MTH 451 HW 1

Lodewyk Jansen van Rensburg

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1 TB 1.3

Suppose $R \in \mathbb{C}^{mxm}$ is upper triangular and non-singular/ invertable.
Then R being non-singular $\Rightarrow det(R) = r_{11}r_{22}...r_{mm} \neq 0 \Rightarrow r_{11}r_{22}...r_{mm} \neq 0$.

Firstly, let A denote the inverse of R. That is $I = AR$. Then ,

$$\vec{e_1} = A\vec{r}_1 = \sum_{i=1}^{m}(r_{i1})\vec{a}_i$$

where $r_{i1} \neq 0$ only for $i = 1$, $r_{i1} = 0$ when $i > 1$ because R is upper triangular.
It follows that $\vec{a}_1$ has a non-zero entry in its first component and zero elsewhere in order to produce $\vec{e}_1$. So the first column of A is satisfied.

Secondly, ,

$$\vec{e_2} = A\vec{r}_2 = \sum_{i=1}^{m}(r_{i2})\vec{a}_i$$

where $r_{i2} \neq 0$ when $i = 2$, $r_{i1} = 0$ when $i > 2$ because R is upper triangular.
Then $a_{21} = 0$, and it follows that $\vec{a}_2$ has a non-zero entry in its second component and zero in the $3, 4, ..., m^{th}$ components in order to produce $\vec{e}_2$ using linear combinations of $\vec{a}_1$ and $\vec{a}_2$. So the second column of A is satisfied.

In general,

$$\vec{e_j} = A\vec{r}_j = \sum_{i=1}^{m}(r_{ij})\vec{a}_i$$

where $r_{ij} \neq 0$ when $i = j$, $r_{ij} = 0$ when $i > j$ because R is upper triangular.
For the previous $j - 1$ columns of A, the $j,\ \ j + 1,\ \ ...,\ \ m$ components are zero, therefore it follows that $\vec{a}_j$ has a non-zero entry in its j^{th} component and zero elsewhere in order to produce $\vec{e}_j$ using linear combinations of $\vec{a}_1, \vec{a}_2, ..., \vec{a}_j$. So the j^{th} column of A for $2 < j \leq m$ is satisfied.

In conclusion, since every column of A namely, $\vec{a}_j$ has entries $a_{kj} = 0$ when $k > j$ we have show that A is also upper triangular. $\square$

2 TB 1.4

3 TB 2.1

Q: Show that if a matrix A is both triangular and unitary, then it is diagonal.

From TB 1.3 we know that an invertable upper triangular matrix's inverse is also upper triangular. We start by show that the same is true for invertable lower triangular matrices. I.e that their inverse is a lower triangular matrix.

Part 1

Suppose $A \in \mathbb{C}^{nxn}$ is invertable and lower triangular. Denote R as the inverse of A. I.e RA = I = AR. Then, consider the n^{th} columns of I,

$$\vec{e}_n = R\vec{a}_n = \sum_{i=1}^{n}(a_{i,n})\vec{r}_i$$

Note that $a_{in} \neq 0$ only when $i = n$, because A is lower triangular. Therefore $\vec{r}_n$ must have all zero entries except for the n^{th} component in order to produce $\vec{e}_n$. This shows that the n^{th} column of R is satisfies the requirment for the n^{th} column of a lower triangular matrix.

Now, consider the $n-1^{th}$ columns of I,

$$\vec{e}_{n-1} = R\vec{a}_{n-1} = \sum_{i=1}^{n}(a_{i,n-1})\vec{r}_i$$

Note that $a_{i,n-1} \neq 0$ when $i = n-1$ and , $a_{i,n-1} = 0$ when $1 \leq i < n-1$ because A is lower triangular. Therefore $\vec{r}_{n-1}$ must have non-zero entry in its $n-1^{th}$ component and zero in the $n-2, n-3, ..., 1$ components in order to produce $\vec{e}_{n-1}$ using linear combinations of $\vec{a}_n$ and $\vec{a}_{n-1}$. This shows that the $n-1^{th}$ column of R satisfies the requirement for the $n-1^{th}$ column of a lower triangular matrix.

