On the Connection Between Multipliers of The Drury-Averson Space and The Non-commutative Hardy Space via Realizations

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Outline

Vector Valued Reproducing Kernel Hilbert Space (RKHS)

Let $\mathcal{E}$ denote a coefficient Hilbert space, and Ω a set.

Definition

We say a linear subspace $\mathcal{H}_{\mathcal{E}} \subseteq \mathcal{F}(\Omega, \mathcal{E})$ is a vector-valued RKHS to mean:

1. $\mathcal{H}_{\mathcal{E}}$ has an inner product turning it into a Hilbert space, and
2. all point evaluations are bounded. That is for all $w \in \Omega$, the linear map
$$\Phi(w) : \mathcal{H}_{\mathcal{E}} \to \mathcal{E}$$
given by $f \mapsto f(e)$ is bounded.

Reproducing Kernel

We want to apply Riesz representation as in the scalar case. So we proceed as follows: Fix $w \in \Omega$, and $e \in \mathcal{E}$:

1. $f \mapsto \langle f(w), e\rangle_{\mathcal{E}} : \mathcal{H}_{\mathcal{E}} \to \mathbb{C}$ is a bounded.
2. Apply Riesz Representation to obtain a unique $K(\cdot, w)e \in \mathcal{H}_{\mathcal{E}}$ such that
$$\langle f, K(\cdot, w)e\rangle_{\mathcal{H}_{\mathcal{E}}} = \langle f(w), e\rangle_{\mathcal{E}}.$$
3. One observes that $K(z, w) = \Phi(z)\Phi(w)^*$ for all $z, w \in \Omega$ so that we get a linear map
$$K : \Omega \times \Omega \to B(\mathcal{E})$$
called the **reproducing kernel** of $\mathcal{H}_{\mathcal{E}}$.

Reproducing Kernel (Continued)

Recall that we have a notion of positivity for operator-valued reproducing kernels:

Definition

Given a function $K : \Omega \times \Omega \to B(\mathcal{E})$ we say K is **positive** to mean for any finite number $z_1, ..., z_n \in \Omega$ the matrix

$$\begin{bmatrix} K(z_1, z_1) & \dots & K(z_1, z_n) \\ \dots & & \dots \\ K(z_n, z_1) & \dots & K(z_n, z_n) \end{bmatrix}$$

is positive in $M_n(B(\mathcal{E})) \simeq B(\mathcal{E}^n)$.

It follows immediately from the factorization above that the reproducing kernels are positive.

Vector Valued Moore

We can go the opposite direction first. Given a positive map

$$K : \Omega \times \Omega \to B(\mathcal{E})$$

we use the positivity to define an inner product on the span of $K(\cdot, w)e$ ranging over $w \in \Omega$, and $e \in \mathcal{E}$. The details of this result is known as Moore's Theorem.

Theorem (Vector Valued Moore)[7]

If $K : \Omega \times \Omega \to B(\mathcal{E})$ a positive function. Then there exists a unique $\mathcal{E}$-valued RKHS $\mathcal{H}_{\mathcal{E}}(K)$ on Ω with K as its reproducing kernel. Moreover the span of

$$\left\{ K(\cdot, w)e : w \in \Omega, e \in \mathcal{E} \right\}$$

can be identified with a dense subspace in $\mathcal{H}_{\mathcal{E}}(K)$.

Vector Valued Moore (Consequence)

1. As a consequence of Moore's theorem we see that $K : \Omega \times \Omega \to B(\mathcal{E})$ being positive is equivalent to the existence of an auxiliary RKHS $\mathcal{H}_{\mathcal{E}}(K)$ such that we can factor our map

$$K(z, w) = \Phi(z)\Phi(w)^*$$

for some function $\Phi : \Omega \to B(\mathcal{H}_{\mathcal{E}}(K), \mathcal{E})$.
2. Indeed because if K is positive, the function Φ above is exactly bounded point evaluation from the RKHS $\mathcal{H}_{\mathcal{E}}(K)$ obtained by Moore's theorem.

