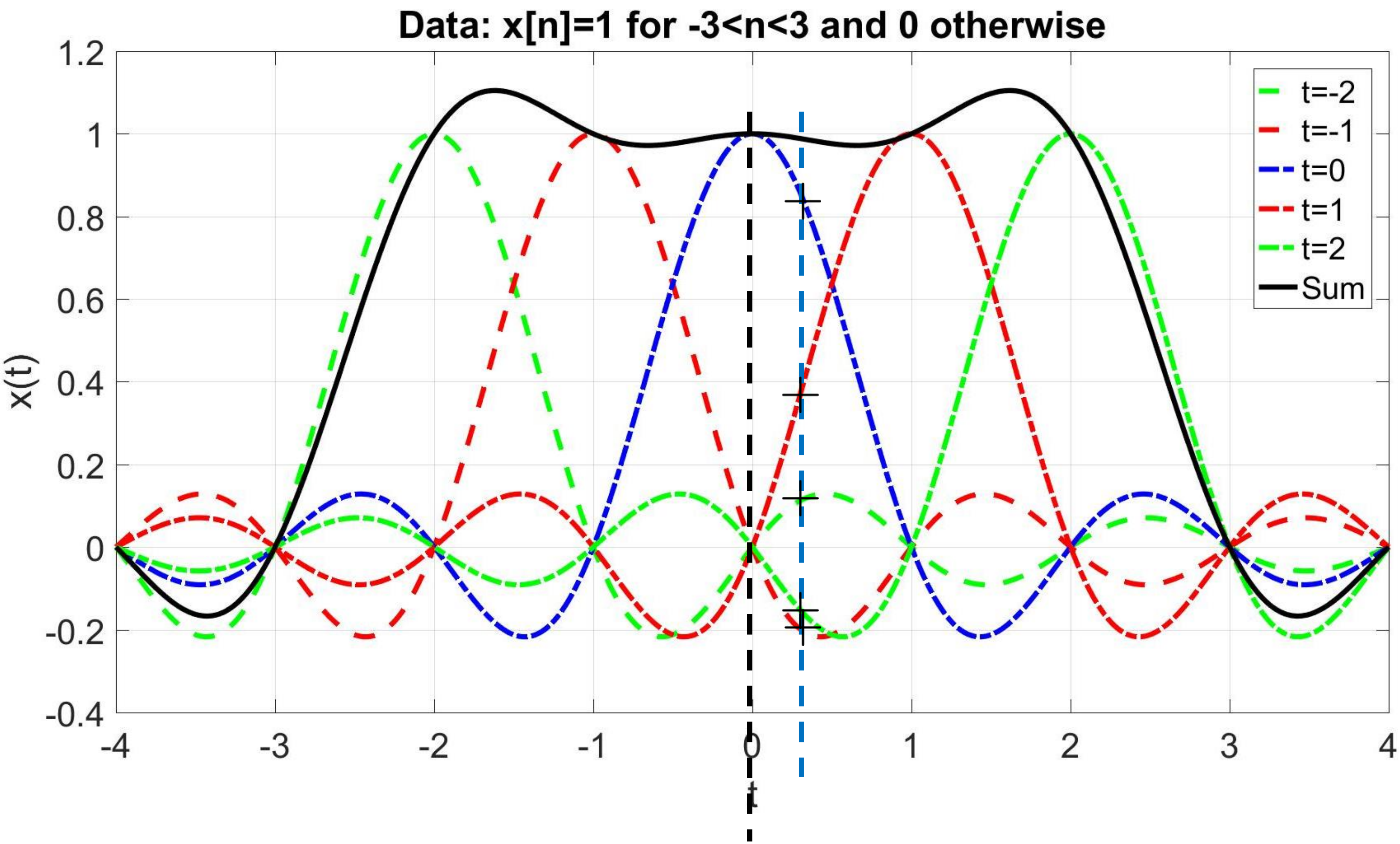
- Any signal $s(t)$ bandlimited to $[-W/2, W/2]$ can be described completely by its samples $\{s(n/W)\}$ at rate W. The signal $s(t)$ can be recovered from its samples using the following interpolation formula

$$s(t) = \sum_{n=-\infty}^{\infty} s\left(\frac{n}{W}\right) g\left(t - \frac{n}{W}\right)$$

where $g(t) = \mathrm{sinc}(Wt)$.

- Book uses $p(t)$ here but I have used $g(t)$ on purpose to differentiate it from the modulating pulse $p(t)$.

