Two-Player Tie-Sharing Games: All Pay Auction Games with Tie-Sharing?

Filip Berntsson

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Abstract

This document explores a specific class of games termed *two-player tie-sharing games*. These involve two players competing over a shared reward by selecting actions from finite sets, with the reward divided equally in the event of a tie. Using systematic numerical simulations and iterative analysis, this work classifies these games based on their reduced action spaces under Iterated Elimination of Strictly Dominated Strategies (IESDS) and the properties of their pure Nash equilibria. The study presents conjectures linking game parameters to strategic outcomes, offering insights into the structural dynamics of tie-sharing games.

This document is not a scientific paper but a personal exploration of *two-player tie-sharing games.* The conjectures are informal observations born from curiosity and experimentation rather than definitive contributions to the field of game theory. While exploratory and unproven, these findings served as an engaging exercise in understanding the structural dynamics of simplified game-theoretic models.

The concept I termed "Two-Player Tie-Sharing Game" is likely known by a different name in existing literature.

Foreword, Recognitions, and A Retrospective Literature Study

This text originates from a project undertaken during my first course in game theory. As such, the concepts and results presented here may be deemed uninteresting, redundant, or incorrect. Any strengths in this work are entirely attributable to the teachers: Claes Andersson, Juan Viguera Diez, and Sofia Cvetkovic Destouni—the latter of whom supervised the project that sparked this exploration. Conversely, all errors, missteps, and oversights are mine alone, reflecting my limited experience and understanding.

The first section reiterates established facts that serve as a foundation for the later sections. This effort arose from a desire to gain clarity and an emotional reaction to the inconsistent notation prevalent in the field. Readers familiar with foundational concepts in game theory may wish to skip the first section. The later sections introduce a class of games forming the base case of a larger simulation project undertaken during the course. My role in this project was to address these base cases, and I may have gone further than necessary.

In hindsight, I wish I had conducted a thorough literature review before embarking on the simulations and pattern recognition that underpin this text. Instead, I now attempt to retroactively situate my work within the broader academic context. My exploration was driven by curiosity and a hands-on approach, likely revisiting ideas already well-established in the field.

Things I Might Have Borrowed (Without Knowing It)

In this text, I introduce a class of games I have termed *two-player tie-sharing games.* These involve two players competing over a shared reward by allocating efforts (or resources) from their finite action sets, with payoffs depending on the comparison of their chosen actions and the reward split equally in the event of a tie.

Several frameworks seem conceptually adjacent to this idea, including:

1. **Colonel Blotto Games:** My games bear superficial resemblance to Colonel Blotto games [Borel(1953)], where players allocate resources across multiple battlefields. However, I focus on a single "battlefield" with a tie-splitting rule, making this a simplified variation.
2. **Bertrand Competition:** The games might also resemble Bertrand competition [Bertrand(1883)], where firms compete by setting prices. My version involves discrete strategies and explicit tie-sharing, more akin to a toy model than a formal extension.
3. **All-Pay Auctions:** Similarities exist with all-pay auctions [Hillman and Riley(1989), Riley and Samuelson(1981)], where participants expend resources regardless of the outcome. However, my games use discrete actions and explicit tie-splitting mechanisms, limiting their scope.

Recent works, such as [Olszewski and Siegel(2023)] and [Carmona and Podczeck(2018)] address games with ties at a much more advanced level. My games share thematic elements with these frameworks but lack the theoretical depth and generality of their contributions.

What I'm Trying to Say

This text is not a scientific paper but a personal exploration of tie-sharing games. While I do not expect it to contribute meaningfully to game theory, the process has been an invaluable learning experience. If the work has any value, it may lie in its potential as a pedagogical example of how tie-splitting mechanisms and discrete strategies interact in small, finite games. My hope is that, even if my conjectures are disproven or redundant, they may spark further thought in someone with a clearer understanding of the field.

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1 Background

This section provides an overview of the established concepts and foundational principles required to understand the ideas presented in the later sections. The ambition is to ensure that readers, regardless of their familiarity with game theory, have the necessary context to follow the discussion. While some of these concepts may be well-known to those experienced in the field, they are included here for clarity and to maintain the self-contained nature of this text.

1.1 Normal-Form Game

The concept of a *Normal-Form Game* was formally introduced by John von Neumann and Oskar Morgenstern in their foundational work *Theory of Games and Economic Behavior* [Von Neumann and Morgenstern(1947)]. Normal-Form Games serve as a central framework for analyzing strategic interactions between decision-makers, providing a mathematical structure for studying how players' choices influence outcomes.

Definition 1.1.1 *(Normal-Form Game)***:** A *Normal-Form Game*, $G = (N, A, u)$, consists of:

- A finite set of *players*, indexed by $N = \{1, 2, \ldots, n\}$.
- An *action space*, $\mathcal{A} = A_1 \times A_2 \times \cdots \times A_n$, where $A_i = \{a_1, \ldots, a_{m_i}\}$ is the set of actions available to Player i.
- A *utility function*, $u : \mathcal{A} \to \mathbb{R}^n$, which maps an action profile $a = (a_1, \ldots, a_n)$ to the vector of payoffs $u(a) = (u_1(a), \ldots, u_n(a))$.

Note 1.1.2: A player is an entity capable of making decisions within the game to maximize their utility. Players may represent humans, machines, biological organisms, or any system that selects actions based on preferences or rules. The term *agent* is often used interchangeably with *player*. For simplicity, we avoid formalizing the concept but aim for this intuitive understanding.

1.2 Actions

Definition 1.2.1 *(Action)***:** An *action* for Player i is any choice $a_i \in A_i$, where A_i is Player i's action set, that is, the set of actions available to Player i, $i \in N$

Example 1.2.2 *(Rock-Paper-Scissors: Action Sets)***:** In this game classic, both players have the same action set,

$$A_1 = A_2 = \{\text{Rock, Paper, Scissors}\}.$$

That is, they can choose from the same set of actions. This does not mean that they need to play the same action, of course.

Example 1.2.3 *(Endgame in Chess: Action sets)***:** Consider the end stages of a game of chess, where "White" plays against "Black." For this example, we assume that White only has one piece remaining: the king, positioned at a1. Black, however, has two pieces remaining: the king, positioned at h8, and a pawn, positioned at a8. The players' sets of actions differ due to the positions and types of pieces: White's available actions are:

$$A_{\text{White}} = \{\text{Ka1-a2}, \text{Ka1-b1}, \text{Ka1-b2}\},$$

where each action corresponds to moving the king from a1 to one of the adjacent squares $(\text{a2}, \text{b1}, \text{b2})$. Black's available actions are

$$A_{\text{Black}} = \{\text{Kh8-g8}, \text{Kh8-g7}, \text{Kh8-h7}, \text{Pa8-a7}\},$$

where the king can move from h8 to one of the adjacent squares $(\text{g8}, \text{g7}, \text{h7})$, or the pawn can move from a8 forward to a7. Here, $A_{\text{White}} \neq A_{\text{Black}}$ both in the number of actions they can choose from, and in that the actions are not the same for the two players.

Note 1.2.4: For readers unfamiliar with algebraic chess notation: Ka1-a2 indicates the White king moving from square a1 to square a2. Similarly, Kh8-g8 indicates the Black king moving from h8 to g8.

