PUSHDOWN AUTOMATA

IF 2124 TEORI BAHASA FORMAL OTOMATA

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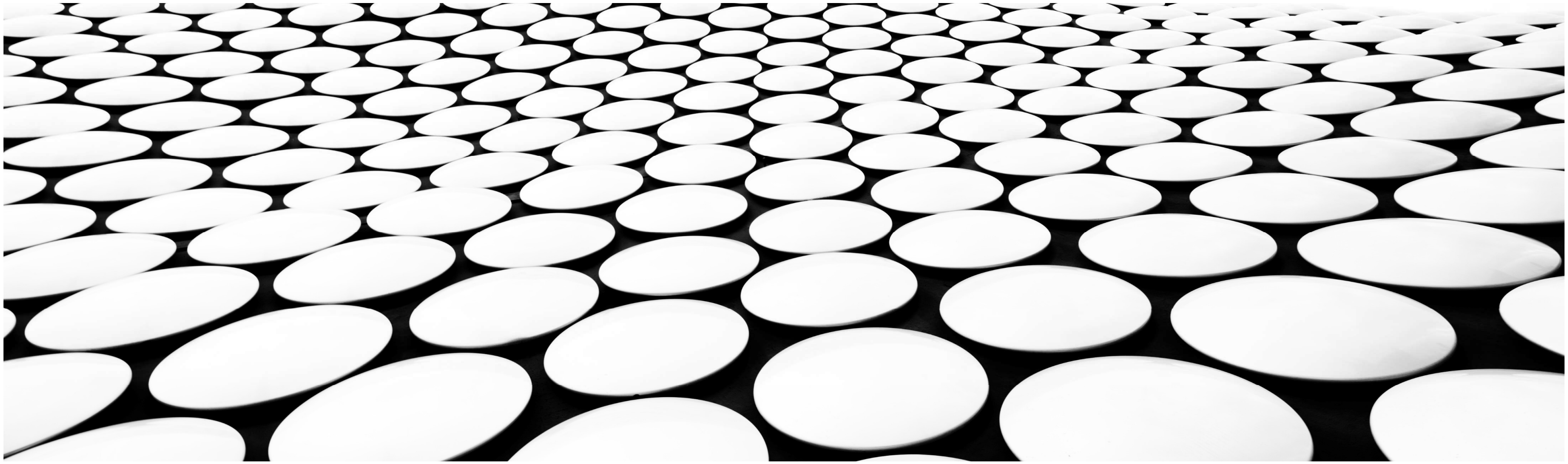
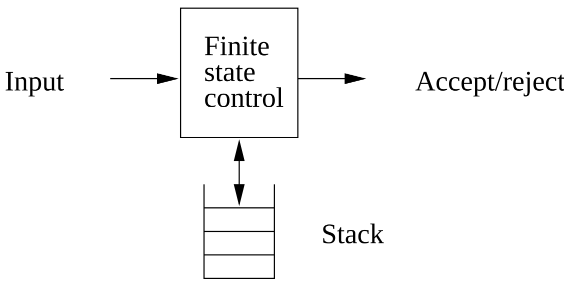


Pushdown Automata

A pushdown automata (PDA) is essentially an ϵ-NFA with a stack.

On a transition the PDA:

1. Consumes an input symbol.
2. Goes to a new state (or stays in the old).
3. Replaces the top of the stack by any string (does nothing, pops the stack, or pushes a string onto the stack)



Example: Let's consider

$$L_{wwr} = \{ww^R : w \in \{0,1\}^*\},$$

with "grammar" $P \to 0P0$, $P \to 1P1$, $P \to \epsilon$. A PDA for L_{wwr} has tree states, and operates as follows:

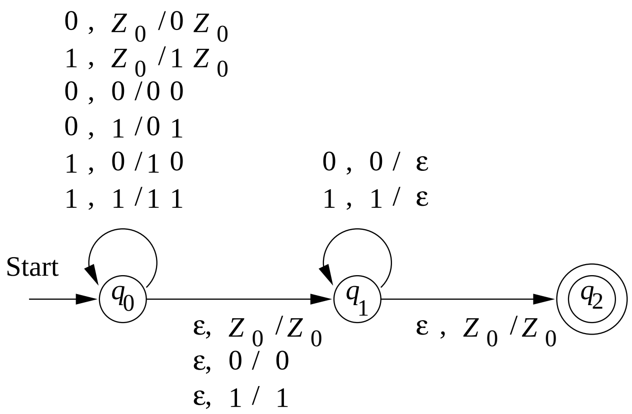
1. Guess that you are reading w. Stay in state 0, and push the input symbol onto the stack.