Multipliers

1. Given $F : \Omega \to B(\mathcal{E}, \mathcal{E}_*)$, and $f \in \mathcal{H}_{\mathcal{E}}$ we define $F\ \ f : \Omega \to \mathcal{E}_*$ by $w \mapsto F(w) \circ f(w)$.
2. We say F is a **multiplier** from $\mathcal{H}_{\mathcal{E}} \to \mathcal{H}_{\mathcal{E}_*}$ to mean that
$$F\ \ f \in \mathcal{H}_{\mathcal{E}_*}$$
for all $f \in \mathcal{H}_{\mathcal{E}}$.
3. The above follows from bounded point evaluation, and an application of the Closed Graph theorem that each multiplier induces a bounded operator.
4. We denote $\mathcal{M}(\mathcal{E})$ the **multiplier algebra** endowed with the operator norm.[1]

[1] Only an algebra when $\mathcal{E}$=$\mathcal{E}_*$

Eigenvector Property

1. A very useful fact in the scalar setting is that the kernel functions are eigenvectors for adjoints of multipliers.
2. We have a similar-type result that says
$$M_F^* K_{\mathcal{E}_*}(\cdot, z)e = K_{\mathcal{E}}(\cdot, z)F(z)^* e \tag{1}$$
for all $z \in \Omega$, $e \in \mathcal{E}_*$.
3. The above follows from an inner product calculation along with the reproducing property, and density of kernel functions.
4. For ease of notation, we will drop the subscripts on the kernel function from here on.

Drury Averson

We will focus on a specific RKHS known as the Drury-Averson space.

1. Denote $K(z,w) = \frac{1}{1-\langle z,w\rangle} : \mathbb{B}^d \times \mathbb{B}^d \to \mathbb{C}$.
2. Form an operator valued kernel $K(z,w) \otimes I_{\mathcal{E}} : \mathbb{B}^d \times \mathbb{B}^d \to B(\mathcal{E})$.

The **Drury-Averson** space (or $\mathcal{E}$-valued version) is the RKHS on $\mathbb{B}^d$ induced by the reproducing kernel $K(z,w) \otimes I_{\mathcal{E}}$, and denoted $\mathcal{H}^2_{\mathcal{E}}$.

Characterization of the Schur Class

Theorem (Ball, Vinnikov, Trent 2001 [2])

Let $F \in \mathcal{M}(\mathcal{E}, \mathcal{E}_*)$. The following are equivalent:

1. F is a contractive multiplier.
2. There exists an auxiliary Hilbert space $\mathcal{H}$, and a *unitary* colligation

$$U = \begin{bmatrix} A_1 & B_1 \\ \dots & \dots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{array}{c} \mathcal{H} \\ \oplus \\ \mathcal{E} \end{array} \to \begin{array}{c} \mathcal{H}^{\oplus^d} \\ \oplus \\ \mathcal{E}_* \end{array}$$

that realizes F. Meaning that for all $z \in \mathbb{B}^d$

$$F(z) = D + C(I - \sum_{i=1}^{d} z_i A_i)^{-1}(\sum_{i=1}^{d} z_i B_i) \tag{2}$$

Characterization of the Schur Class [3] continued

Theorem (Ball, Vinnikov, Trent 2001 [2])

3. The function K_F given by

$$K_F(z,w) = K(z,w) \otimes I_{\mathcal{E}_*} - F(z)(K(z,w) \otimes I_{\mathcal{E}})F(w)^*$$

defines a positive kernel $K_F : \mathbb{B}^d \times \mathbb{B}^d \to B(\mathcal{E}_*)$. That is there exist an auxillary Hilbert space $\mathcal{H}$, and function $H : \mathbb{B}^d \to B(\mathcal{H}, \mathcal{E}_*)$ such that

$$K_F(z,w) = H(z)H(w)^*$$

4. We can obtain a contractive colligation that realizes F.

2

[2] Write down theorem on board

[3] The space of contractive multipliers is referred to as the Schur Class

Proof of Schur Class Characterization

Proof: (1 $\iff$ 3)

The first equivalence follows from the "eigenvector-type" property mentioned earlier. Indeed suppose $||F|| \leq 1$, and let $x_1, ..., x_n \in \Omega$, and $e_1, ..., e_n \in \mathcal{E}_*$.
From contractivity we get

$$\left|\left|\sum_i M_F^* K(\cdot, z_i)e_i\right|\right|^2_{\mathcal{H}_{\mathcal{E}}} \leq \left|\left|\sum_i K(\cdot, z_i)e_i\right|\right|^2_{\mathcal{H}_{\mathcal{E}_*}}.$$

Since $M_F^* K(\cdot, z_i)e_i = K(\cdot, z_i)F(z_i)^* e_i$ we have

$$\left|\left|\sum_i K(\cdot, z_i)F(z_i)^* e_i\right|\right|^2_{\mathcal{H}_{\mathcal{E}}} \leq \left|\left|\sum_i K(\cdot, z_i)e_i\right|\right|^2_{\mathcal{H}_{\mathcal{E}_*}}.$$

