Seri bahan kuliah Algeo #29

Perkalian Geometri
(Bagian 1)

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Bahan kuliah IF2123 Aljabar Linier dan Geometri

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Sumber:

John Vince, *Geometric Algebra for Computer Graphics*. Springer. 2007

Perkalian Vektor

Perkalian vektor yang sudah dipelajari:

1. Perkalian titik (*dot product* atau *inner product*): $\mathbf{a} \cdot \mathbf{b}$
2. Perkalian silang (*cross product*): $\mathbf{a} \times \mathbf{b}$
3. Perkalian luar (*outer product*): $a \wedge b$

Yang akan dipelajari selanjutnya $\rightarrow$ perkalian geometri: ab

$ab = a \cdot b + a \wedge b$

contoh :

$a = 3e_1 + 2e_2$
$b = 5e_1 - 4e_2$
$ab = a \cdot b + a \wedge b$
$= 3(5) + (2)(-4) + (3e_1 + 2e_2) \wedge (5e_1 - 4e_2)$

Perkalian Geometri

- Perkalian geometri dioperasikan pada *multivector* yang mengandung skalar, area, dan volume
- Perkalian geometri ditemukan oleh William Kingdom Clifford (1845 – 1879)

- Perkalian geometri dua buah vektor *a* dan *b* didefinisikan sebagai berikut:

$$\boxed{ab = \underbrace{a \cdot b}_{\text{skalar}} + \underbrace{a \wedge b}_{\text{bivector}}}$$

Sifat-sifat Perkalian Geometri

1. Asosiatif

   (i) $a(bc) = (ab)c = abc$

   (ii) $(\lambda a)b = \lambda(ab) = \lambda ab$

2. Distributif

   (i) $a(b + c) = ab + ac$

   (ii) $(b + c)a = ba + ca$

3. Modulus

   $a^2 = aa = \|a\|^2$

pake contoh yang tadi

$$
\begin{aligned}
a^2 = aa &= a \cdot a + \overset{\text{harusnya } 0}{a \wedge a} \\
&= 3 \cdot 3 + (2)(2) + (3e_1 + 2e_2) \wedge (3e_1 + 2e_2) \\
&= 3^2 + 2^2 + 9(e_1 \wedge e_2) + 6(e_1 \wedge e_2) + 6(e_2 \wedge e_1) + 4(e_2 \wedge e_1) \\
&= 3^2 + 2^2 + 0 + 6(e_1 \wedge e_2) - 6(e_1 \wedge e_2) + 0 \\
&= 3^2 + 2^2 \\
&= \left(\sqrt{3^2 + 2^2}\right)^2 \\
&= \|a\|^2
\end{aligned}
$$

- Bukti untuk 3:

  Misalkan $a = a_1e_1 + a_2e_2$

  maka

$$
\begin{aligned}
a^2 = aa &= a \cdot a + a \wedge a \\
&= a_1a_1 + a_2a_2 + (a_1e_1 + a_2e_2)\wedge(a_1e_1 + a_2e_2) \\
&= a_1{}^2 + a_2{}^2 + a_1a_1(e_1\wedge e_1) + a_1a_2(e_1\wedge e_2) + a_2a_1(e_2\wedge e_1) + a_2a_2(e_2\wedge e_2) \\
&= a_1{}^2 + a_2{}^2 + 0 + a_1a_2(e_1\wedge e_2) + a_2a_1(e_2\wedge e_1) + 0 \\
&= a_1{}^2 + a_2{}^2 + a_1a_2(e_1\wedge e_2) - a_2a_1(e_1\wedge e_2) \\
&= a_1{}^2 + a_2{}^2 + a_1a_2(e_1\wedge e_2) - a_1a_2(e_1\wedge e_2) \\
&= a_1{}^2 + a_2{}^2 + 0 \\
&= a_1{}^2 + a_2{}^2 \\
&= \left(\sqrt{a_1^2 + a_2^2}\right)^2 \\
&= \|a\|^2
\end{aligned}
$$

Contoh 1: Misalkan $a = 3e_1 + 4e_2$ dan $b = 2e_1 + 5e_2$, hitunglah ab dan a^2

Jawaban:

$$ab = a \cdot b + a \wedge b$$

$$= \{(3)(2) + (4)(5)\} + (3e_1 + 4e_2) \wedge (2e_1 + 5e_2)$$

$$= \{6 + 20\} + 6(e_1 \wedge e_1) + 15(e_1 \wedge e_2) + 8(e_2 \wedge e_1) + 20(e_2 \wedge e_2)$$

$$= 26 + (6)(0) + 15(e_1 \wedge e_2) + 8(e_2 \wedge e_1) + (20)(0)$$

$$= 26 + 15(e_1 \wedge e_2) - 8(e_1 \wedge e_2)$$

$$= 26 + 7(e_1 \wedge e_2)$$

$$a^2 = aa = a \cdot a + a \wedge a = \|a\|^2$$

$$= (\sqrt{3^3 + 4^2})^2$$

$$= 3^2 + 4^2$$

$$= 9 + 16$$

$$= 25$$

- Modulus *ab* dihitung dengan dalil Phytagoras sbb:

$$\|ab\|^2 = \|a \cdot b\|^2 + \|a \wedge b\|^2$$

$\|a \cdot b\| = \|a\|\|b\| \cos\theta$

$\|a \wedge b\| = \|a\|\|b\| \sin\theta$

$$= \|a\|^2\|b\|^2 \cos^2\theta + \|a\|^2\|b\|^2 \sin^2\theta$$

$$= \|a\|^2\|b\|^2 (\cos^2\theta + \sin^2\theta)$$

$$= \|a\|^2\|b\|^2 \qquad (\text{sebab } \cos^2\theta + \sin^2\theta = 1)$$

Jadi,

$$\boxed{\|ab\| = \|a\|\|b\|}$$

- Kemudian,

$$ab = a \cdot b + a \wedge b$$
$$ba = b \cdot a + b \wedge a = a \cdot b - a \wedge b$$
$$\begin{aligned} ab - ba &= (a \cdot b + a \wedge b) - (a \cdot b - a \wedge b) \\ &= (a \wedge b) + (a \wedge b) = 2\,(a \wedge b) \end{aligned}$$

Jadi,

$$\boxed{(a \wedge b) = \frac{1}{2}(ab - ba)}$$

- Selanjutnya,

$$ab + ba = (a \cdot b + a \wedge b) + (a \cdot b - a \wedge b) = 2(a \cdot b\,)$$

Jadi,

$$\boxed{(a \cdot b\,) = \frac{1}{2}(ab + ba)}$$

tidak komutatif $\Rightarrow$ $ab \neq ba$

Perkalian geometri vektor-vektor basis

- Vektor-vektor basis satuan standard adalah $e_1, e_2, e_3, \ldots$

$$e_1e_1 = e_1 \cdot e_1 + e_1 \wedge e_1 = 1 + 0 = 1 \quad \rightarrow \quad \boxed{e_1e_1 = e_1^{\,2} = 1}$$

$\rightarrow 0$

- Dengan cara yang sama, maka $\boxed{e_2e_2 = e_2^{\,2} = 1}$ dan $\boxed{e_3e_3 = e_3^{\,2} = 1}$

$e_1 = (1, 0, 0, 0, \ldots)$
$e_2 = (0, 1, 0, 0, \ldots)$
$e_n = (0, \ldots, 1)$

$1.0 + 0.1 = 0$

- Perkalian geometri e_1 dan e_2 :

$\rightarrow 0$

$$e_1e_2 = e_1 \cdot e_2 + e_1 \wedge e_2 = 0 + e_1 \wedge e_2 = e_1 \wedge e_2 \quad \rightarrow \quad \boxed{e_1e_2 = e_1 \wedge e_2}$$

<u>Note</u>: $e_1 \wedge e_2$ dapat diganti dengan notasi e_1e_2 atau e_{12}

$$e_2e_1 = e_2 \cdot e_1 + e_2 \wedge e_1 = 0 + e_2 \wedge e_1 = -e_1 \wedge e_2 \quad \rightarrow \quad \boxed{e_2e_1 = -e_1 \wedge e_2}$$

