Data Structures

23. Heap (Priority Queues)

Motivation

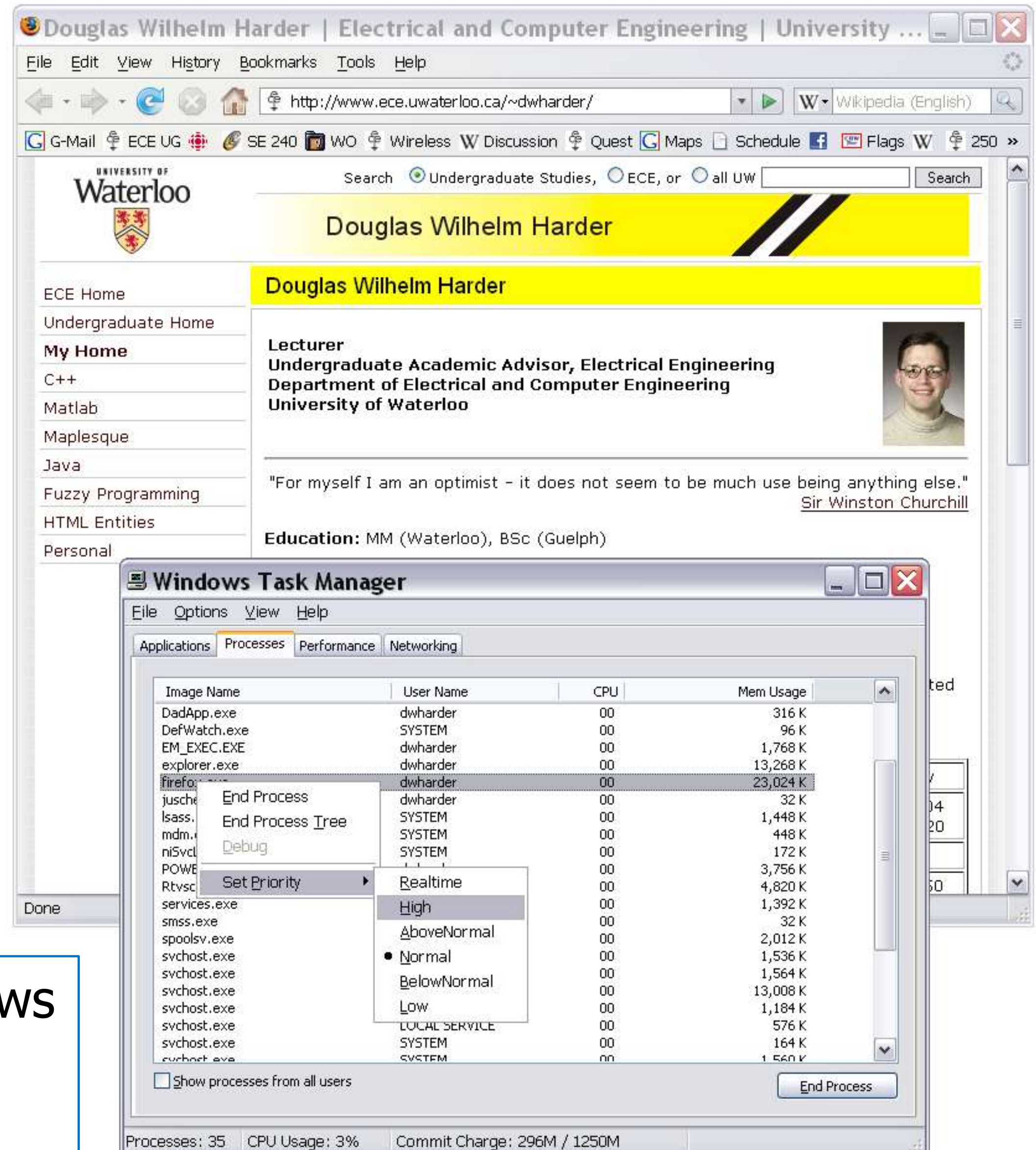
- With queues the order may be summarized by first in, first out

- Some tasks may be more important or timely than others
  - Higher priority

- Priority queues
  - Enqueue objects using a partial ordering based on priority
  - Dequeue that object which has highest priority

Applications Of Priority Queue

- Hold jobs for a printer in order of length
- Store packets on network routers in order of urgency
- Ordering CPU jobs
- Emergency room admission processing



The priority of processes in Windows may be set in the Windows Task Manager

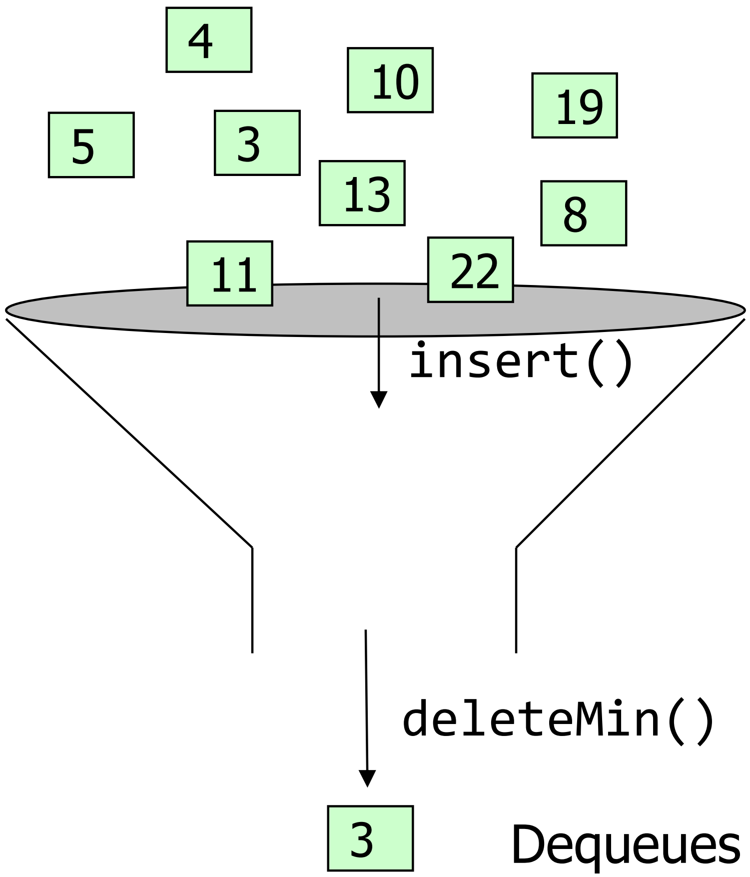
Priority Queue – ADT

- `insert` (i.e., enqueue)
  - Dynamic insert
  - Specification of a priority level (0-high, 1,2.. Low)

- `deleteMin` (i.e., dequeue)
  - Returns the current “highest priority” element in the queue
    - Element with the minimum priority level
  - Deletes that element from the queue

- Performance goal is to make the run time of each operation as close to `O(1)` as possible

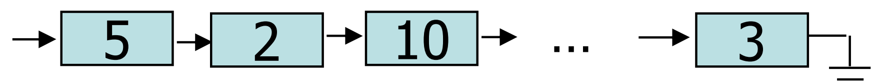
Priority Queue – ADT



Dequeues the next element with the highest priority

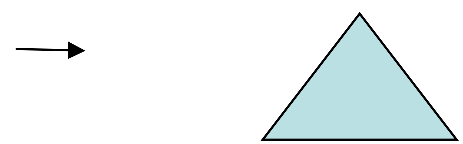
Simple Implementations

- Unordered linked list
  - `Insert` – `O(1)` step
  - `deleteMin` – `O(n)` steps



- Ordered linked list
  - `insert` – `O(n)` steps
  - `deleteMin` – `O(1)` step

- Balanced binary tree, e.g., AVL Tree
  - `insert` – $O(\log_2 n)$ steps
  - `deleteMin` in how many steps?
    - Find min – $O(\log_2 n)$ steps
    - Delete – $O(\log_2 n)$ steps



Can we build a data structure better suited to store and retrieve priorities?

Binary Heap

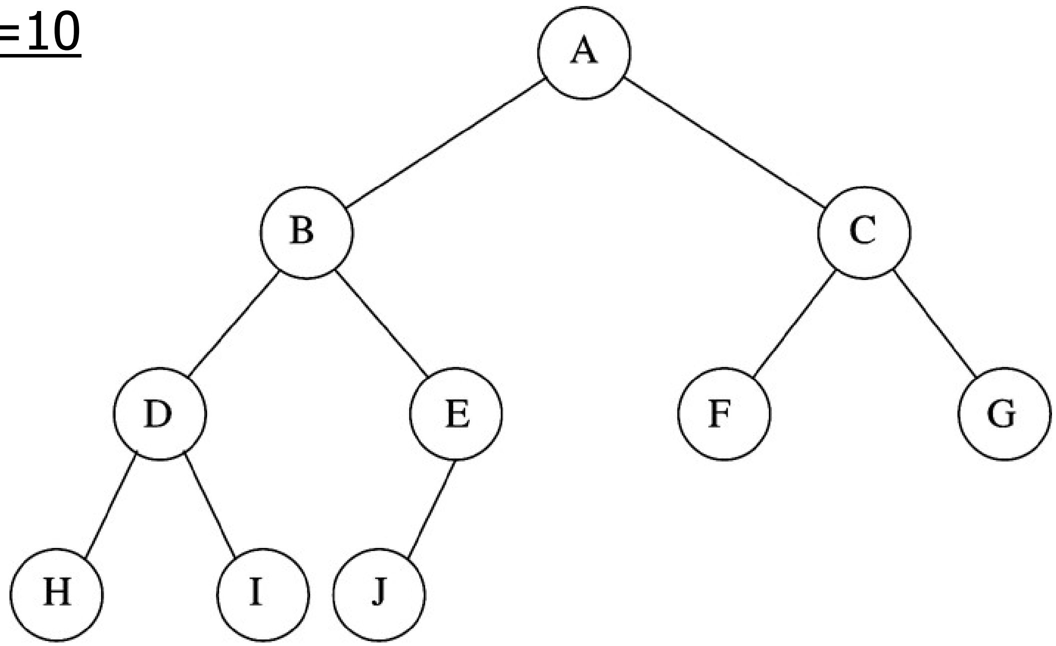
Binary Heap

- A binary heap is a binary tree with two properties
  - Structure property
  - Heap-order property

Binary Heap – Structure Property

- A binary heap is (almost) complete binary tree
  - Each level (except possibly the bottom most level) is completely filled
  - The bottom most level may be partially filled (from left to right)

