Assignment 1

Purvam Jain EE20B101

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1 Assignment-1 Q.3:Matrix Form

Our recurrence equation was of the following form: $u_{n+3} = 3u_{n+1} - 2u_n$ and the known variables are $u_0 = u_1 = u_2 = 1.64$. From these we have equations as

$$u_3 = 3u_2 - 2u_1$$

$$u_4 - 3u_3 = 2u_2$$

$$u_5 - 3u_4 + 2u_3 = 0$$

and this trend continues till

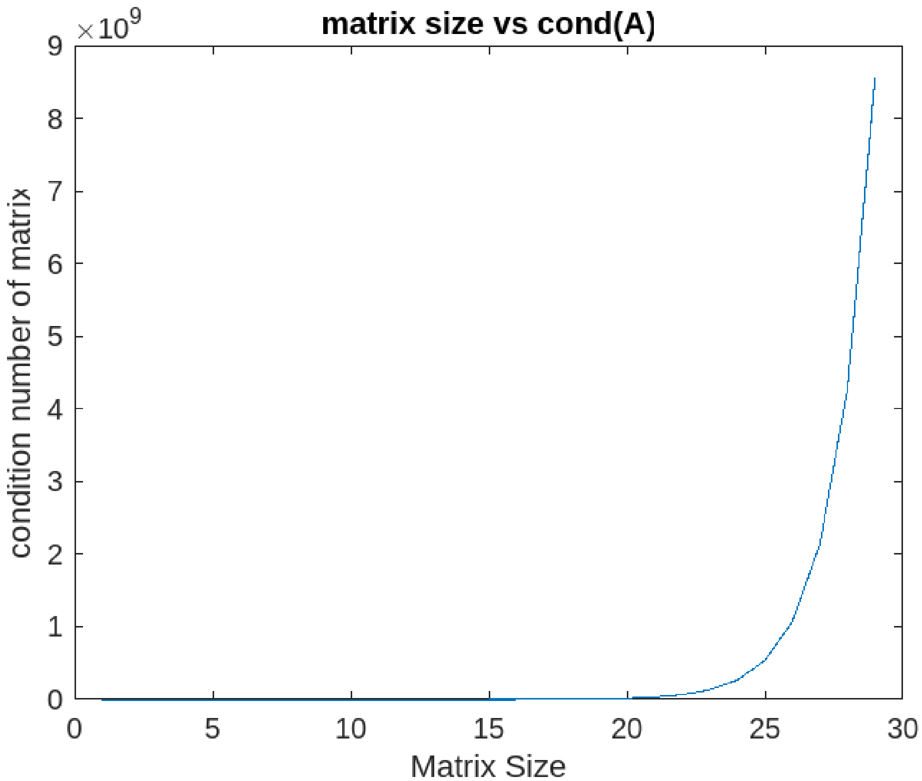
$$u_{n+3} - 3u_{n+1} + 2u_n = 0$$

Our Matrix A_{n*n} will look as follows: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & & & \\ -3 & 1 & 0 & 0 & 0 & \cdots & & & \\ 2 & -3 & 1 & 0 & 0 & 0 & \cdots & & \\ 0 & 2 & -3 & 1 & 0 & 0 & \cdots & & \\ & & \vdots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -3 & 1 \end{bmatrix}$

Column Vector b_n will look as: $\begin{bmatrix} 3u_2 - 2u_1 \\ 2u_2 \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}$ Column Vector u_n will look like:

$\begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ \cdots \\ u_{n+3} \end{bmatrix}$ These satisfy $Au = b$ for the recurrence equation we derived above.

1.1 Condition Number Plot:



Condition number increases exponentially with the size n of matrix.
The accuracy of the solution is inversely related to the condition number. With the increase in condition number we observe a decrease in our accuracy. This is an ill conditioned method!

2 Question 1

Let matrix A be real and symmetric therefore $A^* = A^T = A$ and λ be it's eigen value for the eigen vector x which can be both complex. So we have $Ax = \bar{\lambda}x$. Now taking conjugate transpose on both sides, we get $x^*A = \bar{\lambda}x^*$. Multiplying both sides by x from the right-hand side; $x^*Ax = \bar{\lambda}x^*x$, from this we get $x^*\lambda x = \bar{\lambda}x^*x$. Now we know that $x^*x = \|x\|_2^2$ which is always positive and λ being scalar we can write $x^*\lambda x = \lambda x^*x$. From these two we get $\lambda\|x\|_2^2 = \bar{\lambda}\|x\|_2^2$, which gives $\lambda = \bar{\lambda}$, therefore λ or eigen values have to be real.