<u>Note</u>: $e_2 \wedge e_1$ dapat diganti dengan notasi $-e_1e_2$ atau $-e_{12}$

Soal Latihan dan Jawaban

(Soal UAS 2019)

Jika diketahui tiga buah vektor:

$$\begin{aligned} a &= 2e_1 + 2e_2 + e_3 \\ b &= 3e_1 + 2e_2 - 2e_3 \\ c &= e_1 + 2e_2 - e_3 \end{aligned}$$

Hitunglah :

1). $(a+b)c$ 2). $(a \wedge b)c$ 3). $(a+b) \cdot c$

↳ bisa pake cara $ab = a \cdot b + (a \wedge b)$

1) $a + b = (2e_1 + 2e_2 + e_3) + (3e_1 + 2e_2 - 2e_3) = 5e_1 + 4e_2 - e_3$

$$(a + b)c = (5e_1 + 4e_2 - e_3)(e_1 + 2e_2 - e_3)$$
$$= 5 + 10e_{12} - 5e_{13} + 4e_{21} + 8 - 4e_{23} - e_{31} - 2e_{32} + 1$$
$$= 14 + (10 - 4)e_{12} + (-4 + 2)e_{23} + (5 - 1)e_{31}$$
$$= 14 + 6e_{12} - 2e_{23} + 4e_{31}$$

↳ kali kayak biasa aja

2) $(a \wedge b) = (2e_1 + 2e_2 + e_3) \wedge (3e_1 + 2e_2 - 2e_3)$

$$= (4 - 6)e_{12} + (-4 - 2)e_{23} + (3 + 4)e_{31}$$
$$= -2e_{12} - 6e_{23} + 7e_{31}$$

$(e_1 + 2e_2 - e_3) = 2e_2 - 4e_1 + 2e_{123} - 6e_{123} + 12e_3 + 6e_2 + 7e_3 + 14e_{123} + 7e_1$
$= 3e_1 + 8e_2 + 19e_3 + 10e_{123}$

$$(a \wedge b)c = (-2e_{12} - 2e_{23} + 7e_{31})(e_1 + 2e_2 - e_3)$$
$$= 2e_2 - 4e_1 + 2e_{123} - 2e_{123} + 4e_3 + e_2 + 7e_3 + 14e_{123} + 7e_1$$
$$= (-4 + 7)e_1 + (2 + 1)e_2 + (4 + 7)e_3 + (2 - 2 + 14)e_{123}$$
$$= 3e_1 + 3e_2 + 11e_3 + 14e_{123}$$

3) $(a + b) \cdot c = (5e_1 + 4e_2 - e_3) \cdot (e_1 + 2e_2 - e_3)$

$$= (5)(1) + (4)(2) + (-1)(-1)$$

$$= 5 + 8 + 1$$

$$= 14$$

Sifat-sifat Imajiner *Outer Product*

- Kuadratkan *outer product* dari vektor-vektor basis satuan:

$$
\begin{aligned}
(e_1 \wedge e_2)^2 &= (e_1 \wedge e_2)(e_1 \wedge e_2) \\
&= e_1 \underbrace{\underleftrightarrow{e_2 e_1}}_{-e_1e_2} e_2 \\
&= -e_1e_1e_2e_2 \\
&= -e_1^{\,2}e_2^{\,2} \\
&= -1^2\,1^2 \\
&= -1
\end{aligned}
$$

- Jadi, $\boxed{(e_1 \wedge e_2)^2 = -1}$ $\rightarrow$ mirip dengan imajiner $i^2 = -1$
- Aljabar Geometri memiliki hubungan dengan bilangan kompleks, bahkan juga dengan quaternion, dan dapat melakukan rotasi pada ruang vektor dimensi *n*.

Pseduoscalar

- Elemen-elemen aljabar di dalam aljabar geometri:

  skalar $\rightarrow$ grade-0

  vektor $\rightarrow$ grade-1

  bivector $\rightarrow$ grade-2

  trivector $\rightarrow$ grade-3

  dst
- Di dalam setiap aljabar (aljabar skalar, aljabar vektor, aljabar bivector, dst), elemen paling tinggi dinamakan *pseudoscalar* dan *grade-nya* diasosiasikan dengan dimensi ruangnya.
- Contoh: - di R^2 elemen *pseudoscalar* adalah *bivector* $e_1 \wedge e_2$ dan berdimensi 2.

  - di R^3 elemen *pseudoscalar* adalah *trivector* $e_1 \wedge e_2 \wedge e_3$

Rotasi dengan *Pseudoscalar*

- *Pseudoscalar* dapat digunakan sebagai *rotor* (penggerak rotasi).
- Misalkan *pseudoscalar* di R^2 dilambangkan dengan I, jadi

$$I = e_1 \wedge e_2 = e_1 e_2 = e_{12}$$

pseudoscalar

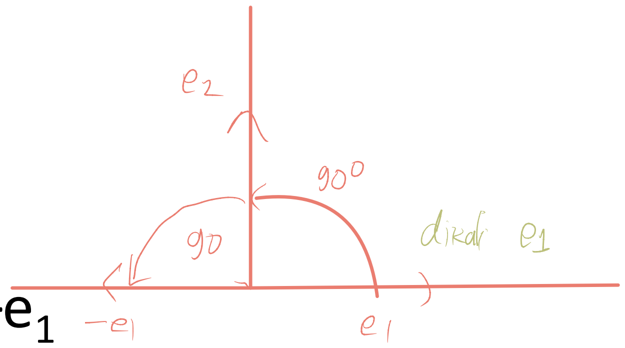
- Perkalian vektor satuan e_1 dan e_2 dengan I:

$$e_1 I = e_1 e_{12} = e_1 e_1 e_2 = e_1^2 e_2 = (1)e_2 = e_2$$

$$e_2 I = e_2 e_{12} = e_2 e_1 e_2 = e_2(-e_2 e_1) = -e_2^2 e_1 = -(1)e_1 = -e_1$$

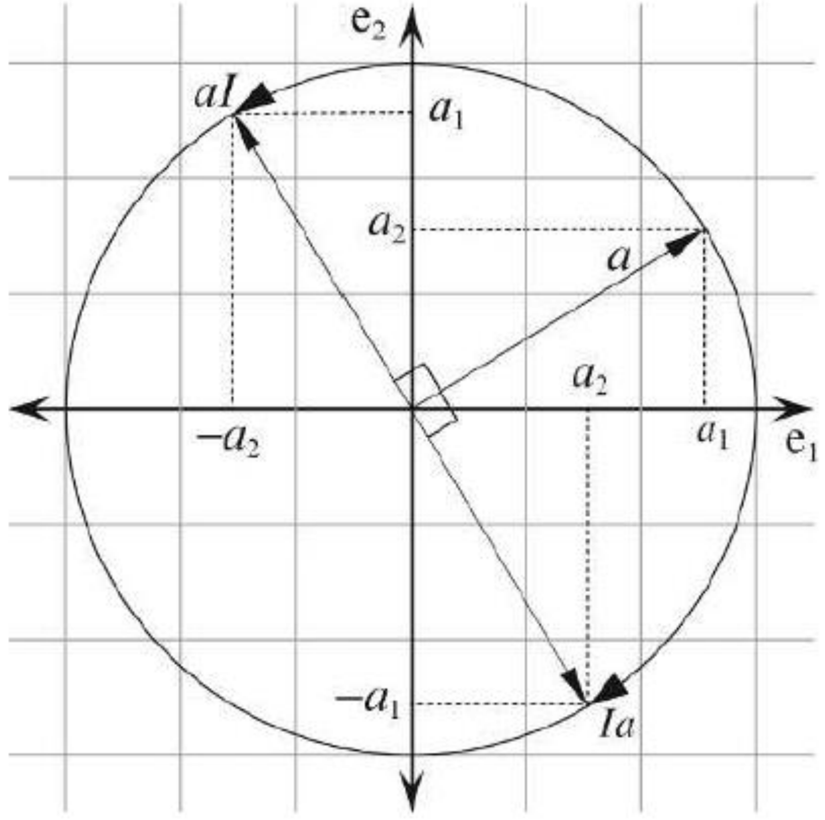
$$-e_1 I = -e_1 e_{12} = -e_1 e_1 e_2 = -e_1^2 e_2 = -(1)e_2 = -e_2$$

$$-e_2 I = -e_2 e_{12} = -e_2 e_1 e_2 = -e_2(-e_2 e_1) = e_2^2 e_1 = (1)e_1 = e_1$$



