RESEARCH STATEMENT

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1. Introduction

In the late 19th century, Cantor showed that the cardinality of the set of real numbers is larger than the cardinality of the set of natural numbers. Hence we have the inequality $2^{\aleph_0} > \aleph_0$. He conjectured that there are no infinite cardinals between these, i.e. that $2^{\aleph_0} = \aleph_1$, and this became known as the *continuum hypothesis* (CH). This was also Hilbert's First Problem, and it was resolved more than half a century later when Cohen developed the technique of forcing to prove that CH is independent of the Zermelo Fraenkel axioms (ZFC), thus paving the way for modern set theory. In fact, Gödel's First Incompleteness Theorem tells us that any effective collection of axioms will leave many questions undecidable by formal proof. However, it is possible to address questions that are undecidable by ZFC using stronger axioms that are independent of ZFC. For example, the Proper Forcing Axiom (PFA), a natural strengthening of Martin's Axiom, resolves the value of the continuum to be $\aleph_2$. More broadly, developments in infinitary combinatorics have shown the ways that specific cardinals like $\aleph_2$, $\aleph_3$, and $\aleph_{\omega+1}$ differ substantially in their properties, resulting in a subtle interplay between outright ZFC theorems and independence phenomena.

There is an important distinction in this area of research between regular and singular cardinals. A cardinal κ is *regular* if it cannot be written as the union of strictly fewer than κ-many cardinals less than κ. The first infinite cardinal $\aleph_0$ (often denoted ω) is regular because finite unions of finite sets are of course finite, and $\aleph_1$ is regular because countable unions of countable sets are countable. This generalizes to $\aleph_n$ for all $n < \omega$. A cardinal is *singular* if it is not regular. The smallest example is $\aleph_\omega$, the first cardinal after the $\aleph_n$'s, which can be written $\bigcup_{n<\omega} \aleph_n$. The cardinal $\aleph_\omega$ has vastly different properties from the cardinals below it, and this cardinal is the subject of some longstanding open questions that will be discussed below. The distinction between regular and singular is especially vital in terms of global patterns of behavior on the class of infinite cardinals. Shortly after Cohen had showed that the value of $2^{\aleph_0}$ was almost entirely unconstrained, Easton proved that the continuum function $\kappa \mapsto 2^\kappa$ can exhibit any reasonable behavior when restricted to the class of regular cardinals [12]. It was expected that Easton's Theorem would be extended to the singular cardinals, but Silver refuted this notion in 1974 by proving that the generalized continuum hypothesis (GCH, which states that $2^\kappa = \kappa^+$ for all infinite cardinals κ) cannot fail for the first time at a singular of uncountable cofinality [50] (see Section 2 for exact definitions). Other phenomena follow this pattern, including recent results of my own (see Theorem 22 below).

The finesse needed for this research stems from the limitations on what combinatorial statements can be proven consistent by forcing. These limitations are especially pronounced when it comes to singular cardinals, whose properties are neither entirely rigid nor entirely malleable. On one hand, there are provable constraints on their behavior that can be derived with the PCF theory developed by Shelah: this technique provides canonical invariants called *scales* that are used to obtain various cardinal bounds within ZFC by showing how singular cardinals interact with the regular cardinals below them. The most famous example is the bound that Shelah obtained by showing that if $\aleph_\omega$ is a strong limit (i.e. a cardinal κ such that $2^\lambda < \kappa$ for all $\lambda < \kappa$) then $2^{\aleph_\omega} < \aleph_{\omega_4}$. The question of what the optimal bound must be is still open, although it is known that it cannot be any lower than $\aleph_{\omega_1}$. Even many of the statements that can be forced depend on large cardinal axioms. In contrast to the Shelah bound, Magidor proved from the consistency of a supercompact cardinal that $\aleph_\omega$ can be strong limit—in fact that we can have $2^{\aleph_n} = \aleph_{n+1}$ for all $n < \omega$ while $2^{\aleph_\omega} > \aleph_{\omega+1}$ [40]. The relevant term here is the *singular cardinal hypothesis* (SCH), which when referring to a cardinal κ is the statement that κ is a singular strong limit such that $2^\kappa > \kappa^+$. It is known that the failure of SCH implies the consistency of large cardinals [17]. In many cases, it is possible to show that the large cardinal assumption is optimal.

The use of large cardinals in these investigations leads to the study of compactness properties and their relationship to cardinal arithmetic. A property P is a compactness property at κ if $P(\kappa)$ implies $P(\lambda)$ for some $\lambda < \kappa$. A typical example is that of *stationary reflection*, which holds at κ if, roughly speaking, large enough sets have large enough initial segments. A property is noncompact if it implies the failure of compactness properties. An important noncompactness property is known as Jensen's *square principle*, which asserts the existence of a sequence of approximations to a closed unbounded set in a cardinal μ even though the approximated set cannot exist. Notable results involve forcing combinations of compactness and non-compactness properties. One example is a question I answered by showing that it is consistent at a singular cardinal κ for a particular square principle denoted $\square_{\kappa,<\kappa}$ to hold without the presence of a PCF-theoretic object called a very good scale, but that $\square_{\kappa,<\kappa}$ nonetheless implies one of the natural combinatorial consequences of a very good scale (see Theorem 13 and Theorem 14 below).

The objects of study and the techniques involved will be discussed Section 2. Section 3 will cover more specific questions in PCF theory. Although there are many questions to consider, this section will present a particular approach of using scales to build square sequences at singular cardinals κ from square sequences on cardinals below κ. Then I will discuss questions that are related to the previous threads but pertain to double successors of regular cardinals (Section 4). The questions discussed in Section 4 are potentially global in nature, yet we must focus on small cardinals because our understanding is in its early stages of development.

2. Definitions, Objects, and Techniques

Now we will go through the basic terminology that we will use for this research statement. The emphasis will be on the motivations for these concepts and how they relate to one another.