3 Question 2

We have $U^T U = I$ and $U \in R^{NxN}$

3.1 Unit Circle:

Let λ be an eigen value of U for eigen vector v. So we have $Uv = \lambda v$, take conjugate transpose on both sides , we get $v^* U^* = \bar{\lambda} v^*$, multiply both the equations, $v^* U^* U v = \lambda \bar{\lambda} v^* v$, Now since U is real symmetric , $U^* = U = U^T$, we get $v^* U^T U v = \|\lambda\|_2^2 v^* v$, which gives , $v^* v = \|\lambda\|_2^2 v^* v$. Therefore $\|\lambda\|_2^2 = 1$. Hence eigen values lie on a unit circle!

3.2 Solution to linear system:

$$Ux = b$$

Multiply both sides by U^T from left:

$$U^T U x = U^T b$$

and given $U^T U = I$;

$$x = U^T b$$

3.3 Norm equivalence:

Using the definition of 2-norm as

$$\|x\|_2^2 = x^* x$$

where x^* is conjugate transpose. Now given $Ux = b$, taking 2-norm squared on both sides;

$$\|Ux\|_2^2 = \|b\|_2^2$$

, using norm definition;

$$\|Ux\|_2^2 = (Ux)^*(Ux) = x^* U^* U x = x^* x = \|x\|_2^2 = \|b\|_2^2$$

Hence Proved.

3.4 1-norm equivalence

Consider $x = [2 \quad 3]^T$ and $U = -1/\sqrt{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ giving $Ux = b = 1/\sqrt{2}[5 \quad -1]^T$. Clearly $\|x\|_1 = 5$ and $\|b\|_1 = 3\sqrt{2}$ which are not equal!

4 Question 3:

(a) Let A and B be two lower triangular matrices and their product be AB = C. Now since A and B are lower triangular , by definition we have $a_{ij} = b_{ij} = 0$ when $i < j$. Let's look at c_{ij} when $i < j$.
$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$, When $k < i < j$ we get $b_{kj} = 0$ hence $c_{ij} = 0$.
When $i < k < j$ we get $a_{ik} = 0$ hence $c_{ij} = 0$.
When $i < j < k$ we again get $a_{ik} = 0$ hence $c_{ij} = 0$.
Therefore C will be a lower triangular matrix.
(b) Let $A \in R^{nxn}$ be a lower triangular matrix. Assuming A^-1 exists , $AA^-1 = I$ where I_n is a nxn identity matrix. By definition Identity matrix is also a lower triangular matrix. Using the above proof we can say that A^-1 will also be lower triangular for product to be lower triangular.
(c)The determinant of lower triangular matrix will be product of its diagonal entries. Let λ_i be its eigen values. We have $det(A - \lambda_i * I) = 0 = det(a_{ii} - \lambda_i)$. Therefore we get $\lambda_i = a_{ii}$.

5 Question 4:

Using interpolation inequality which follows directly from the graph of e^x;

$$e^{at+(1-t)b} \leq te^a + (1-t)e^b$$

where $t \in [0, 1]$, Now given $1/p + 1/q = 1$, where $p > 1$, put $t = 1/p$ in interpolation inequality to get ,

$$ab \leq a^p/p + b^q/q$$

Take $z_i = x_i/\|x\|_p$ and $w_i = y_i/\|y\|_p$

$$|z_i w_i| = |z_i||w_i| \leq |z_i|^p/p + |w_i|^q/q$$

Summing the above elements from $i = 1ton$ and substituting back the values of z_i, w_i we get,

$$\sum_{i=1}^{n} |x_i y_i| \leq (\sum_{i=1}^{n} |x_i|^p)^{1/p} (\sum_{i=1}^{n} |y_i|^q)^{1/q} = \|x\|_p \|y\|_p$$

. Cauchy Schwartz will be a special case of this when p=q=2!