2. Guess that you're in the middle of ww^R. Go spontanteously to state 1.

3. You're now reading the head of w^R. Compare it to the top of the stack. If they match, pop the stack, and remain in state 1. If they don't match, go to sleep.

4. If the stack is empty, go to state 2 and accept.

The PDA for L_{wwr} as a transition diagram:



PDA formally

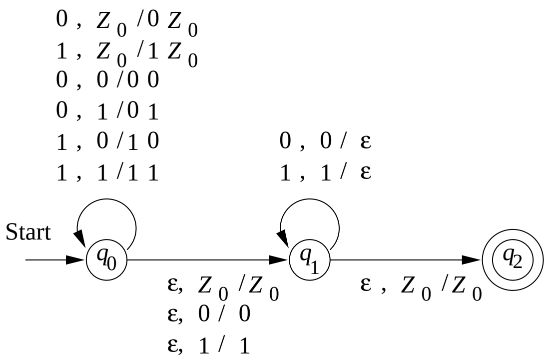
A PDA is a seven-tuple:

$$P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F),$$

where

- Q is a finite set of states,
- Σ is a finite *input alphabet*,
- Γ is a finite *stack alphabet*,
- $\delta : Q \times \Sigma \cup \{\epsilon\} \times \Gamma \to 2^{Q \times \Gamma^*}$ is the *transition function*,
- q_0 is the *start state*,
- $Z_0 \in \Gamma$ is the *start symbol* for the stack, and
- $F \subseteq Q$ is the set of *accepting states*.

Example: The PDA



is actually the seven-tuple

$$P = (\{q_0, q_1, q_2\}, \{0,1\}, \{0,1,Z_0\}, \delta, q_0, Z_0, \{q_2\}),$$

where δ is given by the following table (set brackets missing):

	$0,Z_0$	$1,Z_0$	0,0	0,1	1,0	1,1	ϵ,Z_0	$\epsilon,0$	$\epsilon,1$
$\rightarrow q_0$	$q_0,0Z_0$	$q_0,1Z_0$	$q_0,00$	$q_0,01$	$q_0,10$	$q_0,11$	q_1,Z_0	$q_1,0$	$q_1,1$
q_1			q_1,ϵ			q_1,ϵ	q_2,Z_0		
$\star q_2$									

Instantaneous Descriptions

A PDA goes from configuration to configuration when consuming input.

To reason about PDA computation, we use *instantaneous descriptions* of the PDA. An ID is a triple

$$(q, w, \gamma)$$

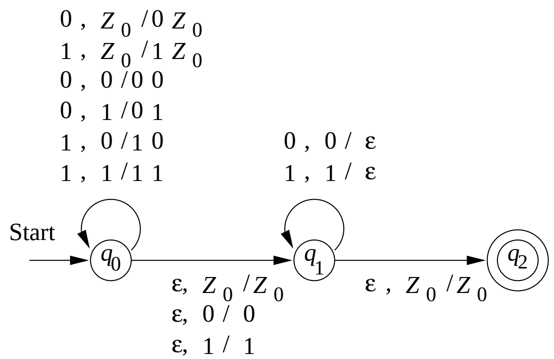
where q is the state, w the remaining input, and γ the stack contents.

Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA. Then $\forall w \in \Sigma^*, \beta \in \Gamma^*$:

$$(p, \alpha) \in \delta(q, a, X) \Rightarrow (q, aw, X\beta) \vdash (p, w, \alpha\beta).$$

We define $\overset{*}{\vdash}$ to be the reflexive-transitive closure of $\vdash$.