Problem With Sinc Pulse



Time-offset=0.3

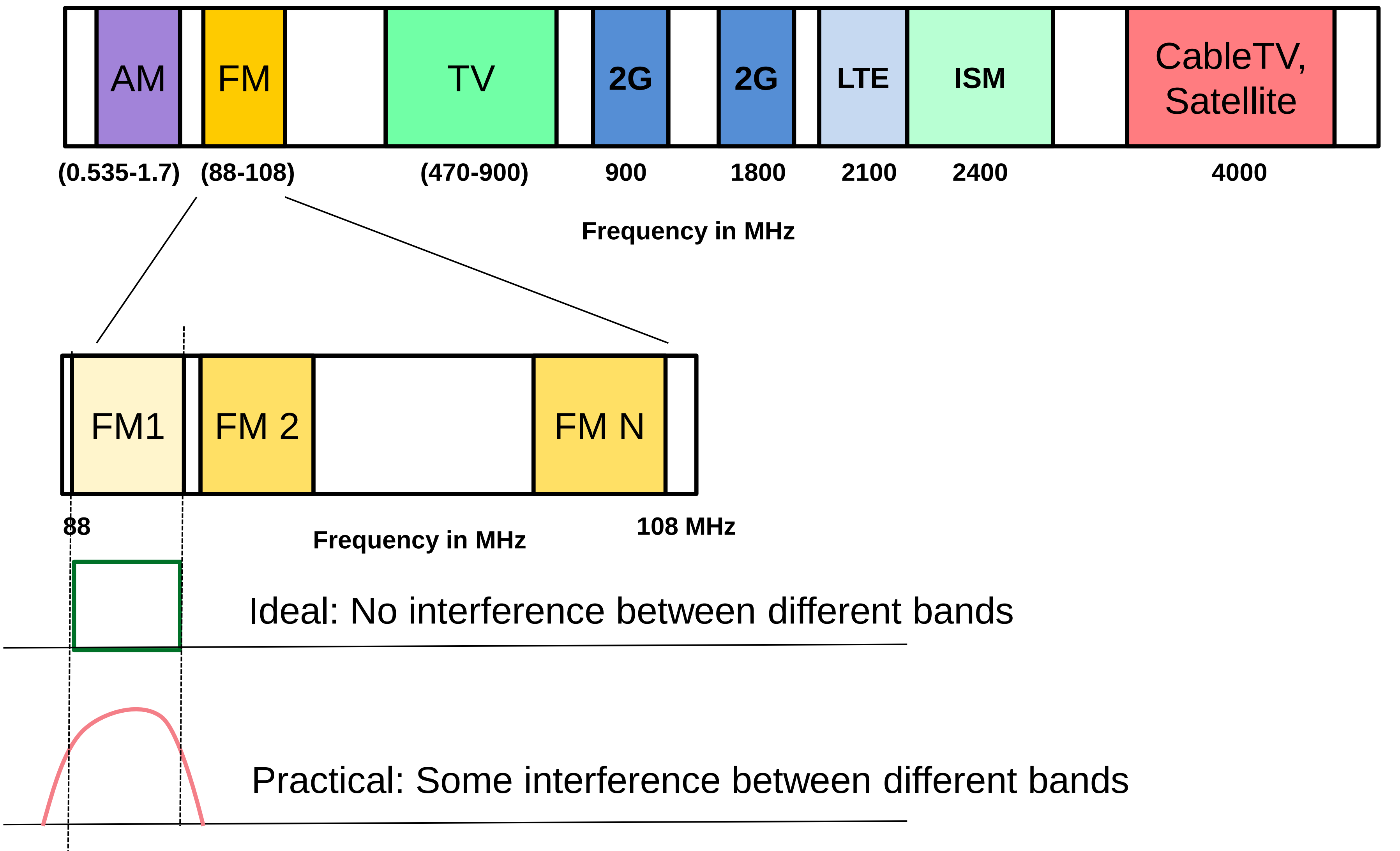
Design for Bandlimited Channels:

Nyquist Criteria for pulse shaping!

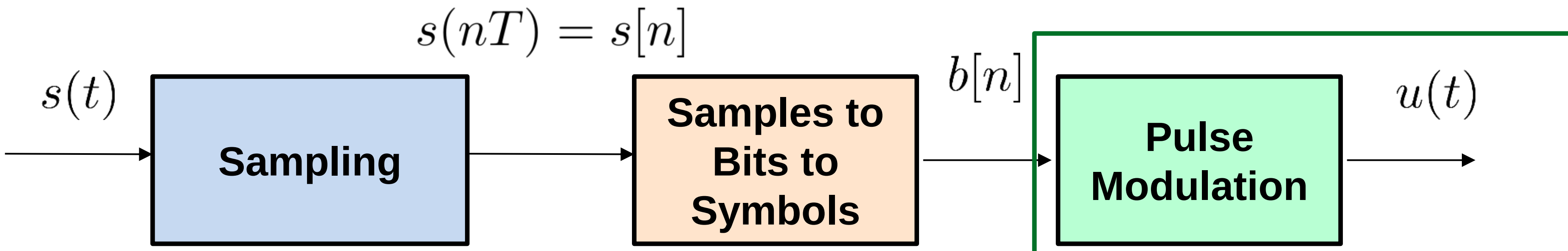
Bandlimited Channels

- Consider that we are given 20 MHz bandwidth at a carrier frequency of 2.4 GHz.
- Any signal that we send on this passband has a complex envelop from -10 MHz to 10 MHz.
- In general, for a passband signal of bandwidth W, we have corresponding complex-baseband signal spanning $[-W/2, W/2]$.
- Note that *sinc* and *sine* pulses are not strictly bandlimited.

