Assignment 3

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Contents

1 Question 1:

Given $A = \begin{bmatrix} 11 & -5 \\ 2 & 10 \end{bmatrix}$
(i) $\|A\|_1$ = Matrix norm derived from L1 norm is maximum of sum of absolute values of columns that is **15**.
(ii) $\|A\|_2$ = Matrix norm derived from actor L2 norm is maximum singular value of A that is **12.649110640674**.
(iii) $\|A\|_\infty$ = Matrix norm derived from L∞ norm is maximum of sum of absolute values of rows that is **16**.
(iv) $\|A\|_{1*}$ = Applying L1 vector norm on matrix will be sum of absolute values of all elements that is **28**.
(v) $\|A\|_{2*}$ = Applying L2 vector norm also known as Forbenius norm of matrix we get **15.811388300842**.
(vi) $\|A\|_{2*}$ = Applying L∞ vector norm , that is maximum absolute value in matrix as, **11**.

2 Question 2:

$$A = \begin{bmatrix} -1 & -1 & 2 & -4 \\ 2 & -2 & -2 & 6 \\ -1 & -9 & 7 & -8 \\ 2 & 6 & -3 & 11 \end{bmatrix}$$

LU Factorization:

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

$$L_1A = \begin{bmatrix} -1 & -1 & 2 & -4 \\ 0 & -4 & 2 & -2 \\ 0 & -8 & 5 & -4 \\ 0 & 4 & 1 & 3 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$L_2L_1A = \begin{bmatrix} -1 & -1 & 2 & -4 \\ 0 & -4 & 2 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$$L_3 L_2 L_1 A = \begin{bmatrix} -1 & -1 & 2 & -4 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse of Lower Triangular Matrices:

$$L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}$$

$$L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$L_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

Final LU decomposition:

$$L = L_1^{-1} L_2^{-1} L_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ -2 & -1 & 3 & 1 \end{bmatrix}$$

$$U = L_3 L_2 L_1 A = \begin{bmatrix} -1 & -1 & 2 & -4 \\ 0 & -4 & 2 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

With Partial Pivoting:

$$T_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \; L_1 T_1 A = \begin{bmatrix} 2 & -2 & -2 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -10 & 6 & -5 \\ 0 & 8 & -1 & 5 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2/10 & 1 & 0 \\ 0 & 8/10 & 0 & 1 \end{bmatrix} \; L_2 T_2 L_1 T_1 A = \begin{bmatrix} 2 & -2 & -2 & 6 \\ 0 & -10 & 6 & -5 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 19/5 & 1 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/19 & 1 \end{bmatrix} L_3T_3L_2T_2L_1T_1A = \begin{bmatrix} 2 & -2 & -2 & 6 \\ 0 & -10 & 6 & -5 \\ 0 & 0 & 19/5 & 1 \\ 0 & 0 & 0 & 1/19 \end{bmatrix}$$

$$L_1^{'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1 \end{bmatrix} L_2^{'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 8/10 & 1 & 0 \\ 0 & -2/10 & 0 & 1 \end{bmatrix} L_3^{'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/19 & 1 \end{bmatrix}$$

$$L_1^{'-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1/2 & 0 & 0 & 1 \end{bmatrix} L_2^{'-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -8/10 & 1 & 0 \\ 0 & 2/10 & 0 & 1 \end{bmatrix} L_3^{'-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/19 & 1 \end{bmatrix}$$

Final LU Factorization with partial pivoting:

$$U = L_3T_3L_2T_2L_1T_1A = \begin{bmatrix} 2 & -2 & -2 & 6 \\ 0 & -10 & 6 & -5 \\ 0 & 0 & 19/5 & 1 \\ 0 & 0 & 0 & 1/19 \end{bmatrix}$$

$$P = T_3T_2T_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$L = L_1^{'-1}L_2^{'-1}L_3^{'-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 1 & -8/10 & 1 & 0 \\ -1/2 & 2/10 & -1/19 & 1 \end{bmatrix}$$

3 Question 3:

We need to prove backward instability of Cramer's rule with a 2 dimensional example. Suppose system of linear equations as $Ax = b$ where , $A = \begin{bmatrix} 0.999 & 3 \\ 0.5 & 1.5 \end{bmatrix}$ and $b = \begin{bmatrix} 7.497 \\ 3.75 \end{bmatrix}$. This set of equations will have the exact solution $x = \begin{bmatrix} 3 \\ 1.5 \end{bmatrix}$ when done by hand on paper. Now , let's apply Cramer's rule. Assuming that our machine has a precision of 3 significant digits or 3 decimal places. This is taken for illustration purpose as this can be extrapolated to higher precision machines easily.