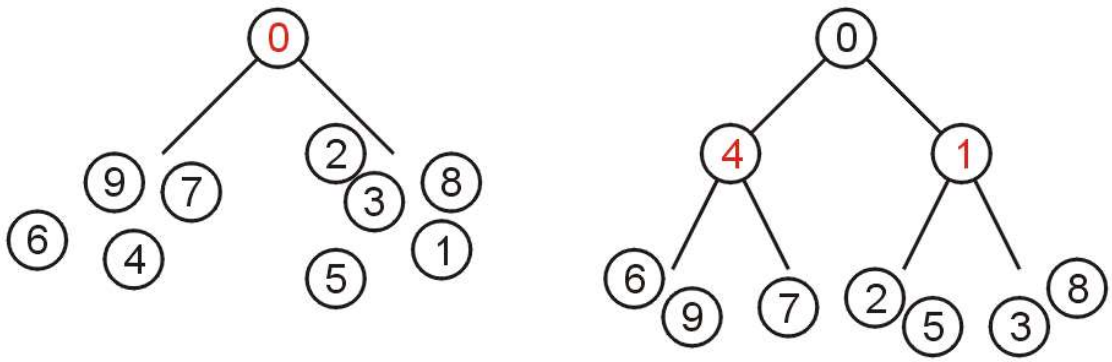
N=10



Every level (except last) saturated

Binary Heap – Heap-Order Property

- Min-Heap property
  - Key associated with the root is less than or equal to the keys associated with either of the sub-trees (if any)
  - Both of the sub-trees (if any) are also binary min-heaps



- Properties of min-heap
  - A single node is a min-heap
  - Minimum key always at root
  - For every node `X`, `key(parent(X)) ≤ key(X)`
  - No relationship between nodes with similar key

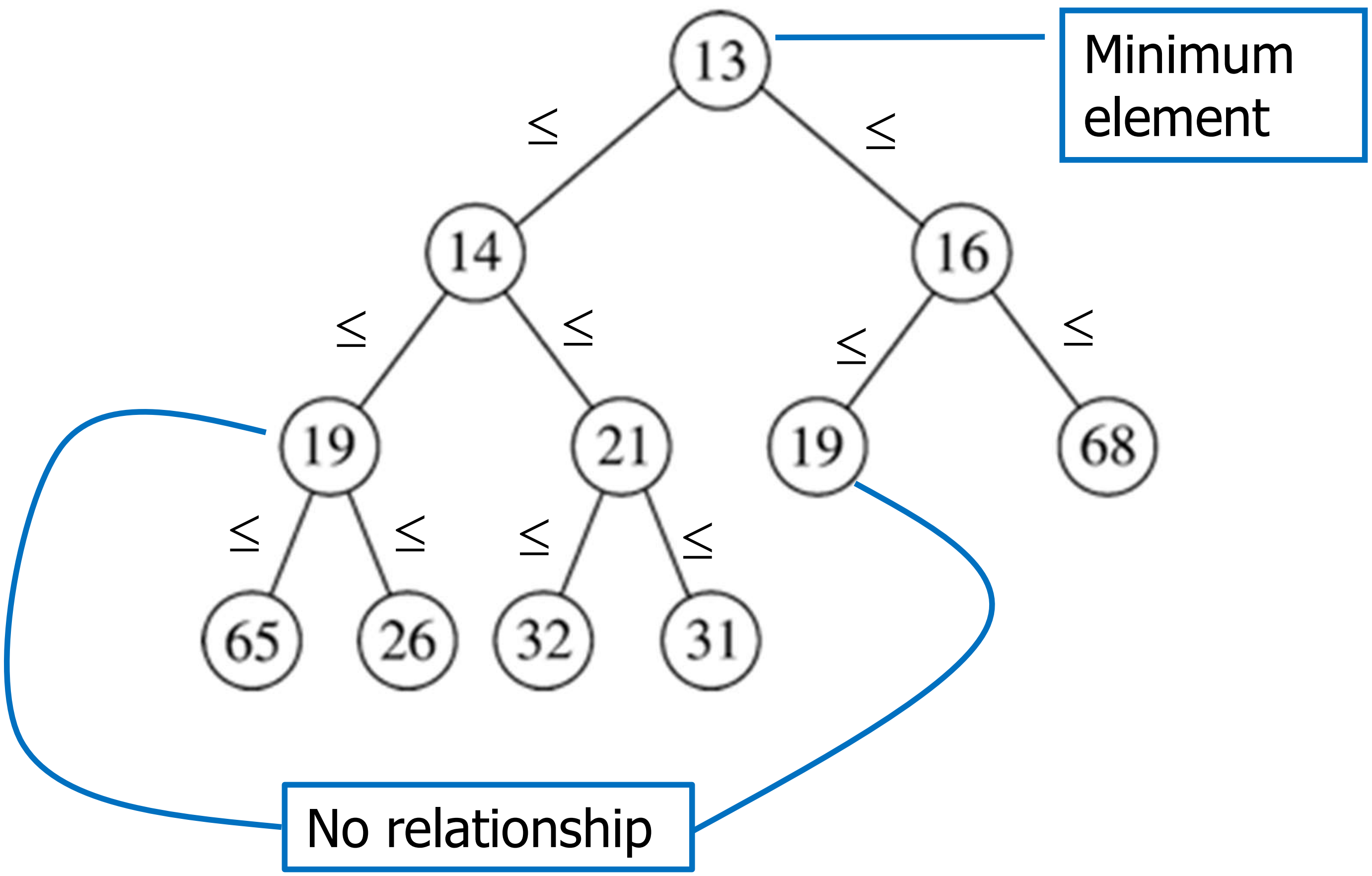
Binary Heap – Heap-Order Property

- Max-Heap property
  - Maximum key at the root
  - For every node `X`, `key(parent(X)) ≥ key(X)`

- `Insert` and `deleteMin` must maintain heap-order property

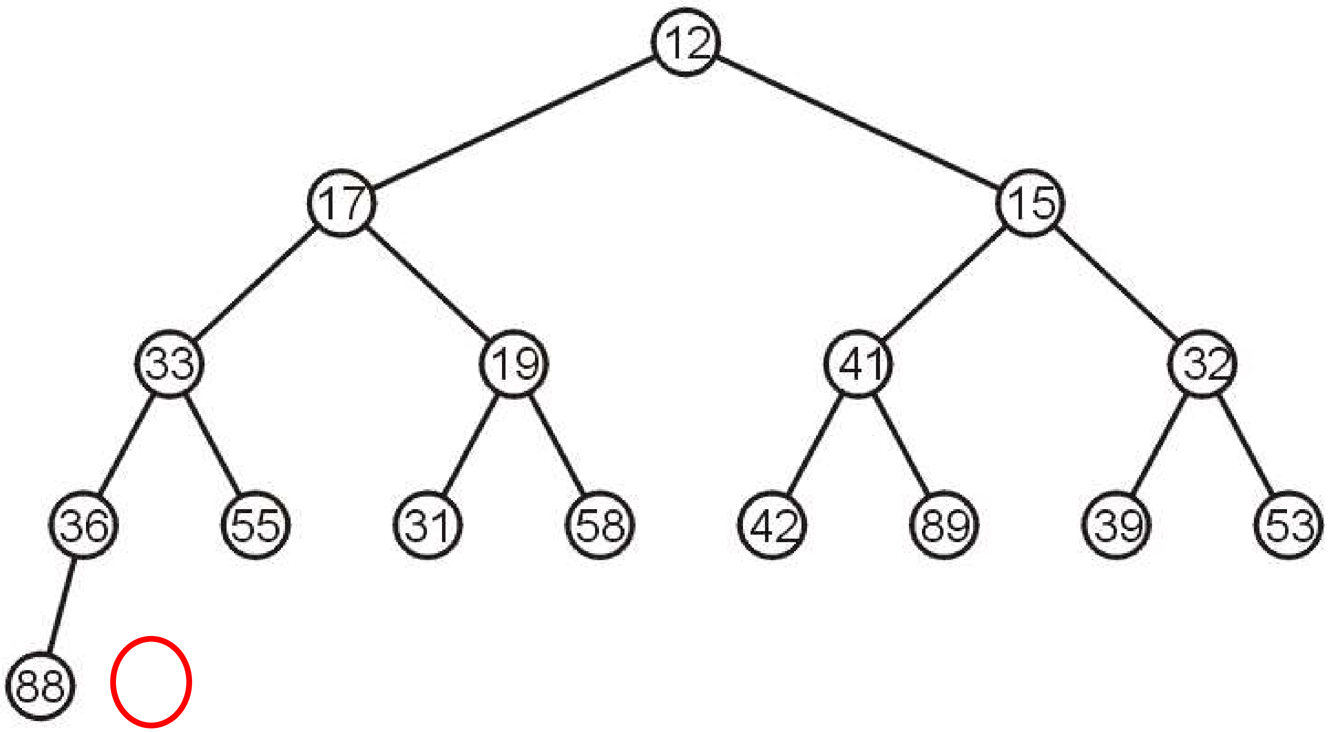
Heap-Order Property – Example

- Min-Heap



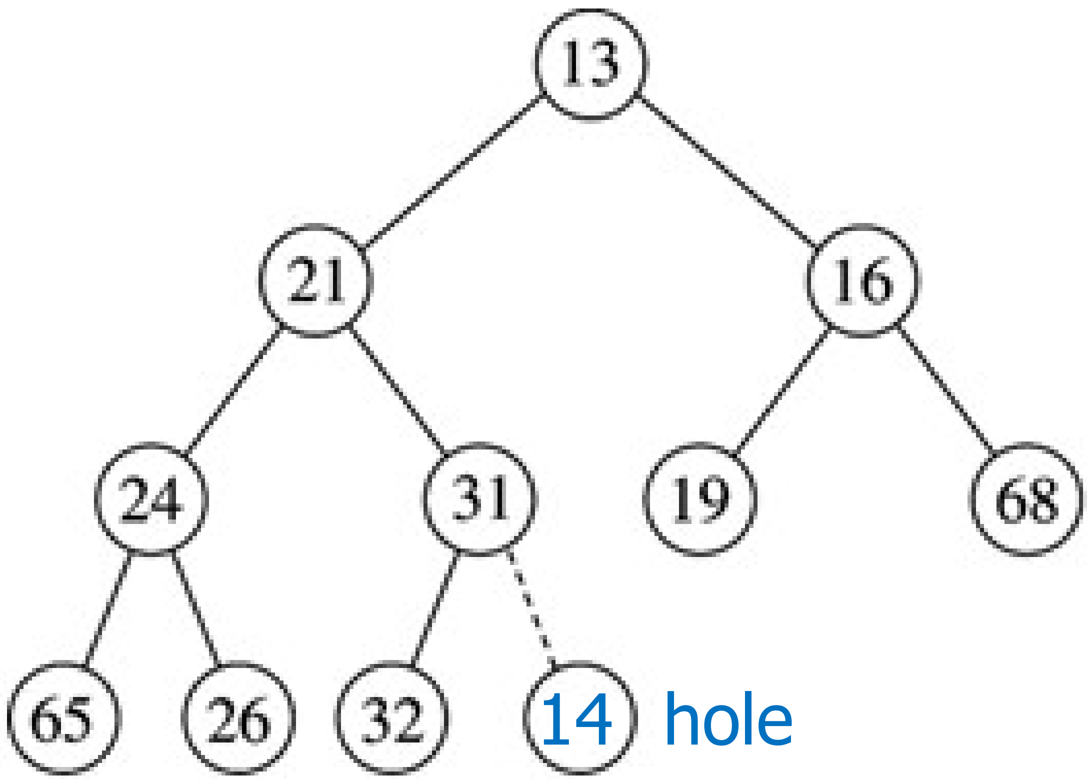
Heap Operations – `insert`

- Insert new element into the heap at the next available slot ("hole")
  - Maintaining (almost) complete binary tree
- Percolate the element up the heap while heap-order property not satisfied