In general, consider the k^{th} columns of I, $1 < k \leq n$,

$$\vec{e}_k = R\vec{a}_k = \sum_{i=1}^{n}(a_{i,k})\vec{r}_i$$

Note that $a_{i,k} \neq 0$ when $i = k$ and , $a_{i,k} = 0$ when $1 \leq i < k$ because A is lower triangular. Therefore $\vec{r}_k$ must have non-zero entry in its k^{th} component and zero in the $k, k-1, ..., 1$ components in order to produce $\vec{e}_k$ using linear combinations of $\vec{a}_n$, $\vec{a}_{n-1}$, ..., $\vec{a}_k$. This shows that the k^{th} column of R satisfies the requirement for the k^{th} column of a lower triangular matrix.

Note the first column of R can have any entries and R will still be lower triangular. Along with TB 1.3, this shows that if a triangular matrix is invertable, then its inverse have the same triangle type/ shape.

Part 2

Suppose $B \in \mathbb{C}^{nxn}$ is a unitary and triangular. By definition of a unitary $b_{ij} = \overline{b_{ji}}$.

If B is upper triangular, then by TB 1.3 we know B^{-1} is upper triangular. Since B is a unitary we have $B^{-1} = B^* \Rightarrow b_{ij} = \overline{b_{ji}} = 0$ when $i > j$ because B is upper triangular and $\overline{b_{ji}} = 0$ when $j > i$ because B^{-1} is upper triangular. Therefore B is a diagonal matrix.

If B is lower triangular, then by above we know B^{-1} is lower triangular. Since B is a unitary we have $B^{-1} = B^* \Rightarrow b_{ij} = \overline{b_{ji}} = 0$ when $i < j$ because B is lower triangular and $\overline{b_{ji}} = 0$ when $j < i$ because B^{-1} is lower triangular. Therefore B is a diagonal matrix.

Both cases show that if B is a unitary and triangular, then B is a diagonal matrix.

4 TB 2.3 and TB 2.4

See bottom of the page.

5 TB 3.3 a, b, c, and d.

a. Show $\|\vec{x}\|_\infty \leq \|\vec{x}\|_2$ for $\vec{x} \in \mathbb{C}^m$.

Let $x_\alpha = max_{i=1}^m\{x_i\}$, then

$$\begin{aligned} \|\vec{x}\|_\infty &= x_\alpha \\ &\leq \sqrt{|x_1|^2 + |x_2|^2 + ... + |x_\alpha|^2 + ... + |x_m|^2} \\ &= \|\vec{x}\|_2 \end{aligned} \tag{1}$$

Example: When we choose $\vec{x}_k \in \mathbb{C}^m$ such that it has a non zero entry in its k^{th}($x_k \neq 0$) component and zero elsewhere. Then,

$$\|\vec{x}_k\|_\infty = |x_k|$$

and

$$\|\vec{x}_k\|_2 = \sqrt{|x_1|^2 + ... + |x_k|^2 + ... + |x_m|^2} = \sqrt{0 + ... + |x_k|^2 + ... + 0} = |x_k|$$

b. Show $\|\vec{x}\|_2 \leq \sqrt{m}\ \ \|\vec{x}\|_\infty$ for $\vec{x} \in \mathbb{C}^m$.

Let $x_\alpha = max_{i=1}^m\{x_i\}$, then

$$\begin{aligned} \|\vec{x}\|_2 &= x_\alpha \\ &= \sqrt{|x_1|^2 + |x_2|^2 + ... + |x_\alpha|^2 + ... + |x_m|^2} \\ &\leq \sqrt{|x_\alpha|^2 + |x_\alpha|^2 + ... + |x_\alpha|^2 + ... + |x_\alpha|^2} \\ &= \sqrt{m|x_\alpha|^2} \\ &= \sqrt{m}\ \ |x_\alpha| \\ &= \sqrt{m}\ \ \|\vec{x}\|_\infty \end{aligned} \tag{2}$$

Example: When we choose $\vec{y} \in \mathbb{C}^m$ such that it has a single non zero entry in its all its components denoted $\beta \neq 0$. Then,

$$\|\vec{y}\|_2 = \sqrt{\sum_{i=1}^{m} |y_i|^2} = \sqrt{m\beta^2} = \sqrt{m}\ \ |\beta|$$

and

$$\sqrt{m}\ \ \|\vec{y}\|_\infty = \sqrt{m}\ \ max_{i=1}^{m}|y_i| = \sqrt{m}\ \ |\beta|$$

c See bottom of page.

d See bottom of page.