Example: On input 1111 the PDA



has the following computation sequences:

$$
\begin{tikzcd}
(q_0, 1111, Z_0) \arrow[d] \arrow[dr] & & \\
(q_0, 111, 1Z_0) \arrow[d] \arrow[dr] & (q_1, 1111, Z_0) \arrow[r] & (q_2, 1111, Z_0) \\
(q_0, 11, 11Z_0) \arrow[d] \arrow[dr] & (q_1, 111, 1Z_0) \arrow[r] & (q_1, 11, Z_0) \arrow[d] \\
(q_0, 1, 111Z_0) \arrow[d] \arrow[dr] & (q_1, 11, 11Z_0) \arrow[dr] & (q_2, 11, Z_0) \\
(q_0, \varepsilon, 1111Z_0) \arrow[d] & (q_1, 1, 111Z_0) \arrow[d] & (q_1, 1, 1Z_0) \arrow[d] \\
(q_1, \varepsilon, 1111Z_0) & (q_1, \varepsilon, 11Z_0) & (q_1, \varepsilon, Z_0) \arrow[d] \\
 & & (q_2, \varepsilon, Z_0)
\end{tikzcd}
$$

The following properties hold:

1. If an ID sequence is a legal computation for a PDA, then so is the sequence obtained by adding an additional string at the end of component number two.

2. If an ID sequence is a legal computation for a PDA, then so is the sequence obtained by adding an additional string at the bottom of component number three.

3. If an ID sequence is a legal computation for a PDA, and some tail of the input is not consumed, then removing this tail from all ID's result in a legal computation sequence.

Theorem 6.5: $\forall w \in \Sigma^*, \beta \in \Gamma^*$:

$$(q, x, \alpha) \overset{*}{\vdash} (p, y, \beta) \Rightarrow (q, xw, \alpha\gamma) \overset{*}{\vdash} (p, yw, \beta\gamma).$$

Proof: Induction on the length of the sequence to the left.

Note: If $\gamma = \epsilon$ we have proerty 1, and if $w = \epsilon$ we have property 2.

Note2: The reverse of the theorem is false.

For property 3 we have

Theorem 6.6:

$$(q, xw, \alpha) \overset{*}{\vdash} (p, yw, \beta) \Rightarrow (q, x, \alpha) \overset{*}{\vdash} (p, y, \beta).$$

Acceptance by final state

Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA. The *language accepted by P by final state* is

$$L(P) = \{w : (q_0, w, Z_0) \overset{*}{\vdash} (q, \epsilon, \alpha),\ q \in F\}.$$

Example: The PDA on slide 183 accepts exactly L_{wwr}.

Let P be the machine. We prove that $L(P) = L_{wwr}$.

($\supseteq$-*direction.*) Let $x \in L_{wwr}$. Then $x = ww^R$, and the following is a legal computation sequence

$$(q_0, ww^R, Z_0) \overset{*}{\vdash} (q_0, w^R, w^R Z_0) \vdash (q_1, w^R, w^R Z_0) \overset{*}{\vdash}$$

$$(q_1, \epsilon, Z_0) \vdash (q_2, \epsilon, Z_0).$$

($\subseteq$-*direction.*)

Observe that the only way the PDA can enter q_2 is if it is in state q_1 with an empty stack.

Thus it is sufficient to show that if $(q_0, x, Z_0) \overset{*}{\vdash} (q_1, \epsilon, Z_0)$ then $x = ww^R$, for some word w.

We'll show by induction on $|x|$ that

$$(q_0, x, \alpha) \overset{*}{\vdash} (q_1, \epsilon, \alpha) \;\; \Rightarrow \;\; x = ww^R.$$

Basis: If $x = \epsilon$ then x is a palindrome.

Induction: Suppose $x = a_1 a_2 \cdots a_n$, where $n > 0$, and the IH holds for shorter strings.

Ther are two moves for the PDA from ID (q_0, x, α):

Move 1: The spontaneous $(q_0, x, \alpha) \vdash (q_1, x, \alpha)$. Now $(q_1, x, \alpha) \overset{*}{\vdash} (q_1, \epsilon, \beta)$ implies that $|\beta| < |\alpha|$, which implies $\beta \neq \alpha$.

Move 2: Loop and push $(q_0, a_1 a_2 \cdots a_n, \alpha) \vdash (q_0, a_2 \cdots a_n, a_1 \alpha)$.

In this case there is a sequence

$(q_0, a_1 a_2 \cdots a_n, \alpha) \vdash (q_0, a_2 \cdots a_n, a_1 \alpha) \vdash \cdots \vdash (q_1, a_n, a_1 \alpha) \vdash (q_1, \epsilon, \alpha)$.