The acute reader might have noticed that in the above examples, another thing differs: Rock-Paper-Scissors is a simultaneous game, where both players act at the same time, while Chess is a turn-based game. *In this text, we only consider simultaneous games.*

Definition 1.2.5 *(Action Profile)***:** An *action profile* $a = (a_1, a_2, \ldots, a_n)$ specifies one action for each player and is an element of the action space $\mathcal{A}$.

Example 1.2.6 *(Rock-Paper-Scissors: Action Profiles)***:** Each action profile represents one possible combination of actions chosen by the players during a single instance of the game. For Rock-Paper-Scissors there are many different action profiles. Three examples are (Rock, Scissors), (Rock, Paper) and (Scissors, Scissors). In the first action profile, Player 1 wins, since Rock beats Scissors, in the second Player 2 wins, since Paper beats Rock, and the third action profile results in a draw.

Worded differently: An action profile is a realization of a strategy profile, which is a collection of random variables (allowing ourselves some leeway and working with the definition that a random variable is a function that maps from the probability distribution to the action space, rather than to the real number line), $S = (S_1, S_2, ..., S_n)$ - see the definition of mixed strategies and strategy profiles.

Definition 1.2.7 *(Action Space)***:** The *action space*, $\mathcal{A}$, is the set of all possible action profiles. Formally:

$$\mathcal{A} = A_1 \times A_2 \times \cdots \times A_n,$$

where $A_i = \{a_1, \dots, a_{m_i}\}$ is the set of actions available to Player i. Each element of A is an action profile $a = (a_1, a_2, \dots, a_n)$, which specifies one action for each player.

The action space $\mathcal{A} = A_1 \times A_2 \times \cdots \times A_n$ represents all possible combinations of actions that players can choose. Each element $a \in A$ is an action profile specifying one action for each player.

Example 1.2.8 *(Rock-Paper-Scissors: Action Space)***:** In Rock-Paper-Scissors, the action space $\mathcal{A}$ is the Cartesian product of the players' action sets:

$$\mathcal{A} = \{\text{Rock},\ \text{Paper},\ \text{Scissors}\} \times \{\text{Rock},\ \text{Paper},\ \text{Scissors}\}.$$

Thus the action space consists of all possible action profiles. Since Rock-Paper-Scissors is a relatively simple game, it is possible to list all possible action profiles:

$$\begin{aligned}\mathcal{A} = \{&(\text{Rock},\ \text{Rock}), (\text{Rock},\ \text{Paper}), (\text{Rock},\ \text{Scissors}),\\ &(\text{Paper}, \text{Rock}), (\text{Paper},\ \text{Paper}), (\ \text{Paper},\ \text{Scissors}),\\ &(\text{Scissors}, \text{Rock}), (\text{Scissors},\ \text{Paper}), (\text{Scissors},\ \text{Scissors})\}.\end{aligned}$$

This exhaustive list shows all possible games of Rock-Paper-Scissors.

Note 1.2.9: For games more complicated than Rock-Paper-Scissors, the action space can be extremely large.

1.3 Utility and Payoff

In game theory, the concepts of utility and payoff are central to understanding player preferences and strategic outcomes. The utility function, also introduced formally by Neumann and Morgenstern, quantifies a player's satisfaction or reward based on the chosen actions of all players in a game. These utilities can then be organized into a payoff matrix, a standard representation of player outcomes in two-player games, which became a cornerstone of the field.

This section introduces utility functions and payoff matrices, grounding the discussion with concrete examples like Rock-Paper-Scissors, a widely used illustrative game that highlights these concepts in an accessible manner.

Definition 1.3.1 *(Utility Function)***:** The *utility function* $u(a)$ maps an action profile a to a vector of real number such that $u_i(a)$, represents Player i's utility (or payoff). Formally:

$$u_i : A \to \mathbb{R}, \quad u_i(a) = u_i(a_1, \dots, a_n).$$

Example 1.3.2 *(Rock-Paper-Scissors: Utility Function)***:** In Rock-Paper-Scissors, each player can either win, lose, or draw depending on their chosen actions and the rules of the game. The utility function $u(a) = (u_1(a), u_2(a))$ is constructed by the utility functions for for Player 1, $u_1(a)$, and Player 2, $u_2(a)$. Given our understanding of the game Rock-Paper-Scissors we can be define them as follows: For Player 1:

$$u_1(a) = \begin{cases} 1 & \text{if Player 1 wins,} \\ 0 & \text{if the game is a draw,} \\ -1 & \text{if Player 1 loses.} \end{cases}$$

For Player 2:

$$u_2(a) = \begin{cases} 1 & \text{if Player 2 wins,} \\ 0 & \text{if the game is a draw,} \\ -1 & \text{if Player 2 loses.} \end{cases}$$

Often it is simpler two write these cases together as

$$u(a) = \begin{cases} (1, -1) & \text{if Player 1 wins,} \\ (0, 0) & \text{if the game is a draw,} \\ (-1, 1) & \text{if Player 2 wins.} \end{cases}$$

Note 1.3.3: Since Rock-Paper-Scissors is symmetric, the utility functions in the example above are identical up to player index. This symmetry implies that the payoffs for corresponding actions are equal but opposite in sign for the players. However, this does not hold for all games. For instance, consider a variation where Player 1 derives more satisfaction from winning with *Rock* than with *Scissors*, while Player 2's payoffs remain unchanged. In such a case, Player 1's utility function $u_1(a)$ might assign a higher payoff to winning with *Rock* (e.g., 2 points) than winning with *Scissors* (e.g., 1 point), breaking the symmetry of the utility functions.

Definition 1.3.4 *(Payoff Matrix)***:** The *payoff matrix* for a two-player normal-form game is a tabular representation of the utility function for all possible action profiles. Formally:

- Let $A_1 = \{a_1^1, a_2^1, \dots, a_{m_1}^1\}$ be the set of actions for Player 1.
- Let $A_2 = \{a_1^2, a_2^2, \dots, a_{m_2}^2\}$ be the set of actions for Player 2.
- The payoff matrix consists of two matrices:

  $$P_1(i, j) = u_1(a_i^1, a_j^2), \quad P_2(i, j) = u_2(a_i^1, a_j^2),$$

  where $P_1(i, j)$ represents Player 1's utility, and $P_2(i, j)$ represents Player 2's utility for the action profile (a_i^1, a_j^2).

In general, for n-player games, the payoff matrix generalizes to a higher-dimensional array, where each entry corresponds to the vector of utilities $(u_1(a), u_2(a), \dots, u_n(a))$ for an action profile $a \in A$.

Example 1.3.5 *(Rock-Paper-Scissors: Payoff Matrix)***:** The payoff matrix for Rock-Paper-Scissors, where both players have the same action set $A_1 = A_2 = \{\text{Rock}, \text{Paper}, \text{Scissors}\}$, is:

	Rock	Paper	Scissors
Rock	$(0, 0)$	$(-1, 1)$	$(1, -1)$
Paper	$(1, -1)$	$(0, 0)$	$(-1, 1)$
Scissors	$(-1, 1)$	$(1, -1)$	$(0, 0)$

Here, each cell $(u_1(a), u_2(a))$ corresponds to the payoff pair for Player 1 and Player 2 for the given action profile a. For example:

- If $a = (\text{Rock}, \text{Paper})$, then $u(a) = (-1, 1)$.
- If $a = (\text{Scissors}, \text{Rock})$, then $u(a) = (-1, 1)$.
- If $a = (\text{Paper}, \text{Paper})$, then $u(a) = (0, 0)$.