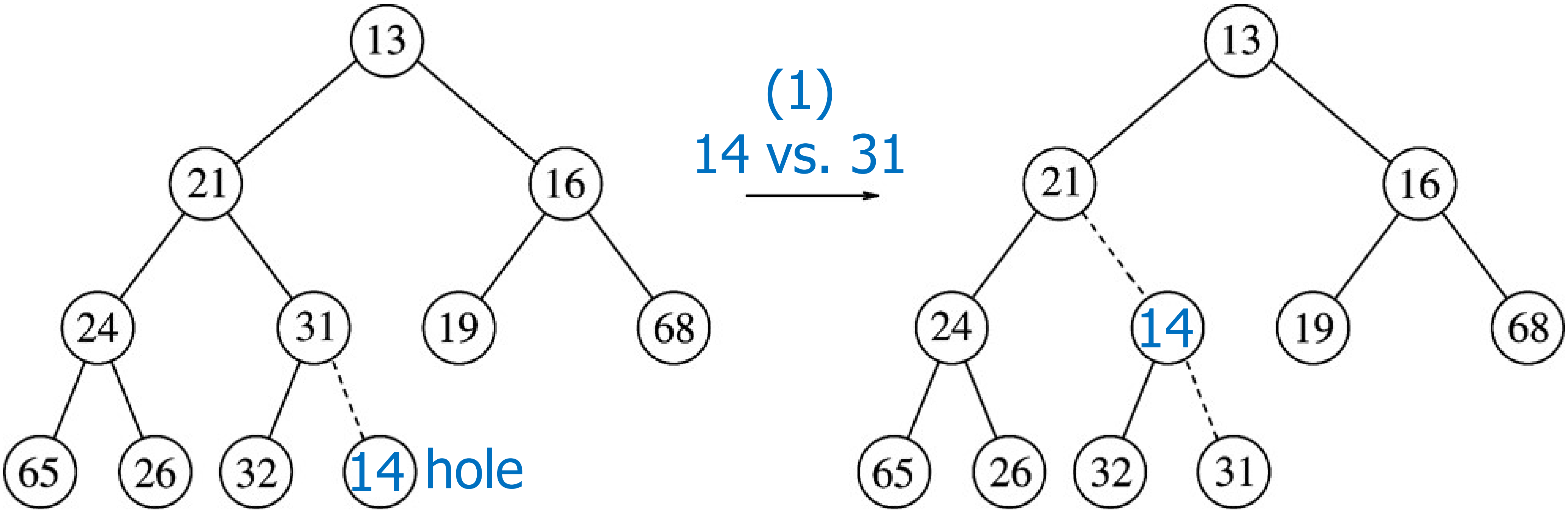
Heap Insert – Example

- Insert 14



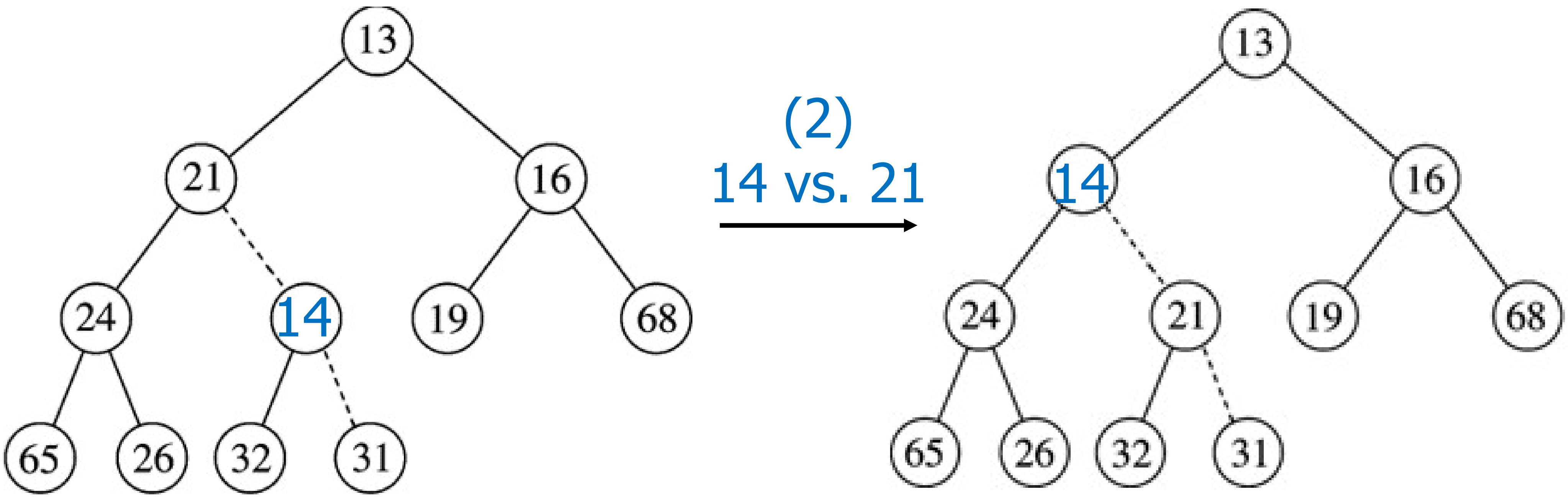
Heap Insert – Example

- Insert 14



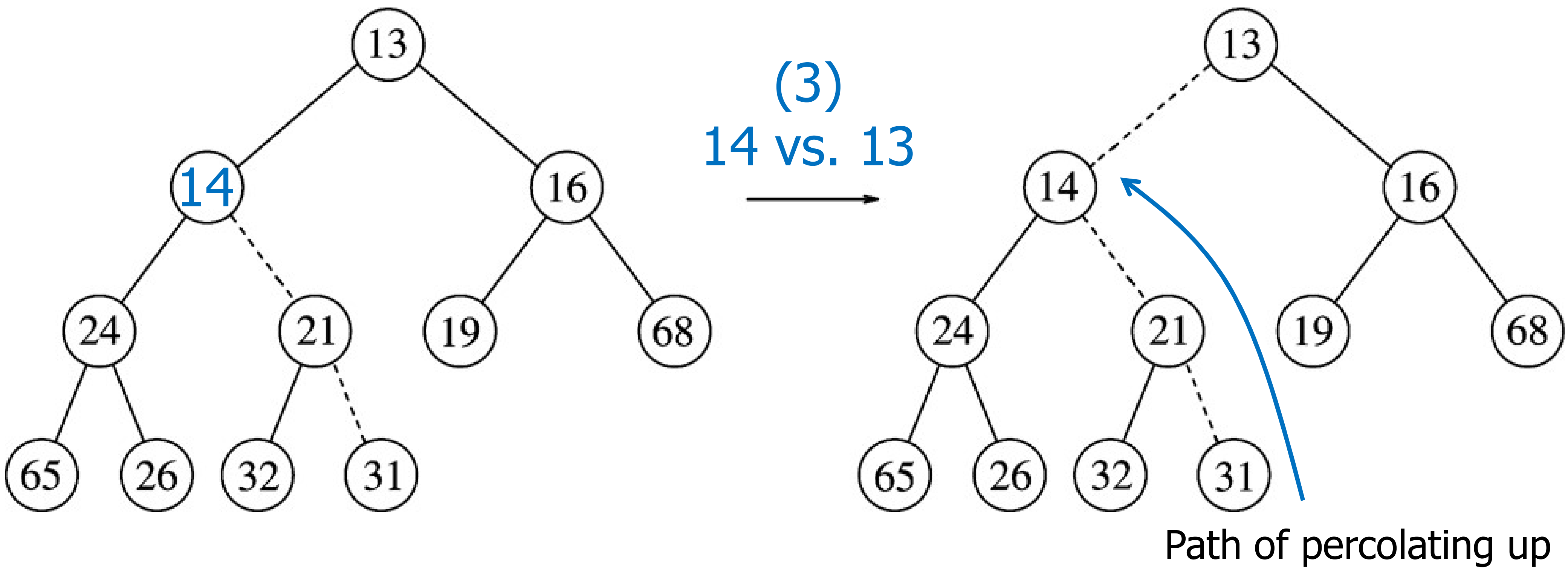
Heap Insert – Example

- Insert 14



Heap Insert – Example

- Insert 14

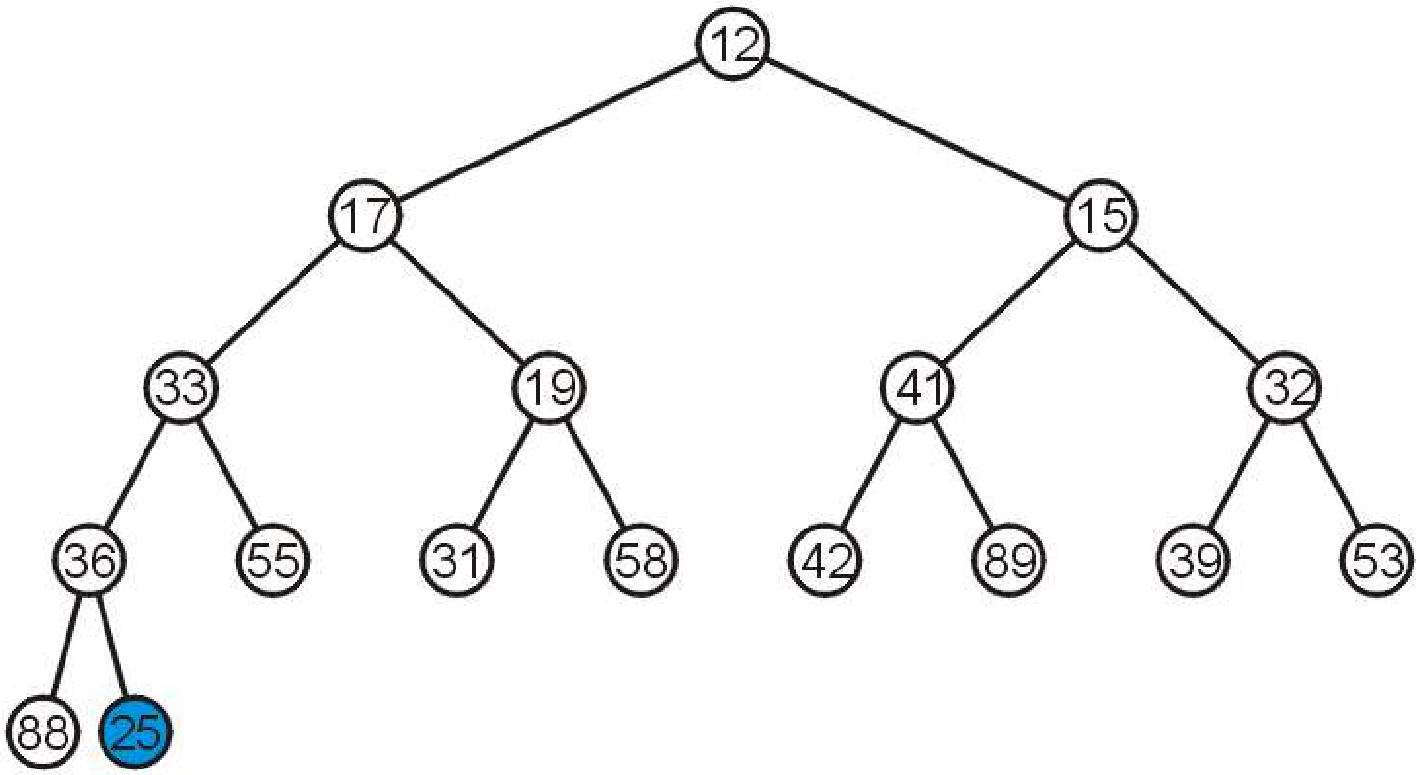