- Perkalian vektor $a = a_1\mathrm{e}_1 + a_2\mathrm{e}_2$ dengan I:

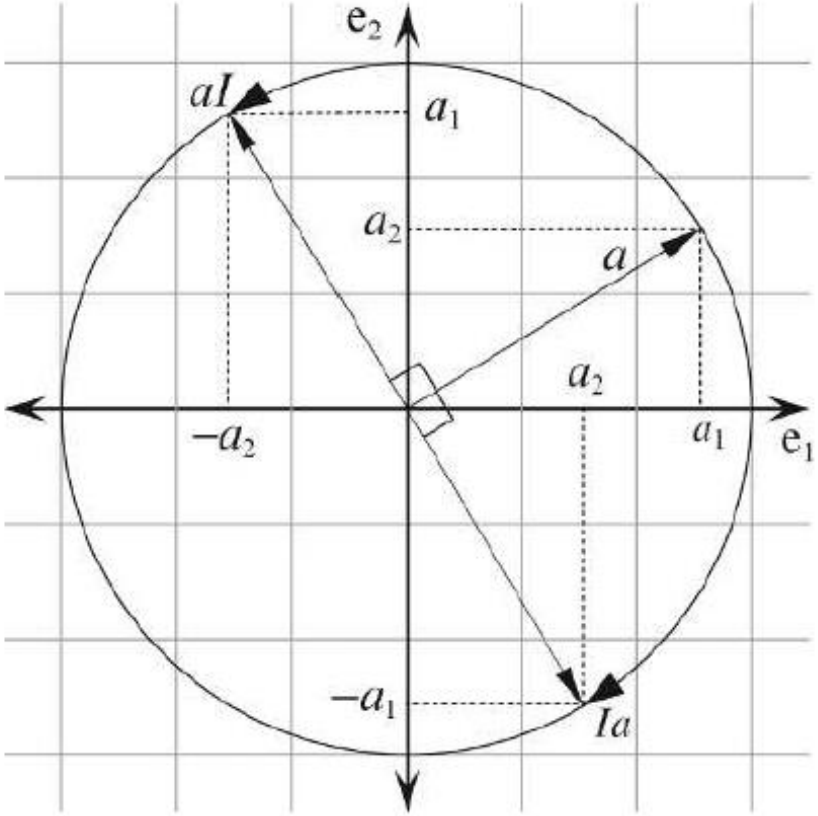
$$
\begin{aligned}
aI &= a\mathrm{e}_1\mathrm{e}_2 \\
&= (a_1\mathrm{e}_1 + a_2\mathrm{e}_2)\mathrm{e}_1\mathrm{e}_2 \\
&= a_1\mathrm{e}_1^2\mathrm{e}_2 + a_2\mathrm{e}_2\mathrm{e}_1\mathrm{e}_2 \\
&= a_1\mathrm{e}_2 - a_2\mathrm{e}_2^2\mathrm{e}_1 \\
&= -a_2\mathrm{e}_1 + a_1\mathrm{e}_2
\end{aligned}
$$



yang sama dengan memutar vektor sejauh 90 derajat berlawanan arah jarum jam.

- Perkalian vektor *I* dengan $a = a_1e_1 + a_2e_2$:

$$
\begin{aligned}
Ia &= e_1e_2a \\
&= e_1e_2(a_1e_1 + a_2e_2) \\
&= a_1e_1e_2e_1 + a_2e_1e_2^2 \\
&= -a_1e_2 + a_2e_1 \\
&= a_2e_1 - a_1e_2
\end{aligned}
$$



yang sama dengan memutar vektor sejauh 90 derajat searah jarum jam.

- Jadi,

$$aI = -Ia$$

- Perkalian vektor dengan *pseudoscalar* tidak komutatif.

TABLE 8.1

	Products in $\mathbb{R}^2$		
Type	Product	Absolute Value	Notes
inner	$e_1 \cdot e_1$	1	$e_2 \cdot e_2 = e_1 \cdot e_1$
outer	$e_1 \wedge e_1$	0	$e_2 \wedge e_2 = e_1 \wedge e_1$
geometric	e_1^2	1	$e_2^2 = e_1^2$
			$e_1 I = -I e_1$
inner	$e_1 \cdot e_2$	0	$e_2 \cdot e_1 = e_1 \cdot e_2$
outer	$e_1 \wedge e_2$	1	$e_1 \wedge e_2 = -(e_2 \wedge e_1)$
geometric	$e_1 e_2$	1	$e_{12} = -e_{21}$
			$e_{12} = I$
			$I^2 = -1$
inner	$a \cdot a$	$\| a \|^2$	
outer	$a \wedge a$	0	
geometric	a^2	$\| a \|^2$	
inner	$a \cdot b$	$\| a \| \| b \| \cos\theta$	$a \cdot b = \frac{1}{2}(ab + ba)$
		$a_1 b_1 + a_2 b_2$	
outer	$a \wedge b$	$\| a \| \| b \| \sin\theta$	$a \wedge b = \frac{1}{2}(ab - ba)$
		$a_1 b_2 - a_2 b_1$	$a \wedge b = (a_1 b_2 - a_2 b_1) e_1 \wedge e_2$
geometric	ab	$\| a \| \| b \|$	$ab = a \cdot b + a \wedge b$
			$aI = -Ia$

Hubungan antara vektor, bivector, dan bilangan kompleks

- Diberikan vektor $a = a_1 e_1 + a_2 e_2$ dan $b = b_1 e_1 + b_2 e_2$ di R^2, maka

$$
\begin{aligned}
ab &= (a_1 e_1 + a_2 e_2)(b_1 e_1 + b_2 e_2) \\
&= a_1 b_1 e_1^2 + a_1 b_2 e_{12} + a_2 b_1 e_{21} + a_2 b_2 e_2^2 \\
&= a_1 b_1 + a_2 b_2 + a_1 b_2 e_{12} - a_2 b_1 e_{12} \\
&= \underbrace{(a_1 b_1 + a_2 b_2)}_{a \cdot b} + \underbrace{(a_1 b_2 - a_2 b_1)}_{a \wedge b} e_{12} \\
&= \underbrace{(a_1 b_1 + a_2 b_2)}_{\text{skalar}} + \underbrace{(a_1 b_2 - a_2 b_1) I}_{\text{bivector}}
\end{aligned}
$$

- Perhatikan bahwa

$$ab = (a_1 b_1 + a_2 b_2) + (a_1 b_2 - a_2 b_1)I$$

ekivalen dengan bilangan kompleks Z = p + qi.

- Jadi, kita dapat membentuk bilangan yang ekivalen dengan bilangan kompleksZ yang dibentuk dengan mengkombinasikan skalar dengan *bivector*:

$$Z = a_1 + a_2 \mathrm{e}_{12} = a_1 + a_2 I$$

yang dalam hal ini a_1 adalah bagian riil dan a_2 bagian imajiner.