2.1. Cofinality.

Regarding the distinction between regular and singular cardinals:

Definition 1. If κ is a cardinal, the *cofinality* of κ is the least cardinal λ such that κ can be written $\kappa = \bigcup_{i<\lambda} \kappa_i$ where $\kappa_i < \kappa$ for all $i < \lambda$. In this case we denote $\mathrm{cf}(\kappa) = \lambda$. Given a set of ordinals X, we denote $\{\alpha \in X : \mathrm{cf}(\alpha) = \tau\} = X \cap \mathrm{cof}(\tau)$.

Hence a cardinal κ is regular if and only if $\mathrm{cf}(\kappa) = \kappa$. König's Theorem holds that $\kappa^{\mathrm{cf}\,\kappa} > \kappa$ for all κ, so for example $\mathrm{cf}(2^{\aleph_0}) = \mathrm{cf}(\aleph_0^{\aleph_0}) > \aleph_0$. This is why the introduction states that the value of the continuum is "almost entirely" unconstrained in Cohen's result.

2.2. Forcing.

Forcing is a standard method in set theory for generating models. The classical example is Cohen's forcing to add a new real number. This is typically denoted $(\mathrm{Add}(\aleph_0), \leq_{\mathrm{Add}(\aleph_0)})$, and it consists of partial binary functions $\aleph_0 \rightharpoonup \{0,1\}$. We let $p \leq_{\mathrm{Add}(\aleph_0)} q$ if and only if $p \supseteq q$. Hence the elements of the poset are approximations of some function $\aleph_0 \to \{0,1\}$. If V is the model in which $\mathrm{Add}(\aleph_0)$ is defined, then we consider a filter G that meets all sets $D \in V$ that are dense in the natural topology (this requires an argument). Then it can be argued that $\bigcup G : \aleph_0 \to \{0,1\}$ is a real that is not in the ground model V.

To prove the consistency of the failure of CH, we use the poset $\mathbb{P} = \mathrm{Add}(\aleph_0, \aleph_2) := \{f : \aleph_0 \times \aleph_2 \rightharpoonup \{0,1\} | \operatorname{dom} f \text{ is finite}\}$. This poset is used to add $\aleph_2$-many reals. The delicate aspect of forcing in this case is that we need to ensure that in $V[G]$ there are no new surjections from $\aleph_1^V$ to $\aleph_2^V$ (the respective cardinalities as defined in V), so that the $\aleph_2$ of $V[G]$ is the same as the $\aleph_2$ from the ground model V.

Since forcing was originally introduced, the technique has developed in sophistication and has been used to produce a great variety of independence results.

2.3. Compactness and Reflection Properties.

Recall the mention of stationary reflection in the introduction—let us be more precise here:

Definition 2. A stationary set $S \subseteq \delta$ *reflects* at $\alpha < \delta$ if $\mathrm{cf}\,\alpha > \omega$ (for non-triviality) and $S \cap \alpha$ is stationary in α. It *reflects* if it reflects at some $\alpha < \delta$. We say that the *stationary reflection property* holds for a stationary S if every stationary subset $T \subseteq S$ reflects. and denote this $\mathrm{Refl}(S)$.

As with all properties being discussed, there are variations to consider. The stationary reflection property can be strengthened by considering multiple stationary sets:

Definition 3. We say that a sequence of stationary subsets $\langle S_i : i < \theta \rangle$ of δ *reflect simultaneously* at $\alpha < \delta$ if $\operatorname{cf} \alpha > \omega$ and $S_i \cap \alpha$ is stationary for *all* $i < \theta$. If all sequences of θ-many stationary subsets of some S reflect, we write $\mathrm{Refl}(\theta, S)$.

2.4. **Large Cardinals.** Many phenomena in set theory demonstrably require large cardinals to be understood. Oftentimes large cardinals have several equivalent definitions. Here is an example that uses filters:

Definition 4. An uncountable cardinal κ is *measurable* if there is an ultrafilter $U \subseteq P(\kappa)$ such that all elements of U have cardinality κ, and such that for all sequences $\langle X_\xi : \xi < \lambda \rangle$ of strictly fewer than κ-many elements of U, we have $\bigcap_{\xi<\lambda} X_\xi \in U$.

Observe that ω fits this definition except for the uncountability requirement. This is a pattern with large cardinals: they usually generalize the behavior of ω. More broadly, large cardinals exhibit compactness properties by their nature. For example, if κ is measurable, then the simultaneous stationary reflection principle $\mathrm{Refl}(\lambda, \kappa)$ holds for all $\lambda < \kappa$.

We can also define large cardinals in terms of embeddings:

Definition 5. A cardinal κ is λ-*supercompact* if $\lambda \geq \kappa$ and there is an elementary embedding $j : V \to M \subset V$ such that $j \restriction V_\kappa = \mathrm{id} \restriction V_\kappa$, $j(\kappa) > \lambda$, and $M^\lambda \subset M$. A cardinal κ is *supercompact* if it is λ-supercompact for all $\lambda \geq \kappa$.

Supercompact cardinals are stronger than measurable cardinals and are for the most part necessary for studying singular cardinals and their successors.

2.5. **Squares.** Square principles are canonical instances of noncompactness. The principle $\square_\kappa$ was distilled by Jensen in the 1970's and was used to find Souslin trees in L, Gödel's canonical inner model [25]. Generally, $\square_\kappa$ represents the combinatorics of L since $\square_\kappa$ holds in L for all cardinals κ. In contrast, it is known that the failure of $\square_\kappa$ for a singular κ implies the consistency of some very substantial large cardinals [45], but the exact large cardinal strength is open.