✓ Heap order property

✓ Structure property

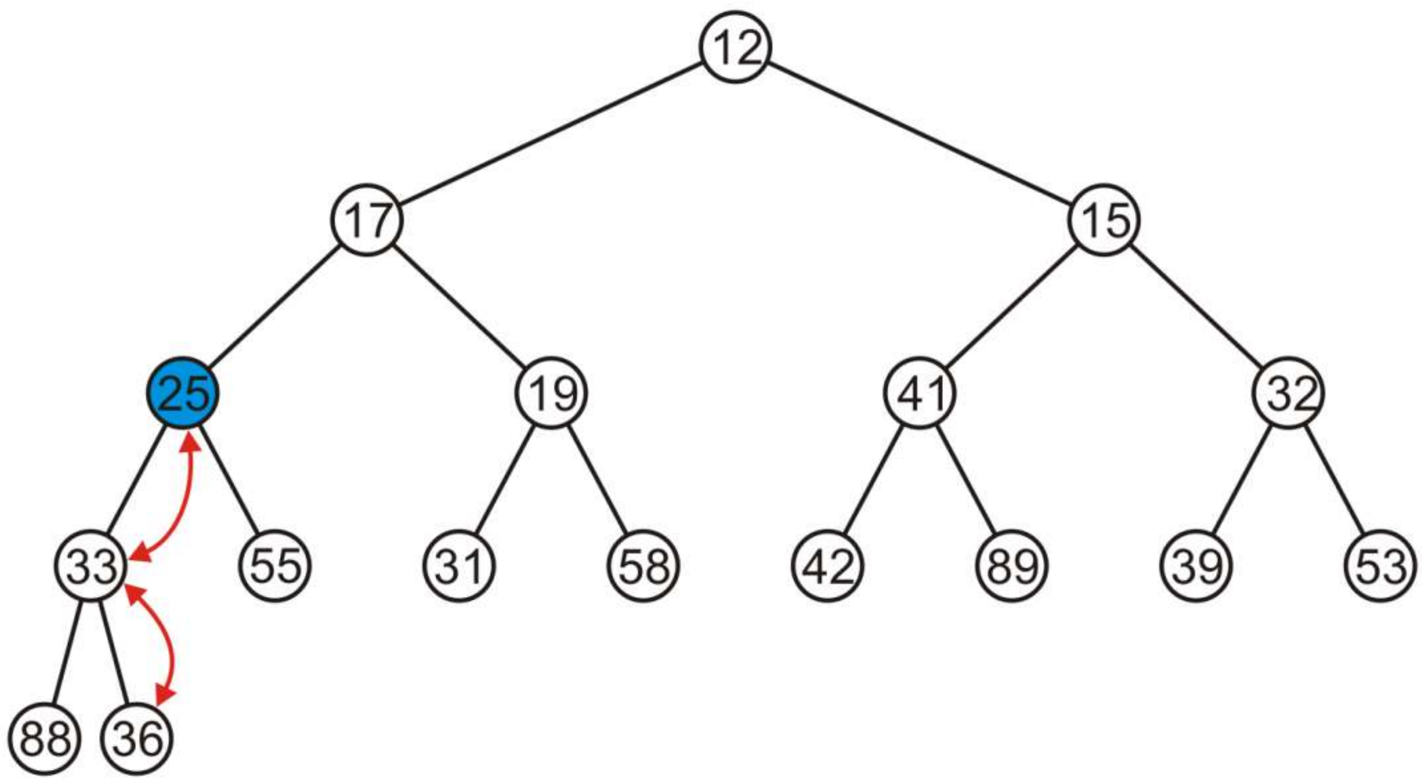
Heap Insert – Example

- Insert 25



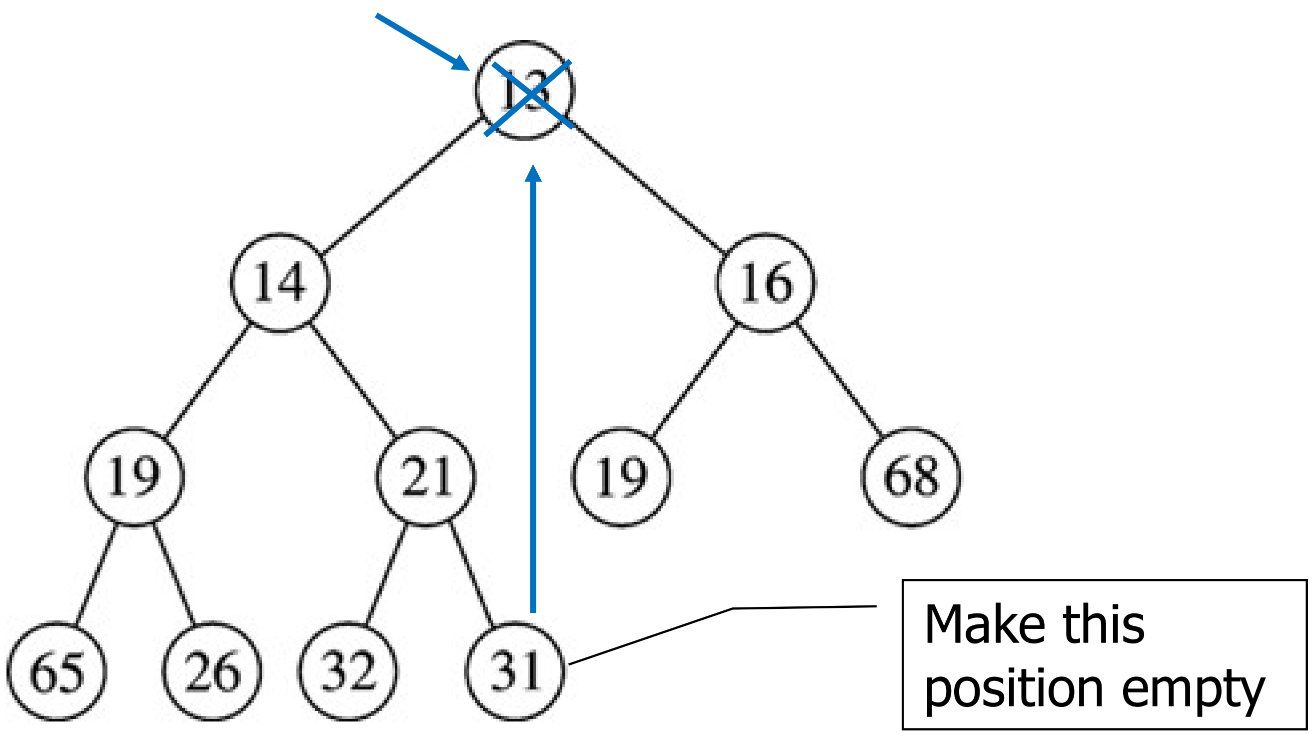
Heap Insert – Example

- Percolate 25 up into its appropriate location
  - The resulting heap is still a complete tree