Example 1.3.6 *(Rock-Paper-Scissor with Preferences: Payoff Matrix)***:** Consider a variation of Rock-Paper-Scissors where Player 1 prefers winning with Rock over winning with Scissors. The modified payoff matrix becomes:

	Rock	Paper	Scissors
Rock	$(0,0)$	$(-1,1)$	$(2,-1)$
Paper	$(1,-1)$	$(0,0)$	$(-1,1)$
Scissors	$(-1,1)$	$(1,-1)$	$(0,0)$

Here, Player 1's utility for $a = (\text{Rock}, \text{Scissors})$ is increased to 2, reflecting their preference for winning with Rock over other options. For example:

- If $a = (\text{Rock}, \text{Scissors})$, then $u(a) = (2, -1)$.
- If $a = (\text{Scissors}, \text{Rock})$, then $u(a) = (-1, 1)$.
- If $a = (\text{Paper}, \text{Paper})$, then $u(a) = (0, 0)$.

Note 1.3.7: The payoff matrix provides all the information needed to describe a two-player Normal-Form Game, as long as the game satisfies certain conditions. Specifically, it assumes that both players:

- Have a finite number of actions to choose from.
- Act simultaneously, without knowing the other player's action in advance.
- Have full knowledge of the game's structure, including the available actions and payoffs for both players.

The rows and columns of the payoff matrix represent the players' actions, while each cell contains the payoffs for the corresponding action profile. However, the payoff matrix does not apply to more complex games, such as those involving more than two players, infinite actions, or sequential moves. In such cases, additional tools or representations are required.

1.4 Strategies

Some decision-making scenarios cannot be adequately captured by simple, deterministic actions. In such cases, it may be beneficial to consider assigning probabilities to different actions, allowing for more flexible and strategic behavior. This concept of *strategies* was formalized by John Nash in his seminal work on equilibrium points in games, where he introduced the notion of mixed strategies as a way to model probabilistic decision-making [Nash Jr(1950)].

Definition 1.4.1 *(Mixed Strategy)***:** A *mixed strategy* S_i for Player i is a probability distribution over their set of actions A_i. Let $A_i = \{a_1, \ldots, a_{m_i}\}$; then a mixed strategy is a vector $S_i = \{(a_1, p_1), \ldots, (a_{m_i}, p_{m_i})\}$, where $p_j \geq 0$ for all j and:

$$\sum_{j=1}^{m_i} p_j = 1.$$

Here, p_j represents the probability that Player i chooses action a_j. A mixed strategy generalizes the concept of a pure strategy by allowing randomness in a player's decision-making process.

Definition 1.4.2: A *pure strategy* is a special case of a mixed strategy, where the probability distribution assigns all weight to a single action. Formally, a pure strategy s_i for Player i is a mixed strategy $S_i = \{(a_1, p_1), \ldots, (a_{m_i}, p_{m_i})\}$ such that $p_j = 1$ for some j and $p_k = 0$ for all $k \neq j$. For example, if $S_i = \{(a_1, 1), (a_2, 0), \ldots, (a_{m_i}, 0)\}$, then S_i corresponds to the pure strategy $s_i = a_1$.

Note 1.4.3: Pure strategies are deterministic and correspond to specific actions, while mixed strategies introduce randomness into decision-making by assigning probabilities to multiple actions.

Example 1.4.4 *(Rock-Paper-Scissors: Strategies)***:** In Rock-Paper-Scissors, playing *Rock* 100% of the time is a pure strategy, while choosing each action (*Rock, Paper, Scissors*) with equal probability (33.3%) is a mixed strategy.

Pure strategies are typically denoted with lowercase letters, such as s_i, while mixed strategies are denoted with uppercase letters, such as S_i. This convention helps differentiate between deterministic and probabilistic strategies.

In some texts, no distinction is made between an action and a pure strategy. Maintaining this distinction mirrors the separation in probability theory between a deterministic value, say the number 3, and an *almost surely constant random variable* that takes the value 3 with probability 1. This distinction not only ensures conceptual clarity but also allows pure strategies to be meaningfully defined as a special case of mixed strategies — a definition that would otherwise be nonsensical.

Definition 1.4.5 *(Strategy Profile)***:** A *strategy profile* specifies a strategy (pure or mixed) for each player. Formally, a strategy profile S is a tuple $S = (S_1, S_2, \dots, S_n)$, where S_i is the strategy of Player i. The set of all possible strategy profiles is called the *strategy profile space* and is denoted by $\mathcal{S}$. Formally:

$$\mathcal{S} = S_1 \times S_2 \times \cdots \times S_n,$$

where S_i is the strategy space for Player i. Each element $S \in \mathcal{S}$ corresponds to a specific combination of strategies chosen by all players.

Note 1.4.6: The concept of strategy profiles is the conceptual extension of action profiles — by allowing players to use mixed strategies. A pure strategy profile always produces the same action profile (just as a almost surely constant random variable always takes the same value), while a mixed strategy profile represents a probability distribution over all possible action profiles (mirroring the random variable that has a distribution over a discrete set of values). For example, in Rock-Paper-Scissors, a mixed strategy profile might specify that both players choose each of their actions with equal probability.

Definition 1.4.7 *(Utility Function for Strategies)***:** The utility function for Player i under a mixed strategy profile $S = (S_1, \dots, S_n)$ is the expected utility, defined as:

$$u_i(S) = E[u_i(a) \mid S = (S_1, \dots, S_n)] = \sum_{a \in \mathcal{A}} P(a \mid S) u_i(a),$$

where:

- $\mathcal{A} = A_1 \times A_2 \times \cdots \times A_n$ is the action space.
- $P(a \mid S) = \prod_{j=1}^{n} p_j(a_j)$ is the probability of the action profile $a = (a_1, \dots, a_n)$ occurring given the mixed strategy profile $S = (S_1, \dots, S_n)$.
- $u_i(a)$ is the utility of Player i for the action profile a.

The expected utility reflects the average payoff for Player i, weighted by the probabilities of all action profiles under the mixed strategy profile S.

The utility function for mixed strategies computes the weighted average of payoffs over all possible action profiles, based on their probabilities. For instance, in Rock-Paper-Scissors, if both players use mixed strategies with equal probabilities for all actions, the expected utility for each player is 0, reflecting the fairness of the game under random play.

1.5 Domination

Definition 1.5.1 *(Dominated Action)***:** An action $a_i \in A_i$ for Player i is *dominated* if there exists another action $a'_i \in A_i$ such that:

$$u_i(a'_i, a_{-i}) \geq u_i(a_i, a_{-i}) \quad \text{for all } a_{-i} \in A_{-i},$$

with strict inequality for at least one a_{-i}. Conversely, a'_i is said to *dominate* a_i. A dominated action is never part of a rational player's optimal strategy.

Domination can be used to simplify games by systematically eliminating dominated actions. In some games, iteratively removing dominated actions for all players, a process called *Iterated Elimination of Strictly Dominated Strategies (IESDS)*, can lead to a unique solution. However, not all games are fully solvable through this process.

