Propositional Logic: Proof Method

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Contents

- Review
- Propositional Logic → Logical Entailment
  - Proof Method

Review

- Computational Logic
  - **Propositional Logic:**
    - Sintax → Simple sentence, Compound Sentence
    - Semantics → interpretation, evaluation, reverse evaluation, types of compund sentence
    - Logical Entailment
      - Semantic Reasoning

Proof Method

- Proof of a conclusion from set of premises:
  - Sequence of sentences terminating in conclusion in which each item is either a premise, an instance of axiom schema, or the result of applying a rule of inference to earlier items in sequence.
  - Base: Applied Rule of Inference to premises

Proof Method

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  - Base: Applied Rule of Inference to premises

- A rule of inference:
  - a pattern of reasoning consisting of premises and conclusions.

Proof Method

- More rule of inference:

1. Modus ponens

$p \rightarrow q$

p

$\therefore q$

2. Modus tollens

$p \rightarrow q$

$\sim q$

$\therefore \sim p$

3. Disjunctive syllogism

$p \vee q$

$\sim p$

$\therefore q$

4. Simplification

$p \wedge q$

$\therefore p$, boleh q juga :)

Proof Method

- More rule of inference:

5. Addition

p

$\therefore p \vee q$

6. Conjunction

p

q

$\therefore p \wedge q$

7. Hypothetical syllogism

$p \rightarrow q$

$q \rightarrow r$

$\therefore p \rightarrow r$

8. Resolution

$p \vee q$

$\sim p \vee r$

$\therefore q \vee r$

Rules of Replacement

- Associativity → disjunction, conjunction, equivalence
- Commutativity → disjunction, conjunction, equivalence
- Distributivity: $p \lor (r \land q) \leftrightarrow (p \lor q) \land (p \lor q)$
- Double Negation: $\neg\neg p \leftrightarrow p$
- De Morgan's Law:

  $\neg(p \land q) \leftrightarrow (\neg p) \lor (\neg q)$

  $\neg(p \lor q) \leftrightarrow (\neg p) \land (\neg q)$
- Transposition: $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
- Material Implication: $(p \rightarrow q) \leftrightarrow (\neg p \lor q)$
- Tautology: $(p \land p) \leftrightarrow p$ or $(p \lor p) \leftrightarrow p$

Example

Premises:

1. $p \rightarrow q$
2. $m \rightarrow p \vee q$

Conclusion: $m \rightarrow q$

$\{p \rightarrow q, m \rightarrow p \vee q\} \models m \rightarrow q$?

3. $\sim \cancel{p} \vee q$ Material Implication 1
4. $\sim m \vee \cancel{p} \vee q$ Material Implication 2

coret²

5. $\sim m \vee q$ Resolution 3,4
6. $m \rightarrow q$ Material Implication 5

cuman kebalik, material implication bisa bolak balik

Jadi dapat dibuktikan $\{p \rightarrow q, m \rightarrow p \vee q\} \models m \rightarrow q$

Exercise

Premis:

1. Kalau mahasiswa malas belajar dan sering bolos kuliah, mahasiswa tidak lulus ujian. $(malas \wedge bolos) \rightarrow \neg lulus$
2. Kalau mahasiswa tidak lulus ujian, orang tuanya akan marah. $\neg lulus \rightarrow marah$
3. Mahasiswa sering bolos kuliah tetapi orang tuanya tidak marah

Kesimpulan: $bolos \wedge \neg marah$

Mahasiswa tidak malas belajar. $\neg malas$.

Gunakan proposisi:

- malas: Mahasiswa malas belajar ; bolos: Mahasiswa sering bolos kuliah
- lulus: Mahasiswa lulus ujian ; marah: Orang tua mahasiswa marah

malas = p
bolos = q
lulus = r
maran = s

$$(p \land q) \Rightarrow r$$
$$\neg r \Rightarrow s$$
$$q \land \neg s$$
$$\therefore \neg p$$

Proving without premises

- No premise → no place to apply rules of inference
- Facts: valid sentences → true for all interpretations
- How to prove $p \rightarrow (q \rightarrow p)$ is a valid sentence?
- Requires: rule of inference without premises
- Example: axiom schemata