Definition 6 (Schimmerling [46])**.** If $\lambda \leq \kappa$, then we say that $\square_{\kappa,\lambda}$ *holds* if there is a $\square_{\kappa,\lambda}$-*sequence*, which is a sequence $\langle \mathcal{C}_\alpha : \alpha \in \lim(\kappa^+) \rangle$ such that for all $\alpha \in \lim(\kappa^+)$: *(1)* $\mathcal{C}_\alpha$ consists of clubs $C \subset \alpha$ of order-type less than or equal to κ; *(2)* $\forall C \in \mathcal{C}_\alpha, \forall \beta \in \lim C$, $C \cap \beta \in \mathcal{C}_\beta$; *(3)* $1 \leq |\mathcal{C}_\alpha| \leq \lambda$.

The principles $\square_{\kappa,<\lambda}$ for $\lambda \leq \kappa$ are defined similarly, except that the third clause reads $1 \leq |\mathcal{C}_\alpha| < \lambda$. We typically denote $\square_{\kappa,1}$ as $\square_\kappa$ and call it *square*, and we denote $\square_{\kappa,\kappa}$ as $\square_\kappa^*$ and call it *weak square*.

Note in particular that $\square_{\kappa,\lambda}$ is in fact a statement about κ^+, and that the fundamental property of $\square_{\kappa,\lambda}$-sequences is that no such sequence can have a *thread*: There is no club $D \subseteq \kappa^+$ such that for all $\alpha \in \lim D$, $D \cap \alpha \in \mathcal{C}_\alpha$. If α were the $(\kappa+\omega)^{\text{th}}$ point of such a club, then we would have $\kappa + \omega = \mathrm{ot}(D \cap \alpha) \leq \kappa$. Moreover, $\square_\kappa$ implies that the failure of $\mathrm{Refl}(S)$ for all stationary $S \subseteq \kappa^+$, i.e. $\square_\kappa$ directly contradicts a compactness property. This is why squares are noncompactness principles.

A weakening of $\square_{\kappa,\lambda}$ replaces the restriction on the order type of the clubs $C \in \mathcal{C}_\alpha$ with the requirement that there is no thread:

Definition 7. Let λ and μ be regular cardinals such that $\lambda < \mu$. Then $\langle \mathcal{C}_\alpha : \alpha \in \lim(\mu) \rangle$ is a $\square(\mu, \lambda)$*-sequence* if the following hold for all $\alpha \in \lim(\mu)$: *(1)* $\mathcal{C}_\alpha$ consists of clubs in α; *(2)* For all $C \in \mathcal{C}_\alpha$ and all $\beta \in \lim(C)$, $C \cap \beta \in \mathcal{C}_\beta$; *(3)* $1 \leq |\mathcal{C}_\alpha| \leq \lambda$; *(4)* There is no club $D \subset \mu$ such that for all $\alpha \in \lim(D)$, $D \cap \alpha \in \mathcal{C}_\alpha$.

If there is a $\square(\mu, \lambda)$-sequence, then we say that $\square(\mu, \lambda)$ *holds*.

There is another technical weakening of $\square^*_\kappa$ that we mention here (without explicit definition) called the *approachability property*, usually denoted $\kappa^+ \in I[\kappa^+]$ when it holds. It was introduced by Shelah to study the preservation of stationary sets in forcing extensions, and for our purposes it is notable as a very weak noncompactness property.

2.6. **Scales.** Scales are the primary object of study in PCF theory.

Definition 8. Let $\langle \kappa_i : i < \mathrm{cf}\,\kappa \rangle$ be a strictly increasing sequence of regular cardinals converging to a singular κ. A *product* on κ consists of functions $f \in \prod_{i < \mathrm{cf}\,\kappa} \kappa_i$ where $f : \mathrm{cf}\,\kappa \to \mathrm{ON}$ and for all $i < \mathrm{cf}\,\kappa$, $f(i) < \kappa_i$. We will also use the term "product" to refer to the sequence $\langle \kappa_i : i < \mathrm{cf}\,\kappa \rangle$ of regular cardinals.

If $f, g \in \prod_{i < \mathrm{cf}\,\kappa} \kappa_i$, we write $f <^* g$ if g is greater than f in the ordering of eventual domination, meaning that there is some $j < \mathrm{cf}\,\kappa$ such that if $i \geq j$ then $f(i) < g(i)$.

A *scale* on a singular cardinal κ is a sequence of functions $\langle f_\alpha : \alpha < \kappa^+ \rangle$ in $\prod_{i < \mathrm{cf}\,\kappa} \kappa_i$ such that: *(1)* for all $\alpha < \beta$, $f_\alpha <^* f_\beta$; *(2)* for all $g \in \prod_{i < \mathrm{cf}\,\kappa} \kappa_i$, there is some $\alpha < \kappa^+$ such that $g <^* f_\alpha$. (In other words, $\langle f_\alpha : \alpha < \kappa^+ \rangle$ is strictly $<^*$-increasing and cofinal in $\prod_{i < \mathrm{cf}\,\kappa} \kappa_i$.)

The fundamental property of scales is due to Shelah: For every singular cardinal κ, some product $\prod_{i < \mathrm{cf}\,\kappa} \kappa_i$ on κ carries a scale [49]. We are specifically interested in varieties of well-behaved scales:

Definition 9. Given a scale $\vec{f}$ in a product $\prod_{i < \mathrm{cf}\,\kappa} \kappa_i$, the limit ordinal $\alpha < \kappa^+$ is *good* if there is some unbounded $A \subset \alpha$ such that $\mathrm{ot}\, A = \mathrm{cf}\,\alpha$ and some index $j < \mathrm{cf}\,\kappa$ such that for $i \geq j$, $\langle f_\beta(i) : \beta \in A \rangle$ is strictly increasing. The ordinal α is *very good* if A is a club. A *good scale* is a scale with a club $D \subseteq \kappa^+$ such that every $\alpha \in D$ with $\mathrm{cf}\,\alpha > \mathrm{cf}\,\kappa$ is good, a *bad scale* is a scale that is not good, and a *very good scale* is a scale with a club $D \subseteq \kappa^+$ such that every $\alpha \in D$ with $\mathrm{cf}\,\alpha > \mathrm{cf}\,\kappa$ is very good.