6 Question 5:

$$\|x\|_2^2 = \sum_{i=1}^{n} x_i^2$$

$$\|x\|_1^2 = (\sum_{i=1}^{n} |x_i|)^2 = \sum_{i=1}^{n} x_i^2 + 2|x_i||x_j| \cdots$$

, Clearly $\|x\|_2^2 \leq \|x\|_1^2$ due to additional positive terms contributed by product terms , and since both norms are positive , $\|x\|_2 \leq \|x\|_1$ Using Cauchy Schwartz with $x_i = x_i and y_i = 1$,

$$|\sum_{i=1}^{n} x_i|^2 \leq \sum_{i=1}^{n} |x_i|^2 \sum_{i=1}^{n} |1|$$

$$|\sum_{i=1}^{n} x_i|^2 \leq \sum_{i=1}^{n} |x_i|^2 n$$

$$\|x\|_1 \leq \sqrt{n}\|x\|_2$$

Hence Proved. Now $\|x\|_\infty = max(x_i)$,
Using above we have $\|x\|_2^2 \leq \|x\|_1^2 \leq n * max(x_i)^2 = n\|x\|_\infty$. And maximum element of x will obviously be less than equal to sum of all elements hence we get $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$.

7 Question 6:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & -1 \\ -2 & 3 & -1 & -3 & 0 \\ -1 & 5 & 2 & 1 & -1 \end{bmatrix}$$

(a) **Row Space:** Clearly row3 = row1 + row2 so Row space will span(row1,row2). We can find the orthonormal basis using gram-schmidt orthoganalization. Orthonormal basis would be $span([0.18, 0.36, 0.54, 0.72, -0.18]^T, [-0.38, 0.85, 0.01, -0.36, -0.08]^T)$.
Dimension of row space = 2
(b) **Column Space:** 7col3 = 11col1 + 5col2 , col4 = col1 + col3 , 6col5 = col2 + col4 , therefore column space span(col1,col2) and orthonormal basis for these would be $span([0.41, -0.82, -0.41]^T, [0.71, 0, 0.71]^T)$
Dimension of column space = 2
(c) **Null Space:**Using Gaussian Elimination we get $A = \begin{bmatrix} 1 & 0 & 11/7 & 18/7 & -3/7 \\ 0 & 1 & 5/7 & 5/7 & -2/7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
Null space will $span([-11/7, -5/7, 1, 0, 0]^T, [-18/7, -5/7, 0, 1, 0]^T, [3/7, 2/7, 0, 0, 1]^T)$
And an orthonormal basis for these will be:
$[-0.79, -0.36, 0.5, 0, 0]^T; [-0.45, 0.06, -0.67, 0.59, 0]^T; [0, 0.13, 0.1, 0.1, 0.98]^T$
Dimension of null space = 3
(d) **Left Null space:** The null space will span([-1,-1,1]) and it's dimension will be 1. (e) **Rank of matrix:** Rank of the matrix is 2.

8 Question 7:

(A) **Multiplication:**
Let's define our function to be $f(x) = uv; x = [u, v]^T$

$$\kappa = \frac{\|J(x)\|\|x\|}{\|f(x\|}$$

where $J(x)$ is Jacobian matrix. Using this we will get

$$\kappa = \frac{u^2 + v^2}{uv}$$

. Now if u and v are positive numbers then $\kappa \geq 2$ using AM-GM inequality. Therefore multiplication is well well-conditioned problem for positive numbers.
(B) **Division:**
Let us consider the singular linear equation $ax = b$:

$$\kappa = lim_{r\to 0} sup(|\delta x|/|x|r)$$

$$(a + \delta a)(x + \delta x) = (b + \delta b)$$

Using this we have ,

$$\delta x/x = \frac{\delta b/b - \delta a/a}{1 + \delta a/a}$$

From this we can conclude that,

$$\delta x/x \leq 2r/1 - r$$

From this it follows that $\kappa = 2$. Hence division of two numbers is a well conditioned problem.

9 Question 8:

Given A_{mn} , B_{nm} and C_{nm} and linear system $Ax = b$, $rank(A) = r$
(A) **r<min(m,n)**: In this case, neither left inverse nor right inverse will exist. This is because the matrix is rank-deficient and has linearly dependent rows/-columns. So it is impossible to construct a left or right inverse. In this case if b belongs to column space of A then we will have infinitely many solutions as we have more unknowns than equations. But if B doesn't lie in column space of A we will not have any solutions!
(B) **r=min(m,n):**
Case 1(m¡n): In this case right inverse will exist since A is full rank (=m) but no left inverse exists. Infinite solutions exists if b is in column space of A , otherwise no solution exists.
Case 2(m¿n): In this case rank(A) = n , so right inverse doesn't exist , but a left inverse exists as it is full rank wrt to it. Unique solution exists if b is in column space of A , otherwise no solution exists.
Case 3(m=n): In this case both right and left inverse will exist and a unique solution also exists if b is in column space of A otherwise no solution exists.