Thus $a_1 = a_n$ and

$$(q_0, a_2 \cdots a_n, a_1 \alpha) \overset{*}{\vdash} (q_1, a_n, a_1 \alpha).$$

By Theorem 6.6 we can remove a_n. Therefore

$$(q_0, a_2 \cdots a_{n-1}, a_1 \alpha \overset{*}{\vdash} (q_1, \epsilon, a_1 \alpha).$$

Then, by the IH $a_2 \cdots a_{n-1} = yy^R$. Then $x = a_1 yy^R a_n$ is a palindrome.

Acceptance by Empty Stack

Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA. The *language accepted by P by empty stack* is

$$N(P) = \{w : (q_0, w, Z_0) \overset{*}{\vdash} (q, \epsilon, \epsilon)\}.$$

Note: q can be any state.

Question: How to modify the palindrome-PDA to accept by empty stack?

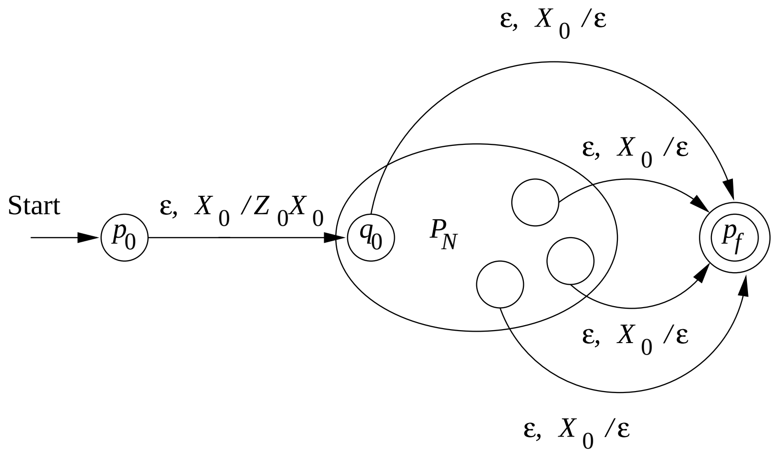
From Empty Stack to Final State

Theorem 6.9: If $L = N(P_N)$ for some PDA $P_N = (Q, \Sigma, \Gamma, \delta_N, q_0, Z_0)$, then $\exists$ PDA P_F, such that $L = L(P_F)$.

Proof: Let

$P_F = (Q \cup \{p_0, p_f\}, \Sigma, \Gamma \cup \{X_0\}, \delta_F, p_0, X_0, \{p_f\})$

where $\delta_F(p_0, \epsilon, X_0) = \{(q_0, Z_0X_0)\}$, and for all $q \in Q, a \in \Sigma \cup \{\epsilon\}, Y \in \Gamma : \delta_F(q, a, Y) = \delta_N(q, a, Y)$, and in addition $(p_f, \epsilon) \in \delta_F(q, \epsilon, X_0)$.



We have to show that $L(P_F) = N(P_N)$.

($\supseteq$*direction.*) Let $w \in N(P_N)$. Then

$$(q_0, w, Z_0) \overset{*}{\underset{N}{\vdash}} (q, \epsilon, \epsilon),$$

for some q. From Theorem 6.5 we get

$$(q_0, w, Z_0X_0) \overset{*}{\underset{N}{\vdash}} (q, \epsilon, X_0).$$

Since $\delta_N \subset \delta_F$ we have

$$(q_0, w, Z_0X_0) \overset{*}{\underset{F}{\vdash}} (q, \epsilon, X_0).$$

We conclude that

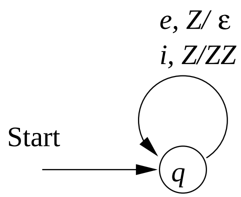
$$(p_0, w, X_0) \underset{F}{\vdash} (q_0, w, Z_0X_0) \overset{*}{\underset{F}{\vdash}} (q, \epsilon, X_0) \underset{F}{\vdash} (p_f, \epsilon, \epsilon).$$

($\subseteq$*direction.*) By inspecting the diagram.