- Vektor a dapat dikonversi menjadi bilangan kompleks Z sebagai berikut. Diberikan vektor a adalah $a = a_1 e_1 + a_2 e_2$, maka

$$e_1 a = e_1(a_1 e_1 + a_2 e_2) = a_1 e_1^2 + a_2 e_1 e_2 = a_1 + a_2 I$$

Jadi,

$$\boxed{e_1 a = Z}$$

- Kalau urutan perkaliannya dibalik sebagai berikut:

$$a e_1 = (a_1 e_1 + a_2 e_2) e_1 = a_1 e_1^2 + a_2 e_2 e_1 = a_1 - a_2 I$$

maka hasilnya adalah bilangan kompleks sekawan (conjugate) $\bar{Z}$.

$$\boxed{a e_1 = \bar{Z}}$$

Soal Latihan Mandiri

1. (Soal UAS 2018)

Diberikan tiga buah vektor:

$$
\begin{aligned}
a &= 2e_1 + e_2 + e_3 \\
b &= 3e_1 + 5e_2 - 2e_3 \\
c &= -e_1 + 2e_2 - e_3
\end{aligned}
$$

hitunglah :

1). $a(b \wedge c)$ 2). $a \cdot (b \wedge c)$ 3). $a(b + c)$

2. (Soal UAS 2019)

Jika $I_n = e_{123\ldots n}$, adalah *pseudoscalar* di $\mathbb{R}^n$, tuliskan ekspresi berikut dalam bentuk yang paling sederhana:

1). $I_1 I_2 I_3$ 2). $e_1 I_2 I_3 I_4 I_5$ 3). $(I_3)^4 (I_2)^2 I_3 I_2$

$e_1 e_1 e_2 e_1 e_2 e_3$

$e_1^2 - e_1 e_2 e_2 e_3$

$e_1^2 - e_1 e_2^2 e_1 = -e_1 e_3 = -e_{13}$

3. (Soal UAS 2018)

Misalkan a adalah sebuah vektor $5e_1 - 2e_2$. Bagaimana cara merotasikan vektor a searah jarum jam sebesar 90° dengan *pseudo-scalar*. Tentukan bayangan a (misalkan a').

Multivector

- ***Multivecto*r** adalah objek yang mengandung skalar, vektor, bivector, dan objek lain yang dihasilkan dengan perkalian geometri.

- *Multivector* dapat dijumlahkan atau dikalikan seperti objek-objek geometri lainnya

- *Multivector* di R^2 mengandung skalar, vektor, dan *bivector*.

- *Multivector* di R^3 mengandung skalar, vektor, *bivector*, dan trivector.

- Dan seterusnya untuk *multivector* di ruang dimensi yang lebih tinggi.

Multivector di R^2

- *Multivector* di R^2 merupakan kombinasi linier dari skalar, vektor, dan *bivector.* Elemen-elemen di dalam *multivector* diresumekan pada tabel berikut:

TABLE 8.2

Element	Symbol	Grade
1 scalar	λ	0
2 vectors	$\{e_1, e_2\}$	1
1 unit bivector	$e_1 \wedge e_2 = e_{12}$	2

- Multivector A di R^2 dinyatakan sebagai

$$A = \underbrace{\lambda_0}_{\text{skalar}} + \underbrace{\lambda_1 e_1 + \lambda_2 e_2}_{\text{vektor}} + \underbrace{\lambda_3 (e_1 \wedge e_2)}_{\text{bivector}}$$

Contoh 1: Diberikan dua buah *multivector* A dan B sebagai berikut:

$$A = 4 + 3e_1 + 4e_2 + 5e_{12}$$

$$B = 3 + 2e_1 + 3e_2 + 4e_{12}$$

(i) Penjumlahan

$$A + B = 7 + 5e_1 + 7e_2 + 9e_{12}$$

$$A - B = 1 + e_1 + e_2 + e_{12}$$

(ii) Perkalian

$$AB = (4 + 3e_1 + 4e_2 + 5e_{12})(3 + 2e_1 + 3e_2 + 4e_{12})$$

(lakukan perkalian suku-suku seperti biasa,

dan gunakan $e_1^2 = e_2^2 = 1$, $e_{21} = -e_{12}$, $e_{12}^2 = -1$)

$$= 10 + 16e_1 + 26e_2 + 32e_{12} \quad \text{(tunjukkan!!)}$$

Rotasi Vektor di R^2

- Kembali ke bilangan kompleks

$$z = a + bi$$

- Rotasi bilangan kompleks z sejauh ϕ berlawanan arah jarum jam adalah:

$$z' = ze^{i\phi}$$

yang dalam hal ini,

$$e^{i\phi} = \cos\phi + i\sin\phi \qquad \text{(formula Euler)}$$

- Karena $i^2 = I^2 = -1$, maka

$$e^{I\phi} = \cos\phi + I\sin\phi$$

sehingga

$$z' = ze^{I\phi}$$

- Jika Z adalah *multivector* yang terdiri dari scalar dan *bivector*, yang identik dengan bilangan kompleks z:

$$Z = a_1 + a_2 e_{12} \quad \text{(identik dengan } z = a + bi \text{)}$$

maka

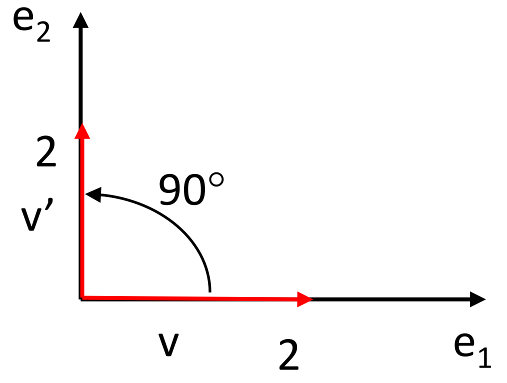
$$Z' = Ze^{I\phi}$$

- Untuk vektor $v = a_1e_1 + a_2e_2$, dapat dibuktikan bahwa rotasi v sejauh ϕ menghasilkan vektor bayangan:

$$\boxed{v' = ve^{I\phi}}$$

Contoh 2: Misalkan $v = 2e_1$ diputar 90 derajat berlawanan arah jarum jam, maka

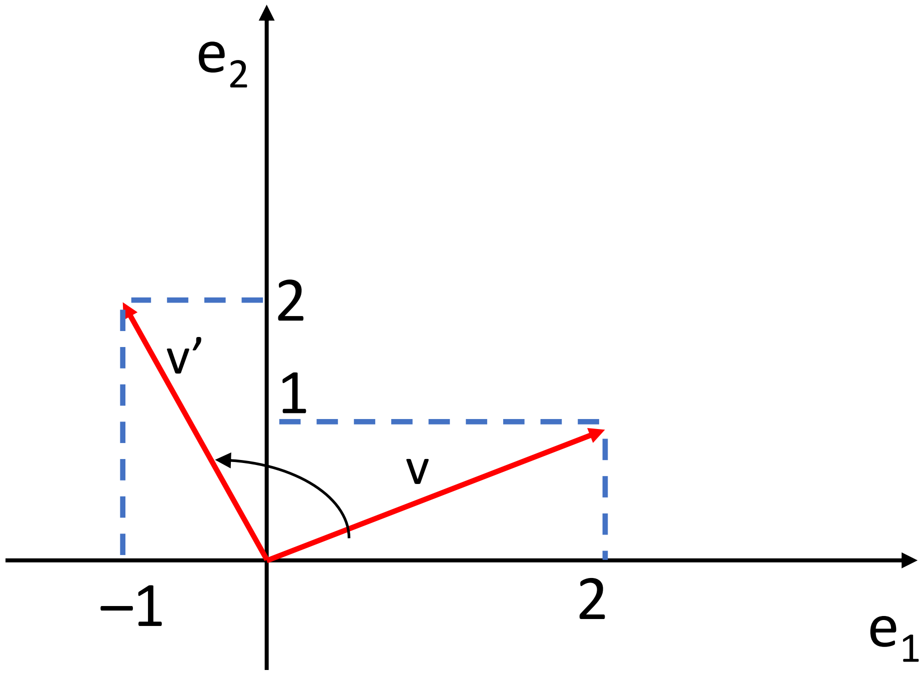
$$\begin{aligned} v' &= ve^{I\phi} = 2e_1e^{I\phi} \\ &= 2e_1(\cos 90^\circ + I \sin 90^\circ) \\ &= 2e_1(0 + I) = 2e_1 I \\ &= 2e_1e_{12} \qquad (\text{ingat, } I = e_1 \wedge e_2 = e_{12} = e_1e_2 \text{ }) \\ &= 2e_1e_1e_2 = 2e_1^2e_2 = 2(1)^2e_2 = 2e_2 \end{aligned}$$



Contoh 3: Tentukan bayangan vektor $v = 2e_1 + e_2$ yang diputar 90 derajat berlawanan arah jarium jam.