Example 1.5.2 *(Dominated Actions in the Prisoner's Dilemma)***:** The *Prisoner's Dilemma* is a classic example of a two-player Normal-Form Game, illustrating the conflict between individual rationality and collective welfare. The game is defined as follows:

- **Players:** Two players, referred to as Player 1 and Player 2.
- **Actions:** Each player has two possible actions:

$$A_1 = A_2 = \{\text{Cooperate (C), Defect (D)}\}.$$

- **Payoffs:** The payoffs for each player depend on their own action and the action of the other player. The utility function is represented by the following payoff matrix:

	Cooperate (C)	Defect (D)
Cooperate (C)	$(3,3)$	$(0,5)$
Defect (D)	$(5,0)$	$(1,1)$

Here, the first value in each cell represents Player 1's utility, and the second value represents Player 2's utility. The payoffs correspond to the following outcomes:

- If both players cooperate (C,C), they each receive a moderate payoff of 3.
- If one player defects (D) while the other cooperates (C), the defector gains a high payoff of 5, while the cooperator receives 0.
- If both players defect (D,D), they each receive a lower payoff of 1.

Now consider Player 1's decision-making process. Player 1's payoffs for each possible action of Player 2 are as follows:

- If Player 2 cooperates (C):

$$u_1(C,C) = 3, \quad u_1(D,C) = 5.$$

  Defecting (D) yields a higher payoff than cooperating (C).

- If Player 2 defects (D):

$$u_1(C, D) = 0, \quad u_1(D, D) = 1.$$

  Again, defecting (D) yields a higher payoff than cooperating (C).

In both cases, Player 1 achieves a strictly higher utility by defecting than by cooperating, regardless of Player 2's action. Therefore, *Defect (D)* strictly dominates *Cooperate (C)* for Player 1. A similar analysis applies to Player 2, for whom *Defect (D)* also strictly dominates *Cooperate (C)*.

As a result, the only rational outcome under iterated elimination of strictly dominated strategies (IESDS) is mutual defection (D, D), with payoffs $(1, 1)$. This outcome highlights the dilemma: while mutual cooperation (C, C) is better for both players collectively, individual incentives drive both players to defect, resulting in a worse outcome.

Note 1.5.3: Dominated actions simplify analysis by identifying suboptimal choices that rational players would never choose. However, not all games allow elimination of dominated actions, and some games may require mixed strategies to reach equilibrium.

Note 1.5.4: The elimination of dominated actions refines the action space, leaving only the rational choices for each player. This refinement is a key step towards identifying equilibrium concepts such as Nash Equilibria, where no player has an incentive to deviate from their chosen strategy.

Definition 1.5.5 *(Nash Equilibrium)***:** A *Nash Equilibrium* is a strategy profile $S = (S_1, S_2, \ldots, S_n)$ such that no player can improve their payoff by unilaterally changing their strategy. Formally, S is a Nash Equilibrium if:

$$u_i(S_i, S_{-i}) \geq u_i(S_i', S_{-i}) \quad \text{for all } S_i' \in S_i \text{ and all } i \in N,$$

where S_{-i} represents the strategies of all players other than Player i.

2 Introducing the Base Case

In this section, we introduce a specific class of games that we have chosen to call *two-player tie-sharing games*. These games involve two players competing for a shared reward by selecting actions from finite sets, with the reward being divided equally in the event of a tie. Despite their simplicity, these games exhibit intriguing strategic dynamics worth exploring.

The conjectures proposed in the subsequent section pertain exclusively to these two-player tie-sharing games. They are the product of numerical simulations and personal exploration, marking the author's first attempt to derive general insights in game theory. While no counterexamples to these conjectures have yet been found, they are presented as exploratory ideas rather than established results. Proving these conjectures may well be straightforward, but we lack the expertise to make definitive claims. We therefore invite scrutiny and further investigation from those more experienced in the field.

2.1 Two-Player Tie-Sharing Games

Definition 2.1.1 *(Two-Player Tie-Sharing Game)***:** A *two-player tie-sharing game* is defined by the following components:

- A maximal reward $R > 0$.
- Action sets for the two players, denoted as $A_A = \{0, 1, \ldots, M_A\}$ for Player A and $A_B = \{0, 1, \ldots, M_B\}$ for Player B.

In a stage game, each player selects one action, $s_A \in A_A$ for Player A and $s_B \in A_B$ for Player B, forming an action profile $a = (s_A, s_B)$.

The utility function for the game, $\pi(S) = (\pi_A(S), \pi_B(S))$, assigns payoffs based on the players' chosen actions and is given by:

$$\pi(S) = \begin{cases} (R - s_A, -s_B) & \text{if } s_A > s_B, \\ (-s_A, R - s_B) & \text{if } s_A < s_B, \\ \left(\frac{R}{2} - s_A, \frac{R}{2} - s_B\right) & \text{if } s_A = s_B. \end{cases}$$

We denote such a game by (R, M_A, M_B).

Note 2.1.2: A key strategy in this game is to invest just enough to exceed your opponent's action, as this maximizes your reward while minimizing costs. However, when the available reward R is small relative to the cost of investment, there is little incentive to increase your action further.

Definition 2.1.3 *(Low-resource Game)***:** We call a two player tie-sharing game a *low-resource game* if $R = 1$.

Example 2.1.4 *(Low-Resource Game:* $(1, 2, 2)$*)***:** Consider the low-resource two-player tie-sharing game with parameters $(R, M_A, M_B) = (1, 2, 2)$. The initial payoff matrix is:

	0	1	2
0	$(0.5, 0.5)$	$(0.0, 0.0)$	$(0.0, -1.0)$
1	$(0.0, 0.0)$	$(-0.5, -0.5)$	$(-1.0, -1.0)$
2	$(-1.0, 0.0)$	$(-1.0, -1.0)$	$(-1.5, -1.5)$

We now iteratively eliminate strictly dominated strategies:

1. Player A's strategy $s_A = 1$ is strictly dominated by $s_A = 0$:

	0	1	2
0	$(0.5, 0.5)$	$(0.0, 0.0)$	$(0.0, -1.0)$
2	$(-1.0, 0.0)$	$(-1.0, -1.0)$	$(-1.5, -1.5)$

2. Player A's strategy $s_A = 2$ is strictly dominated by $s_A = 0$:

	0	1	2
0	$(0.5, 0.5)$	$(0.0, 0.0)$	$(0.0, -1.0)$

3. Player B's strategy $s_B = 1$ is strictly dominated by $s_B = 0$:

	0	2
0	$(0.5, 0.5)$	$(0.0, -1.0)$

4. Player B's strategy $s_B = 2$ is strictly dominated by $s_B = 0$:

	0
0	$(0.5, 0.5)$

After iterated elimination of strictly dominated strategies (IESDS), the game reduces to the single outcome ($s_A = 0, s_B = 0$), which is the unique pure-strategy Nash equilibrium with payoffs $(0.5, 0.5)$.

In low-resource games, the small reward $R = 1$ ensures that action costs dominate strategic considerations. As a result, players have no incentive to invest beyond the minimal action, leading to the unique equilibrium where both players choose $s_A = s_B = 0$.

Definition 2.1.5 *(Symmetric Game)***:** A two-player tie-sharing game is called *symmetric* if both players have the same maximum action set size, i.e., $M_A = M_B = M$. Symmetry implies that both players face identical strategic opportunities. Games where $M_A \neq M_B$ are correspondingly referred to as *asymmetric*.