6 Questions

a See end of document.

b See end of document.

c See end if document.

7 Problem

Let $A \in \mathbb{C}^{mxn}$

We show that $\|A\|_\infty = \max_{1\leq j\leq m} \sum_{i=1}^{n} |a_{ji}|$.

Now

$$\|A\|_\infty = \max_{\|\vec{x}\|_\infty} \|A\vec{x}\|_\infty \ \ ,\ \ by\ \ definition\ \ of \|A\|_\infty$$

Let the k^{th} row be the row with the largest sum $1 \leq k \leq m$

Then,

$$\begin{aligned}
\|A\|_\infty &= |(x_1)a_{k1} + (x_2)a_{k2} + ... + (x_n)a_{kn}| \\
&\leq |(x_1)a_{k1}| + |(x_2)a_{k2}| + ... + |(x_n)a_{kn}| \\
&= |x_1||a_{k1}| + |x_2||a_{k2}| + ... + |x_n||a_{kn}| \\
&\leq |a_{k1}| + |a_{k2}| + ... + |a_{kn}| \qquad\qquad , |x_i| \leq 1 \\
&= \max_{1\leq j\leq m} \sum_{i=1}^{n} |a_{ji}|
\end{aligned} \tag{3}$$

Also consider $\vec{y} = [1, 1, ..., 1]^T \in \mathbb{C}^n$, note $\|\vec{y}\|_\infty = 1$.

$$\begin{aligned}
\max_{1\leq j\leq m} \sum_{i=1}^{n} |a_{ji}| &= \|\vec{a}_1 + \vec{a}_2 + ... + \vec{a}_n\|_\infty \\
&= \|A\vec{y}\|_\infty
\end{aligned} \tag{4}$$

Exercises Lec 2.

self adjoint

Q 2.3 Let A ~~be~~ an $\in \mathbb{C}^{m \times m}$ be hermitian.
An eigenvector of A is a non-zero vector $\vec{v} \in \mathbb{C}^{m \times 1}$ such that $A\vec{v} = \lambda\vec{v}$ for some eigenvalue $\lambda \in \mathbb{C}$.

(a) Start

(a) Prove that all eigenvalues of A are real.
* We need to show that $\lambda \in \mathbb{R}$.

Suppose $\vec{v} \in \mathbb{C}^m$ is any eigenvector of A with corresponding eigen value $\lambda \in \mathbb{C}^m$.

(Notation: let $\langle \vec{x}, \vec{y} \rangle = \vec{y}^* \vec{x}$ denote the inner product

Now $\langle A\vec{v}, \vec{v} \rangle = \overline{\langle \vec{v}, A\vec{v} \rangle}$, conjugate symmetry

$\Rightarrow \vec{v}^* A\vec{v} = \overline{(A\vec{v})^* \vec{v}}$

$\Rightarrow \vec{v}^* A\vec{v} = \overline{\vec{v}^* A^* \vec{v}}$, property of adjoint

$\Rightarrow \vec{v}^* A\vec{v} = \overline{\vec{v}^* A\vec{v}}$, since $A = A^*$.

$\Rightarrow \vec{v}^* \lambda\vec{v} = \overline{\vec{v}^* \lambda\vec{v}}$, $\vec{v}$ is an eigenvector of A

$\Rightarrow \langle \lambda\vec{v}, \vec{v} \rangle = \overline{\langle \lambda\vec{v}, \vec{v} \rangle}$

$\Rightarrow \lambda \langle \vec{v}, \vec{v} \rangle = \overline{\lambda \langle \vec{v}, \vec{v} \rangle}$, linearity in the first component

$\Rightarrow \lambda \langle \vec{v}, \vec{v} \rangle = \overline{\lambda} \langle \vec{v}, \vec{v} \rangle$, conjugate symmetry.