Jawaban:

$$v' = ve^{I\phi} = (2e_1 + e_2)\, e^{I\phi}$$
$$= (2e_1 + e_2)\,(\cos 90^\circ + I \sin 90^\circ)$$
$$= (2e_1 + e_2)(0 \; + I)$$
$$= (2e_1 + e_2)(I)$$
$$= (2e_1 + e_2)(e_{12})$$
$$= 2e_1e_1e_2 + e_2e_1e_2$$
$$= 2e_1{}^2e_2 - e_2{}^2e_1$$
$$= 2(1)^2e_2 - (1)^2e_1$$
$$= - e_1 + 2e_2$$



Latihan

- Diberikan sebuah vektor v = $4e_1 - 3e_2$, tentukan bayangan vektor setelah

  (a) diputar sejauh 45 derajat berlawanan arah jarum jam

  (b) diputar sejauh 120 derajat berlawaban arah jarum

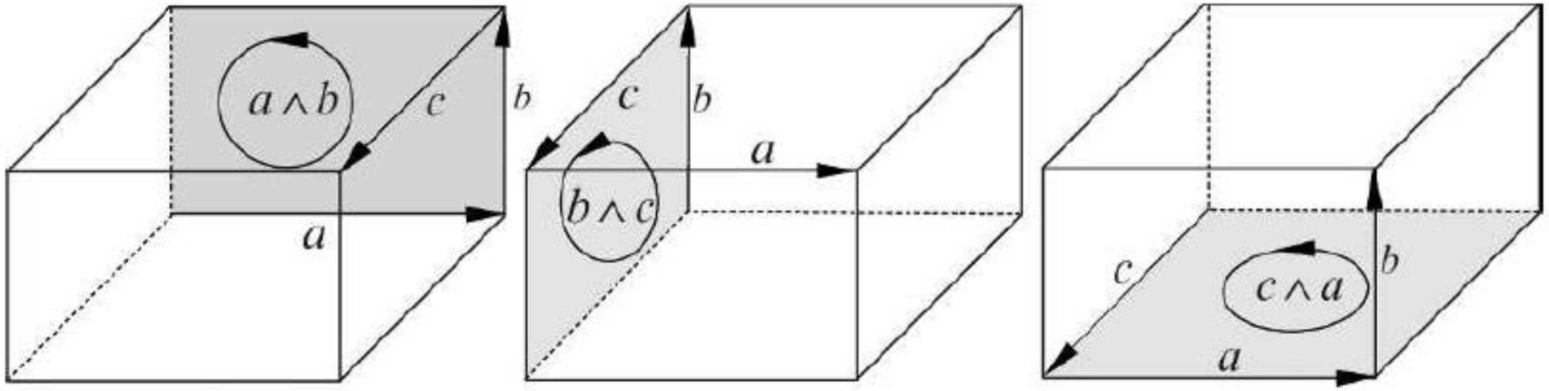
  (c) diputar sejauh 90 searah jarum jam

Trivector

- Pada materi sebelumnya (Algeo 22) sudah disinggung tentang *trivector*, yaitu objek berbentuk:

$$a \wedge b \wedge c$$

- Interpretasi geometri *trivector* adalah menyatakan volume *parallelpiped* yang dibentuk oleh vector *a*, *b*, dan *c*



(a) $(a \wedge b) \wedge c$. (b) $(b \wedge c) \wedge a$ (c) $(c \wedge a) \wedge b$.

- Ketiga buah volume tersebut identik:

$$(a \wedge b) \wedge c = (b \wedge c) \wedge a = (c \wedge a) \wedge b.$$

- Misalkan

$$a = a_1 \mathrm{e}_1 + a_2 \mathrm{e}_2 + a_3 \mathrm{e}_3$$

$$b = b_1 \mathrm{e}_1 + b_2 \mathrm{e}_2 + b_3 \mathrm{e}_3$$

$$c = c_1 \mathrm{e}_1 + c_2 \mathrm{e}_2 + c_3 \mathrm{e}_3$$

maka

$$a \wedge b \wedge c = (a_1 \mathrm{e}_1 + a_2 \mathrm{e}_2 + a_3 \mathrm{e}_3) \wedge (b_1 \mathrm{e}_1 + b_2 \mathrm{e}_2 + b_3 \mathrm{e}_3) \wedge (c_1 \mathrm{e}_1 + c_2 \mathrm{e}_2 + c_3 \mathrm{e}_3)$$

$$
\begin{aligned}
a \wedge b \wedge c &= (a_1 \mathrm{e}_1 + a_2 \mathrm{e}_2 + a_3 \mathrm{e}_3) \wedge (b_1 \mathrm{e}_1 + b_2 \mathrm{e}_2 + b_3 \mathrm{e}_3) \wedge (c_1 \mathrm{e}_1 + c_2 \mathrm{e}_2 + c_3 \mathrm{e}_3) \\
&= \begin{pmatrix} a_1 b_1 \mathrm{e}_1 \wedge \mathrm{e}_1 + a_1 b_2 \mathrm{e}_1 \wedge \mathrm{e}_2 + a_1 b_3 \mathrm{e}_1 \wedge \mathrm{e}_3 + \\ a_2 b_1 \mathrm{e}_2 \wedge \mathrm{e}_1 + a_2 b_2 \mathrm{e}_2 \wedge \mathrm{e}_2 + a_2 b_3 \mathrm{e}_2 \wedge \mathrm{e}_3 + \\ a_3 b_1 \mathrm{e}_3 \wedge \mathrm{e}_1 + a_3 b_2 \mathrm{e}_3 \wedge \mathrm{e}_2 + a_3 b_3 \mathrm{e}_3 \wedge \mathrm{e}_3 \end{pmatrix} \wedge (c_1 \mathrm{e}_1 + c_2 \mathrm{e}_2 + c_3 \mathrm{e}_3) \\
&= \begin{pmatrix} a_1 b_2 \mathrm{e}_1 \wedge \mathrm{e}_2 - a_1 b_3 \mathrm{e}_3 \wedge \mathrm{e}_1 - a_2 b_1 \mathrm{e}_1 \wedge \mathrm{e}_2 + \\ a_2 b_3 \mathrm{e}_2 \wedge \mathrm{e}_3 + a_3 b_1 \mathrm{e}_3 \wedge \mathrm{e}_1 - a_3 b_2 \mathrm{e}_2 \wedge \mathrm{e}_3 \end{pmatrix} \wedge (c_1 \mathrm{e}_1 + c_2 \mathrm{e}_2 + c_3 \mathrm{e}_3) \\
a \wedge b \wedge c &= \begin{pmatrix} (a_1 b_2 - a_2 b_1) \mathrm{e}_1 \wedge \mathrm{e}_2 + (a_2 b_3 - a_3 b_2) \mathrm{e}_2 \wedge \mathrm{e}_3 \\ +(a_3 b_1 - a_1 b_3) \mathrm{e}_3 \wedge \mathrm{e}_1 \end{pmatrix} \wedge (c_1 \mathrm{e}_1 + c_2 \mathrm{e}_2 + c_3 \mathrm{e}_3) \\
&= (a_1 b_2 - a_2 b_1) c_3 \mathrm{e}_{123} + (a_2 b_3 - a_3 b_2) c_1 \mathrm{e}_{123} + (a_3 b_1 - a_1 b_3) c_2 \mathrm{e}_{123} \\
&= ((a_2 b_3 - a_3 b_2) c_1 + (a_3 b_1 - a_1 b_3) c_2 + (a_1 b_2 - a_2 b_1) c_3) \mathrm{e}_{123} \\
&= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \mathrm{e}_{123}
\end{aligned}
$$