Note 2.1.6: For asymmetric games, we can assume without loss of generality that $M_A < M_B$. This assumption simplifies the analysis by reducing redundant cases where player roles could be swapped without changing the game dynamics.

We will now examine two symmetric games, both with $M = 3$, to highlight the diverse behaviors that can arise even within symmetric setups. The first game demonstrates how symmetry and Iterated Elimination of Strictly Dominated Strategies (IESDS) can reduce the game to a unique pure-strategy Nash equilibrium. In contrast, the second game is irreducible from the start, exhibiting cycling best responses that prevent the existence of any pure-strategy Nash equilibrium.

Example 2.1.7 *(Symmetric Game $(8, 3, 3)$)***:** Consider the symmetric game with parameters $(R, M, M) = (8, 3, 3)$. The initial payoff matrix is:

	0	1	2	3
0	$(4, 4)$	$(0, 7)$	$(0, 6)$	$(0, 5)$
1	$(7, 0)$	$(3, 3)$	$(-1, 6)$	$(-1, 5)$
2	$(6, 0)$	$(6, -1)$	$(2, 2)$	$(-2, 5)$
3	$(5, 0)$	$(5, -1)$	$(5, -2)$	$(1, 1)$

By iterated elimination of strictly dominated strategies (IESDS), we proceed as follows:

1. $s_A = 0$ is strictly dominated by $s_A = 3$. The updated payoff matrix is:

	0	1	2	3
1	$(7, 0)$	$(3, 3)$	$(-1, 6)$	$(-1, 5)$
2	$(6, 0)$	$(6, -1)$	$(2, 2)$	$(-2, 5)$
3	$(5, 0)$	$(5, -1)$	$(5, -2)$	$(1, 1)$

2. $s_B = 0$ is strictly dominated by $s_B = 3$. The updated payoff matrix is:

	1	2	3
1	$(3, 3)$	$(-1, 6)$	$(-1, 5)$
2	$(6, -1)$	$(2, 2)$	$(-2, 5)$
3	$(5, -1)$	$(5, -2)$	$(1, 1)$

3. $s_A = 1$ is strictly dominated by $s_A = 3$. The updated payoff matrix is:

	1	2	3
2	$(6, -1)$	$(2, 2)$	$(-2, 5)$
3	$(5, -1)$	$(5, -2)$	$(1, 1)$

4. $s_B = 1$ is strictly dominated by $s_B = 3$. The updated payoff matrix is:

	2	3
2	$(2, 2)$	$(-2, 5)$
3	$(5, -2)$	$(1, 1)$

5. $s_A = 2$ is strictly dominated by $s_A = 3$. The updated payoff matrix is:

	2	3
3	$(5, -2)$	$(1, 1)$

6. $s_B = 2$ is strictly dominated by $s_B = 3$. The final payoff matrix is:

	3
3	$(1, 1)$

After removing dominated strategies at each step, the game collapses to the single outcome $(s_A = 3, s_B = 3)$, which is a unique pure-strategy Nash equilibrium.

Example 2.1.8 *(Symmetric Game* $(4, 3, 3)$*)***:** Next, we examine the symmetric game with parameters $(R, M, M) = (4, 3, 3)$. The payoff matrix is:

	0	1	2	3
0	$(2, 2)$	$(0, 3)$	$(0, 2)$	$(0, 1)$
1	$(3, 0)$	$(1, 1)$	$(-1, 2)$	$(-1, 1)$
2	$(2, 0)$	$(2, -1)$	$(0, 0)$	$(-2, 1)$
3	$(1, 0)$	$(1, -1)$	$(1, -2)$	$(-1, -1)$

In this game, no strategy is strictly dominated for either player. Indeed, each player's best response cycles among the strategies depending on what the opponent plays:

- If Player B plays 0, A's best response is 1.
- If Player B plays 1, A's best response is 2.
- If Player B plays 2, A's best response is 3.
- If Player B plays 3, A's best response is 0.

A similar cycle emerges for Player B given A's strategy. Consequently, there is no pure-strategy Nash equilibrium, and the game is *irreducible* under IESDS.

In both of the above examples, we have $M_A = M_B = 3$, and the variation lies in how the reward R compares to the players' cost ceilings M_A and M_B. By exploring the family of games $(R, 3, 3)$ for $R = 2, 3, 4, \ldots$, we observe distinct patterns. Specifically:

- For $R = 2$, the outcome $(3, 3)$ is strictly dominated and eliminated, leaving the action sets $\{0, 1, 2\}$ for both players.
- For $R = 3, 4, 5, 6$, the game remains irreducible under IESDS, with no strategies being eliminated.
- For $R > 6$, the game reduces so that $\{3\} \times \{3\}$ becomes the remaining action space.

As per Example 2.1.4, for low-resource games ($R = 1$), investing anything beyond 0 yields no net benefit because the cost outweighs the split reward. As a result, both players invest 0, leading to the unique equilibrium with payoffs $\left(\frac{1}{2}, \frac{1}{2}\right)$. This same reasoning applies to games of the form $(1, C, C)$ for any integer C, reducing the action space to $\{0\} \times \{0\}$. This behavior was illustrated in Example 2.1.4 and applies to both symmetric and asymmetric cases, including $(1, 3, 3)$. Thus, we have fully characterized the behavior of $(R, 3, 3)$ for $R = 1, 2, 3, 4, \ldots$.

Further analysis shows that this behavior is not unique to $M = 3$. Instead, it generalizes to symmetric games with any M. In the next section, we will introduce a set of six classification rules that describe the IESDS reduction of all two-player tie-sharing games: one rule for low-resource games, two for symmetric games, and three for asymmetric games. Before delving into these results, we will first establish some notation.

2.2 Introducing Signatures

Definition 2.2.1 *(Signature)***:** The *signature* of a two-player tie-sharing game is a compact notation used to represent the action sets of the players. For Player A with an action set $A_A = \{i, \ldots, j\}$ and Player B with an action set $A_B = \{k, \ldots, l\}$, where $i, j, k, l \in \{0, 1, 2, \ldots\}$, the signature is denoted as:

$$(\{i : j\} \times \{k : l\})$$

Here, i and j are the minimum and maximum actions available to Player A, respectively, while k and l are the minimum and maximum actions available to Player B. This notation assumes that the action sets for both players consist of consecutive integers, which holds for all games considered in this text.

Example 2.2.2: Recall Example 2.1.7, where we analyzed the symmetric game $(R, M, M) = (8, 3, 3)$. The initial payoff matrix was:

	0	1	2	3
0	$(4, 4)$	$(0, 7)$	$(0, 6)$	$(0, 5)$
1	$(7, 0)$	$(3, 3)$	$(-1, 6)$	$(-1, 5)$
2	$(6, 0)$	$(6, -1)$	$(2, 2)$	$(-2, 5)$
3	$(5, 0)$	$(5, -1)$	$(5, -2)$	$(1, 1)$

This corresponds to the signature $(\{0 : 3\} \times \{0 : 3\}) = \{0, 1, 2, 3\} \times \{0, 1, 2, 3\}$. After applying Iterated Elimination of Strictly Dominated Strategies (IESDS), the game was reduced to the following payoff matrix:

	3
3	$(1, 1)$

The reduced action sets correspond to the signature $(\{3 : 3\} \times \{3 : 3\}) = \{3\} \times \{3\}$.