Schemata

- Schema: expression satisfying the grammatical rules of our language → occurs meta-variables in the expression
  - $\varphi \rightarrow (\psi \rightarrow \varphi)$ ini pasti benar

    kalo ini benar
- Instance of sentence schema: substituting the occurrences of metavariables, form legal expressions

$$
\begin{array}{lll}
p \Rightarrow (p \Rightarrow p) & p \Rightarrow (p \wedge p \Rightarrow p) & p \Rightarrow (p \vee p \Rightarrow p) \\
p \Rightarrow (q \Rightarrow p) & p \Rightarrow (p \wedge q \Rightarrow p) & p \Rightarrow (p \vee q \Rightarrow p) \\
p \Rightarrow (r \Rightarrow p) & p \Rightarrow (p \wedge r \Rightarrow p) & p \Rightarrow (p \vee r \Rightarrow p) \\
q \Rightarrow (p \Rightarrow q) & \dots & \dots \\
q \Rightarrow (q \Rightarrow q) & & \\
q \Rightarrow (r \Rightarrow q) & & \\
r \Rightarrow (p \Rightarrow r) & & \\
r \Rightarrow (q \Rightarrow r) & & \\
r \Rightarrow (r \Rightarrow r) & &
\end{array}
$$

Axiom

- Axiom:
  - Proposition that is believed to be true
  - base assumption for proving
  - valid proposition
- Example: $p \rightarrow (q \rightarrow p)$

Standard Axiom Schemata

- Implication Introduction (II):
  $A \rightarrow (B \rightarrow A)$
- Implication Distribution (ID):
  $A \rightarrow (B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
- Contradiction Realization (CR):
  $(A \rightarrow \sim B) \rightarrow ((A \rightarrow B) \rightarrow \sim A)$
  $(\sim A \rightarrow B) \rightarrow ((\sim A \rightarrow \sim B) \rightarrow A)$

Standard Axiom Schemata (2)

- Equivalence (EQ):
  $(A \leftrightarrow B) \rightarrow (A \rightarrow B)$
  $(A \leftrightarrow B) \rightarrow (B \rightarrow A)$
  $(A \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow (A \leftrightarrow B))$

- Other operators:
  $(A \leftarrow B) \leftrightarrow (B \rightarrow A)$
  $(A \vee B) \leftrightarrow (\sim A \rightarrow B)$
  $(A \wedge B) \leftrightarrow \sim(\sim A \vee \sim B)$

Proofs

- Prove all logical consequences from any set of premises
  - Standard axiom schemata
  - Modus Ponen

Example

- Whenever p is true, q is true. Whenever q is true, r is true. Prove that whenever p is true, r is true.
- Premis: $p \rightarrow q$, $q \rightarrow r$
- Konklusi: $p \rightarrow r$

1.	$p \rightarrow q$	premise
2.	$q \rightarrow r$	premise
3.	$(q \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r))$	II
4.	$(p \rightarrow (q \rightarrow r))$	Modus Ponen 2,3
5.	$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$	ID
6.	$(p \rightarrow q) \rightarrow (p \rightarrow r)$	Modus Ponen 4,5
7.	$p \rightarrow r$	Modus Ponen 1,6

Exercise Axiom Schemata

1. Premises: $p \rightarrow q$, $q \rightarrow r$
   Prove conclusion: $(p \rightarrow \sim r) \rightarrow \sim p$

II: $A \rightarrow (B \rightarrow A)$

ID: $A \rightarrow (B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

CR: $(A \rightarrow \sim B) \rightarrow ((A \rightarrow B) \rightarrow \sim A)$
$(\sim A \rightarrow \sim B) \rightarrow ((\sim A \rightarrow B) \rightarrow A)$

EQ: $(A \leftrightarrow B) \rightarrow (A \rightarrow B)$
$(A \leftrightarrow B) \rightarrow (B \rightarrow A)$
$(A \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow (A \leftrightarrow B))$

O: $(A \leftarrow B) \leftrightarrow (B \rightarrow A)$
$(A \vee B) \leftrightarrow (\sim A \rightarrow B)$
$(A \wedge B) \leftrightarrow \sim(\sim A \vee \sim B)$

$\{p \rightarrow q,\ q \rightarrow r\} \models (p \rightarrow \sim r) \rightarrow \sim p$