Proof of Schur Class Characterization

Expanding out the inner product we obtain

$$\sum_{i,j} \langle (K(z_i, z_j) - F(z_i)K(z_i, z_j)F(z_j)^*))e_j, e_i\rangle_{\mathcal{E}_*} \geq 0.$$

As required to show positivity of

$$K(z, w) - F(z)K(z, w)F(w)^* = \frac{I - F(z)F(w)^*}{1 - \langle z, w\rangle}.$$

For the converse, we can reverse the calculation done above, and since the span of kernel functions is dense in $\mathcal{H}_{\mathcal{E}_*}$ it follows that F^* is contractive, and hence F is contractive.

Lurking Isometry Step

Proof: (3 $\implies$ 2)

Since K_F is positive, we can apply Moore's theorem to obtain a Hilbert space $\mathcal{H}_{\mathcal{E}_*}(K_F) = \mathcal{H}$, and a function $H : \mathbb{B}^d \to B(\mathcal{H}_{\mathcal{E}_*}, \mathcal{E}_*)$ such that for

$$\frac{I_{\mathcal{E}_*} - F(z)F(w)^*}{1 - \langle z, w \rangle} = H(z)H(w)^* \text{ for all } z, w \in \mathbb{B}^d.$$

Reorganize the equation, and rewrite the inner product in terms of rows and columns operators to obtain

$$I_{\mathcal{E}_*} + \left(\begin{bmatrix} \overline{z_1} \\ \dots \\ \overline{z_d} \end{bmatrix} H(z)^* \right)^* \begin{bmatrix} \overline{w_1} \\ \dots \\ \overline{w_d} \end{bmatrix} H(w)^* = H(z)H(w)^* + F(z)F(w)^* \tag{3}$$

Proof of Schur Class Characterization

The equation (3) above is what will allow us to well defined linear map acting isometrically on a subspace of $\mathcal{H}^{\oplus^d} \oplus \mathcal{E}_*$. Indeed define

$$\mathcal{D}_0 := \operatorname{span}\left\{ \begin{bmatrix} \overline{w_1} H(w)^* e_* \\ \cdots \\ \overline{w_d} H(w)^* e_* \\ e_* \end{bmatrix} : w \in \mathbb{B}^d, e_* \in \mathcal{E}_* \right\} \subseteq \mathcal{H}^{\oplus^d} \oplus \mathcal{E}_*$$

and define V_0^* on $\mathcal{D}_0$ by the linear map that sends

$$\begin{bmatrix} \overline{w_1} H(w)^* e_* \\ \cdots \\ \overline{w_d} H(w)^* e_* \\ e_* \end{bmatrix} \mapsto \begin{bmatrix} H(w)^* e_* \\ F(w)^* e_* \end{bmatrix} \subseteq \mathcal{H} \oplus \mathcal{E}.$$

Proof of Schur Class Characterization

Using (3) in the following computation we obtain

$$\left\| \begin{bmatrix} H(w)^* e_* \\ F(w)^* e_* \end{bmatrix} \right\|^2 = \langle (H(w)H(w)^* + F(w)F(w)^*) e_*, e_* \rangle_{\mathcal{E}_*}$$

$$= \langle e_*, e_* \rangle_{\mathcal{E}_*} + \langle \begin{bmatrix} \overline{w_1} H(w)^* e_* \\ \cdots \\ \overline{w_d} H(w)^* e_* \end{bmatrix}, \begin{bmatrix} \overline{w_1} H(w)^* e_* \\ \cdots \\ \overline{w_d} H(w)^* e_* \end{bmatrix} \rangle_{\mathcal{H}^{\oplus d}}$$

$$= \left\| \begin{bmatrix} \overline{w_1} H(w)^* e_* \\ \cdots \\ \overline{w_d} H(w)^* e_* \\ e_* \end{bmatrix} \right\|^2 \tag{4}$$

Proof of Schur Class Characterization

1. Now V_0^* extends uniquely to an isometry on the closure of $\mathcal{D}_0$ in $\mathcal{H}^{\oplus^d} \oplus \mathcal{E}_*$.
2. Observe that for any isometric extension W of V_0^* a quick calculation shows

$$W(\overline{\mathcal{D}_0}^{\perp}) \subseteq V_0^*(\overline{\mathcal{D}_0})^{\perp}.$$

3. This means the one obstacle in extending to a unitary is one of dimension. That is if $\dim(\overline{\mathcal{D}_0}^{\perp}) > \dim(V_0^*(\overline{\mathcal{D}_0})^{\perp})$.
4. We can resolve the problem by direct summing on a Hilbert space to the co-domain such that the dimension match.