Note 2.2.3: Introducing a notation to explicitly represent the IESDS reduction process allows for a concise description of how games evolve under strategy elimination. This will facilitate future classification and analysis.

Definition 2.2.4 *(IESDS Reduction Notation)***:** Let (R, M_A, M_B) denote a two-player tie-sharing game. Then, the notation

$$(R, M_A, M_B) \xrightarrow{\text{IESDS}} (\{a : b\} \times \{c : d\})$$

indicates that the game (R, M_A, M_B), after applying Iterated Elimination of Strictly Dominated Strategies (IESDS), is reduced to the action space described by the signature $(\{a : b\} \times \{c : d\})$.

Note 2.2.5: While the signature $(\{a : b\} \times \{c : d\})$ encodes the structure of the reduced action sets, it does not convey information about the reward structure in the remaining game. However, when combined with the maximal reward R, the signature fully describes the resulting game.

3 Method and Conjectures

This section is dedicated to systematically classifying two-player tie-sharing games based on two key aspects: the signatures they reduce to under Iterated Elimination of Strictly Dominated Strategies (IESDS) and the existence, count, and location of their pure Nash equilibria. These classifications aim to establish general rules that link the game parameters (R, M_A, M_B) to both the structure of the reduced action spaces and the equilibrium behavior. This framework offers a structured approach to understanding the strategic dynamics and fundamental properties of two-player tie-sharing games.

3.1 Method

The conjectures and rules presented here are the result of iterative exploration and numerical simulations. To perform Iterated Elimination of Strictly Dominated Strategies (IESDS), I utilized functionality from the open-source package `Game Theory Python Package` by Carlos Gómez avalable at `https://github.com/carlosgoe/game-theory`. This package provided tools for implementing IESDS and analyzing game-theoretic structures, forming the foundation for my numerical experiments.

For each game, I recorded the signature of the action space after applying IESDS and analyzed patterns in the reduced structures. These observations revealed recurring trends in how games simplify and how equilibrium configurations emerge, leading to the formulation of general rules connecting game parameters to reduced forms and equilibrium structures.

These initial insights were tested across a variety of parameter combinations to evaluate their consistency. While no counterexamples have been identified, the results remain exploratory. Additionally, similar predictive rules were devised for the number and location of pure Nash equilibria within the action space. The full codebase and simulation details for my extensions are available at `https://github.com/wikefjol/two-player-tie-sharing-games`.

3.2 Conjectures

The first step in this classification involves understanding how the action spaces simplify under IESDS. This simplification is captured in six distinct rules, derived from observing the structural changes in the games during the elimination process. Since these rules apply to two-player games, we can assume, without loss of generality, that $M_A \leq M_B$; otherwise, the players' roles can simply be swapped. These rules are categorized into three broad groups based on the key properties of the game:

- **Low-Resource Games:**
  - Games where $R = 1$. (**Rule 1**)
- **Symmetric Non-Low-Resource Games:**
  - Games where $M < \frac{R}{2}$. (**Rule 2**)
  - Games where $M \geq \frac{R}{2}$. (**Rule 3**)

- **Asymmetric Non-Low-Resource Games:**
  - Games where $M_A < \lceil \frac{R}{2} \rceil$. (**Rule 4**)
  - Games where $\lceil \frac{R}{2} \rceil \leq M_A < R$. (**Rule 5**)
  - Games where $R \leq M_A$. (**Rule 6**)

The second step builds upon this classification to explore equilibrium behavior. Specifically, it establishes a direct relationship between the game parameters and the existence, count, and location of pure Nash equilibria. This relationship is governed by three additional rules:

- For low-resource games ($R = 1$), there is a single Nash equilibrium where both players select $s = (0, 0)$. (**Rule A:**)
- For games with $R = 2$, four Nash equilibria exist, corresponding to the corners of the unit square in the action space. (**Rule B:**)
- For games with $R > 2$, a single Nash equilibrium exists if the game is symmetric with $M < \lfloor R/2 \rfloor$; otherwise, no pure Nash equilibria exist. (**Rule C:**)

Together, these conjectures provide a unified framework for predicting reduced action spaces and equilibrium behavior, significantly reducing the computational complexity of analyzing two-player tie-sharing games.

Conjecture 3.2.1 *(Rules for Classifying Reduced Action Spaces)***:**
Let (R, M_A, M_B) be a two-player tie-sharing game. In the cases where $M_A \neq M_B$ we can assume $M_A < M_B$ without loss of generality. Under Iterated Elimination of Strictly Dominated Strategies (IESDS), these games can be classified into the following cases:

Rule 1: For low-resource games (R, M_A, M_B),

$$(R, M_A, M_B) \xrightarrow{\text{IESDS}} (\{0 : 0\} \times \{0 : 0\}).$$

Rule 2: For symmetric games (R, M, M) with $M < \frac{R}{2}$,

$$(R, M, M) \xrightarrow{\text{IESDS}} (\{M : M\} \times \{M : M\}).$$

Rule 3: For symmetric games (R, M, M) with $M \geq \frac{R}{2}$,

$$(R, M, M) \xrightarrow{\text{IESDS}} (\{0 : \min(M, R)\} \times \{0 : \min(M, R)\}).$$

Rule 4: For asymmetric games (R, M_A, M_B) with $M_A < \lceil \frac{R}{2} \rceil$,

$$(R, M_A, M_B) \xrightarrow{\text{IESDS}} (\{0 : M_A\} \times \{1 : M_A + 1\}).$$

Rule 5: For asymmetric games (R, M_A, M_B) with $\lceil \frac{R}{2} \rceil \leq M_A < R$,

$$(R, M_A, M_B) \xrightarrow{\text{IESDS}} (\{0 : T\} \times \{0 : T + 1\}),$$

where $T = \min(M_A, R)$.

Rule 6: For asymmetric games (R, M_A, M_B) with $R \leq M_A$,

$$(R, M_A, M_B) \xrightarrow{\text{IESDS}} (\{0 : R\} \times \{0 : R\}).$$

Conjecture 3.2.2 *(Pure Nash Equilibria Counts)***:**
Let (R, M_A, M_B) be a two-player tie-sharing game. Assume $M_A \neq M_B$ with $M_A < M_B$ without loss of generality. Let N_{PN} denote the number of pure Nash equilibria in the reduced action spaces classified above. The following cases hold:

Rule A: If $R = 1$, then

$$N_{PN} = 1,$$

and the pure strategy is $s = (0, 0)$.

Rule B: If $R = 2$, then

$$N_{PN} = 4,$$

corresponding to the corners of the unit square in the action space: $s^1 = (0, 0), s^2 = (0, 1), s^3 = (1, 0), s^4 = (1, 1)$.

Rule C: If $R > 2$, then

$$N_{PN} = \begin{cases} 1 & \text{if the game is symmetric and } M < \lfloor R/2 \rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

If a Nash equilibrium exists, it is the pure strategy $s = (M, M)$.