1.	$p \rightarrow q$	premise
2.	$q \rightarrow r$	premise
3.	$(q \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r))$	II
4.	$(p \rightarrow (q \rightarrow r))$	Modus Ponen 2,3
5.	$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$	ID
6.	$(p \rightarrow q) \rightarrow (p \rightarrow r)$	Modus Ponen 4,5
7.	$p \rightarrow r$	Modus Ponen 1,6
8.	$(p \rightarrow r) \rightarrow ((p \rightarrow \sim r) \rightarrow \sim p)$	CR
9.	$(p \rightarrow \sim r) \rightarrow \sim p$	Modus Ponen 7,8

Provability

- $A \vdash B$
  - A conclusion is said to be *provable* from a set of premises (written $\vdash$) if and only if there is a finite proof of the conclusion from the premises using only Modus Ponens and the Standard Axiom Schemata.
- Previous Example: $\{p \rightarrow q, q \rightarrow r\} \vdash (p \rightarrow r)$
- $(A \vdash B) \leftrightarrow (A \models B)$

Deduction Theorems

A |- $(B \rightarrow C)$ iff $A \cup \{B\}$ |- C

Example:

$\{p \rightarrow q, q \rightarrow r\}$ |- $(p \rightarrow r)$

$\{p \rightarrow q, q \rightarrow r, p\}$ |- r

Review

- $\Delta \models \varphi$
  - Set of premises Δ logically entails a conclusion φ iff every interpretation that satisfies the premises also satisfies the conclusion
- Propositional Logic: **Propositional entailment**
- Semantic reasoning:
  - Truth table
  - Validity checking
  - Unsatisfiability checking
- Proof Method:
  - Rules of Inference
  - Axiom schemata

Exercise 1

- Prove:

  $\{p \to q, q \to r\} \models (q \to r) \to ((p \to \sim r) \to \sim p)$

  using:

  - Axiom schemata, Rule of Inference (without Deduction Theorems)

Exercise 2

1. Sesuatu di laut yang tidak diperhatikan, bukan putri duyung.
2. Sesuatu di laut yang dicatat pada log, berarti layak untuk diingat.
3. Sesuatu yang saya lihat di laut, tidak ada yang layak untuk diingat.
4. Sesuatu yang saya perhatikan di laut, adalah sesuatu yang saya catat di log.

Buktikan bahwa kesimpulan: "**Sesuatu yang saya lihat di laut, bukanlah putri duyung**", dapat diturunkan dari kumpulan fakta tersebut dengan memanfaatkan kaidah inferensi saja.

Exercise 2 (2)

Gunakan proposisi sebagai berikut:

- n: sesuatu di laut yang saya (di) perhatikan;
- m: putri duyung;
- l: sesuatu di laut yang dicatat di log;
- r: sesuatu di laut yang layak untuk diingat;
- i: sesuatu yang saya lihat di laut.

Nilai:

1. Pengubahan ke kalimat logika proposisi (premis dan kesimpulan)
2. Pembuktian

Exercise 3

Buktikan bahwa kesimpulan $\mathbf{(\sim r \to (\sim q \wedge \sim p)) \to ((p \to \sim r) \to \sim p)}$ dapat ditarik dari kumpulan fakta $\{p \to q, q \to r\}$ dengan memanfaatkan *axiom schema* dan *modus ponen* saja .

Exercise 4

Misal Γ dan Δ adalah kumpulan kalimat dalam logika proposisi, kemudian ψ dan φ adalah sebuah kalimat dalam logika proposisi. Tentukan tiap pernyataan di bawah ini benar atau salah.

- Jika $\Gamma \not\models \psi$ maka $\Gamma \models \neg\psi$.
- Jika $\Gamma \models \varphi$ dan $\Delta \models \varphi$ maka $\Gamma \cup \Delta \models \varphi$.
- Jika $\Gamma \models \varphi$ dan $\Delta \models \varphi$ maka $\Gamma \cap \Delta \models \varphi$.
- Jika $\Gamma \models \varphi$ dan $\Delta \not\models \varphi$ maka $\Gamma \cup \Delta \models \varphi$.
- Jika $\Gamma \vdash \psi$ maka $\Gamma \models \psi$.
- Jika $\Gamma \cup \neg\psi$ valid, maka $\Gamma \models \psi$.

THANK YOU