Proof of Schur Class Characterization

1. That is we extend V_0^* to a unitary

$$V^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \tilde{\mathcal{H}}^{\oplus^d} \oplus \mathcal{E}_* \to \tilde{\mathcal{H}} \oplus \mathcal{E}$$

where $\mathcal{H}$ can be identified as a subspace of the Hilbert space $\tilde{\mathcal{H}}$.

2. Next will will use V_0^*, and how it acts on $\mathcal{D}_0$ to show for all $z \in \mathbb{B}^d$

$$F(z)^* = D^* + (\sum_{i=1}^{d} B_i^* z_i^*)(I_{\tilde{\mathcal{H}}} - \sum_{i=1}^{d} A_i^* z_i^*)^{-1} C^*$$

3. But first we outline an argument on why the inverse exists that we will return to later.

Proof of Schur Class Characterization

1. Fix take $z \in \mathbb{B}_d$, and define
$$Z(z) = \begin{bmatrix} z_1 I_{\tilde{\mathcal{H}}} & \dots & z_d I_{\tilde{\mathcal{H}}} \end{bmatrix} : \tilde{\mathcal{H}}^{\oplus^d} \to \tilde{\mathcal{H}}.$$
One sees that $||Z(z)||^2 = \sum |z_i|^2 < 1$.
2. Then since A and $Z(z)$ just operators between Banach spaces we get
$$||Z(z)A|| \leq ||Z(z)|| \;\; ||A|| < 1.$$
3. By standard C^*-theory we know $(I_{\tilde{\mathcal{H}}} - Z(z)A)^{-1}$ exist in $B(\tilde{\mathcal{H}})$ and is given by norm limit geometric series
$$(I_{\tilde{\mathcal{H}}} - Z(z)A)^{-1} = \sum_{n=0}^{\infty}(Z(z)A)^n = \sum_{n=0}^{\infty}(\sum_{i=1}^{d} z_i A_i)^n$$

Proof of Schur Class Characterization

3. Since V^* is an extension of V_0^* we obtain the following system of equations: Fix $w \in \mathbb{B}^d$, $e_* \in \mathcal{E}_*$

$$\begin{bmatrix} A_1^* & \dots & A_d^* & C^* \\ B_1^* & \dots & B_d^* & D^* \end{bmatrix} \begin{bmatrix} \overline{w_1} H(w)^* e_* \\ \dots \\ \overline{w_d} H(w)^* e_* \\ e_* \end{bmatrix} = \begin{bmatrix} H(w)^* e_* \\ F(w)^* e_* \end{bmatrix}$$

4. Which turns into

$$\begin{aligned} (\sum_{i=1}^{d} A_i^* \overline{z_i}) H(w)^* e_* + C^* e_* &= H(z)^* e_* \\ (\sum_{i=1}^{d} B_i^* \overline{z_i}) H(w)^* e_* + D^* e_* &= F(z)^* e_* \end{aligned} \tag{5}$$

Proof of Schur Class Characterization

5. Solve for $H(w)^* e_*$ in the first equation to obtain

$$H(w)^* e_* = (I_{\tilde{\mathcal{H}}} - \sum_{i=1}^{d} A_i^* \overline{z_i})^{-1} C^* e_*$$

6. Substitute $H(w)^* e_*$ into the second equation to obtain

$$F(w)^* e_* = D^* e_* + (\sum_{i=1}^{d} B_i^* \overline{w_i})(I_{\tilde{\mathcal{H}}} - \sum_{i=1}^{d} A_i^* \overline{w_i})^{-1} C^* e_*.$$

7. Since this hold for all e_* we have equality in $B(\mathcal{E}_*)$, and lastly take adjoins to obtain

$$F(w) = D + C(I_{\tilde{\mathcal{H}}} - \sum_{i=1}^{d} w_i A_i)^{-1} (\sum_{i=1}^{d} w_i B_i)$$

as required.