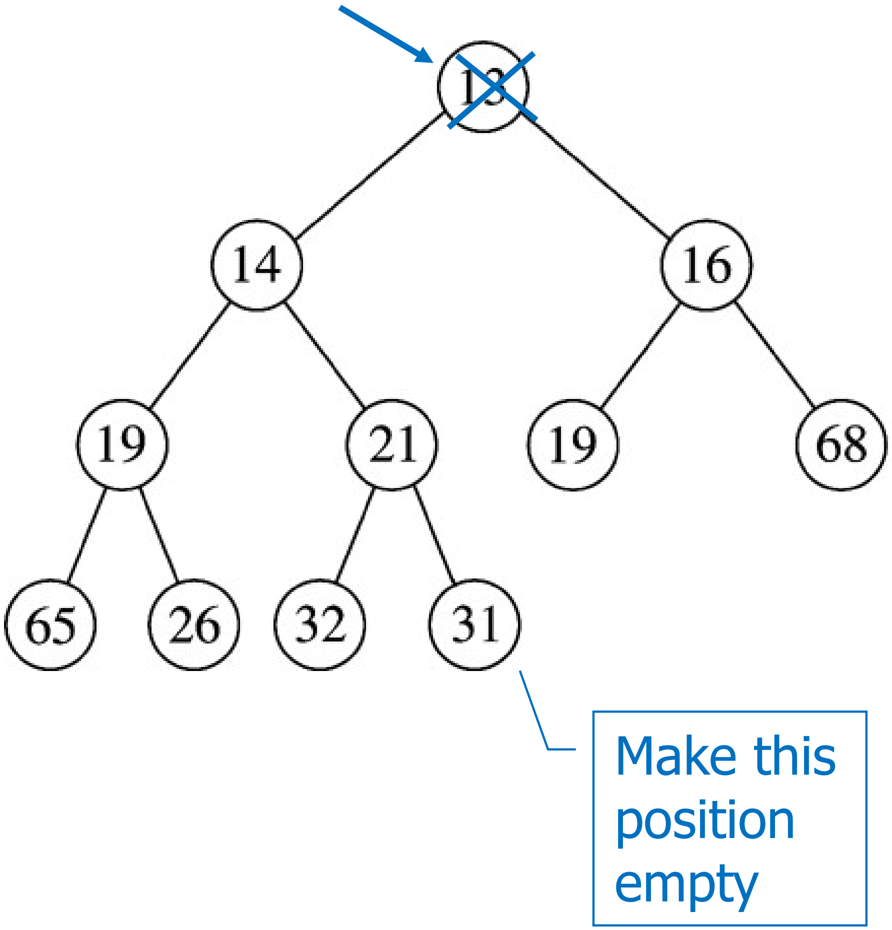


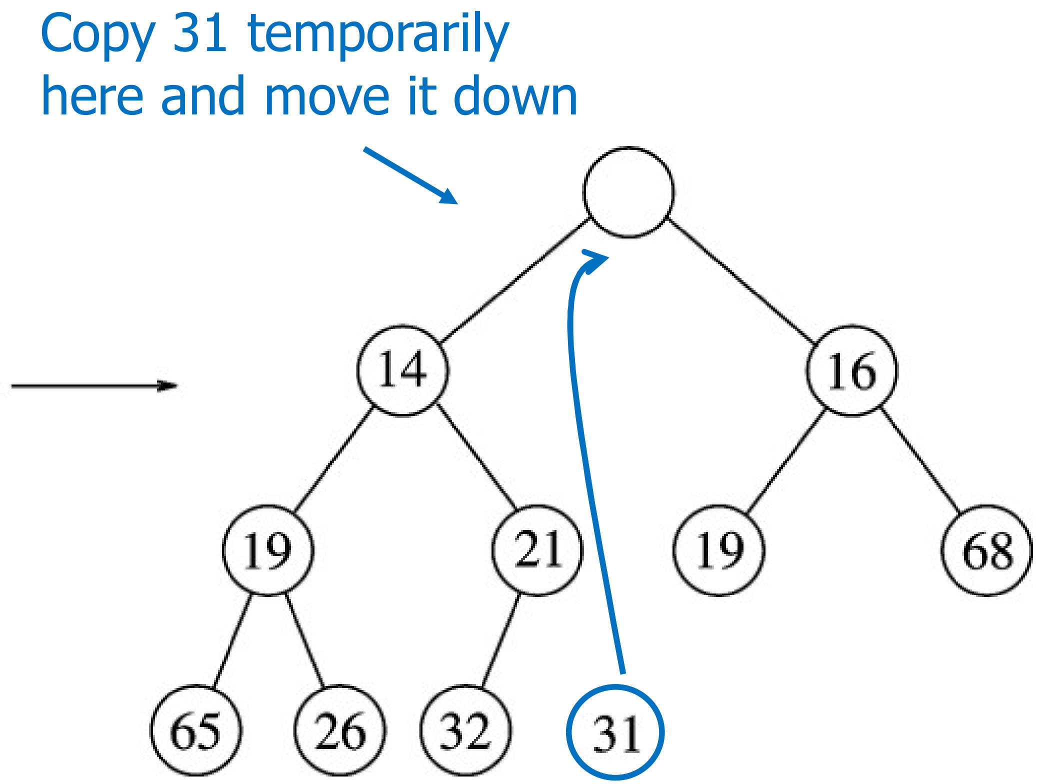
Heap Operation – `deleteMin`

- Minimum element is always at the root
  - Return the element at the root and delete it
- Heap decreases by one in size
- Move last element of the tree into hole at root
- Percolate down while heap-order property not satisfied



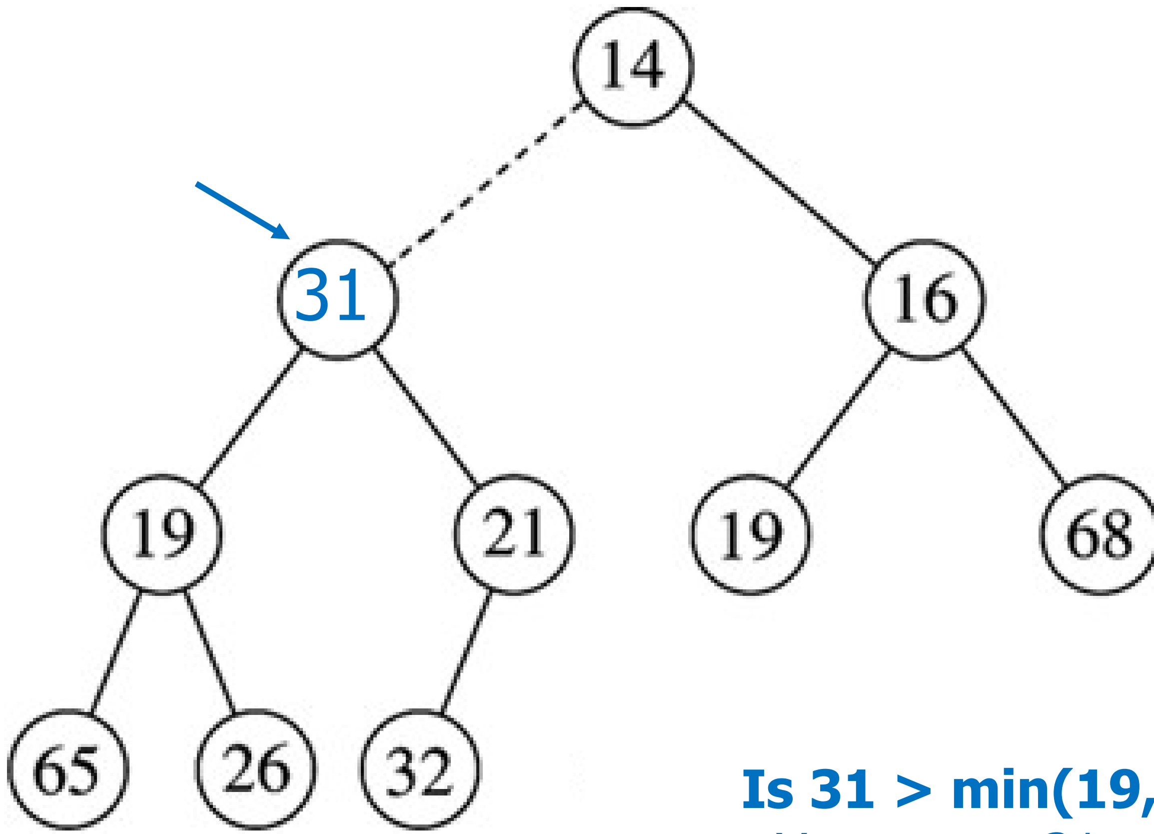
`deleteMin` – Example





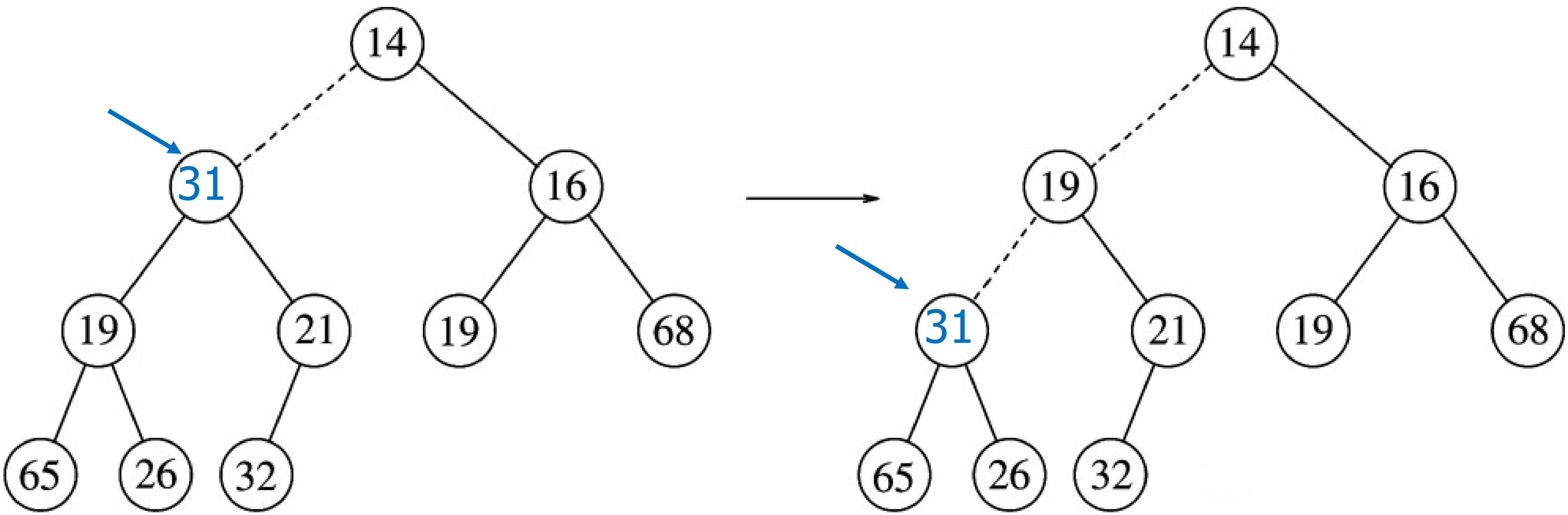
Is 31 > min(14,16)?
- Yes - swap 31 with min(14,16)

`deleteMin` – Example



Is 31 > min(19,21)?
- Yes - swap 31 with min(19,21)

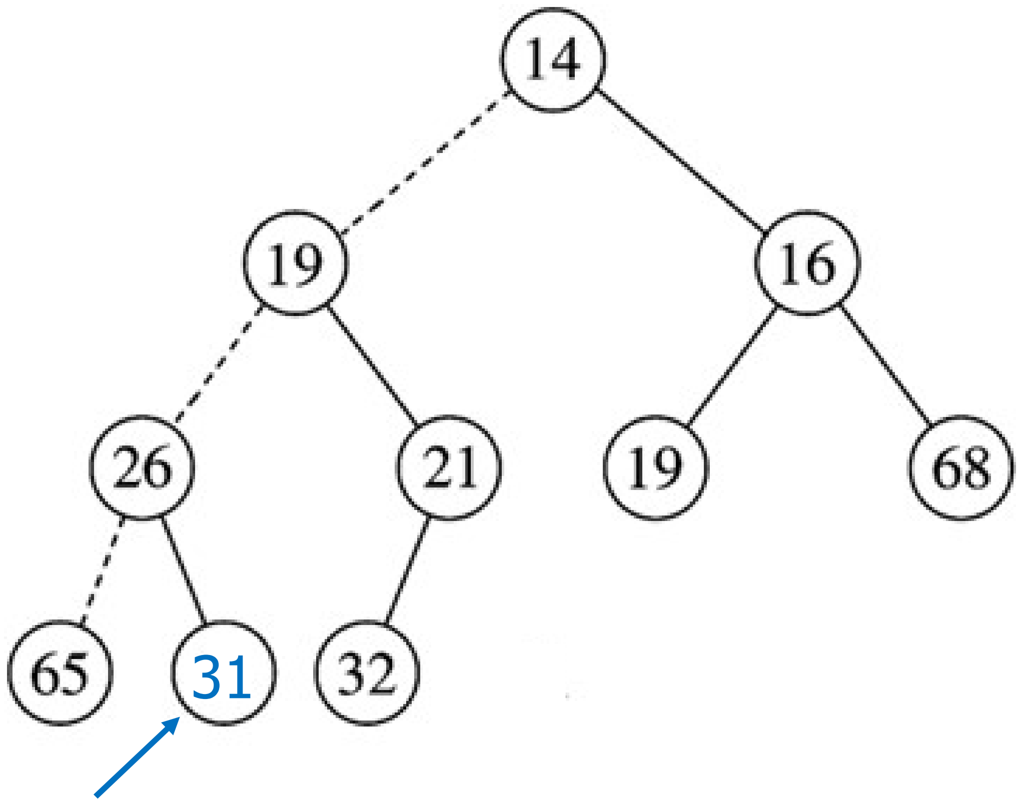
deleteMin – Example



Is 31 > min(19,21)?
- Yes - swap 31 with min(19,21)