$\Rightarrow \lambda \|\vec{v}\|^2 = \overline{\lambda} \|\vec{v}\|^2$

$\vec{v} \neq \vec{0} \Rightarrow \|\vec{v}\|^2 \neq 0.$

Thus $\lambda = \overline{\lambda}$. This only holds when $\lambda \in \mathbb{R}$.

Therefore all eigenvalues of the hermitian matrix A is real.

End.

2.3 b)

* Note that in the definition of a self adjoint matrix, it implicitly says that the matrix is square. $A^* = A$, otherwise transpose won't w.

Hermitian Matrices and their Eigen values
(self adjoint)

Let $A \in \mathbb{C}^{m \times m}$ and let A be self-adjoint (hermitian)

By def $A = A^*$.

In exercise 2.3 [TB] we proved that all eigen values of A must be real.

Main idea for this page. [Here we show why the eigen vectors of A with distinct eigen values are orthogonal.]

Start: Let $\vec{x}, \vec{y}$ be two eigen vectors of A with corresponding eigen values λ_x, λ_y, respectively. let $\lambda_x \neq \lambda_y$. Ie. distinct eigen values.

* We show that $\langle \vec{x}, \vec{y} \rangle = 0$ to show that $\vec{x}$ and $\vec{y}$ are orthogonal.

Now
$$
\begin{aligned}
0 &= \langle A\vec{x}, A\vec{y} \rangle - \langle A\vec{x}, A\vec{y} \rangle \\
&= \langle A\vec{x}, A\vec{y} \rangle - \overline{\langle A\vec{y}, A\vec{x} \rangle} && \text{, conjugate sym} \\
&= (A\vec{y})^* A\vec{x} - \overline{(A\vec{x})^* A\vec{y}} \\
&= \vec{y}^* A^* A\vec{x} - \overline{\vec{x}^* A^* A \vec{y}} && \text{, property of adjoint.} \\
&= \vec{y}^* A A\vec{x} - \overline{\vec{x}^* A A \vec{y}} && \text{, } A \text{ is self adj.} \\
&= \vec{y}^* \lambda_x^2 \vec{x} - \overline{\vec{x}^* \lambda_y^2 \vec{y}} \\
0 &= \langle \lambda_x^2 \vec{x}, \vec{y} \rangle - \overline{\lambda_y^2 \langle \vec{y}, \vec{x} \rangle} \\
0 &= (\lambda_x^2) \langle \vec{x}, \vec{y} \rangle - \overline{\lambda_y}^2 \overline{\langle \vec{y}, \vec{x} \rangle} && \text{property of the conjugate} \\
0 &= (\lambda_x^2) \langle \vec{x}, \vec{y} \rangle - \overline{\lambda_y}^2 \langle \vec{x}, \vec{y} \rangle \\
0 &= (\lambda_x^2 - \overline{\lambda_y}^2)(\langle \vec{x}, \vec{y} \rangle) \\
0 &= (\lambda_x^2 - \lambda_y^2)(\langle \vec{x}, \vec{y} \rangle) && \text{, Since } \lambda_y \in \mathbb{R},\ \overline{\lambda_y} = \lambda_y. \\
0 &= \langle \vec{x}, \vec{y} \rangle && \text{, Since } \lambda_x \neq \lambda_y \Rightarrow \lambda_x^2 - \lambda_y^2 \neq 0
\end{aligned}
$$

This shows that eigen ~~values~~ vectors with distinct eigen values are orth

5c)

Proof. $\|A\|_\infty = \sup_{\substack{\vec{x} \in \mathbb{C}^n \\ \vec{x} \neq \vec{0}}} \dfrac{\|A\vec{x}\|_\infty}{\|\vec{x}\|_\infty}$

$= \sup_{\|\vec{x}\|_\infty = 1} \|A\vec{x}\|_\infty$, note $|x_i| \leq 1$

Let the k^{th} row have the largest sum, then:

$\|A\|_\infty = |x_1 a_{k1} + x_2 a_{k2} + \dots + x_n a_{kn}|$

$= \left| \sum_{i=1}^{n} x_i a_{ki} \right|$, rewriting

$= \sqrt{\left| \sum_{i=1}^{n} x_i a_{ki} \right|^2}$

$\leq \sqrt{\left| \sum_{i=1}^{n} x_i a_{1i} \right|^2 + \dots + \left| \sum_{i=1}^{n} x_i a_{ki} \right|^2 + \dots + \left| \sum_{i=1}^{n} x_i a_{mi} \right|^2}$

$= \|A\vec{x}\|_2$, by definition of $\|\cdot\|_2$

$\leq \|A\|_2 \|\vec{x}\|_2$, by property of $\|\cdot\|_2$ for matrices.

$\leq \|A\|_2 \sqrt{n}$, $\|\vec{x}\|_\infty = 1 \Rightarrow |x_i| \leq 1$ $\forall\, 1 \leq i \leq n \Rightarrow \|x\|_2 \leq \sqrt{n}$.

$= \sqrt{n}\, \|A\|_2$

Therefore $\|A\|_\infty \leq \sqrt{n}\, \|A\|_2$ ∎

Proof 5d)

$$\|A\|_2 = \sup_{\|\vec{x}\|_2=1} \|A\vec{x}\|_2$$

$$= \sup_{\|\vec{x}\|_2=1} \left\| \begin{bmatrix} \sum_{i=1}^{n} x_i a_{1i} \\ \sum_{i=1}^{n} x_i a_{2i} \\ \vdots \\ \sum_{i=1}^{n} x_i a_{mi} \end{bmatrix} \right\|_2$$

let $\vec{x}$ be the vector of max stretching.

Then

$$\|A\|_2 = \left\| \begin{bmatrix} \sum_{i=1}^{n} x_i a_{1i} \\ \vdots \\ \sum_{i=1}^{n} x_i a_{mi} \end{bmatrix} \right\|_2$$

$$= \sqrt{\left|\sum_{i=1}^{n} x_i a_{1i}\right|^2 + \dots + \left|\sum_{i=1}^{n} x_i a_{mi}\right|^2}$$

(let $\left|\sum_{i=1}^{n} x_i a_{ki}\right|$ be the largest).

Then $$\leq \sqrt{m \left|\sum_{i=1}^{n} x_i a_{ki}\right|^2}$$

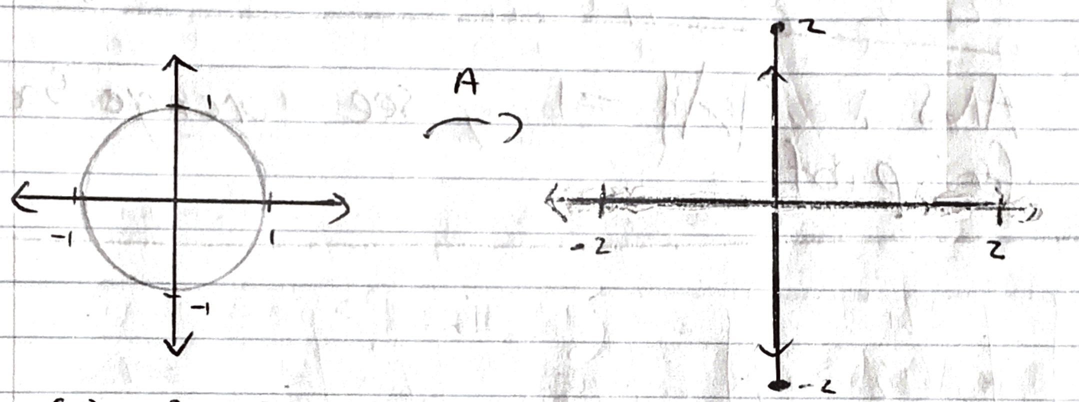
$$= \sqrt{m} \left|\sum_{i=1}^{n} x_i a_{ki}\right|$$

$$= \sqrt{m} \|A\|_\infty$$

I.e. $$\|A\|_2 \leq \sqrt{m} \|A\|_\infty$$ ∎

For each of the following matrices, sketch the image $\{A\vec{x} \mid \vec{x} \in S_2\}$ of the closed unit ball $S_2 \subseteq \mathbb{R}^2$. (Q6)
Give a geometric interpretation of $\|A\|_2$ in each case.