Proof of Schur Class Characterization

(2 $\implies$ 1)
Suppose that we have a unitary colligation that realizes our multiplier F. Expanding $UU^* = I$ we obtain:

$$\begin{bmatrix} AA^* + BB^* & AC^* + BD^* \\ CA^* + DB^* & CC^* + DD^* \end{bmatrix} = \begin{bmatrix} I_{\tilde{\mathcal{H}}} & 0 \\ 0 & I_{\mathcal{E}_*} \end{bmatrix} \tag{6}$$

This in turns gives us

1. $I_{\tilde{\mathcal{H}}} - AA^* = BB^*$
2. $I_{\mathcal{E}_*} - DD^* = CC^*$
3. $-DB^* = CA^*$
4. $-BD^* = AC^*$

Proof of Schur Class Characterization

Fix $z \in \mathbb{B}^d$, and we will show $I_{\mathcal{E}_*} - F(z)^* F(w) \geq 0$ [4].
Indeed, expand $I_{\mathcal{E}_*} - F(z)^* F(z)$, use the resolvent identity, and the four inequalities above to obtain

$$\begin{aligned} & I_{\mathcal{E}_*} - F(z)F(w)^* \\ & = C(I_{\tilde{\mathcal{H}}} - \sum z_i A_i)^{-1}(I - \langle z, w\rangle)(I_{\tilde{\mathcal{H}}} - \sum A_i^* w_i^*)^{-1} C^* \\ & = (I - \langle z, w\rangle) H(z) H(w)^* \\ & \geq 0 \end{aligned} \tag{7}$$

where $H(z) = C(I_{\tilde{\mathcal{H}}} - \sum z_i A_i)^{-1}$. Which concludes our proof that $||F|| \leq 1$. $\square$

[4] Showing $||F(z)|| \leq 1$ not sufficient for positive kernel.

Robustness of Transfer Functions

1. A key step in the proof was the invertibility of

$$I_H - \sum_{i=1}^{d} z_i A_i = I_{\tilde{\mathcal{H}}} - Z(z)A.$$

2. All we needed was that:
   - A is a **column contraction**
   - $Z(z) = \begin{bmatrix} z_i I_{\tilde{\mathcal{H}}} & \dots & z_d I_{\tilde{\mathcal{H}}} \end{bmatrix}$ a **strict row contraction**
3. This means that for any strict row-contraction $Z = \begin{bmatrix} Z_1 & \dots & Z_d \end{bmatrix}$ where $Z_i \in \mathcal{A}$ for some operator algebra $\mathcal{A}$, we have invertability of

$$I - ZA = I_{\mathcal{A}} \otimes I_{\tilde{\mathcal{H}}} - \sum_{i=1}^{d} Z_i \otimes A_i$$

Robustness of Transfer Functions

1. For example suppose $A = \begin{bmatrix} A_1 \\ \dots \\ A_d \end{bmatrix}$ is a column contraction as in the proof.
2. Let $X = \begin{bmatrix} X_1 & \dots X_d \end{bmatrix}$ be a d-tuple of $n \times n$ matrices such $||XX^*|| = ||X_1X_1^* + \dots + X_dX_d^*|| < 1$. That is assume X is a strict row contraction.
3. By a similar argument used at level 1 (i.e. $\mathbb{B}^d$), we can show that

$$I_n \otimes I_{\tilde{\mathcal{H}}} - \sum_{i=1}^{d} X_i \otimes A_i$$

is invertable in $M_n(B(\tilde{\mathcal{H}}))$, where $\otimes$ denotes the Kronecker product.

Non-Commutative Setting

The Non-Commutative Setting

Free Monoid on d-generators

1. Fix an integer $d \geq 1$. A **word** of length n is any finite string of letters $w = w_1 ... w_n$ where $w_i \in \{1, 2..., d\}$.
2. Let $\mathbb{F}_d$ denote the **free monoid on d generators**, where elements are words, the operation is the concatenation of words, and the neutral element is the empty word $\emptyset$.
3. We have map from $\mathbb{F}_d$ to itself namely **transposition** where $w^T = w_n w_{n-1} ... w_1$.
4. Given a non-commutative in-determinant $z = (z_1, ..., z_d)$, we write
$$z^w = z_{w_1} z_{w_2} ... z_{w_n}$$
for example if $d \geq 4$, and $w = 11421$ we have
$$z^w = z_1^2 z_4 z_2 z_1.$$

Formal Power Series

1. We denote the set of all formal power series with coefficients in $\mathcal{E}$ by

$$\mathcal{E}\langle\langle z\rangle\rangle := \left\{ \sum_{\alpha\in\mathbb{F}_d} f_\alpha z^\alpha : f_\alpha \in \mathcal{E} \right\}.$$

2. $\mathcal{E}\langle z\rangle$ denotes all formal power series with finite support (i.e. polynomials).
3. Given another non-commuting indeterminate $w = (w_1, \dots, w_d)$ we denote $\mathcal{E}\langle\langle z, w\rangle\rangle$ the formal power series in z and w.