The existence of a good scale on a singular λ is a non-compactness property. If κ is supercompact and $\mathrm{cf}\,\lambda < \kappa < \lambda$, then every scale on λ is bad. On the other hand, if κ is singular and $\square^*_\kappa$ holds, then all scales on κ are good.

3. Singular Cardinals, Canonical Structure, and Compactness

The first section of this statement focuses on singular cardinals and their compactness properties. The general framework is this:

Problem 10. *Given a singular cardinal κ, consider the interdependence of the compactness properties below κ, the compactness properties of κ^+, and the types of scales that κ carries. Consider how this interdependence changes for particular values of κ.*

This problem is integral to the singular cardinals problem. It has been investigated by Cummings, Foreman, and Magidor [7, 8], and it relates to some questions in a survey by Foreman [13]. So far, I have made some progress with my own research.

3.1. Very good scales and $\aleph_\omega$.

If $2^{\aleph_\omega} = \aleph_{\omega+1}$ and $\aleph_\omega$ is a strong limit, then $\aleph_{\omega+1}^{\aleph_\omega} = \aleph_{\omega+1}$, and it is known that $\square^*_{\aleph_{\omega+1}}$ holds in this case (see [28]). This leads to what is known as Woodin's Question, a general form of which asked whether $\square^*_\kappa$ follows if κ is a singular cardinal where the *singular cardinal hypothesis* $\mathsf{SCH}(\kappa)$ fails, i.e. if κ is a singular strong limit and $2^\kappa > \kappa^+$. (This is connected to a conjecture of Magidor which holds that it is consistent that the *tree property* holds for all regular cardinals $\kappa \geq \aleph_2$.) A strong version of this question clarifies the role of the notion of compactness. If GCH holds below κ, then $\square^*_\lambda$ holds for all regular $\lambda < \kappa$, so asking whether $\square^*_\kappa$ can fail in this case is asking whether κ is *compact with respect to* $\square^*_\lambda$ for $\lambda < \kappa$.

Cummings, Foreman, and Magidor devised a program towards proving that $\neg\mathsf{SCH}(\kappa) \implies \square^*_\kappa$. It consisted roughly of the following three steps: (1) isolate PCF properties that follow from square principles; (2) show that they imply the existence of *Aronszajn trees* (which can be expressed in terms of square principles); (3) show that they are implied by the failure of SCH [13]. In 2008, Gitik and Sharon established the consistency, relative to the existence of a supercompact cardinal, of $\neg\mathsf{SCH}(\aleph_{\omega^2}) \wedge \neg\square^*_{\aleph_{\omega^2}}$ [19], which was a surprising development given that the original program sought a direct ZFC proof. (This result was soon strengthened to obtain the tree property at $\aleph_{\omega^2+1}$ [43, 51].) However, the use of $\aleph_{\omega^2}$ appears to be necessary.

This state of affairs leaves Woodin's Question at $\aleph_\omega$ open:

Question 11. *Does the failure of* $\mathsf{SCH}(\aleph_\omega)$ *imply* $\square^*_{\aleph_\omega}$*?*

Some barriers to a negative solution were identified by Sinapova and Unger [52, 53], so we must consider the possibility that Question 11 has a positive answer.

This line of research is also connected to the question of the Shelah Bound. One variant that underlines its connection to Woodin's Question is the following:

Question 12. *Does the failure of* $\mathsf{SCH}(\aleph_\omega)$ *(or perhaps the failure of* GCH *for the first time at* $\aleph_\omega$*) imply the existence of a partial* $\square^*_{\aleph_\omega}$*-sequence on* $\aleph_{\omega+1} \cap \mathrm{cof}(\geq \aleph_3)$*?*

Hence any progress in this direction will be extremely valuable.

My early work builds on the 2001 paper of Cummings, Foreman, and Magidor called "Squares, Scales, and Stationary Reflection" in which the authors pursued the program mentioned above, classifying the relationships between the combinatorial principles that can consistently hold at singular cardinals and their successors [6]. In particular, Cummings et al. proved that if κ is singular, then $\square_{\kappa,\lambda}$ for $\lambda < \kappa$ implies the consistency of a very good scale on κ. They also proved, assuming the existence of ω-many supercompact cardinals, that $\square^*_{\aleph_\omega}$ (i.e. $\square_{\aleph_\omega,\aleph_\omega}$) is consistent with the absence of a very good scale on $\aleph_\omega$, specifically by getting compatibility with some simultaneous reflection. Cummings et al. then raised the question of whether $\square_{\kappa,<\kappa}$ for singular κ implies that there is a very good scale on κ. I resolved this question in the negative:

Theorem 13 (Levine)**.** *Assuming the consistency of a supercompact cardinal, it is consistent that $\square_{\aleph_\omega,<\aleph_\omega}$ holds while $\aleph_\omega$ does not carry a very good scale* [33]*.*

The model for Theorem 13 involves collapsing a single supercompact cardinal κ to be $\aleph_2$ and then adding a $\square_{\aleph_\omega,<\aleph_\omega}$-sequence by initial segments. The argument works because of a stationary preservation lemma that takes advantage of the width of the square sequence. I also obtained the following, which sharpened a result of Cummings and Schimmerling [10].

Theorem 14 (Levine)**.** *If κ is singular and $\square_{\kappa,<\kappa}$ holds, then there is a sequence $\langle S_i : i < \operatorname{cf}\kappa\rangle$ of stationary subsets of κ^+ that do not reflect simultaneously* [33]*.*

Since proving these results, I have continued to work on questions regarding the goodness of scales [30, 32, 33, 39].