Motivation


AM
FM
TV
2G
2G
LTE
ISM
CableTV, Satellite
(0.535-1.7)
(88-108)
(470-900)
900
1800
2100
2400
4000
Frequency in MHz
FM1
FM 2
FM N
88
108 MHz
Frequency in MHz
Ideal: No interference between different bands
Practical: Some interference between different bands

Design for Bandlimited Channels



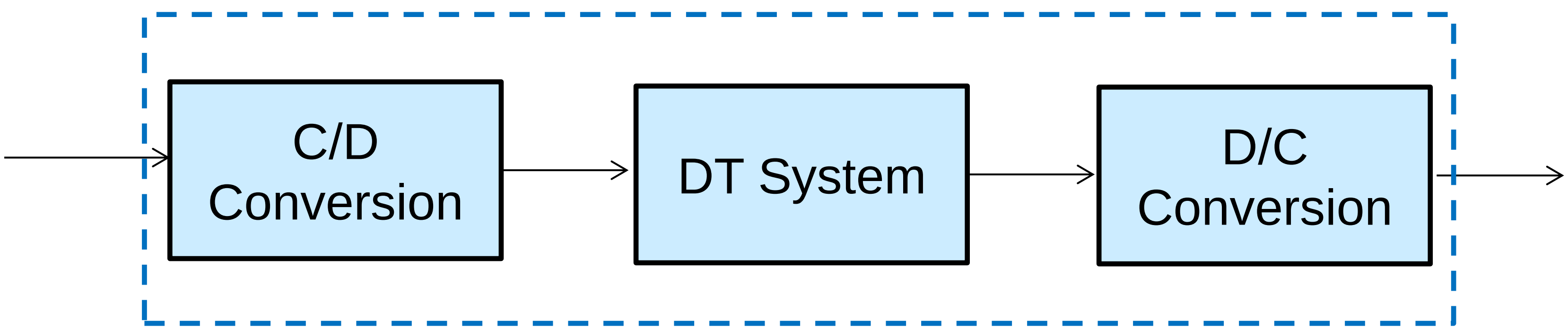
$$s(t) = \sum_{n=-\infty}^{\infty} s\left(\frac{n}{W}\right) g\left(t - \frac{n}{W}\right)$$

$$u(t) = \sum_{n} b[n] p(t - nT)$$

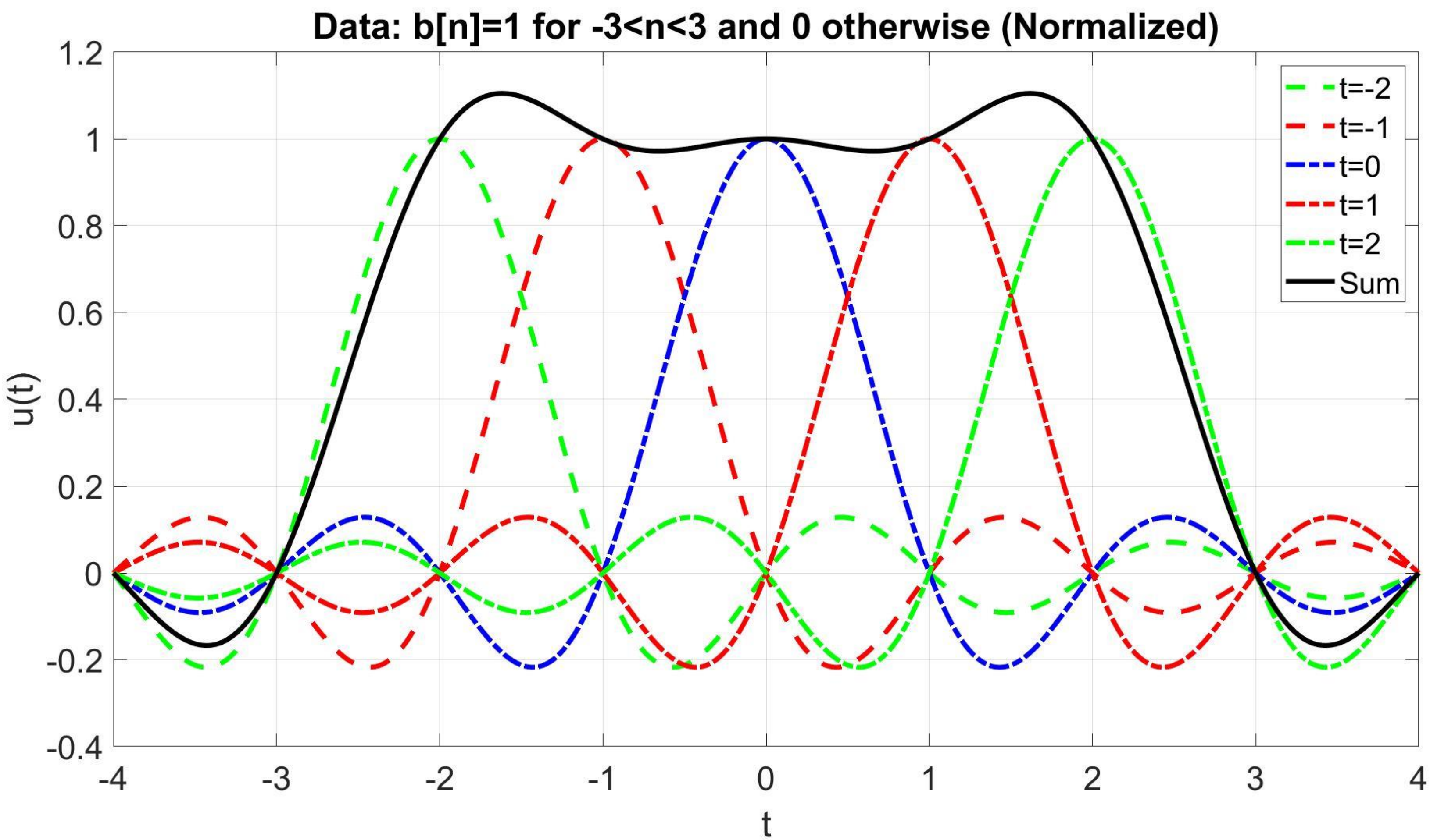
$$S_u(f) = \frac{|P(f)|^2}{T} \sigma_b^2$$