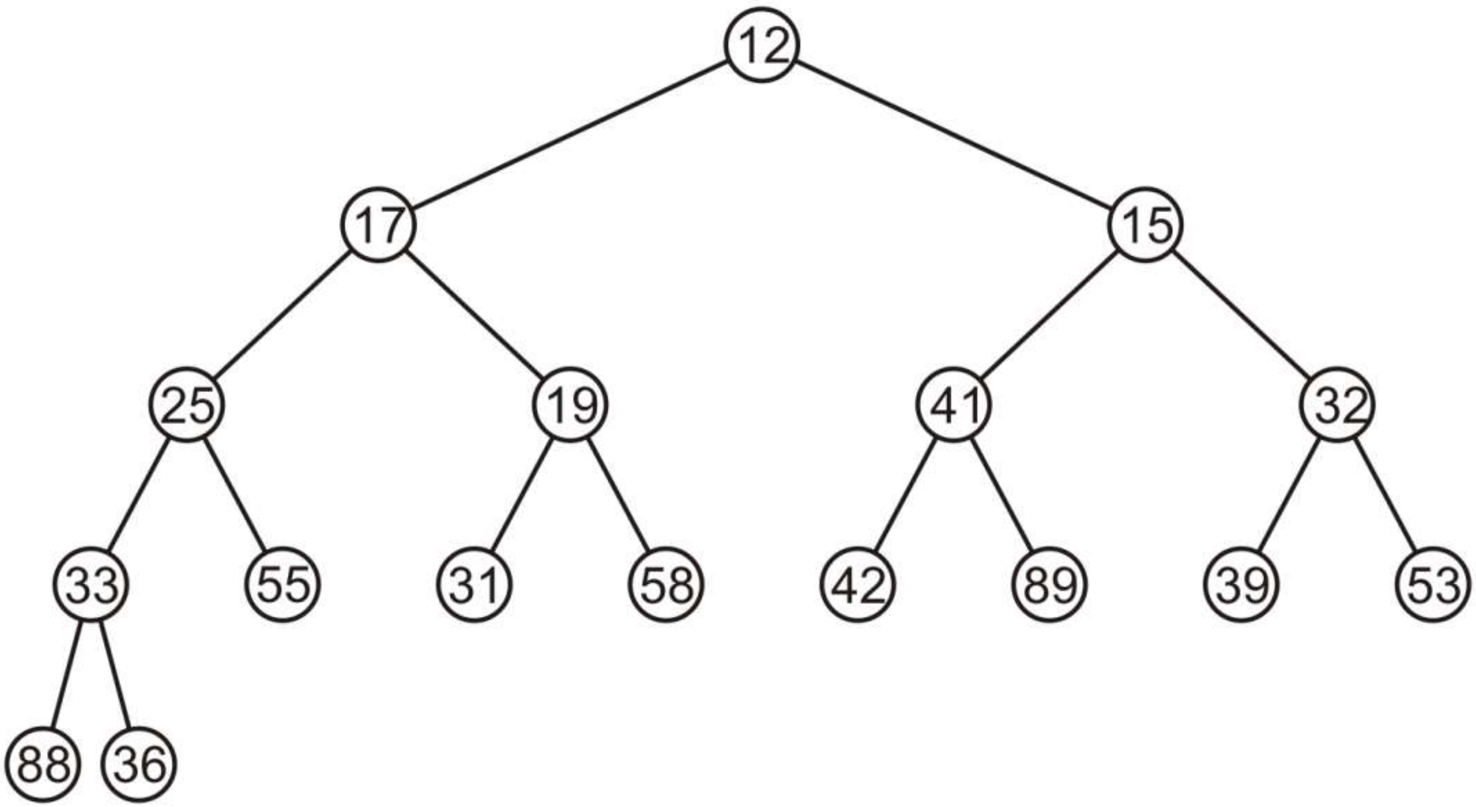
Is 31 > min(65,26)?
- Yes - swap 31 with min(65,26)