One natural question when considering Question 11 and Question 12 is whether or not very good scales are unavoidable when SCH fails:

Question 15. *Does the failure of $\mathsf{SCH}(\kappa)$ imply the existence of a very good scale on κ?*

It is known that the failure of $\mathsf{SCH}(\kappa)$ implies the existence of a so-called *better* scale on κ (an intermediate between a good and a very good scale) [4], so this is a question that could go either way. It would be natural to specify the question further—for example, to assume that the cofinality of κ is countable—but here we can discuss how this question pertains to varieties of Prikry extensions. Many known models of the failure of $\mathsf{SCH}(\kappa)$ have very good scales on κ because of boundedness arguments for Prikry forcings that go back to work of Jech [23].[1] These arguments could perhaps generalize. On the other hand, there do exist Prikry-style forcings developed by Gitik for obtaining the failure of $\mathsf{SCH}(\kappa)$ where it is not known whether κ carries a very good scale. These are called *extender-based forcing*, and their purpose is to expand the powerset of an already singular κ [18]. These models are

[1]It should be noted that Assaf Sharon produced a model with a singular κ such that $2^{\operatorname{cf}\kappa} < \kappa$, $2^\kappa > \kappa^+$, and there is no very good scale on κ. However, κ is not a strong limit in Sharon's model [47].

potential counterexamples for Question 15. There are also several technical questions about these forcing extensions that are still open.

Before moving forward, we mention a related result which shows how investigations of very good scales can feed into other natural questions. Cummings and Foreman proved that if κ is singular and $\square_\kappa$ holds, then there is always a product of regular cardinals on κ that does not carry a very good scale [5]. This is in contrast to good scales: an interleaving argument tells us that if a product has a good scale, then all scales on that product are good. But it is strange that $\square_\kappa$, which implies that all products on κ carry some very good scale, would be used as a hypothesis of a theorem that produces a scale that is not very good. Working with Assaf Rinot, we removed the hypothesis of $\square_\kappa$:

Theorem 16 (Levine-Rinot)**.** [39] *It is a theorem of* ZFC *that for every singular κ and every product $\prod_{i<\operatorname{cf}\kappa}\kappa_i$ on κ, there is a scale $\vec{f}$ on κ that is not very good.*

To prove this theorem, we proved a generalization of Solovay's Partition Theorem, which states that for any regular uncountable κ and any stationary subset $S \subset \kappa$, there is a partition $S = \coprod_{\xi<\kappa} S_\xi$ of stationary sets $S_\xi \subset \kappa$ [24]. The following is a specific form of our partition theorem:

Theorem 17 (Levine-Rinot)**.** [39] *Suppose λ is a singular strong limit, and suppose $S \subseteq \lambda^+$ is a stationary set such that there is another stationary set $T \subseteq \lambda^+$ with the following properties:* (1) *$T \subseteq \lambda^+ \cap \operatorname{cof}(\theta)$ for $\theta \neq \operatorname{cf}\lambda$;* (2) *$\forall \rho \in T$, S reflects in ρ;* (3) *$\forall \rho \in T$, ρ is a good point of $\vec{f}$. Then S can be partitioned in θ-many disjoint stationary sets that reflect simultaneously.*

The fact that Theorem 17 already has an application regarding very good scales suggests that further applications are in the waiting. It is open whether the hypothesis on good points is necessary.

It is natural to focus on $\aleph_\omega$ here. Question 15 has a complement in the following:

Question 18. *If $\aleph_\omega$ is a strong limit and there is a very good scale on $\aleph_\omega$, does $\square^*_{\aleph_\omega}$ hold? Would a better scale be enough? More generally, can we find* ZFC *theorems pertaining to $\aleph_\omega$ with similar implications?*

Positive answers to both Questions 15 and 18 would give a poof of $\neg\mathsf{SCH}(\aleph_\omega) \implies \square^*_{\aleph_\omega}$. We note that $\square^*_{\aleph_\omega}$ can be obtained without a very good scale on $\aleph_\omega$ [6], and that the Gitik-Sharon model has a very good scale $\aleph_{\omega^2}$ in spite of the fact that $\square^*_{\aleph_{\omega^2}}$ fails. Hence, Question 18 is interesting on its own because it asks what makes $\aleph_\omega$ distinct from other singulars of countable cofinality.

The point here is that $\aleph_\omega$ does have some properties that are peculiar to it alone. Let me give three examples. The first has to do with reflection: if every stationary subset of $\aleph_{\omega+1}$ reflects, then for all $n < \omega$, every stationary subset of $\aleph_{\omega+1}$ reflects to a point of cofinality at least n [27]. The key is that every stationary set S has a derivative S' consisting of reflection points of S, and these derivatives are the basis of an induction argument. (Very good points of a scale are analogous to reflection points because both characterize an ordinal α by considering all clubs through that ordinal, so there is some potential of making an induction argument for a positive answer to Question 18 using very good points.) The second property, which is a corollary of the first, is that if all stationary subsets of $\aleph_{\omega+1}$ reflect, then the approachability property holds at $\aleph_{\omega+1}$. The third property is that if a scale on $\aleph_\omega$ has an exact upper bound g (a PCF-theoretic term related to good points) such that $g(n)$ has uncountable cofinality for large n, then the scale is automatically good [4]. Some initial work towards Theorem 30 below also makes use of the specificity of $\aleph_\omega$.

3.2. **Singular cardinals of uncountable cofinality and the structure of models.** Here we take a brief digression from $\aleph_\omega$ for reasons that will become apparent. The work I have done keeps returning to the theme that the properties of singular cardinals of uncountable cofinality contrast sharply with the properties of singulars of countable cofinality. (Recall the differences between Silver and Magidor's results on SCH mentioned in the introduction.) Here we will find more surprising differences that relate to Problem 10.