Minimum bandwidth dictated by Sampling
W=1/T

Bandwidth occupancy can be designed based on P(f) and independent of S(f) (FT of s(t)).



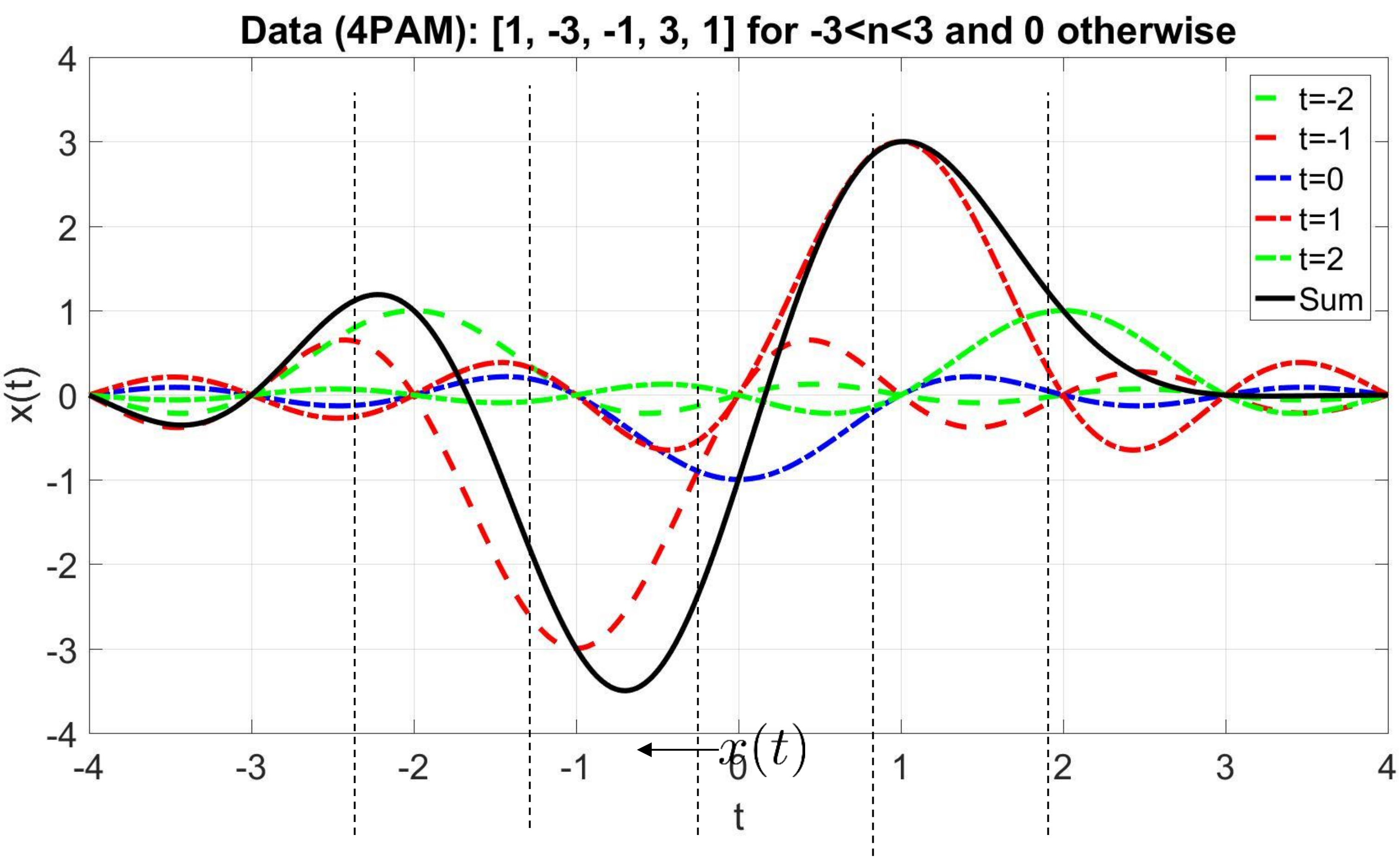
Sum of Sinc Pulses: Ex. 1



$$u(t) = \sum_{n=-\infty}^{\infty} u\left(\frac{n}{W}\right)\mathrm{sinc}\left(t - \frac{n}{W}\right) = \sum_{n=-\infty}^{\infty} b[n]\mathrm{sinc}\left(t - \frac{n}{W}\right)$$

- Normalized W=1=1/T
- *p(t)* is sinc function

Signal as Sum of Sinc Pulses: Ex. 2