`deleteMin` – Example



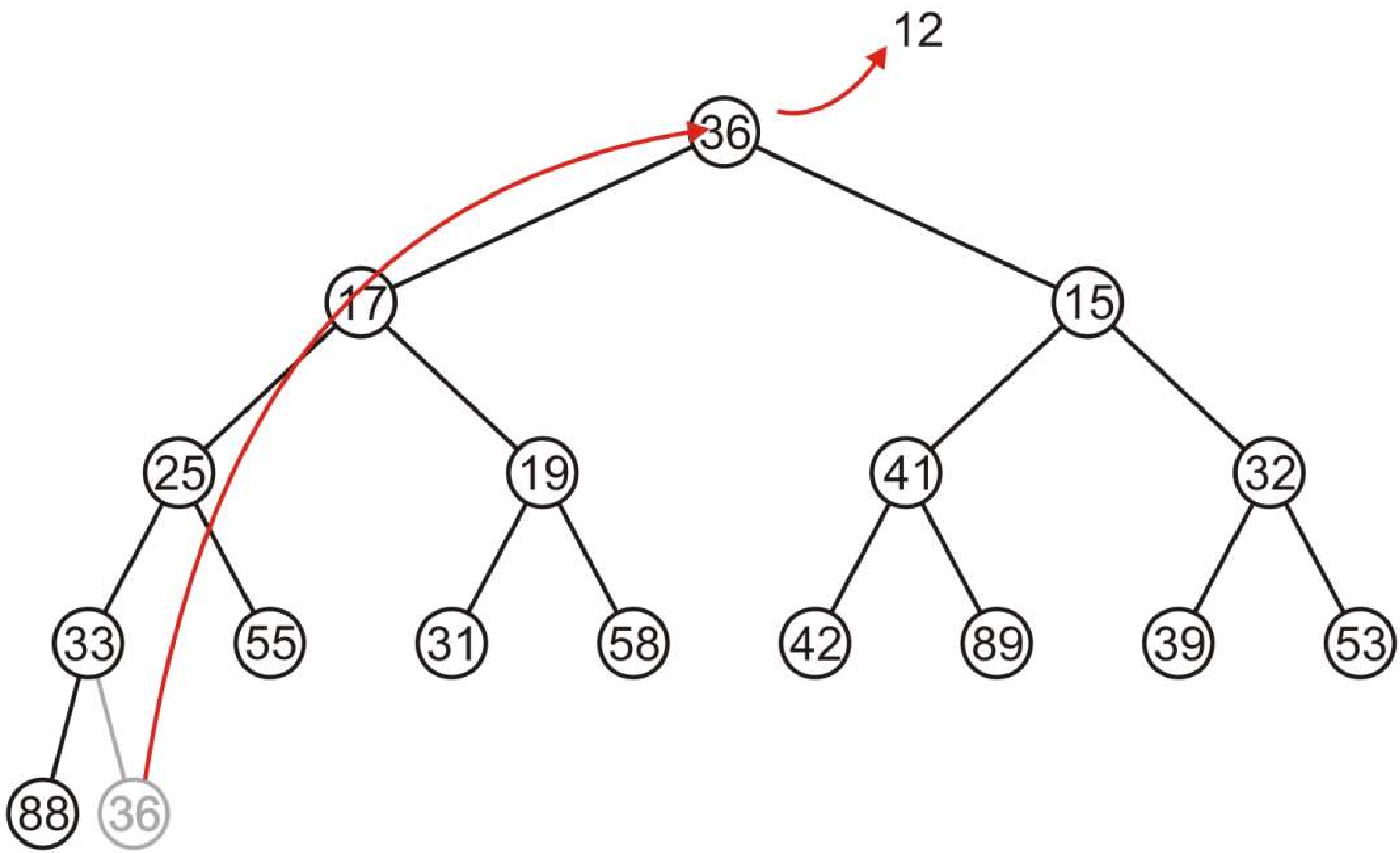
`deleteMin` – Example

- `deleteMin` will dequeue element 12 from the top



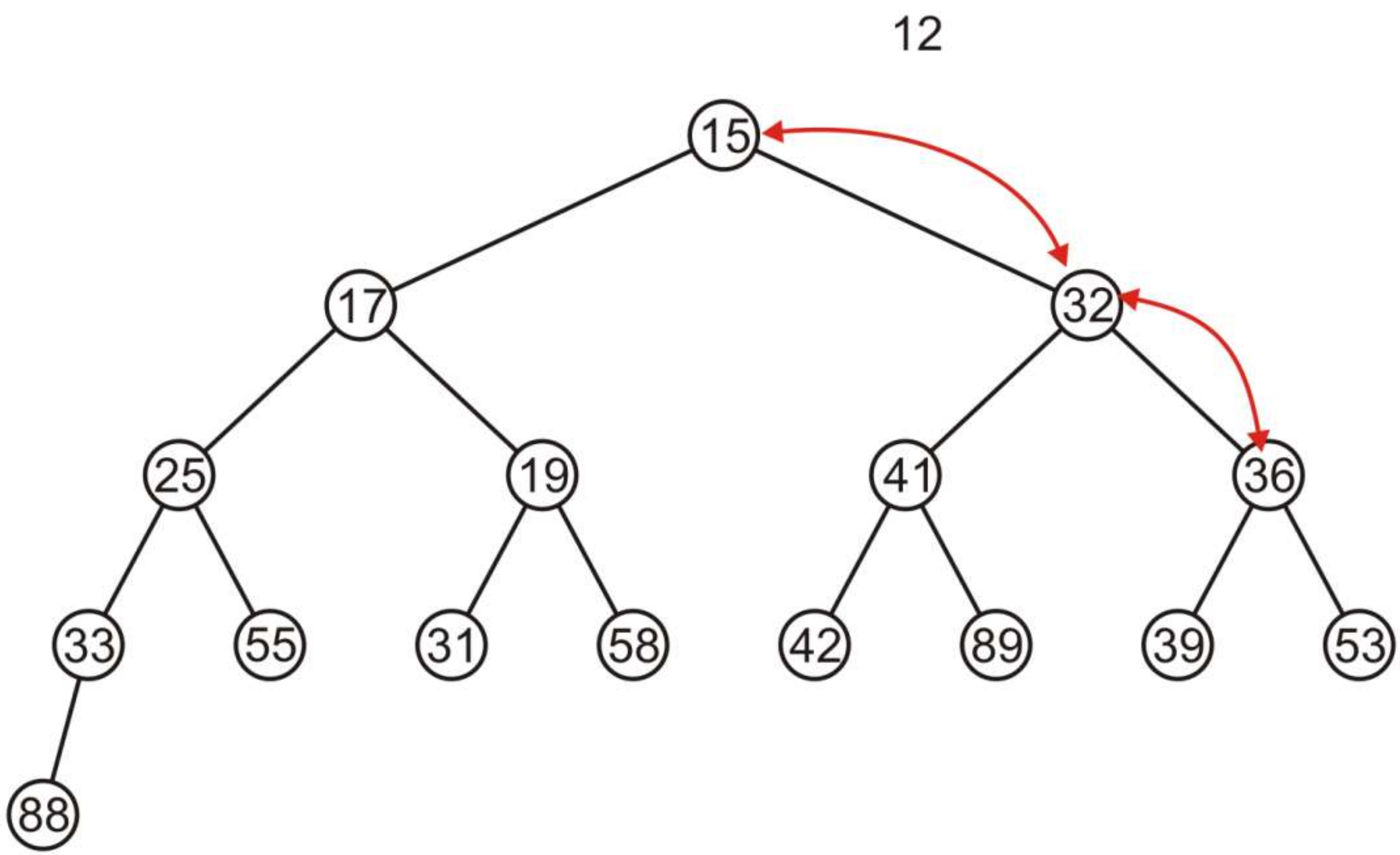
`deleteMin` – Example

- Copy the last entry in the heap to the root



`deleteMin` – Example

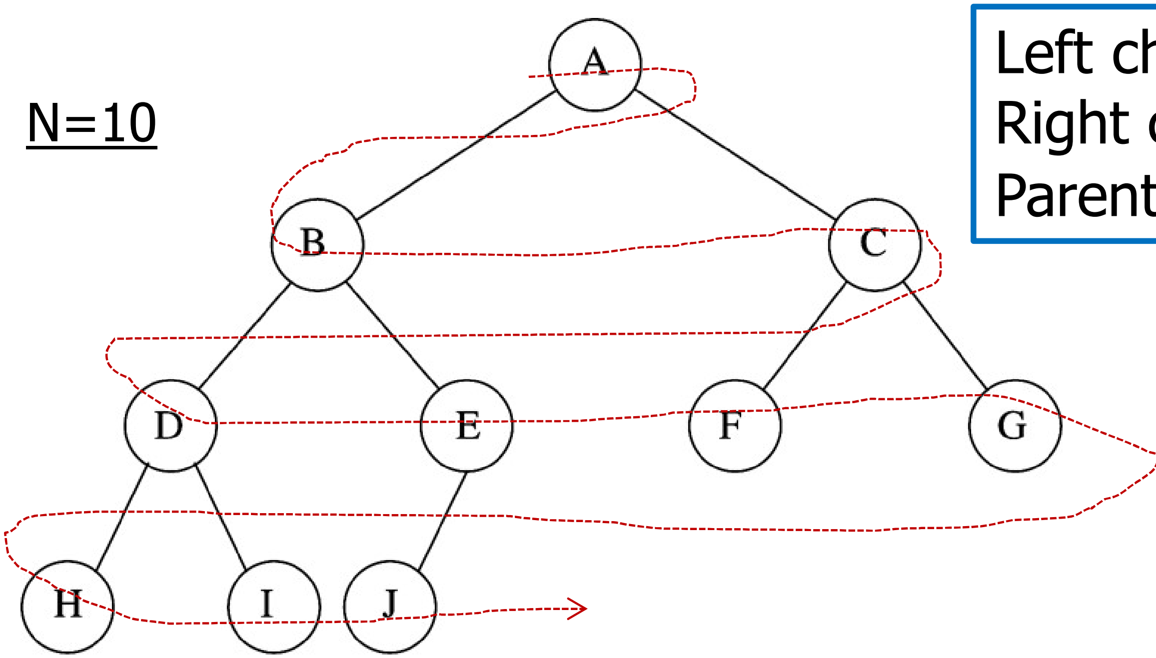
- Percolate 36 down swapping it with the smallest of its children
  - Halt when both children are larger



Array-Based Implementation Of Binary Tree

Left child(i) = at position 2i
Right child(i) = at position 2i + 1
Parent(i) = at position $\lfloor i / 2 \rfloor$

N=10



Array representation:

0	1	2	3	4	5	6	7	8	9	10	11	12	13
	A	B	C	D	E	F	G	H	I	J			

i

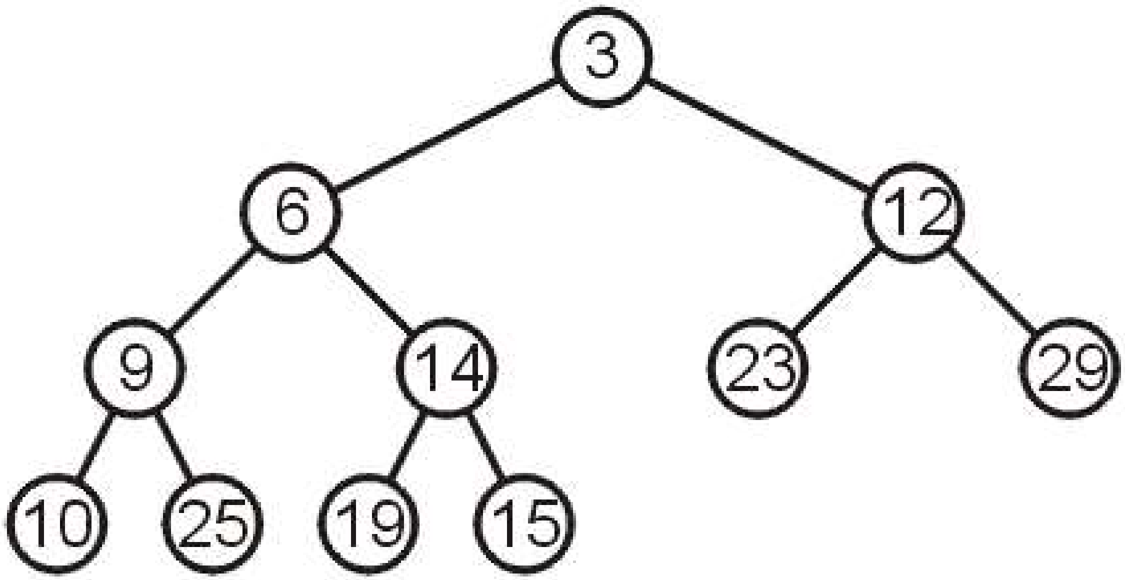
2i + 1

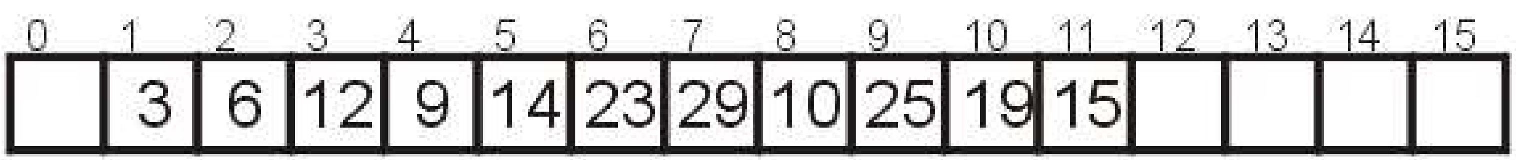
$\lfloor i / 2 \rfloor$

2i

Array-Based Implementation Of Binary Heap

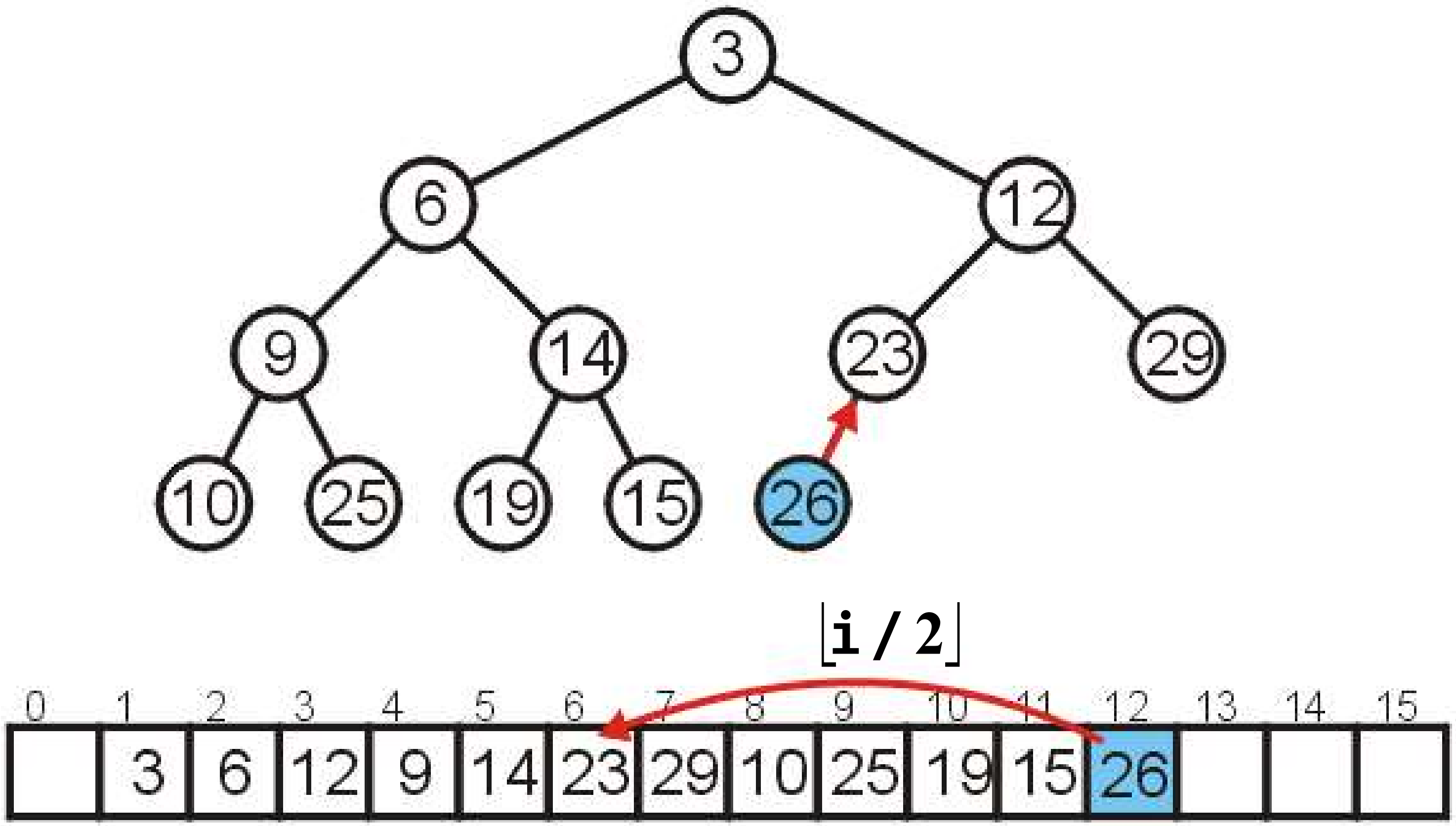
- Consider the following heap, both as a tree and in its array representation