My work with Sy-David Friedman was motivated by the following:

Question 19 (Friedman)**.** *Suppose F is two-valued function on the class of all cardinals $\geq \aleph_1$. Is it possible to obtain a model such that $\square_\kappa$ holds if and only if $F(\kappa) = 1$, or are there ZFC restrictions on the global behavior of $\square_\kappa$?*

In other words, is it possible to Eastonize square? By work of Cummings, Foreman, and Magidor, it is consistent that $\square_{\aleph_n}$ holds for all $n < \omega$ while $\square_{\aleph_\omega}$ fails [7].

Our results focused on the global behavior of the stationary reflection property $\mathsf{SR}(S)$. We proved that, assuming the consistency of a supercompact cardinal, ZFC imposes only trivial restrictions on the global behavior of $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ for a fixed $n < \omega$. These are: (1) $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ holds vacuously if $\kappa \leq \aleph_n$; (2) $\mathsf{SR}(\aleph_{n+1} \cap \mathrm{cof}(\aleph_n))$ fails; (3) if λ is a singular cardinal, then $\mathsf{SR}(\lambda \cap \mathrm{cof}(\aleph_n))$ holds if and only if $\mathsf{SR}(\mathrm{cf}\,\lambda \cap \mathrm{cof}(\aleph_n))$ holds.

Theorem 20 (Friedman-Levine)**.** [15] *Suppose χ is a supercompact cardinal in V such that GCH holds above χ. Let F be a definable 2-valued function on the class of regular cardinals $\geq \chi$. Then there is a forcing extension $W \supset V$ in which $\chi = \aleph_{n+2}$, cofinalities $\geq \chi$ are preserved, GCH is preserved above χ, and for all regular $\kappa \in W$ such that $\kappa \geq \aleph_{n+2}$, $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ holds if and only if $F(\kappa) = 1$.*

For instance, we can require that for all regular $\kappa \geq \aleph_{n+2}$, $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ holds if and only if κ is the successor of a singular cardinal. Hence $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ for different κ is noncompact from a global perspective. In particular Theorem 20 implies that there is no Silver's Theorem for stationary reflection, although we will see shortly that the situation is different when we ask the same question about square sequences. Moreover, it is the only result I am aware of in which the ZFC restrictions on the global behavior of a compactness property are completely settled.

One can also ask for ZFC constraints if we do not fix the cofinalities:

Question 21. *Suppose that F is a function on the class of regular cardinals to itself such that $F(\kappa)^+ < \kappa$ for all κ. Is it consistent that $\mathsf{SR}(\kappa \cap \mathrm{cof}(\lambda))$ holds if and only if $F(\kappa) = \lambda$?*

I did in fact find additional ZFC constraints for singulars of uncountable cofinality:

Theorem 22 (Levine)**.** [34] *Suppose that κ is a singular strong limit of cofinality $\lambda > \omega$ such that for some stationary set $S \subseteq \kappa$, $\square_\delta^*$ holds for all $\delta \in S$ and $\prod_{\delta \in S} \delta^+$ carries a good scale. Then $\square_\kappa^*$ holds.*

This result shows that even if a positive answer to Question 19 is feasible (in the sense that square *can* be Eastonized), it is likely to be very complicated. It is clear that Theorem 22 is significant: Assaf Rinot wrote a note building on this work, applying it to a question of Shelah [44]. It also pertains to a question of Golshani:

Question 23 (Golshani)**.** [21] *Is there a Silver's Theorem for special Aronszajn trees? In other words, if κ is a singular of uncountable cofinality and $S := \{\delta < \kappa : \square_\delta^* \text{ holds}\}$ is stationary, then does $\square_\kappa^*$ necessarily hold?*

In light of Theorem 22, this reduces to:

Question 24. *Suppose that κ is a singular strong limit of uncountable cofinality λ such that $S := \{\delta < \kappa : \square_\delta^* \text{ holds}\}$ is stationary and of order-type λ. Does $\prod_{\delta \in S} \delta^+$ carry a good scale?*

Question 23 is important not only because it relates to Question 19, but would bear an impact on the program discussed at the beginning of Section 3. It is interesting because it applies the concept of Silver's theorem to a wider range of objects. One challenge in obtaining a negative answer in particular is the fact that the go-to technique would be to interleave square sequences in a Prikry forcing, yet this creates a problem with guiding generics and hence the chain condition.

3.3. **Connections to Tree Forcings.** This section brings us back to $\aleph_\omega$. We will discuss some results on singular tree forcings obtained in joint work with Heike Mildenberger. Our work focused on functions added when forcing with perfect trees, which provides another angle of approach for the types of questions discussed earlier in Section 3.

Let us establish some notation. Given a singular κ and a sequence $\langle \kappa_i : i < \mathrm{cf}\,\kappa \rangle$ of regular cardinals converging to κ, conditions are perfect trees $T \subset \prod_{i < \mathrm{cf}\,\kappa} \kappa_i$. The poset $\mathbb{M}_\kappa$ will be the "Miller version" of Namba-tree forcing, meaning we consider all perfect trees. On the other hand, $\mathbb{L}_\kappa$ is the "Laver version", meaning that we only consider trees for which every node splits past the stem of the tree. These versions of tree forcings differ starkly, particularly regarding the boundedness of newly-added functions. (The difference was first employed by Laver in the context of the reals when he proved the consistency of the Borel Conjecture.) Studying these trees for singular cardinals, we answered a question of Dobrinen, Hathaway, and Prikry [11] about the behavior of singular Namba forcings. Specifically, we found a contrast to Prikry's theorem that if $\mathrm{cf}(\kappa) = \omega$ and $\forall \nu < \kappa$, $\nu^\omega < \kappa$, then for any $\nu < \kappa$, $\mathbb{M}_\kappa$ is (ω, ν)-distributive.

