Under-Actuated Manipulator Control (is hard)

& other revelations in manipulator kinematics

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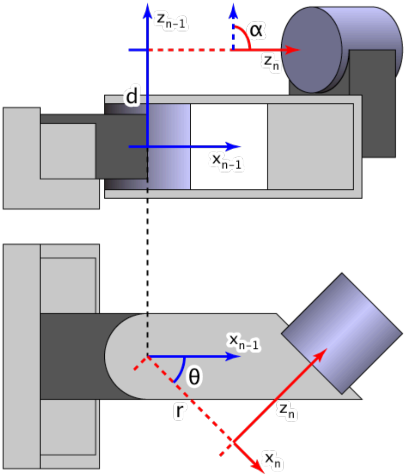
Contents

1 Representing the Arm

DH Parameters

Denavit Hartenberg (DH) parameters describe each joint using four values $a_i, \alpha_i, d_i, \theta_i$ and produce forward kinematics through a fixed transform pattern. They were popular because they give a compact, minimal representation.



Issues with DH Parameters

- Frame assignment rules are unintuitive and heavily constrained.
- Small mistakes in axis alignment break the model.
- Hard to insert tool frames, offsets, or calibration transforms.

Some people have tried to bridge the gap between modern methods like URDF and DH, but for this project we used a different approach.

ETS

- Elementary Transform Sequence (ETS) represents each joint as a separate transform.
- More intuitive frame assignment.
- Simply compose transforms to get forward kinematics.

$$^{0}T_{e} = \prod_{\mathrm{i}=1}^{n} E_{i(\eta_i)}$$

> This tutorial [...] deliberately avoids the commonly-used Denavit-Hartenberg parameters which we feel confound pedagogy.
>
> — arxiv.org/abs/2207.01796

2 Metrics to Visualize Reachability

Manipulability

$$\omega = \sqrt{\det(J * J^T)}$$

But for our 5-DOF arm, `J.shape() == (6, 5)`, so $J * J^T$ is not full rank and $\det(J * J^T) = 0$ always.

So instead, we use the singular values of `J` to define manipulability as:

$$\omega = \sqrt{\prod_{\mathrm{i}=1}^{6} \sigma_i}$$

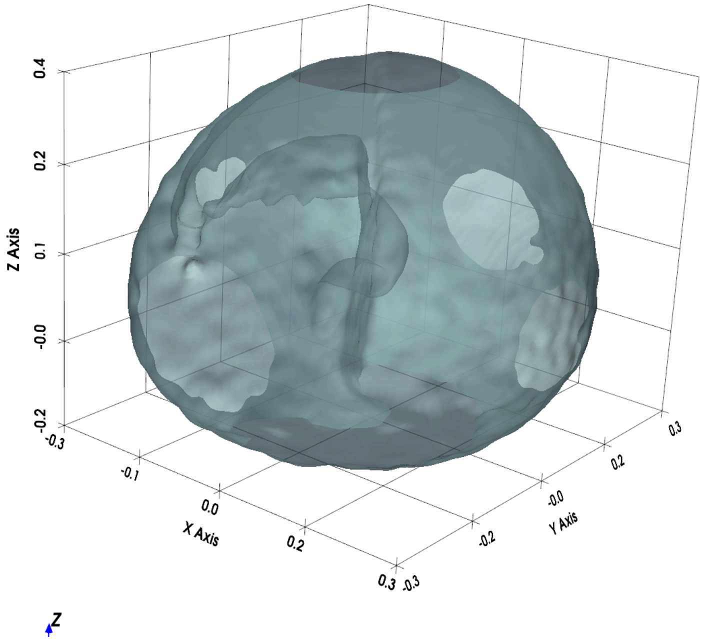
Condition

Another useful metric is the condition number of the Jacobian, defined as the ratio of the maximum and minimum singular values:

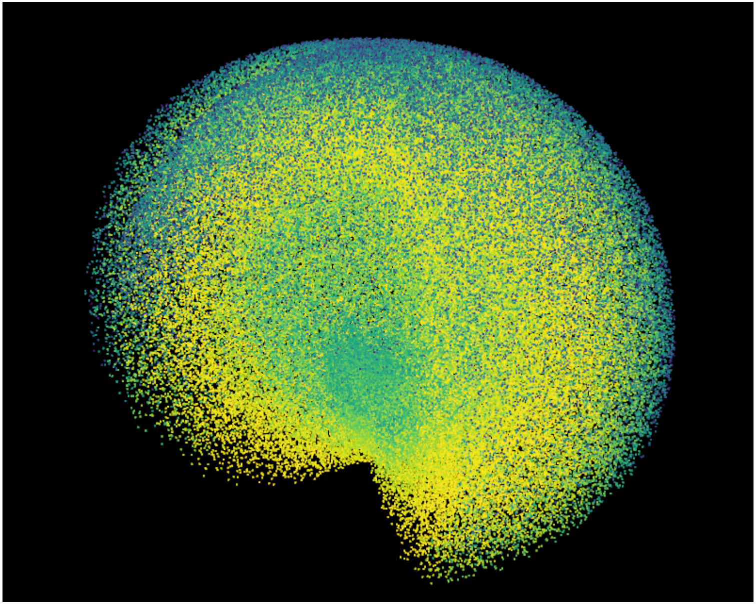
$$\kappa = \frac{\sigma_{\max}}{\sigma_{\min}}$$