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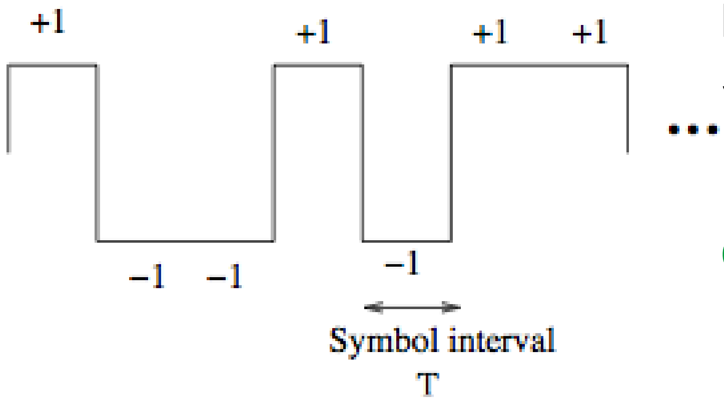
- Normalized W=1=1/T
- *p(t)* is sinc function

What is Inter Symbol Interference (ISI)?

Time Domain *p(t)*

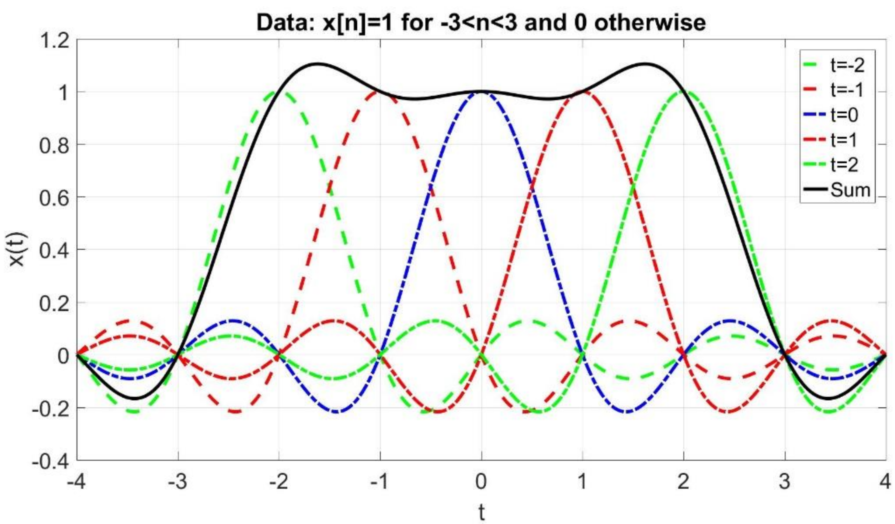
Rectangular Pulse



Freq. Domain *P(f)*
Sinc

No ISI=> SmallTiming offset does not cause issues

Sinc Pulse



Freq. Domain *P(f)*
Rect. Pulse

ISI=> SmallTiming offset does not cause issues

No ISI at sampling instances though!