Theorem 25 (Levine-Mildenberger)**.** *Suppose* $\mathrm{cf}(\kappa) = \lambda > \omega$ *and* κ *is a* λ*-strong limit. Then* $\mathbb{M}_\kappa$ *and* $\mathbb{L}_\kappa$ *are not* $(\lambda, 2)$*-distributive.*[2]

Hence there is yet another strong difference between singulars of countable versus uncountable cofinality. We further explored this area, finding a number of additional peculiarities, suggesting that there is a wide variety of behaviors as specific parameters change:

Theorem 26 (Levine-Mildenberger)**.** [37] *Let* κ *be a singular cardinal of cofinality* ω_1*. Then* $\mathrm{Add}(\omega_1, 1) \lessdot \mathbb{L}_\kappa$*.*

Theorem 27 (Levine-Mildenberger)**.** [37] *Let* κ *be a singular cardinal of cofinality* $\lambda > \omega$*. Then* $\mathbb{M}_\kappa$ *is not* $(\omega, \cdot, \lambda^+)$*-distributive.*

This work on added functions has precedent, particularly in a widely-known paper of Bukovský and Copláková. First, note that the Lévy collapse $\mathrm{Col}(\aleph_0, < \aleph_{\omega+1})$ forces $\aleph_{\omega+1}^V = \aleph_1^{V[\mathrm{Col}(\omega, <\aleph_{\omega+1})]}$, and it can be argued that $\mathrm{Col}(\aleph_1, < \aleph_{\omega+1})$ forces $|\aleph_{\omega+1}^V| = \aleph_1$. But then, how different are $\aleph_{\omega+1}$ and $\aleph_2$ from the perspective of forcing? Bukovský Copláková posed the following:

Question 28. [1] *Is there an extension* $W \supseteq V$ *such that* $\aleph_1^V = \aleph_1^W$ *and* $\aleph_{\omega+1}^V = \aleph_2^W$*?*

This question is especially important because it is connected to the question of the Shelah Bound in the sense that it asks obliquely about good points of cofinality $\aleph_2$. It has been investigated by Cummings [3], who showed among other things that large cardinals are necessary. Both the context of the Bukovský-Copláková paper and the results of Cummings hint that it would make sense to force with a tree forcing over a model of Martin's Maximum (the strongest possible strengthening of Martin's Axiom). Moreover, Question 28 is related to the Namba forcing used by Magidor—for this section we will denote it as $\mathbb{L}$ because it is a "Laver version" of Namba forcing—to show that Martin's Maximum implies that scales on $\aleph_\omega$ are bad [9]. There are variations of singular Namba forcing that collapse the $\aleph_n$'s to have size $\aleph_1$, so a positive result would require one to argue that $\aleph_{\omega+1}^V$ remains a regular cardinal in the extension.

There are ways of breaking down this investigation. A related question is in a footnote of Cummings-Magidor:

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[2]Gitik observed that this follows if we assume the non-existence of $0^\#$ and apply Jensen's covering lemma, but our context is motivated by the presence of large cardinals implying the existence of $0^\#$.

Conjecture 29 (Cummings-Magidor)**.** [9] *If* MM *holds then* $\mathbb{L}^{\mathrm{stat}}_{\aleph_\omega}$ *(where splitting is into stationary sets) collapses* $\aleph_{\omega+1}$ *to be an ordinal of cardinality and cofinality* $\aleph_1$*.*

This can be viewed as a necessary sub-question of the Bukovský-Copláková question. I myself conjecture that Conjecture 29 is false. Similarly, Cummings, Foreman, and Magidor proved that if $\square_{\aleph_n}$ holds for $n < \omega$ then all good points of cofinality $> \aleph_1$ are approachable [8]. But what about points of cofinality $\aleph_1$?

Theorem 30 (Levine-Mildenberger)**.** *It is consistent modulo large cardinals that every scale on* $\aleph_\omega$ *is good,* $\square_{\aleph_n}$ *holds for all* $n < \omega$*, and the approachability property for* $\aleph_{\omega+1}$ *fails* [38].

This used ideas for Namba forcing that I have recently developed [35].

4. Successors of Regular Cardinals and the Specificity of $\aleph_2$ and $\aleph_3$

Next, we focus on double successors of regular cardinals. This relates to themes already discussed such as square principles and approachability, but some of the specific issues are quite different. For example, we consider different varieties of forcing, and it often is more manageable to get equiconsistency results with successors of regular cardinals.

The literature is rich in exploring the differences between $\aleph_1$ and $\aleph_2$. One on hand, $\aleph_1$ is definitively noncompact, which $\aleph_2$ might or might not be compact depending on the model one is working with. Also, many results pertaining to compactness of $\aleph_2$ generalize to $\aleph_n$ for $n \geq 2$, with few pertaining to $\aleph_3$ specifically (see e.g. [42] for an exception). Hence, our guiding motivation is this:

Problem 31. *Can we find properties that truly differ for specific double successors of regulars larger than* $\aleph_2$*?*

4.1. **Square sequences and** $\aleph_2$**.**

A particularly compelling question was raised by Hayut and Lambie-Hanson [20]. They proved that for all regular $\lambda \geq \aleph_2$, $\square(\lambda, <\omega)$ implies the failure of $\mathrm{Refl}(2, \lambda)$. They also proved modulo large cardinals that $\square(\lambda, \omega)$ is compatible with the finite stationary reflection property $\mathrm{Refl}(<\omega, \lambda)$.

Question 32. *Does* $\square(\aleph_2, \aleph_0)$ *imply* $\neg\mathrm{Refl}(\omega, \aleph_2 \cap \mathrm{cof}(\aleph_0))$*?*

Like successors of singular cardinals, the question of which square properties hold at a given cardinal tells us a lot about the model in question in terms of its relationship to canonical inner models like Gödel's model L.