(A) $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, $A: \mathbb{R}^2 \to \mathbb{R}^2$



$\{\vec{x} \in \mathbb{R}^2 \mid \|\vec{x}\|_2 \leq 1\}$ $\qquad$ $\{A\vec{x} \in \mathbb{R}^2 \mid \vec{x} \in S_2\}$

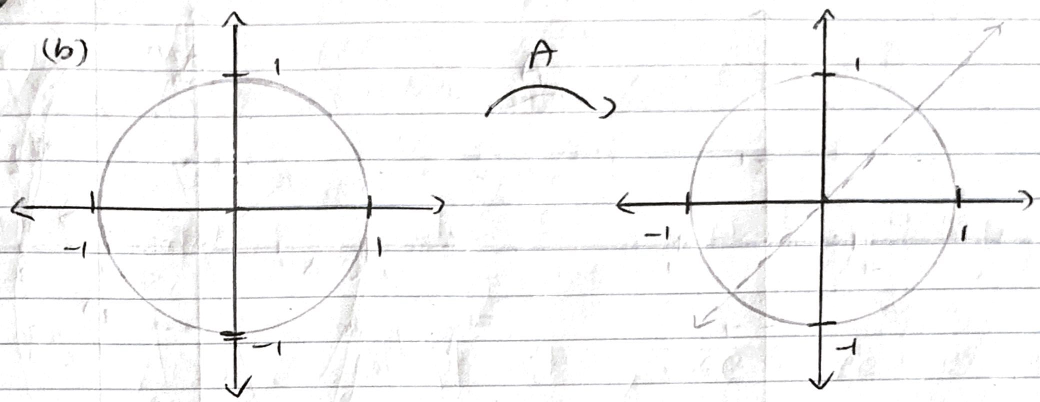
Because the first column of A is a zero vector, we lose ~~any~~ all information about the ~~x~~ ~~co~~ first component of the vector.
The 2^{nd} column $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ streches all the y-components.
Since the largest values in the y component is (1), and (-1), that is where
The vector $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are the ones amplified by A the most.

The transformation A projects and stretches the vectors onto the line $x = 0$.

$\|A\|_2 = 2$.

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $A : \mathbb{R}^2 \to \mathbb{R}^2$

(b)



$S_2 = \{ \vec{x} \in \mathbb{R}^2 \mid \|\vec{x}\|_2 \leq 1 \}$ $\qquad$ $\{ A\vec{x} \mid \vec{x} \in S_2 \}$

Now $A = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{E_1} \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{E_2}$

Then $\forall \vec{x} \in S_2$, $E_2\vec{x}$ is the vector where the sign of the x-component changes. i.e. E_1 is a reflection accross the line $x=0$. Reflecting the sphere around the line $x=0$ leaves the sphere unchanged.

Then $\forall \vec{x} \in \{ E_2\vec{y} \mid \vec{y} \in S_2 \}$, the transformation E_1 swaps the x and y components of $\vec{x}$.

Geometrically, this is a reflection around the line $y = x$. This also leaves the unit sphere unchanged, by symmetry.