Pseudoscalar trivector satuan

- *Pseudoscalar* di R^2 (*bivector*):

$$I = e_1 \wedge e_2 = e_{12} = e_1e_2$$

$$I^2 = (e_1 \wedge e_2)^2 = -1$$

- *Pseudoscalar* di R^3 (*trivector*):

$$I = e_1 \wedge e_2 \wedge e_3 = e_{123} = e_1e_2e_3$$

$$I^2 = (e_1 \wedge e_2 \wedge e_3)^2 = (e_1e_2e_3)^2$$

$$= e_1e_2e_3e_1e_2e_3 = e_1e_2e_1e_3e_3e_2$$

$$= e_1e_2e_1e_2 = -1$$

- Sudah dibahas sebelumnya bahwa

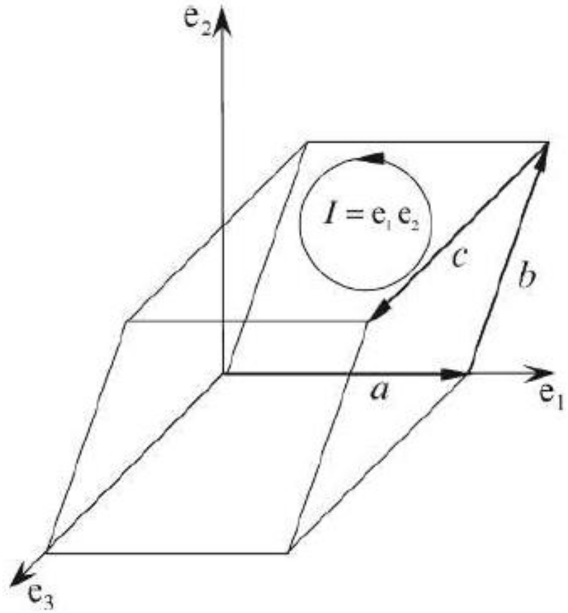
$$a \wedge b \wedge c = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} e_{123}$$

maka volume *parallelpiped* adalah V = $\|a \wedge b \wedge c\|$

Contoh 4: Misalkan $a = 2e_1 \quad b = 0.5e_1 + 2e_2 \quad c = 3e_3.$

maka volume *parallelpiped* adalah

$$\begin{aligned} V &= \|a \wedge b \wedge c\| \\ &= \|2e_1 \wedge (0.5e_1 + 2e_2) \wedge 3e_3\| \\ &= \|4e_{12} \wedge 3e_3\| \\ &= \|12e_{123}\| \\ V &= 12. \end{aligned}$$



Latihan

Diberikan tiga buah vektor di R^3 sebagai berikut:

$$a = 3e_1 + 4e_2 + 5e_3$$

$$b = 2e_1 + 3e_2 + 4e_3$$

$$c = e_1 - 3e_2 - 2e_3$$

Tentukan volume *parallelpiped* yang dibentuk oleh vektor *a*, *b*, dan *c*.

Latihan (UAS 2022)

Menggunakan vektor $a = e_1 + 2e_2 - 2e_3$; $b = 2e_1 - e_2 + e_3$; $c = e_1 + 3e_2 + e_3$, hitunglah volume bangun ruang yang dibentuk oleh tiga vektor tersebut. (*nilai 5*)

Perkalian vektor basis satuan standard di R^3

- Vektor basis satuan standard di R^3 adalah e_1, e_2, dan e_3.
- Hasil perkalian vektor satuan standard dengan dirinya sendiri:

$$e_1^2 = e_2^2 = e_3^2 = 1$$

- *Bivector* satuan standard:

$$e_{12} = e_1 \wedge e_2 \qquad e_{23} = e_2 \wedge e_3 \qquad e_{31} = e_3 \wedge e_1.$$

- Sifat imajiner bivector satuan:

$$e_{12}^2 = (e_1 \wedge e_2)^2 = -1$$
$$e_{23}^2 = (e_2 \wedge e_3)^2 = -1$$
$$e_{31}^2 = (e_3 \wedge e_1)^2 = -1.$$

Perkalian vektor dengan bivector satuan di R^3

- Diberikan vektor di R^3: $a = a_1\mathrm{e}_1 + a_2\mathrm{e}_2 + a_3\mathrm{e}_3$

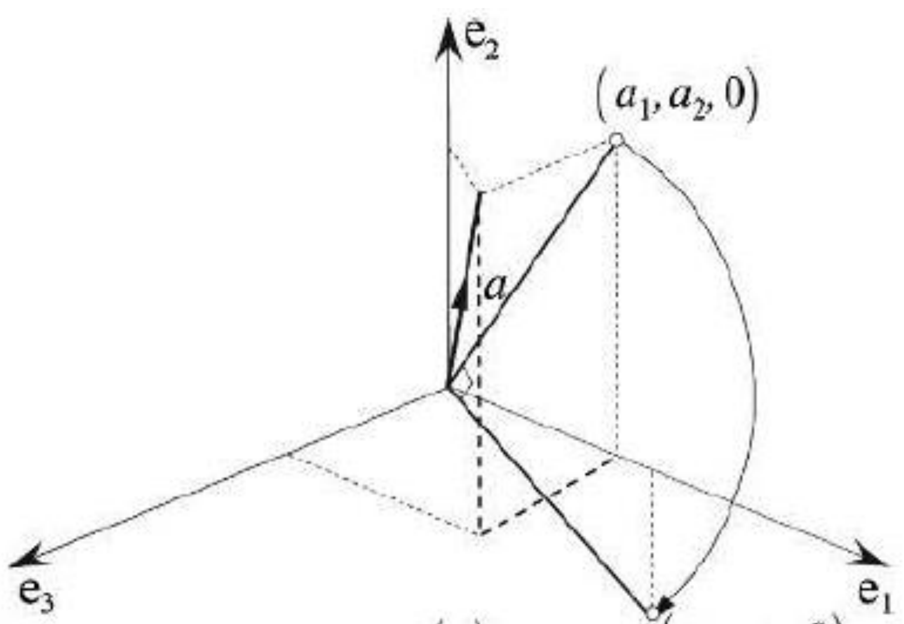
  dan bivector satuan: $\mathrm{e}_{12} = \mathrm{e}_1 \wedge \mathrm{e}_2$

- Perkalian bivector satuan dengan vektor:

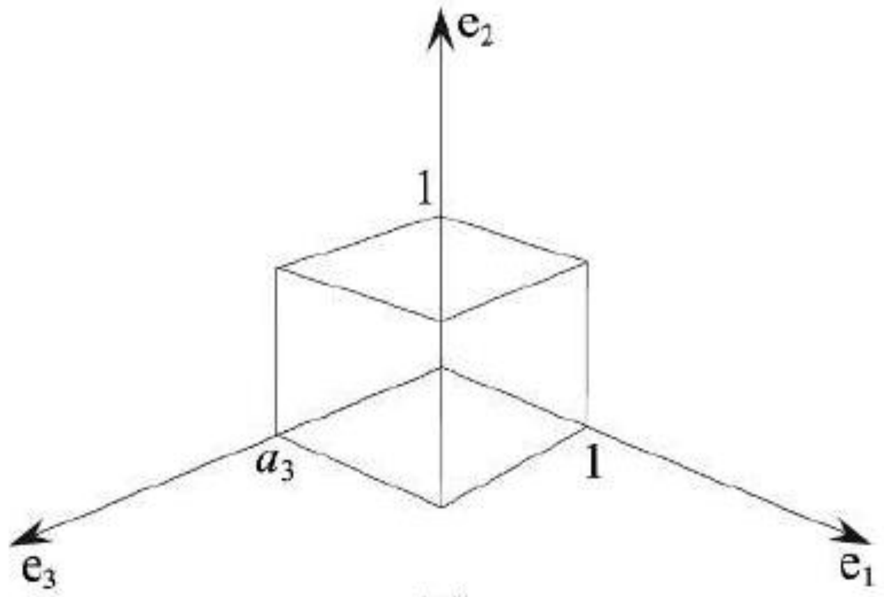
$$\begin{aligned}
\mathrm{e}_{12}a &= a_1\mathrm{e}_{12}\mathrm{e}_1 + a_2\mathrm{e}_{12}\mathrm{e}_2 + a_3\mathrm{e}_{12}\mathrm{e}_3 \\
&= -a_1\mathrm{e}_2 + a_2\mathrm{e}_1 + a_3\mathrm{e}_{123}
\end{aligned}$$

$$\mathrm{e}_{12}a = \underbrace{a_2\mathrm{e}_1 - a_1\mathrm{e}_2}_{\text{vektor}} + \underbrace{a_3\mathrm{e}_{123}}_{\text{volume}}.$$