Non-Commutative Formal RKHS (NFRKHS)

Definition

Let $\mathcal{E}$ be a Hilbert space, and $z = (z_1, ..., z_d)$ non-commuting indeterminants. A linear subspace

$$\mathcal{H}_{\mathcal{E},nc} \subseteq \mathcal{E}\langle\langle z \rangle\rangle$$

is called a **Non-commutative Formal RKHS (NFRKHS)** when:

1. $\mathcal{H}_{\mathcal{E},nc}$ comes equipped with an inner product which turns it into a Hilbert space.
2. For each $v \in \mathbb{F}_d$, the map $\Phi_v : \mathcal{H}_{\mathcal{E},nc} \to \mathcal{E}$ given by

$$\sum_{\alpha \in \mathbb{F}_d} f_\alpha z^\alpha \mapsto f_v$$

is bounded.

How Does Reproducing Kernels Arise?

1. Since coefficients uniquely determine the power-series we obtain a standard vector valued RKHS by viewing the coefficients as functions $(f_\alpha) : \mathbb{F}_d \to \mathcal{E}$.
2. Obtain a vector valued reproducing kernel
$$(\alpha, \beta) \mapsto K_{\alpha,\beta} : \mathbb{F}_d \times \mathbb{F}_d \to B(\mathcal{E}).$$
3. This induces a formal power series
$$K(z, w) = \sum_{\alpha,\beta \in \mathbb{F}_d} K_{\alpha,\beta} z^\alpha w^{\beta^T} \in B(\mathcal{E})\langle\langle z, w \rangle\rangle$$
which is positive in a sense, satisfy a reproducing property (shown on next slide).

NF Reproducing Kernel

For $\mathcal{H} = \mathcal{H}_{\mathcal{E},nc}$

1. For fixed $\alpha \in \mathbb{F}_d$, and $e \in \mathcal{E}$ we denote
$$K_\alpha(z)e := \sum_\beta K_{\alpha,\beta} e \ z^\beta \in \mathcal{H}$$
2. We denote
$$K(\cdot, w)e := \sum_\alpha K_\alpha(z)e \ w^{\alpha^T} \in \mathcal{H}\langle\langle w \rangle\rangle$$
3. And we have a reproducing property
$$\langle f, K(\cdot, w)e\rangle_{\mathcal{H}\times\mathcal{H}\langle\langle w\rangle\rangle} = \langle f(w), e\rangle_{\mathcal{E}\langle\langle w\rangle\rangle\times\mathcal{E}} \tag{8}$$
which holds for all $f \in \mathcal{H}$, $e \in \mathcal{E}$. [5]

[5] Write out definition on black board

NF Reproducing Kernels

Definition

We say the formal power series

$$K(z,w) = \sum_{\alpha,\beta\in\mathbb{F}_d} K_{\alpha,\beta} z^{\alpha} w^{\beta^T} \in B(\mathcal{E})\langle\langle z,w\rangle\rangle$$

is a **Non-commutative Formal reproducing kernel** for $\mathcal{H}_{\mathcal{E},nc}$ when:

1. $K_{\alpha}(z)e \in \mathcal{H}_{\mathcal{E},nc}$ for all $\alpha \in \mathbb{F}_d$ and $e \in \mathcal{E}$.
2. $K(z,w)$ satisfy the reproducing property in (8).

It can be shown that each NF reproducing kernel uniquely determines a NF RKHS just (see [3] for details).