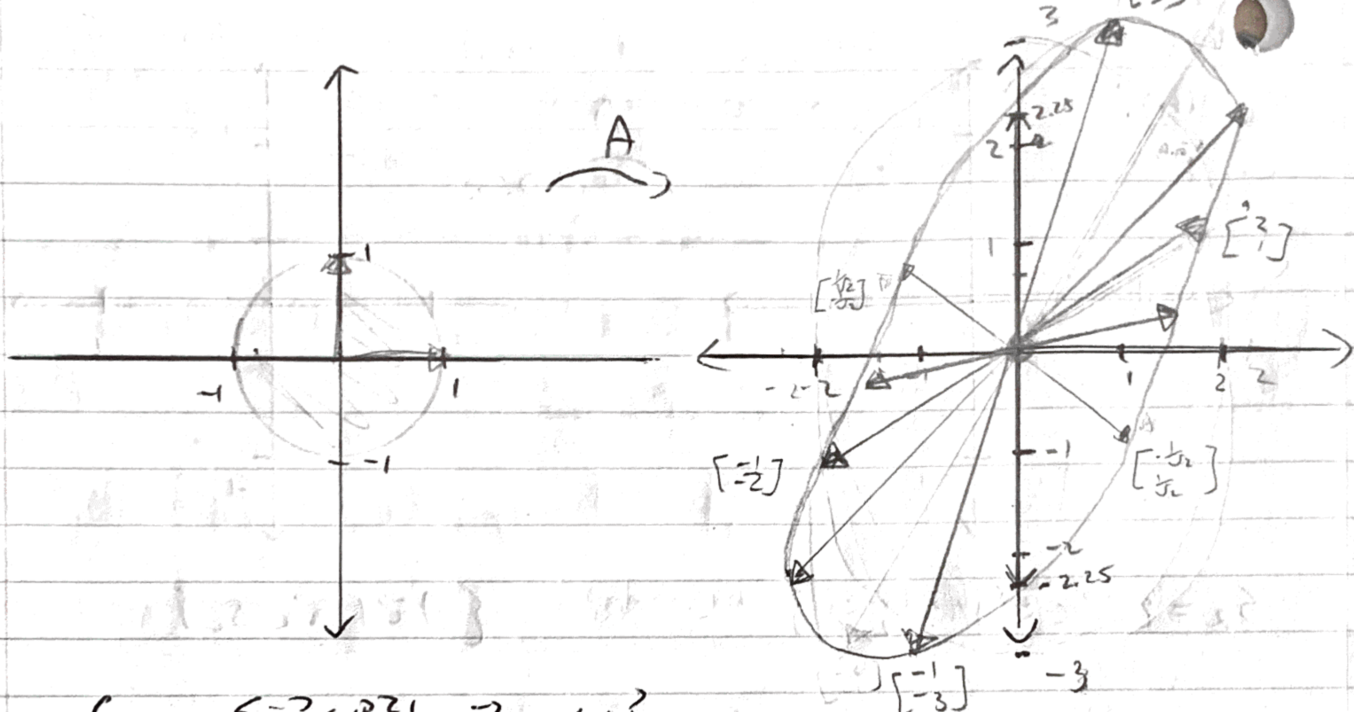
In conclusion A can be broken up into two reflection accross straight lines through the origin.

Therefore $S_2 = \{ A\vec{x} \mid \vec{x} \in S_2 \}$.

$\|A\|_2 = 1$

$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} =$

(c)



$S_2 = \{\vec{x} \in \mathbb{R}^2 \mid \|\vec{x}\|_2 \le 1\}$

$\{A\vec{x} \mid \vec{x} \in S_2\}$

- $A\vec{e_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $A\begin{bmatrix} 1/\sqrt2 \\ 1/\sqrt2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt2} + \frac{1}{\sqrt2} \\ 1/\sqrt2 + \frac{3}{\sqrt2} \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt2} \\ 3/\sqrt2 \end{bmatrix}$
- $A\vec{e_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$
- $A(-\vec{e_1}) = -\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $A\begin{bmatrix} \frac{1}{\sqrt2} \\ \frac{-1}{\sqrt2} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt2} + \frac{-1}{\sqrt2} \\ 1/\sqrt2 + \frac{-3}{\sqrt2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt2} \\ \frac{-2}{\sqrt2} \end{bmatrix}$

- $A\begin{bmatrix} -\sqrt{\frac{3}{4}} \\ \frac{1}{4} \end{bmatrix} \approx \begin{bmatrix} -1.48 \\ -0.12 \end{bmatrix}$

$A\begin{bmatrix} -\frac{2.25}{5} \\ \frac{4.5}{5} \end{bmatrix} \approx \begin{bmatrix} 0 \\ 2.25 \end{bmatrix}$

And $\|A\|_2 =$

Question 6.c)