3.3 Additional examples

Note 3.3.1: The following examples demonstrate the application of the conjectures:

- **Reduced Action Spaces (Conjecture 1):**
  - Rule 1: Example 2.1.4.
  - Rule 2: Example 2.1.7.
  - Rule 3: Example 2.1.8.
  - Rule 4: Example 3.3.6.
  - Rule 5: Example 3.3.7.
  - Rule 6: Example 3.3.8.
- **Pure Nash Equilibria (Conjecture 2):**
  - Rule A: Example 2.1.4.
  - Rule B: Example 3.3.5.
  - Rule C: Example 2.1.7, Example 3.3.6, Example 3.3.7.

These examples illustrate the conjectures' applicability and provide intuition for further analysis.

Note 3.3.2: While an example cannot serve as a formal proof, well-chosen examples illustrate the utility and applicability of these conjectures, offering intuition and context for further analysis.

Example 3.3.3 *(Applying the Conjectures: Asymmetric Game* $(8, 4, 7)$*)***:** Consider the game $(8, 4, 7)$. This is an asymmetric non-low-resource game, so rules 4–6 from the first conjecture apply. With $M_A = 4$ and $R = 8$, we find $\lceil R/2 \rceil = 4 \leq M_A < R$, so Rule 5 applies. Thus,

$$(8, 4, 7) \xrightarrow{\text{IESDS}} (\{0 : 4\} \times \{0 : 5\}).$$

For the number of pure Nash equilibria, we use the second conjecture. Since $R > 2$ and the game is asymmetric, Rule C applies, yielding $N_{PN} = 0$. Therefore, there are no pure Nash equilibria for this game.

Example 3.3.4 *(IESDS: Asymmetric Game* $(8, 4, 7)$*)***:** Consider the asymmetric game with parameters $(R, M_A, M_B) = (8, 4, 7)$. The initial payoff matrix is:

	0	1	2	3	4	5	6	7
0	$(4,4)$	$(0,7)$	$(0,6)$	$(0,5)$	$(0,4)$	$(0,3)$	$(0,2)$	$(0,1)$
1	$(7,0)$	$(3,3)$	$(-1,6)$	$(-1,5)$	$(-1,4)$	$(-1,3)$	$(-1,2)$	$(-1,1)$
2	$(6,0)$	$(6,-1)$	$(2,2)$	$(-2,5)$	$(-2,4)$	$(-2,3)$	$(-2,2)$	$(-2,1)$
3	$(5,0)$	$(5,-1)$	$(5,-2)$	$(1,1)$	$(-3,4)$	$(-3,3)$	$(-3,2)$	$(-3,1)$
4	$(4,0)$	$(4,-1)$	$(4,-2)$	$(4,-3)$	$(0,0)$	$(-4,3)$	$(-4,2)$	$(-4,1)$

After applying IESDS and eliminating strictly dominated strategies, the reduced action space has the signature:

$$(\{0 : 4\} \times \{0 : 5\}).$$

Analyzing best responses reveals cyclic behavior, confirming that there are no pure strategy Nash equilibria.

This section presents explicit and detailed examples for each remaining rule in the first conjecture, as Rules 1, 2, and 3 were already illustrated in earlier examples. While these examples are designed to build intuition, they do not constitute formal proofs.

Example 3.3.5 *(Symmetric Game* $(2, 2, 2)$*)***:** Consider the symmetric game with parameters $(R, M, M) = (2, 2, 2)$. The initial payoff matrix is:

	0	1	2
0	$(1,1)$	$(0,1)$	$(0,0)$
1	$(1,0)$	$(0,0)$	$(-1,0)$
2	$(0,0)$	$(0,-1)$	$(-1,-1)$

and there are no strictly dominated strategies for either player, so we note that $(2, 2, 2) \xrightarrow{\text{IESDS}} (\{0 : 2\} \times \{0 : 2\})$, illustrating **Rule 3**. Then we move directly to analyzing best responses.

- Player 1's best responses:
  - To $s_B = 0$: $s_A = 0, 1$,
  - To $s_B = 1$: $s_A = 0, 1, 2$,
  - To $s_B = 2$: $s_A = 0$.
- Player 2's best responses:
  - To $s_A = 0$: $s_B = 0, 1$,
  - To $s_A = 1$: $s_B = 0, 1, 2$,
  - To $s_A = 2$: $s_B = 0$.

The pure strategy Nash equilibria correspond to the action profiles where both players are simultaneously playing their best responses:

$$NE = \{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

This example illustrates **Rule B** from the second conjecture, where $R = 2$. As predicted, the game has four pure strategy Nash equilibria located at the corners of the unit square in the action space.

Example 3.3.6 *(Asymmetric Game* $(3, 1, 2)$*)***:** Consider the asymmetric game with parameters $(R, M_A, M_B) = (3, 1, 2)$. The initial payoff matrix is:

	0	1	2
0	$(1.5, 1.5)$	$(0.0, 2.0)$	$(0.0, 1.0)$
1	$(2.0, 0.0)$	$(0.5, 0.5)$	$(-1.0, 1.0)$

We now proceed with Iterated Elimination of Strictly Dominated Strategies (IESDS):

1. Player 2's strategy $s_B = 0$ is strictly dominated by $s_B = 1$:

	1	2
0	$(0.0, 2.0)$	$(0.0, 1.0)$
1	$(0.5, 0.5)$	$(-1.0, 1.0)$

At this point, no further strictly dominated strategies remain and we can see that $(3, 1, 2) \xrightarrow{\text{IESDS}} (\{0 : 1\} \times \{1 : 2\})$, illustrating **Rule 4**. We note that the best responses are

- Player 1's best responses:
  - To $s_B = 1$: $s_A = 1$ (since $0.5 > 0.0$).
  - To $s_B = 2$: $s_A = 0$ (since $0.0 > -1.0$).
- Player 2's best responses:
  - To $s_A = 0$: $s_B = 1$ (since $2.0 > 1.0$).
  - To $s_A = 1$: $s_B = 2$ (since $1.0 > 0.5$).

and thus no pure strategy Nash equilibria exist, illustrating **Rule C**.

Example 3.3.7 *(Asymmetric Game* $(5, 3, 7)$*)***:** Consider the asymmetric game with parameters $(R, M_A, M_B) = (5, 3, 7)$. The initial payoff matrix is:

	0	1	2	3	4	5	6	7
0	$(2.5, 2.5)$	$(0.0, 4.0)$	$(0.0, 3.0)$	$(0.0, 2.0)$	$(0.0, 1.0)$	$(0.0, 0.0)$	$(0.0, -1.0)$	$(0.0, -2.0)$
1	$(4.0, 0.0)$	$(1.5, 1.5)$	$(-1.0, 3.0)$	$(-1.0, 2.0)$	$(-1.0, 1.0)$	$(-1.0, 0.0)$	$(-1.0, -1.0)$	$(-1.0, -2.0)$
2	$(3.0, 0.0)$	$(3.0, -1.0)$	$(0.5, 0.5)$	$(-2.0, 2.0)$	$(-2.0, 1.0)$	$(-2.0, 0.0)$	$(-2.0, -1.0)$	$(-2.0, -2.0)$
3	$(2.0, 0.0)$	$(2.0, -1.0)$	$(2.0, -2.0)$	$(-0.5, -0.5)$	$(-3.0, 1.0)$	$(-3.0, 0.0)$	$(-3.0, -1.0)$	$(-3.0, -2.0)$

We proceed with Iterated Elimination of Strictly Dominated Strategies (IESDS):