NF Multipliers

We also have multipliers in this formal setting.
A formal power series

$$F = \sum_{\alpha} F_{\alpha} z^{\alpha} \in B(\mathcal{E}, \mathcal{E}_*)\langle\langle z \rangle\rangle$$

can act on elements in $\mathcal{E}\langle\langle z \rangle\rangle$ via a Cauchy product. That is for $f = \sum_{\alpha} f_{\alpha} z^{\alpha} \in \mathcal{E}\langle\langle z \rangle\rangle$ we define

$$F \; f := \sum_{\alpha} \Big(\sum_{\alpha = \beta\theta} F_{\beta} f_{\theta} \Big) z^{\alpha} \in \mathcal{E}_*\langle\langle z \rangle\rangle.$$

NF Multipliers

1. Given two NF RKHS $\mathcal{H}_{\mathcal{E},nc}$, and $\mathcal{H}_{\mathcal{E}_*,nc}$ we say $F \in B(\mathcal{E}, \mathcal{E}_*)\langle\langle z \rangle\rangle$ is a **left-multiplier** from $\mathcal{H}_{\mathcal{E},nc} \to \mathcal{H}_{\mathcal{E}_*,nc}$ to mean that
$$F \ f \in \mathcal{H}_{\mathcal{E}_*,nc}$$
for all $f \in \mathcal{H}_{\mathcal{E},nc}$.
2. Again by an application of the Closed Graph theorem, and continuity of evaluation, one observe that each multipliers induces a bounded operator.
3. When $\mathcal{E} = \mathcal{E}_*$ we denote $\mathcal{M}_{nc}(\mathcal{E})$ the **multiplier algebra** equipped with the operator norm.

NC-Hardy Space

Definition

Let $\mathcal{E}$ be a Hilbert space, and $z = (z_1, ..., z_d)$ non-commuting in-determinants. Define

$$\mathcal{H}^2_{nc,\mathcal{E}} := \left\{ f = \sum_{\alpha \in \mathbb{F}_d} f_\alpha z^\alpha : ||f||^2 := \sum_{\alpha \in \mathbb{F}_d} ||f_\alpha||^2_{\mathcal{E}} < \infty \right\}.$$

1. We equip $\mathcal{H}^2_{nc,\mathcal{E}}$ with the ℓ_2-inner product induced by $\mathcal{E}$:

$$\langle \sum_{\alpha \in \mathbb{F}_d} f_\alpha z^\alpha, \sum_{\beta \in \mathbb{F}_d} g_\beta z^\beta \rangle := \sum_{\alpha \in \mathbb{F}_d} \langle f_\alpha, g_\alpha \rangle_{\mathcal{E}}$$

   and it follows immediately that evaluation functions are bounded turning $\mathcal{H}^2_{nc,\mathcal{E}}$ into a NFRKHS.
2. When $\mathcal{E} = \mathbb{C}$ we refer to $\mathcal{H}^2_{nc}$ as the **"NC-Hardy space"**.

NF Reproducing Kernel

What is the NF-reproducing kernel of $\mathcal{H}^2_{\mathcal{E},nc}$?

1. Consider $K_{\alpha,\beta} = \delta_{\alpha,\beta} \otimes I_{\mathcal{E}} : \mathbb{F}_d \times \mathbb{F}_d \to B(\mathcal{E})$.
2. One sees that $K_\alpha(z)e = ez^\alpha \in \mathcal{H}^2_{\mathcal{E},nc}$, and checks that

$$K_{nc}(z, w) := \sum_{\alpha \in \mathbb{F}_d} z^\alpha w^{\alpha^T}$$

satisfy the reproducing property in (8). Hence we have the NF-reproducing kernel for $\mathcal{H}^2_{\mathcal{E},nc}$.

Characterization of the Non Commutative Schur Class

Theorem (Ball,Vinnikov - 2003,2005 [3],[1])

Let $F \in \mathcal{M}_{NC}(\mathcal{E}, \mathcal{E}_*)$. The following are equivalent:

1. F is a contractive multiplier.
2. There exists an auxiliary Hilbert space $\mathcal{H}$, and a *unitary* colligation

$$U = \begin{bmatrix} A_1 & B_1 \\ \dots & \dots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{matrix} \mathcal{H} \\ \oplus \\ \mathcal{E} \end{matrix} \to \begin{matrix} \mathcal{H}^{\oplus^d} \\ \oplus \\ \mathcal{E}_* \end{matrix}$$

where F can be realises as a formal power series

$$F(z) = D + \sum_{i=1}^{d} (\sum_{\alpha \in \mathbb{F}} CA^{\alpha} B_i z^{\alpha}) z_j = D + C(I - Z(z)A)^{-1} Z(z) B \quad (9)$$

Characterization of the Schur Class
continued

Theorem (Ball,Vinnikov - 2003,2005 [3],[1])

3. The formal power series $K_F \in B(\mathcal{E}_*)\langle\langle z, w\rangle\rangle$ given by

$$K_F(z,w) = K_{nc}(z,w) - F(z)K_{nc}(z,w)F(w)^*$$

defines an NF reproducing kernel.