Nyquist Criterion for ISI avoidance

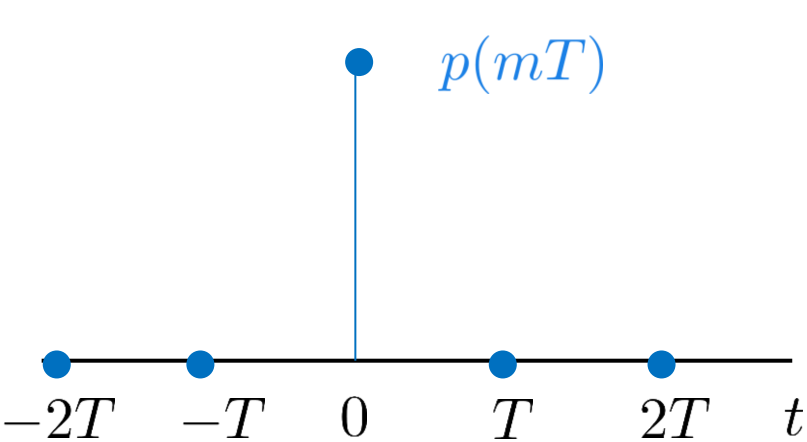
- The pulse $p(t) \leftrightarrow P(f)$ is Nyquist for sampling rate $1/T$ if

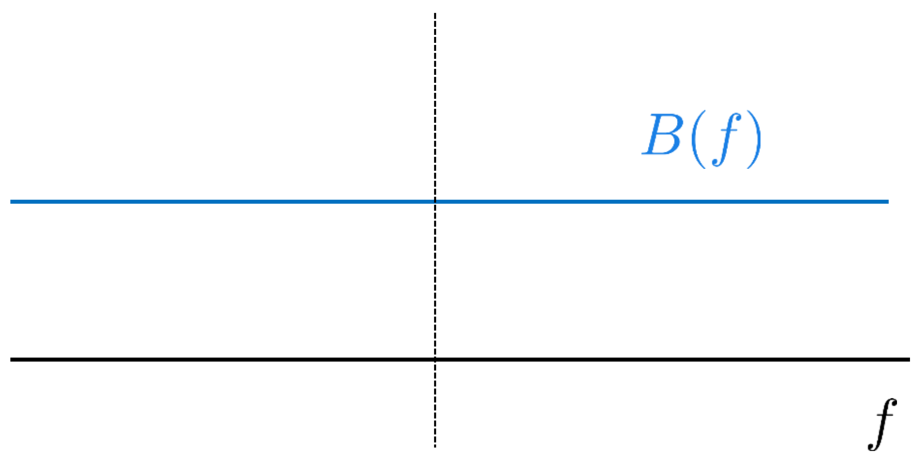
$$p(mT) = \delta_{m0} = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}$$

or equivalently

DT Fourier Transform Pair

$$B(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} P\left(f + \frac{k}{T}\right) = 1 \quad \forall f$$