$\kappa \gg 1$ implies the arm is close to a singularity, while $\kappa \approx 1$ implies good dexterity.

Visualizing the Reachable Workspace

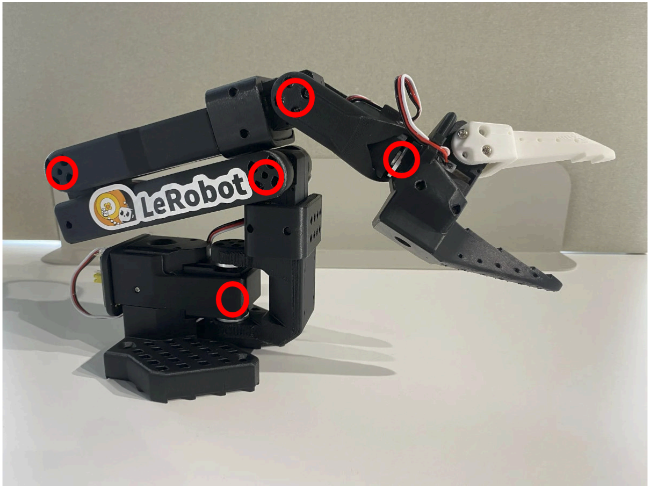


Visualizing the Reachable Workspace (ii)



3 The Problem and its Naive Solution

5-DOF, What are we missing?



There is no end-effector yaw.

Feasibility Projected IK

- 5-DOF arm can only move on a 5D "surface" inside the 6D space
- Project desired pose to closest feasible pose on that surface

$$q^* = \arg\min_q \|W_x(f(q) - x_d)\|^2$$

$$\Delta x \approx J\Delta q$$

$$\Delta x_{\text{achievable}} = JJ^+\Delta x$$

From this we solve for:

Where it Fails

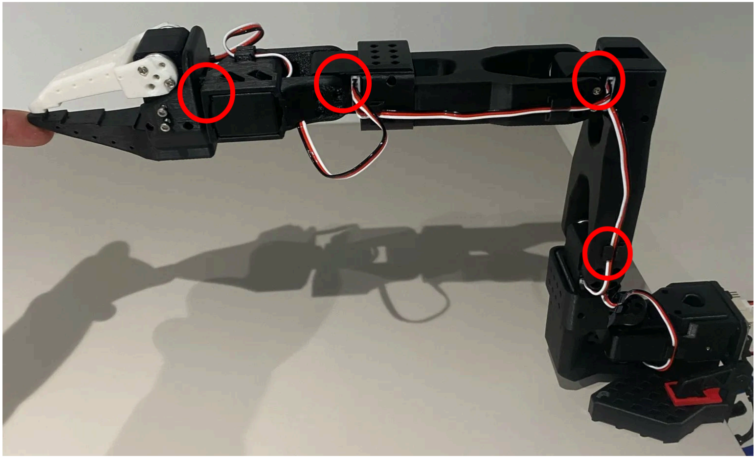
- Because the yaw axis is missing, when the end effector moves off the 2D plane defined by the first joint axis, the IK solver continuously fails to find a solution.
- It barely leaves the 2D plane in practice.
- The “projected” solution is always just on (or very close to) the 2D plane.

4 The Workaround that Works

Constraining the IK

- Instead, we constrain the IK solver.
- Never solve for the first joint angle, so the solver thinks the motion is on a 2D plane.
- Use a geometric IK to get back full position control.
- End effector yaw is still not controllable :(but at least the solver doesnt continuously fail when we leave the 2D plane.

Fool the IK solver



The solver never gets end effector y-axis position or yaw updates, and never updates the first joint angle.

This way the solver doesnt continuously fail when we leave the 2D plane.

Limitations

- Only works for position control, not full 6D pose control.
- End effector yaw is completely uncontrollable, and pitch and roll are constrained to the rotating 2D plane.

5 Demo