4. The colligation U that realizes F can be chosen contractive.

For $F(z) \in B(\mathcal{E}, \mathcal{E}_*)\langle\langle z\rangle\rangle$ we define $F(z)^* := \sum F^*_\alpha z^{\alpha^T}$, and the product is defined by Cauchy products of formal power series.

Intro to NC-Function Theory

1. We define the row-ball

$$\mathbb{B}^d_{row} := \left\{ X = \begin{bmatrix} X_1 & \dots & X_d \end{bmatrix} \in \bigsqcup_{n=1}^{\infty} M_n(\mathbb{C})^d : ||XX^*|| < 1 \right\}$$

2. We have two operations on $\mathbb{B}^d_{row}$. Given $X, Y \in \mathbb{B}^d_{row}$ at level n, and m respectively, and single invertable matrix $S \in M_n(\mathbb{C})$:

$$\begin{aligned} X \oplus Y &:= \begin{bmatrix} X_1 \oplus Y_1 & \dots & X_d \oplus Y_d \end{bmatrix} \\ S^{-1} X S &:= \begin{bmatrix} S^{-1} X_1 S & \dots & S^{-1} X_d S \end{bmatrix} \end{aligned} \tag{10}$$

3. We say f is an **nc-function** on $\mathbb{B}^d_{row}$ to mean f is graded, preserves direct sums, and respects similarities.

Intro To NC-Function Theory

4. We denote $\mathbb{H}^\infty_{row}$ all uniformly bounded nc-funtions on the the row-ball. That is nc functions f on the row ball such that

$$||f||_\infty := \sup_{Z \in \mathbb{B}^d_{row}} ||f(X)|| < \infty$$

where the norm on the right is taken in $M_n(\mathbb{C})$ when X is at level n.

5. For example all co-ordinate function $f_i(Z) = Z_i$ are in $\mathbb{H}^\infty_{row}$, and polynomials are in $\mathbb{H}^\infty_{row}$.
6. It a well know result that for nc-functions: Locally bounded at each level implies analytic at each level. Hence it follows that every function in $\mathbb{H}^\infty_{row}$ is analytic at every level.

NC-Hardy Space As a Space of NC Functions

1. We can view elements of the NC-Hardy space as nc-functions on the row ball.
2. Given a formal power series $\sum_{\alpha \in \mathbb{F}_d} c_\alpha z^\alpha \in \mathcal{H}^2_{nc}$ one can show that for each fixed $X \in \mathbb{B}^d_{row}$
$$\sum_{n=0}^{\infty} \sum_{|\alpha|=n} c_\alpha X^\alpha$$
is norm convergent in $M_n(\mathbb{C})$, where X is at level n.
3. Define $F(X) = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} c_\alpha X^\alpha$, and one uses the fact that polynomials are nc functions to show F is an nc-function.

NC-functions and Left Multipliers

1. The connection between $\mathbb{H}^{\infty}_{row}$ and Left-Multipliers of H^2_{nc}, was first seen by Arias and Popescu working over the operatorial closed unit ball, and independently by Davidson and Pitts in the language of operator algebras.
2. But recently Salomon, Shalit, Shamovich ([8]) formulated the result in the nc-function theory language over the row-ball.

Theorem Salomon, Shalit, Shamovich (2018) [8]

Let Φ be a nc-function on $\mathbb{B}^d_{row}$.
Then Φ is a left-multiplier of $\mathcal{H}^2_{nc}$ if, and only if $\Phi \in \mathbb{H}^{\infty}_{nc}$.

Conclusion

Given a contractive multiplier F of the Drury-Averson space

↓

There exists a transfer function Φ that realizes F

↓

Φ is a nc-function bounded by 1 on the row ball

↓

Φ [6]is a left multiplier of the NC-Hardy Space and agree with F on level 1

Given a contractive multiplier F of the NC-Hardy space

↓

There exists a transfer function Φ that realizes F

↓

Φ restricted to level one is a multiplier of the Drury-Averson space

[1]Although Φ is not unique, we can chose Φ to have the same norm as F.

Some Applications

1. NC-Inner Outer Factorization (Jury, Martin, Shamovich [6])
2. Characterization of the Extreme points of the multiplier algebra of the Drury-Averson space.
   (Jury, Martin, Hartz [5], [4])

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