1. Player 2's strategy $s_B = 5$ is strictly dominated by $s_B = 4$:

	0	1	2	3	4	6	7
0	$(2.5, 2.5)$	$(0.0, 4.0)$	$(0.0, 3.0)$	$(0.0, 2.0)$	$(0.0, 1.0)$	$(0.0, -1.0)$	$(0.0, -2.0)$
1	$(4.0, 0.0)$	$(1.5, 1.5)$	$(-1.0, 3.0)$	$(-1.0, 2.0)$	$(-1.0, 1.0)$	$(-1.0, -1.0)$	$(-1.0, -2.0)$
2	$(3.0, 0.0)$	$(3.0, -1.0)$	$(0.5, 0.5)$	$(-2.0, 2.0)$	$(-2.0, 1.0)$	$(-2.0, -1.0)$	$(-2.0, -2.0)$
3	$(2.0, 0.0)$	$(2.0, -1.0)$	$(2.0, -2.0)$	$(-0.5, -0.5)$	$(-3.0, 1.0)$	$(-3.0, -1.0)$	$(-3.0, -2.0)$

2. Player 2's strategy $s_B = 6$ is strictly dominated by $s_B = 0$:

	0	1	2	3	4	7
0	$(2.5, 2.5)$	$(0.0, 4.0)$	$(0.0, 3.0)$	$(0.0, 2.0)$	$(0.0, 1.0)$	$(0.0, -2.0)$
1	$(4.0, 0.0)$	$(1.5, 1.5)$	$(-1.0, 3.0)$	$(-1.0, 2.0)$	$(-1.0, 1.0)$	$(-1.0, -2.0)$
2	$(3.0, 0.0)$	$(3.0, -1.0)$	$(0.5, 0.5)$	$(-2.0, 2.0)$	$(-2.0, 1.0)$	$(-2.0, -2.0)$
3	$(2.0, 0.0)$	$(2.0, -1.0)$	$(2.0, -2.0)$	$(-0.5, -0.5)$	$(-3.0, 1.0)$	$(-3.0, -2.0)$

3. Player 2's strategy $s_B = 7$ is strictly dominated by $s_B = 0$:

	0	1	2	3	4
0	$(2.5, 2.5)$	$(0.0, 4.0)$	$(0.0, 3.0)$	$(0.0, 2.0)$	$(0.0, 1.0)$
1	$(4.0, 0.0)$	$(1.5, 1.5)$	$(-1.0, 3.0)$	$(-1.0, 2.0)$	$(-1.0, 1.0)$
2	$(3.0, 0.0)$	$(3.0, -1.0)$	$(0.5, 0.5)$	$(-2.0, 2.0)$	$(-2.0, 1.0)$
3	$(2.0, 0.0)$	$(2.0, -1.0)$	$(2.0, -2.0)$	$(-0.5, -0.5)$	$(-3.0, 1.0)$

No further strictly dominated strategies remain, and we can see that $(5, 3, 7) \xrightarrow{\text{IESDS}} (\{0 : 3\} \times \{04 :\})$, illustrating **Rule 5**. We note that the best responses are

- Player 1's best responses:
  - To $s_B = 0$: $s_A = 1$,
  - To $s_B = 1$: $s_A = 2$,
  - To $s_B = 2$: $s_A = 3$,
  - To $s_B = 3, 4$: $s_A = 0$.
- Player 2's best responses:
  - To $s_A = 0$: $s_B = 1$,
  - To $s_A = 1$: $s_B = 2$,
  - To $s_A = 2$: $s_B = 3$,
  - To $s_A = 3$: $s_B = 4$.

and thus no pure strategy Nash equilibria exist, once again illustrating **Rule C**.

Example 3.3.8 *(Asymmetric Game* $(4, 5, 6)$*)***:** Consider the asymmetric game with parameters $(R, M_A, M_B) = (4, 5, 6)$. The initial payoff matrix is:

	0	1	2	3	4	5	6
0	$(2, 2)$	$(0, 3)$	$(0, 2)$	$(0, 1)$	$(0, 0)$	$(0, -1)$	$(0, -2)$
1	$(3, 0)$	$(1, 1)$	$(-1, 2)$	$(-1, 1)$	$(-1, 0)$	$(-1, -1)$	$(-1, -2)$
2	$(2, 0)$	$(2, -1)$	$(0, 0)$	$(-2, 1)$	$(-2, 0)$	$(-2, -1)$	$(-2, -2)$
3	$(1, 0)$	$(1, -1)$	$(1, -2)$	$(-1, -1)$	$(-3, 0)$	$(-3, -1)$	$(-3, -2)$
4	$(0, 0)$	$(0, -1)$	$(0, -2)$	$(0, -3)$	$(-2, -2)$	$(-4, -1)$	$(-4, -2)$
5	$(-1, 0)$	$(-1, -1)$	$(-1, -2)$	$(-1, -3)$	$(-1, -4)$	$(-3, -3)$	$(-5, -2)$

We proceed with Iterated Elimination of Strictly Dominated Strategies (IESDS):

1. Player 1's strategy $s_A = 5$ is strictly dominated by $s_A = 0$:

	0	1	2	3	4	6
0	$(2, 2)$	$(0, 3)$	$(0, 2)$	$(0, 1)$	$(0, 0)$	$(0, -2)$
1	$(3, 0)$	$(1, 1)$	$(-1, 2)$	$(-1, 1)$	$(-1, 0)$	$(-1, -2)$
2	$(2, 0)$	$(2, -1)$	$(0, 0)$	$(-2, 1)$	$(-2, 0)$	$(-2, -2)$
3	$(1, 0)$	$(1, -1)$	$(1, -2)$	$(-1, -1)$	$(-3, 0)$	$(-3, -2)$
4	$(0, 0)$	$(0, -1)$	$(0, -2)$	$(0, -3)$	$(-2, -2)$	$(-4, -2)$

2. Player 2's strategy $s_B = 5$ is strictly dominated by $s_B = 0$:

	0	1	2	3	4
0	$(2, 2)$	$(0, 3)$	$(0, 2)$	$(0, 1)$	$(0, 0)$
1	$(3, 0)$	$(1, 1)$	$(-1, 2)$	$(-1, 1)$	$(-1, 0)$
2	$(2, 0)$	$(2, -1)$	$(0, 0)$	$(-2, 1)$	$(-2, 0)$
3	$(1, 0)$	$(1, -1)$	$(1, -2)$	$(-1, -1)$	$(-3, 0)$
4	$(0, 0)$	$(0, -1)$	$(0, -2)$	$(0, -3)$	$(-2, -2)$

No further strictly dominated strategies remain, and we can see that $(4, 5, 6) \xrightarrow{\text{IESDS}} (\{0 : 5\} \times \{0 : 6\})$, illustrating **Rule 6**. We note that the best responses are:

- Player 1's best responses:
  - To $s_B = 0$: $s_A = 1$,
  - To $s_B = 1$: $s_A = 2$,
  - To $s_B = 2$: $s_A = 3$,
  - To $s_B = 3, 4$: $s_A = 0$.
- Player 2's best responses:
  - To $s_A = 0$: $s_B = 1$,
  - To $s_A = 1$: $s_B = 2$,
  - To $s_A = 2$: $s_B = 3$,
  - To $s_A = 3, 4$: $s_B = 0$.

and thus no pure strategy Nash equilibria exist, illustrating **Rule C** again.

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