Let's design P_N for for cathing errors in strings meant to be in the *if-else*-grammar G

$$S \to \epsilon | SS | iS | iSe.$$

Here e.g. $\{ieie, iie, iei\} \subseteq G$, and e.g. $\{ei, ieeii\} \cap G = \emptyset$.
The diagram for P_N is



Formally,

$$P_N = (\{q\}, \{i, e\}, \{Z\}, \delta_N, q, Z),$$

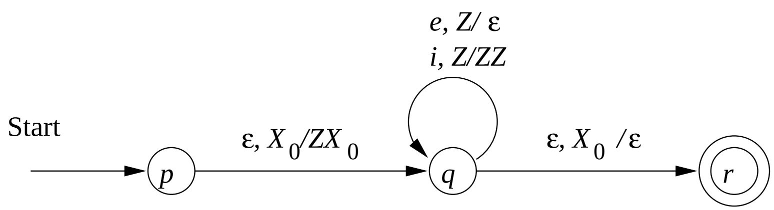
where $\delta_N(q, i, Z) = \{(q, ZZ)\}$,
and $\delta_N(q, e, Z) = \{(q, \epsilon)\}$.

From P_N we can construct

$$P_F = (\{p, q, r\}, \{i, e\}, \{Z, X_0\}, \delta_F, p, X_0, \{r\}),$$

where
$\delta_F(p, \epsilon, X_0) = \{(q, ZX_0)\}$,
$\delta_F(q, i, Z) = \delta_N(q, i, Z) = \{(q, ZZ)\}$,
$\delta_F(q, e, Z) = \delta_N(q, e, Z) = \{(q, \epsilon)\}$, and
$\delta_F(q, \epsilon, X_0) = \{(r, \epsilon)\}$

The diagram for P_F is



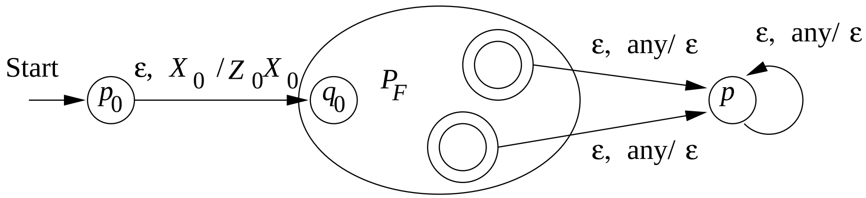
From Final State to Empty Stack

Theorem 6.11: Let $L = L(P_F)$, for some PDA $P_F = (Q, \Sigma, \Gamma, \delta_F, q_0, Z_0, F)$. Then $\exists$ PDA P_n, such that $L = N(P_N)$.

Proof: Let

$$P_N = (Q \cup \{p_0, p\}, \Sigma, \Gamma \cup \{X_0\}, \delta_N, p_0, X_0)$$

where $\delta_N(p_0, \epsilon, X_0) = \{(q_0, Z_0 X_0)\}$, $\delta_N(p, \epsilon, Y) = \{(p, \epsilon)\}$, for $Y \in \Gamma \cup \{X_0\}$, and for all $q \in Q$, $a \in \Sigma \cup \{\epsilon\}, Y \in \Gamma : \delta_N(q, a, Y) = \delta_F(q, a, Y)$, and in addition $\forall q \in F$, and $Y \in \Gamma \cup \{X_0\}$: $(p, \epsilon) \in \delta_N(q, \epsilon, Y)$.



We have to show that $N(P_N) = L(P_F)$.

($\subseteq$-*direction*.) By inspecting the diagram.

($\supseteq$-*direction*.) Let $w \in L(P_F)$. Then

$$(q_0, w, Z_0) \overset{*}{\underset{F}{\vdash}} (q, \epsilon, \alpha),$$

for some $q \in F, \alpha \in \Gamma^*$. Since $\delta_F \subseteq \delta_N$, and Theorem 6.5 says that X_0 can be slid under the stack, we get

$$(q_0, w, Z_0 X_0) \overset{*}{\underset{N}{\vdash}} (q, \epsilon, \alpha X_0).$$

The P_N can compute:

$$(p_0, w, X_0) \underset{N}{\vdash} (q_0, w, Z_0 X_0) \overset{*}{\underset{N}{\vdash}} (q, \epsilon, \alpha X_0) \overset{*}{\underset{N}{\vdash}} (p, \epsilon, \epsilon).$$