- Interpretasi geometrinya adalah, e_{12} menghasilkan efek:
  - (i) merotasi proyeksi vektor *a* pada bidang $e_1 \wedge e_2$ sejauh 90° searah jarum jam
  - (ii) membentuk volume a_3 dengan bidang alasnya $e_1 \wedge e_2$ dan tingginya e_3



(a)



(b)

- Jika urutan perkaliannya dibalik:

$$a\mathrm{e}_{12} = a_1\mathrm{e}_1\mathrm{e}_{12} + a_2\mathrm{e}_2\mathrm{e}_{12} + a_3\mathrm{e}_3\mathrm{e}_{12}$$

$$= a_1\mathrm{e}_2 - a_2\mathrm{e}_1 + a_3\mathrm{e}_{123}$$

$$a\mathrm{e}_{12} = -a_2\mathrm{e}_1 + a_1\mathrm{e}_2 + a_3\mathrm{e}_{123}.$$

- Interpretasi geometrinya adalah, e_{12} menghasilkan efek:
  - (i) merotasi proyeksi vektor *a* pada bidang $e_1 \wedge e_2$ sejauh 90° berlawanan arah jarum jam
  - (ii) membentuk volume a_3 dengan bidang alasnya $e_1 \wedge e_2$ dan tingginya e_3

- Dengan cara yang sama, maka

$$
\begin{aligned}
e_{23}a &= a_1 e_{23} e_1 + a_2 e_{23} e_2 + a_3 e_{23} e_3 \\
&= a_1 e_{123} - a_2 e_3 + a_3 e_2 \\
&= a_3 e_2 - a_2 e_3 + a_1 e_{123}
\end{aligned}
$$

dan

$$a e_{23} = -a_3 e_2 + a_2 e_3 + a_1 e_{123}.$$

- dan

$$
\begin{aligned}
e_{31}a &= a_1 e_{31} e_1 + a_2 e_{31} e_2 + a_3 e_{31} e_3 \\
&= a_1 e_3 + a_2 e_{123} - a_3 e_1 \\
&= a_1 e_3 - a_3 e_1 + a_2 e_{123}
\end{aligned}
$$

dan

$$a e_{31} = -a_1 e_3 + a_3 e_1 + a_2 e_{123}.$$

Latihan

Diberikan dua buah vektor di R^3 sebagai berikut:

$$a = e_1 - 4e_2 + 2e_3$$

$$b = 3e_1 + e_2 - 4e_3$$

Hitunglah $ae_{12} + be_{12}$

$e_{112} - 4e_2e_{12} + 2e_{312} + 3e_{112} + e_{212} - 4e_{312}$

$e_2 + 4e_1 + 2e_{312} + 3e_2 - e_1 - 4e_{312}$

$4e_2 + 3e_1 - 2e_{312}$

$= 4e_2 + 3e_1 - 2e_{123}$

Perkalian vektor dan bivector di R^3

- Diberikan vektor di R^3: $a = a_1 \mathrm{e}_1 + a_2 \mathrm{e}_2 + a_3 \mathrm{e}_3$

  dan bivector: $B = b \wedge c$

- Perkalian geometri *a* dan *B* adalah (pembuktiannya tidak ditunjukkan di sini):

$$\boxed{aB = a \cdot B + a \wedge B}$$

- Perkalian geometri *B* dan *a* adalah (pembuktiannya tidak ditunjukkan di sini):

$$\boxed{Ba = B \cdot a + B \wedge a}$$

- Hubungan keduanya adalah:

$$\boxed{a \cdot B = \frac{1}{2}(aB - Ba)} \qquad \boxed{a \wedge B = \frac{1}{2}(aB + Ba)}$$

Contoh 1: Diberikan tiga buah vektor di R^3 sebagai berikut

$$a = 2e_1 + e_2 - e_3$$

$$b = e_1 - e_2 + e_3$$

$$c = 2e_1 + 2e_2 + e_3.$$

Hitunglah (i) $B = b \wedge c$ (ii) aB (iii) Ba (iv) $a \cdot B$ (v) $a \wedge B$

Jawaban:

(i) $$B = b \wedge c = (e_1 - e_2 + e_3) \wedge (2e_1 + 2e_2 + e_3)$$

$$= 2e_{12} - e_{31} + 2e_{12} - e_{23} + 2e_{31} - 2e_{23}$$

$$B = 4e_{12} - 3e_{23} + e_{31}.$$

(ii) $$aB = (2e_1 + e_2 - 2e_3)(4e_{12} - 3e_{23} + e_{31})$$

$$= 8e_2 - 6e_{123} - 2e_3 - 4e_1 - 3e_3 + e_{123} - 8e_{123} - 6e_2 - 2e_1$$

$$aB = -6e_1 + 2e_2 - 5e_3 - 13e_{123}.$$ $\rightarrow$ vektor + trivector

(iii) $$Ba = (4e_{12} - 3e_{23} + e_{31})(2e_1 + e_2 - 2e_3)$$

$$= -8e_2 + 4e_1 - 8e_{123} - 6e_{123} + 3e_3 + 6e_2 + 2e_3 + e_{123} + 2e_1$$

$$Ba = 6e_1 - 2e_2 + 5e_3 - 13e_{123}.$$ $\rightarrow$ vektor + trivector

(iv) $$a \cdot B = \frac{1}{2}(aB - Ba)$$

$$= \frac{1}{2}(-6e_1 + 2e_2 - 5e_3 - 13e_{123} - 6e_1 + 2e_2 - 5e_3 + 13e_{123})$$

$$= \frac{1}{2}(-12e_1 + 4e_2 - 10e_3)$$

$$a \cdot B = -6e_1 + 2e_2 - 5e_3.$$ $\rightarrow$ vektor

(v) $$a \wedge B = \frac{1}{2}(aB + Ba)$$

$$= \frac{1}{2}(-6e_1 + 2e_2 - 5e_3 - 13e_{123} + 6e_1 - 2e_2 + 5e_3 - 13e_{123})$$

$$a \wedge B = -13e_{123}.$$ $\rightarrow$ trivector

Dari (iv) dan (v) terlihat bahwa:

$$aB = a \cdot B + a \wedge B$$

$$aB = -6e_1 + 2e_2 - 5e_3 - 13e_{123}.$$

yng berarti bahwa *aB* diidentifikasi oleh *inner product* ($a \cdot B$) dan *outer product* ($a \wedge B$)

Latihan (UAS 2022)

Diberikan tiga buah vektor sebagai berikut: $a = e_1 + 2e_2 - 2e_3$; $b = 2e_1 - e_2 + e_3$; dan $c = e_1 + 3e_2 + e_3$, dan $B = b \wedge c$, hitunglah:

a) $a \wedge (b + c)$ (*nilai 5*)

b) $a \cdot B$ (*nilai 5*)