Question 32 is certainly interesting in its own right as well, because $\square_{\aleph_1,\aleph_0}$ does in fact imply $\neg\mathrm{Refl}(\aleph_2 \cap \mathrm{cof}(\aleph_0))$ (see [9]), so this is specifically an issue of "unthreadability", i.e. the implications of the non-existence of a thread as opposed to the order-types being bounded. As is often the case, the difficulty appears to be that lack of available techniques. Since the question specifically asks after stationary reflection, the usual square-forcing techniques do not work (the usual forcings for adding square sequences also add non-reflecting stationary subsets [31]). Therefore, nothing like the argument for Theorem 13 will work.

It appears that there are distinctions to be found when we shift to $\aleph_3$. Caicedo et al. have undertaken an investigation of $\mathbb{P}_{\max}$ forcing and have found interesting questions pertaining to $\square(\aleph_3, \aleph_1)$ [2], for example:

Question 33. *Does the reflection property for subsets of $P_{\omega_1}(\aleph_3)$ imply failure of $\square(\aleph_3, \aleph_1)$?*

By work of Veličković, the answer is yes for $\square(\lambda, \aleph_0)$ given regular λ [55] (see also [54]).

It appears at the moment that there should be a consistency result for Question 32 and perhaps also Question 33, so the right approach is to consider other ways of obtaining $\square(\mu^{++}, \mu)$ for regular μ by forcing. One characteristic to focus on is the size of the conditions. For example, Stanley and Shelah produced a forcing for adding $\square_{\aleph_1}$ with countable conditions [48], so we can ask if there is a natural way to add $\square(\aleph_2, \aleph_0)$ with countable conditions. I found such a forcing in a recent preprint:

Theorem 34 (Levine)**.** [29] *If $V \models \mathsf{CH}$ then there is a forcing $\mathbb{S}$ consisting of countable conditions such that $V[\mathbb{S}] \models \square(\aleph_2, \aleph_0)$. Moreover, if $W \models$ "κ is weakly compact", then* $W[\mathrm{Col}(\aleph_1, <\kappa)][\mathbb{S}] \models \square(\aleph_2, \aleph_0) \wedge \neg\square(\aleph_2, <\aleph_0) \wedge \neg\square_{\aleph_1, \aleph_0}$.

This construction is natural for $\square(\aleph_2, \aleph_0)$ in the sense that the forcing $\mathbb{S}$ adds exactly $\square(\aleph_2, \aleph_0)$. It would be productive to continue experimenting with adding $\square(\aleph_2, \aleph_0)$ by different forcings. There is almost certainly a forcing with finite conditions that adds a $\square(\aleph_2, \aleph_0)$ in the natural way—i.e. in such a way that $\square_{\aleph_1, \aleph_0}$ and $\square(\aleph_2, <\aleph_0)$ fail. We can also consider a more precise question which looks for an advantage in reducing the sizes of the conditions:

Question 35. *Is there a forcing consisting of finite conditions that adds $\square(\aleph_2)$, but in which* $\mathrm{Refl}(\aleph_2 \cap \mathrm{cof}(\omega))$ *holds?*

The point is that this might help avoid the necessity of an iteration, which is difficult to manage when it comes to finite conditions.[3]

4.2. **Disjoint Stationary Sequences.** Now we can look at a property that differs significantly from what we have covered so far: that of *disjoint stationary sequences*. These are sequences $\langle \mathcal{S}_\alpha : \alpha \in S \rangle$ where $S \subset \mu^+ \cap \mathrm{cof}(\mu)$ is stationary and μ is regular, $\mathcal{S}_\alpha \subset P_\mu(\alpha)$ is stationary for all $\alpha \in S$, and $\mathcal{S}_\alpha \cap \mathcal{S}_\beta = \emptyset$ for $\alpha \neq \beta$. A stronger property is the existence of a *disjoint club sequence*, which is a disjoint stationary sequence where there is the requirement that the $\mathcal{S}_\alpha$'s are club in $P_\mu(\alpha)$. Disjoint club sequences were originally used by Krueger to prove the consistency of the non-existence of a thin stationary subset of $P_{\omega_1}(\omega_2)$ [14]. Krueger then fleshed out this area in a series of papers exploring the extent to which there

[3] See the Appalachian Set Theory talk given by Neeman in 2016.

can be disjoint stationary sequences on various cardinals as well as related questions about separating variants of internal approachability.

Krueger raised some interesting open questions, for instance asking whether it is consistent for there to be disjoint stationary sequences on successive cardinals. I resolved this in a recent preprint. Krueger had perfected the use of *mixed-support iterations* in this area, and my idea use variations that more explicitly resemble a forcing due to Mitchell:

Theorem 36 (Levine)**.** [36] *It is consistent up to two Mahlo cardinals that there are disjoint stationary sequences on $\aleph_2$ and $\aleph_3$.*

Using similar methods, Jakob and I proved the following, which answered more questions of Krueger:

Theorem 37 (Jakob-Levine)**.** [22] *It is consistent from ω-many Mahlo cardinals that internally club and internally stationary are distinct for $[H(\theta)]^{\aleph_n}$ for all $n < \omega$ and $\theta \geq \aleph_{n+1}$.*

At this point, the main direction is the following, together with its variants:

Question 38 (Krueger)**.** [26] *Is there consistently a disjoint club sequence on $\aleph_3$?*

It is not even known whether there can be a disjoint stationary sequence on a cardinal $\mu > \aleph_2$ with $2^{\aleph_0} < \mu$. Hence we are brought back to the study of the continuum. There are a number of technical complexities that arise in the pursuit of variations of Question 38 that connect this question to other concepts. Many of Krueger's results in this area rely on Gitik's theorem on the costationarity of the ground model [16]. However, an analog of Gitik's theorem that applies to the questions on $\aleph_3$ requires at least an ω_1-Erdős cardinal as a result of Magidor's covering lemma [41]. This is why an investigation into Question 38 is quite likely to tease out more differences between $\aleph_2$ and $\aleph_3$.

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