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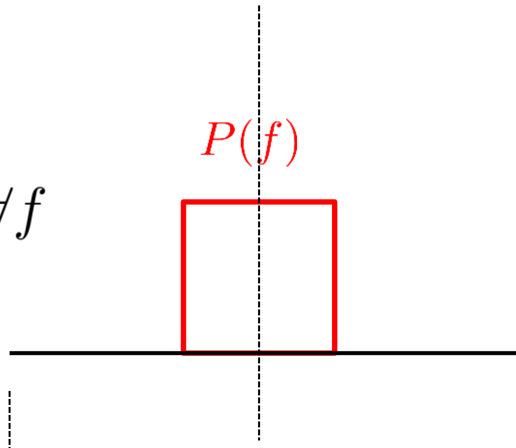
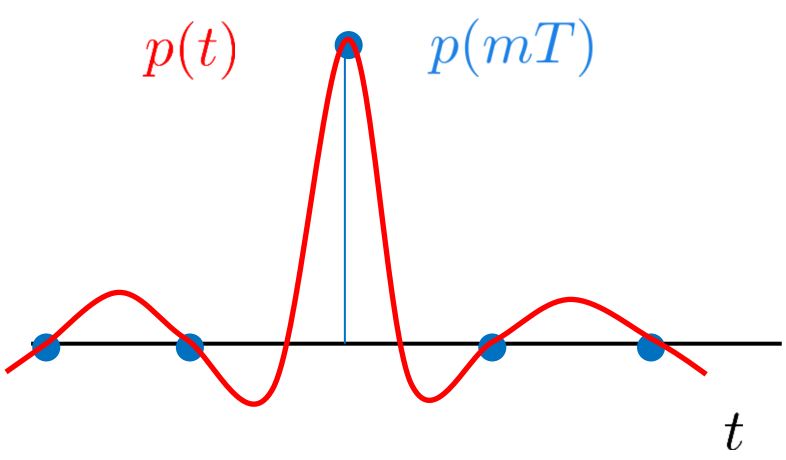
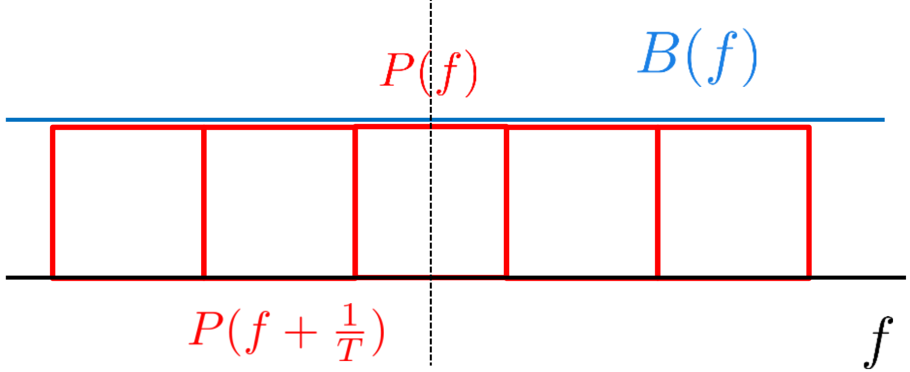
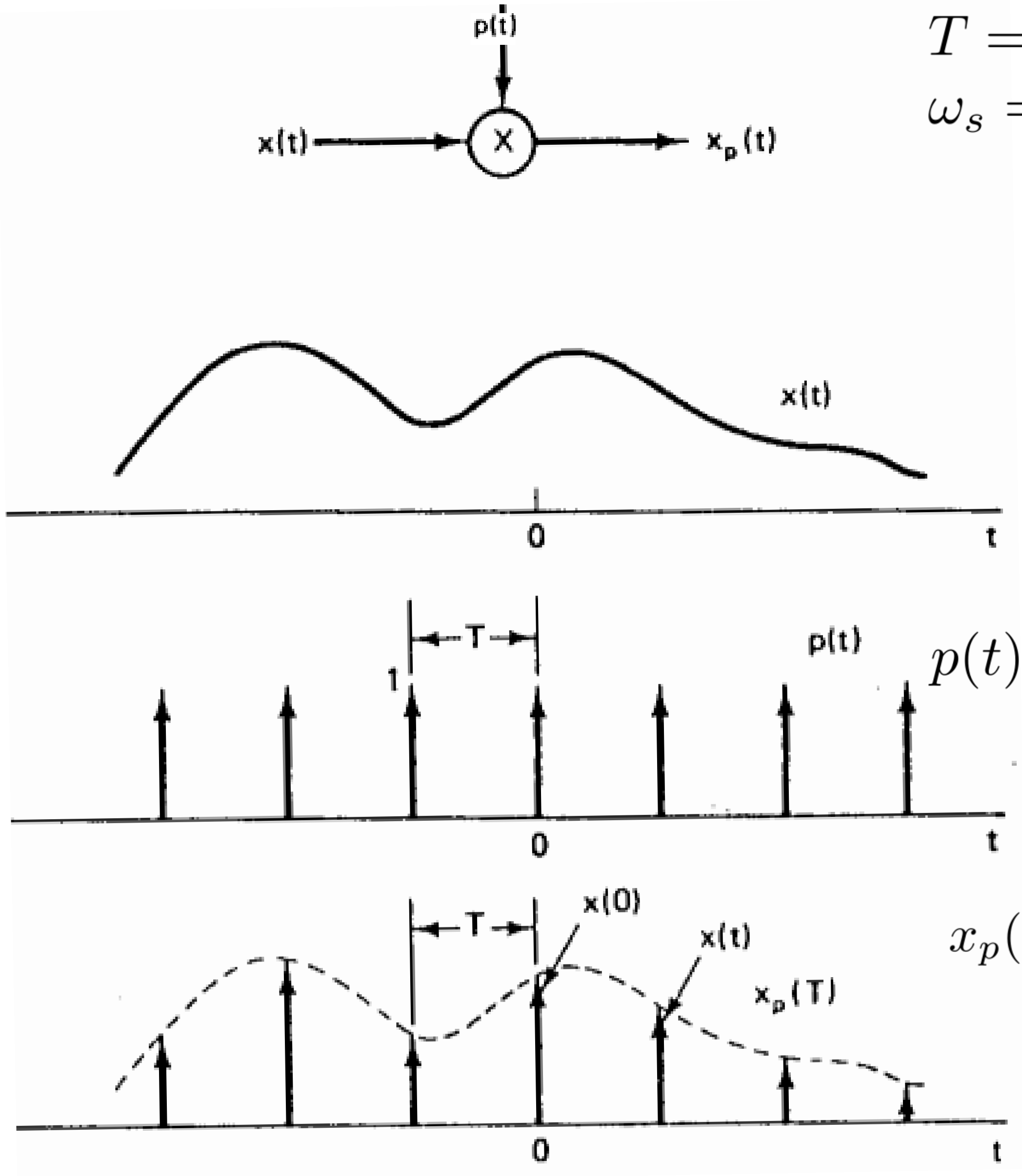