$b + c = 3e_1 + 2e_2 + 2e_3$

a) $a \wedge (b+c) = (e_1 + 2e_2 - 2e_3) \wedge (3e_1 - 2e_2 + 2e_3)$

$= 3 + (-2)e_{12} + 2e_{13} + 6e_{21} - 4 + 4e_{23} - 6e_3e_1 + 4e_{32} + 4$

$= 3 - 8e_{12} + 8e_{13}$

$=$

Perkalian *bivector-bivector* satuan di R^3

$$e_{12}^2 = e_{23}^2 = e_{31}^2 = -1$$

$$e_{12}e_{23} = e_{13} = -e_{31}$$

$$e_{23}e_{31} = e_{21} = -e_{12}$$

$$e_{31}e_{12} = e_{32} = -e_{23}$$

$$e_{12}e_{31} = e_{23}$$

$$e_{23}e_{12} = e_{31}$$

$$e_{31}e_{23} = e_{12}.$$

TABLE 8.4

GP	e_{12}	e_{23}	e_{31}
e_{12}	-1	$-e_{31}$	e_{23}
e_{23}	e_{31}	-1	$-e_{12}$
e_{31}	$-e_{23}$	e_{12}	-1

Contoh cara mendapatkan salah satu hasil di samping:

$$\begin{aligned} e_{31}e_{23} &= e_3e_1e_2e_3 \\ &= -e_3e_1e_3e_2 \\ &= e_3e_3e_1e_2 \\ &= e_3{}^2e_1e_2 \\ &= (1)e_1e_2 = e_1e_2 = e_{12} \end{aligned}$$

Perkalian vektor dan *trivector* di R^3

Perkalian vektor dengan *trivector* menghasilkan *bivector*

$$e_1 e_{123} = e_{23} \qquad e_{123} e_1 = e_{23}$$

$$e_2 e_{123} = e_{31} \quad \Longleftrightarrow \quad e_{123} e_2 = e_{31}$$

$$e_3 e_{123} = e_{12}. \qquad e_{123} e_3 = e_{12}.$$

$\therefore$ Perkalian vektor dengan *trivector* bersifat komutatif

Contoh 2: Diberikan vektor a = $2e_1 + 3e_2 + 4e_3$ dan trivector $B = 5(e_1 \wedge e_2 \wedge e_3) = 5e_{123}$

Hitunglah aB.

Jawaban:

$$
\begin{aligned}
aB &= (2e_1 + 3e_2 + 4e_3)\, 5e_{123} \\
&= 10e_1e_{123} + 15e_2e_{123} + 20e_3e_{123} \\
&= 10e_1e_1e_2e_3 + 15e_2e_1e_2e_3 + 20e_3e_1e_2e_3 \\
&= 10e_2e_3 - 15e_2e_2e_1e_3 - 20e_3e_1e_3e_2 \\
&= 10e_2e_3 - 15e_1e_3 + 20e_3e_3e_1e_2 \\
&= 10e_2e_3 + 15e_3e_1 + 20e_1e_2 \\
&= 20e_1e_2 + 10e_2e_3 + 15e_3e_1 \\
&= 20e_{12} + 10e_{23} + 15e_{31}
\end{aligned}
$$

Perkalian vektor dengan *trivector* menghasilkan tiga buah *bivector*.

Perkalian *bivector* dan *trivector* di R^3

Perkalian *bivector* dengan *trivector* menghasilkan vector

$$e_{12}e_{123} = -e_3 \qquad e_{123}e_{12} = -e_3$$

$$e_{23}e_{123} = -e_1 \quad \Longleftrightarrow \quad e_{123}e_{23} = -e_1$$

$$e_{31}e_{123} = -e_2. \qquad e_{123}e_{31} = -e_2.$$

$\therefore$ Perkalian bivector dengan *trivector* bersifat komutatif

Contoh 3 : Diberikan *bivector* $B = 2e_{12} + 3e_{23} + 4e_{31}$ dan *trivector* $C = 5e_{123}$
Hitunglah *BC*.
Jawaban:

$$\begin{aligned} B5e_{123} &= (2e_{12} + 3e_{23} + 4e_{31})5e_{123} \\ &= -15e_1 - 20e_2 - 10e_3. \end{aligned}$$

Ringkasan perkalian vektor di R^3

TABLE 8.5

Inner product	
Vectors commute	$a \cdot b = b \cdot a$
Vectors and bivectors anticommute	$a \cdot B = -B \cdot a$
	$a \cdot B = \frac{1}{2}(aB - Ba)$
	$a \cdot B = (a \cdot b)c - (a \cdot c)b$
	$B \cdot a = \frac{1}{2}(Ba - aB)$
	$B \cdot a = (a \cdot c)b - (a \cdot b)c$
Outer product	
Vectors anticommute	$a \wedge b = -b \wedge a$
Vectors and bivectors commute	$a \wedge B = B \wedge a$
	$a \wedge B = \frac{1}{2}(aB + Ba)$
	$a \wedge B = abc$
	$B \wedge a = \frac{1}{2}(Ba + aB)$
	$B \wedge a = abc$

Geometric product	
Orthogonal vectors anticommute	$e_{12} = -e_{21}$
Orthogonal bivectors anticommute	$e_{12}e_{23} = -e_{23}e_{12}$
Bivectors square to -1	$e_{12}^2 = e_{23}^2 = e_{31}^2 = -1$
Definition	$ab = a \cdot b + a \wedge b$
Vectors and bivectors anticommute	$aB = -Ba$
	$aB = a \cdot B + a \wedge B$
	$aB = (a \cdot b)c - (a \cdot c)b + abc$
	$Ba = B \cdot a + B \wedge a$
	$Ba = (a \cdot c)b - (a \cdot b)c + abc$
Trivector commutes with all multivectors in the space	$aT = Ta \quad BT = TB$
The pseudoscalar	$e_{123} = I$
Vectors and the pseudoscalar commute	$aI = Ia$
	$aI = a \cdot I$
Duality transformation	$e_{23} = Ie_1$
	$e_{31} = Ie_2$
	$e_{12} = Ie_3$
The trivector squares to -1	$I^2 = -1$

Where a and b are vectors, B is a bivector, and T is a trivector.

Table 8.6

GP	λ	e_1	e_2	e_3	e_{12}	e_{23}	e_{31}	e_{123}
λ	λ^2	λe_1	λe_2	λe_3	λe_{12}	λe_{23}	λe_{31}	λe_{123}
e_1	λe_1	1	e_{12}	$-e_{31}$	e_2	e_{123}	$-e_3$	e_{23}
e_2	λe_2	$-e_{12}$	1	e_{23}	$-e_1$	e_3	e_{123}	e_{31}
e_3	λe_3	e_{31}	$-e_{23}$	1	e_{123}	$-e_2$	e_1	e_{12}
e_{12}	λe_{12}	$-e_2$	e_1	e_{123}	-1	$-e_{31}$	e_{23}	$-e_3$
e_{23}	λe_{23}	e_{123}	$-e_3$	e_2	e_{31}	-1	$-e_{12}$	$-e_1$
e_{31}	λe_{31}	e_3	e_{123}	$-e_1$	$-e_{23}$	e_{12}	-1	$-e_2$
e_{123}	λe_{123}	e_{23}	e_{31}	e_{12}	$-e_3$	$-e_1$	$-e_2$	-1

Keterangan: GP = Geometry Product

Bersambung ke Bagian 2

Rangkuman

$ab = a \cdot b + a \wedge b$

vektor basis standar

$e_1 \wedge e_2 = e_1 e_2 = e_{12}$

$e_1^2 = e_2^2 = e_3^2 = 1$

$e_1 \wedge e_2 = -e_2 \wedge e_1$

pseudoskalar

kalau di R_2 = $e_1 \wedge e_2 = e_{12} = I$

R_3 = $e_1 \wedge e_2 \wedge e_3 = e_{123} = I$

$I^2 = -1$

bilangan kompleks = $z = a + bi$ di aljabar kompleks

maka bil. kompleks di aljabar geometri = $z = a_1 + b_1 I$

$$\begin{aligned}(e_{12})^2 &= e_{12} e_{12} \\ &= -e_1 e_2 e_2 e_1 \\ &= -e_1 (1) e_1 \\ &= -e_1^2 \\ &= -1\end{aligned}$$

$$\begin{aligned}e_{12} e_{23} &= e_1 e_2 e_2 e_3 \\ &= e_1 e_2^2 e_3 \\ &= e_1 (1) e_3 \\ &= e_1 e_3 \\ &= -e_{31}\end{aligned}$$