$det(A) = -0.001$, $det(A|b_1) = -0.004$ that is first column of A replaced by b and $det(A|b_2) = 0.002$, that is second column of A replace by b. From here we get $x = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. The forward error is evidently poor. For backward error let's find $\tilde{b}$ using the new value of x from Cramer's Rule as $Ax = \tilde{b}$.

$\begin{bmatrix} 0.999 & 3 \\ 0.5 & 1.5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 9.996 \\ 5 \end{bmatrix} = \tilde{b}$

Backward error $= \frac{\|b-\tilde{b}\|}{\|b\|} = 0.666$ (Using max norm) , which is 3 orders bigger than our machine's precision and hence proves Cramer's rule to be unstable.

4 Question 4:

$$Lx = b$$

Assuming standard Finite Precision arithmetic
Forward Substitution algorithm: $x_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij}x_j}{l_{ii}}$ Let's look at numerator $s_k = b_k - \sum_{j=1}^{k-1} l_{ij}x_j$.

$$\tilde{s} = fl(b_k - \sum_{j=1}^{k-1} l_{kj})$$

$$\tilde{s} = fl(fl(fl(b_k - fl(l_{k1}x_1)) - fl(l_{k2}x_2)) - fl(l_{k3}x_3)) \cdots$$

$$\tilde{s}^{(j+1)} = (\tilde{s}^j - l_{ij}\tilde{x_j}(1+\delta_j))(1+\delta_j^{'})$$

$$\tilde{s} = b_k \prod_{j=1}^{i-1}(1+\delta_j^{'}) - \sum_{j=1}^{k-1} l_{kj}\tilde{x_j}(1+\delta_j)\prod_{j=j}^{i-1}(1+\delta_j^{'})$$

where for all $|\delta| \leq |\epsilon_m|$

$$fl(x_i l_{ii}) = \frac{\tilde{x_i} l_{ii}}{1+\delta_i} = \tilde{s}$$

Divinding whole equation by coefficient of b_i

$$\frac{\tilde{x_i} l_{ii}}{(1+\delta_i)(1+\delta_1^{'})\cdots(1+\delta_{i-1}^{'})} = b_i - \sum_{j=1}^{i-1} l_{ij}\tilde{x_j}\frac{(1+\delta_j)}{(1+\delta_1^{'})\cdots(1+\delta_{j-1}^{'})}$$

Lower Triangular elements by definition should be:

$$\tilde{x_i}\tilde{l_{ii}} = b_i - \sum_{j=1}^{i-1}\tilde{l_{ij}}\tilde{x_j}$$

From these we get,

$$\tilde{l_{ii}} = \frac{l_{ii}}{(1+\delta_i)(1+\delta_1^{'})\cdots(1+\delta_{i-1}^{'})} = l_{ii}(1+\theta_i)$$

$$|\theta_i| \leq \frac{i\epsilon_m}{1-i\epsilon_m} = \gamma_i$$

$$\tilde{l_{ij}} = \frac{l_{ij}(1+\delta_j)}{(1+\delta_1^{'})\cdots(1+\delta_{j-1}^{'})} = l_{ij}(1+\theta_j)$$

$$|\theta_j| \leq \frac{j\epsilon_m}{1-j\epsilon_m} = \gamma_j$$

This Relationship between θ and n uses the same derivation as was used in Assignment 1 using Taylor series expansion and squared order terms are ignored. Now to evaluate backward error using max norm. Since n is the largest factor possible as component values cancel out we get:

$$\frac{\|\tilde{L} - L\|}{\|L\|} = \theta_n \leq \gamma_n = \frac{n\epsilon_m}{1 - n\epsilon_m} = f(n)\epsilon_m$$