S&S Recap: *Impulse Train Sampling (Time)*



$T = \text{sampling time}$

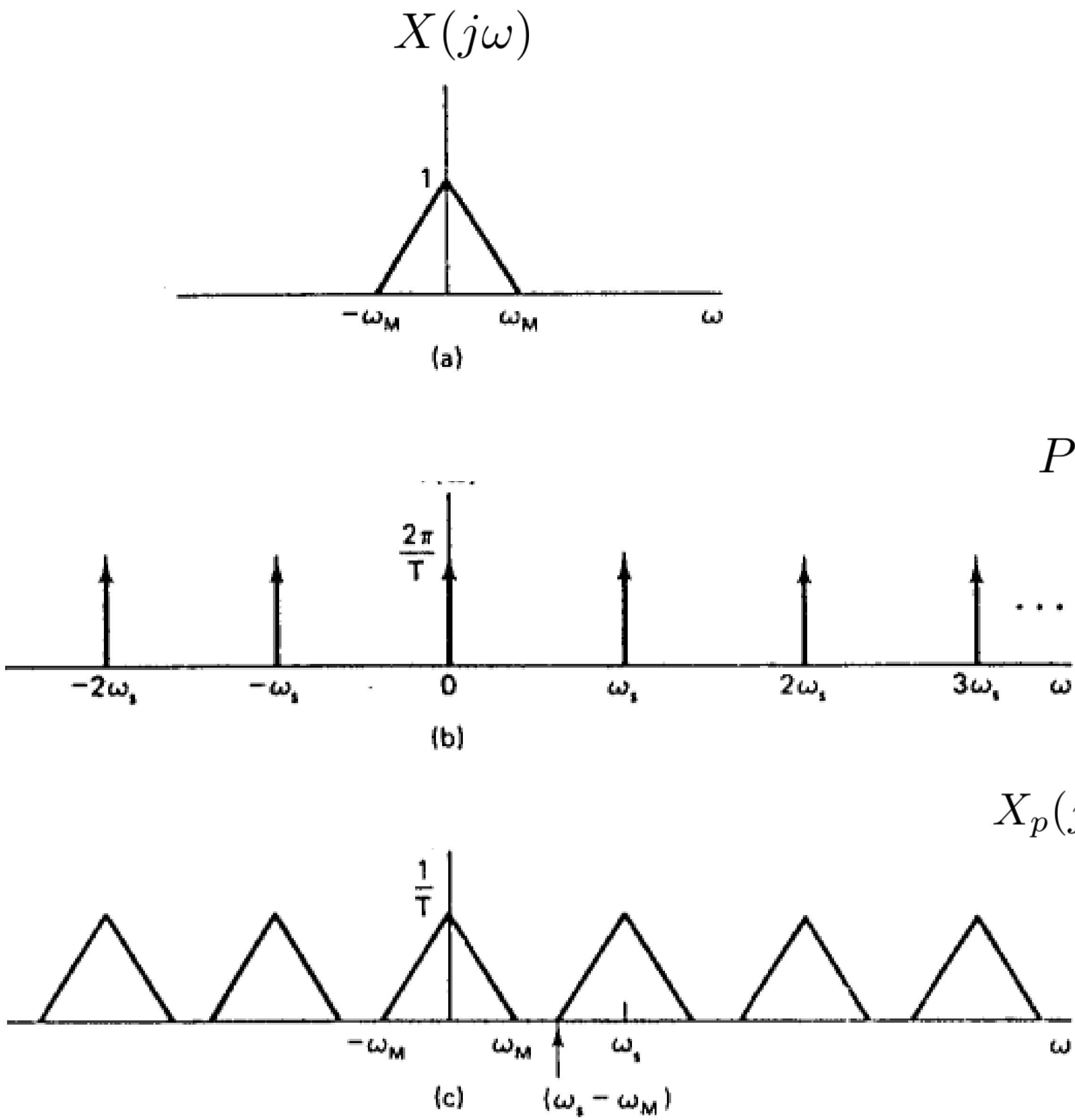
$\omega_s = 2\pi/T = \text{sampling frequency}$

$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$

$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

S&S Recap: *Impulse Train Sampling (Freq.)*



$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{k=\infty} \delta(\omega - k\omega_s)$$

$$\begin{aligned} X_p(j\omega) &= \frac{1}{2\pi} X(j\omega) * P(j\omega) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} P(j\theta) X(j(w-\theta))\, d\theta \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \end{aligned}$$

Theorem 4.5.1: Sampling

- Theorem (Sampling): Consider a signal $s(t)$, sampled at rate $1/T_s$. Let $S(f)$ denote the spectrum of $s(t)$, and let

$$B(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} S(f + \frac{k}{T_s})$$

denotes the sum of translates of the spectrum. Then the following observations hold

1. $B(f)$ is periodic with period $1/T_s$.
2. The samples $s(nT_s)$ are Fourier series for $B(f)$, satisfying

$$s(nT_s) = T_s \int_{-\frac{1}{2T_s}}^{\frac{1}{2T_s}} B(f) e^{j2\pi f n T_s} df$$

$$B(f) = \sum_{n=-\infty}^{\infty} s(nT_s) e^{-j2\pi f n T_s}$$