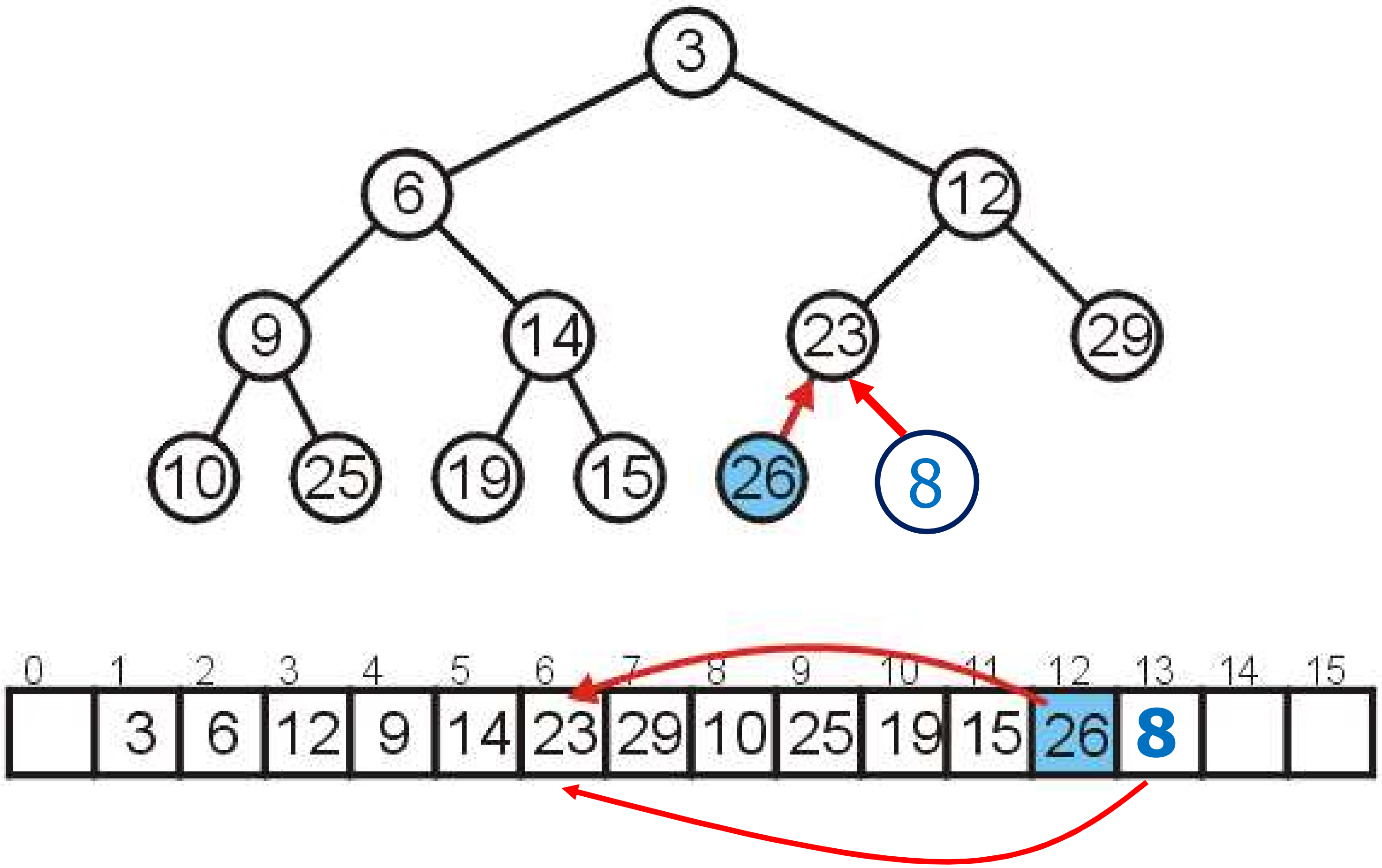
Array-Based Implementation – `insert`

- Inserting 26 requires no changes



Array-Based Implementation – `insert`

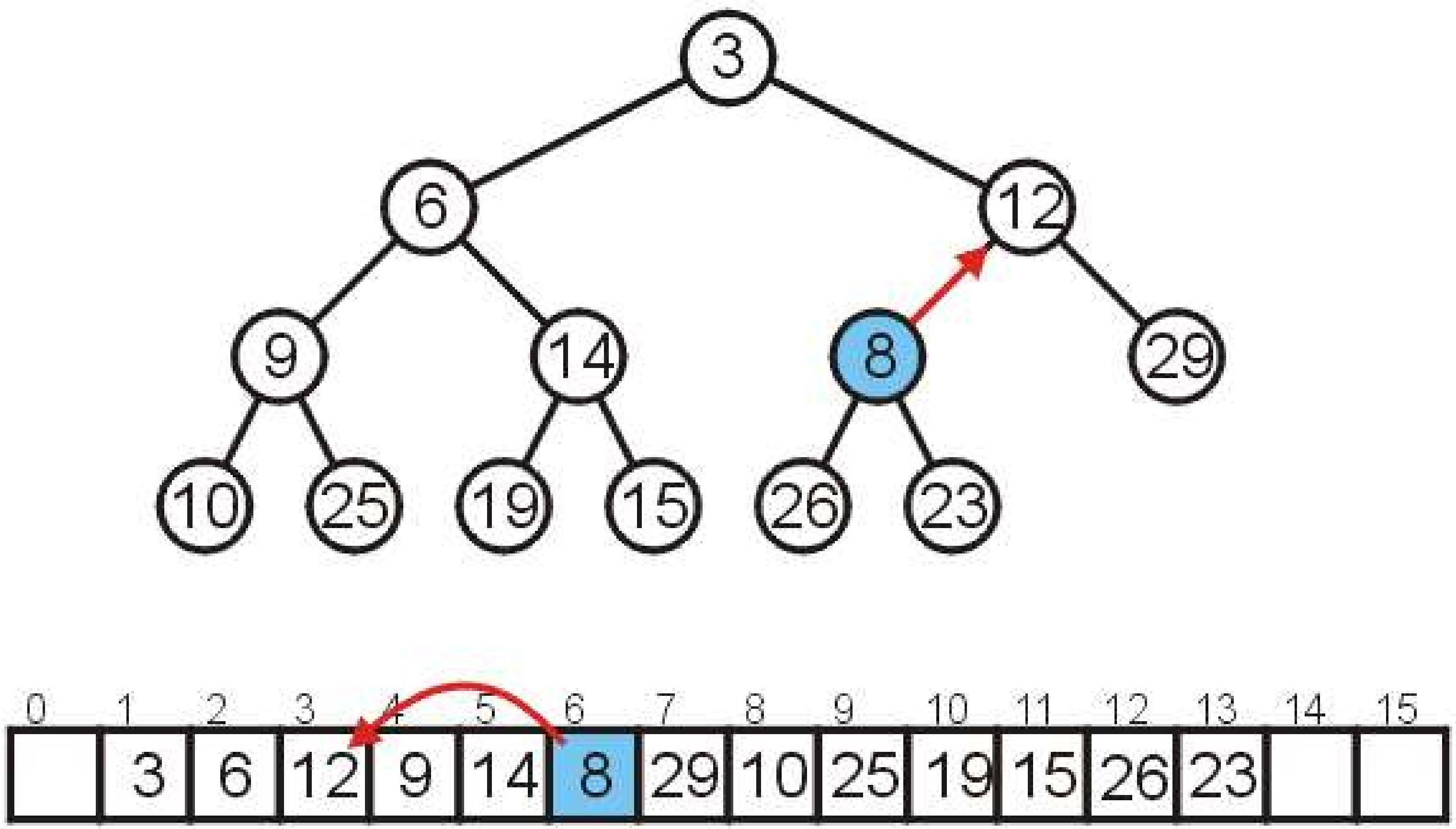
- Inserting 8 requires a few percolations
  - Swap 8 and 23



$\lfloor i / 2 \rfloor$

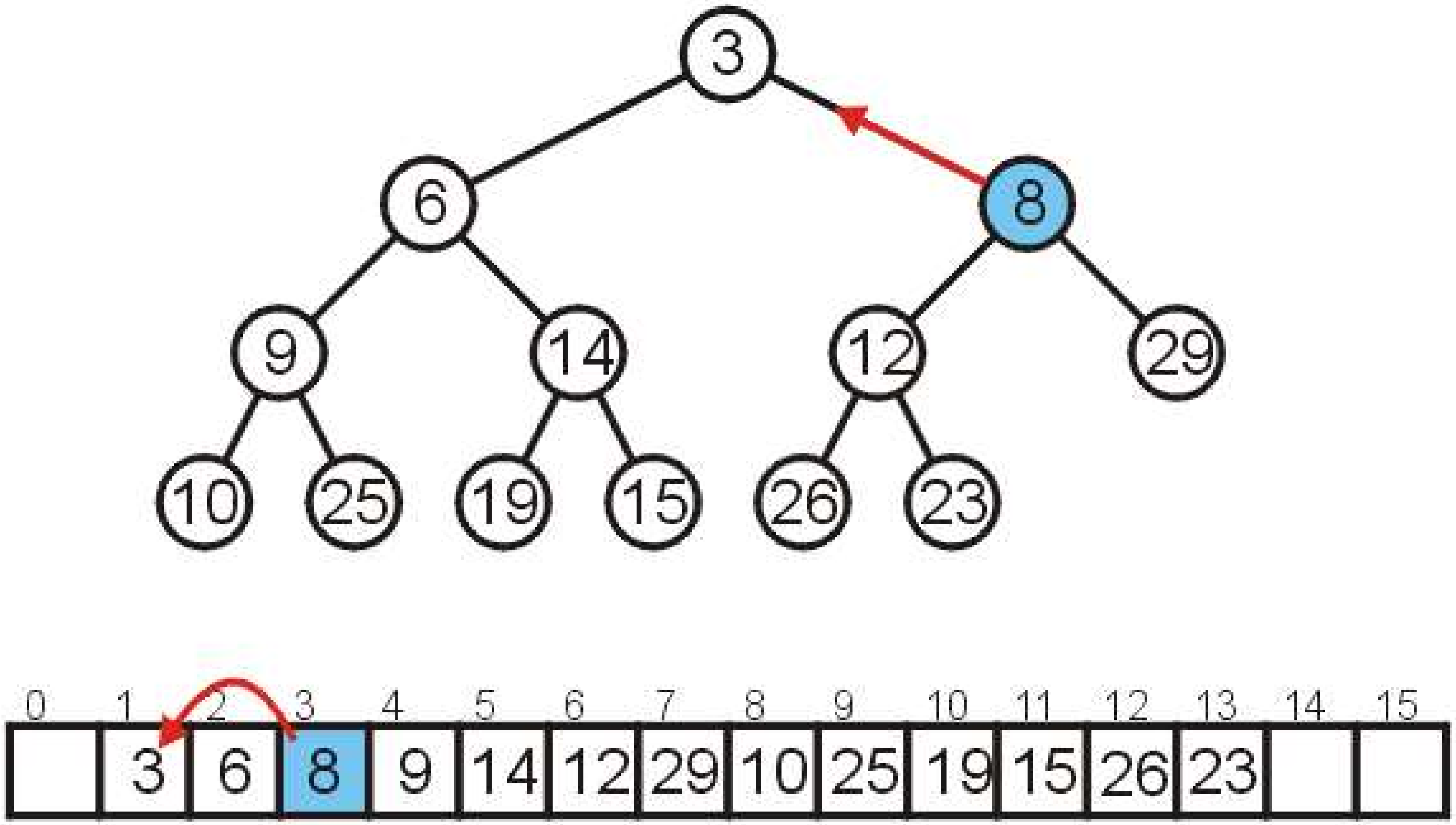
Array-Based Implementation – `insert`

- Swap 8 and 12



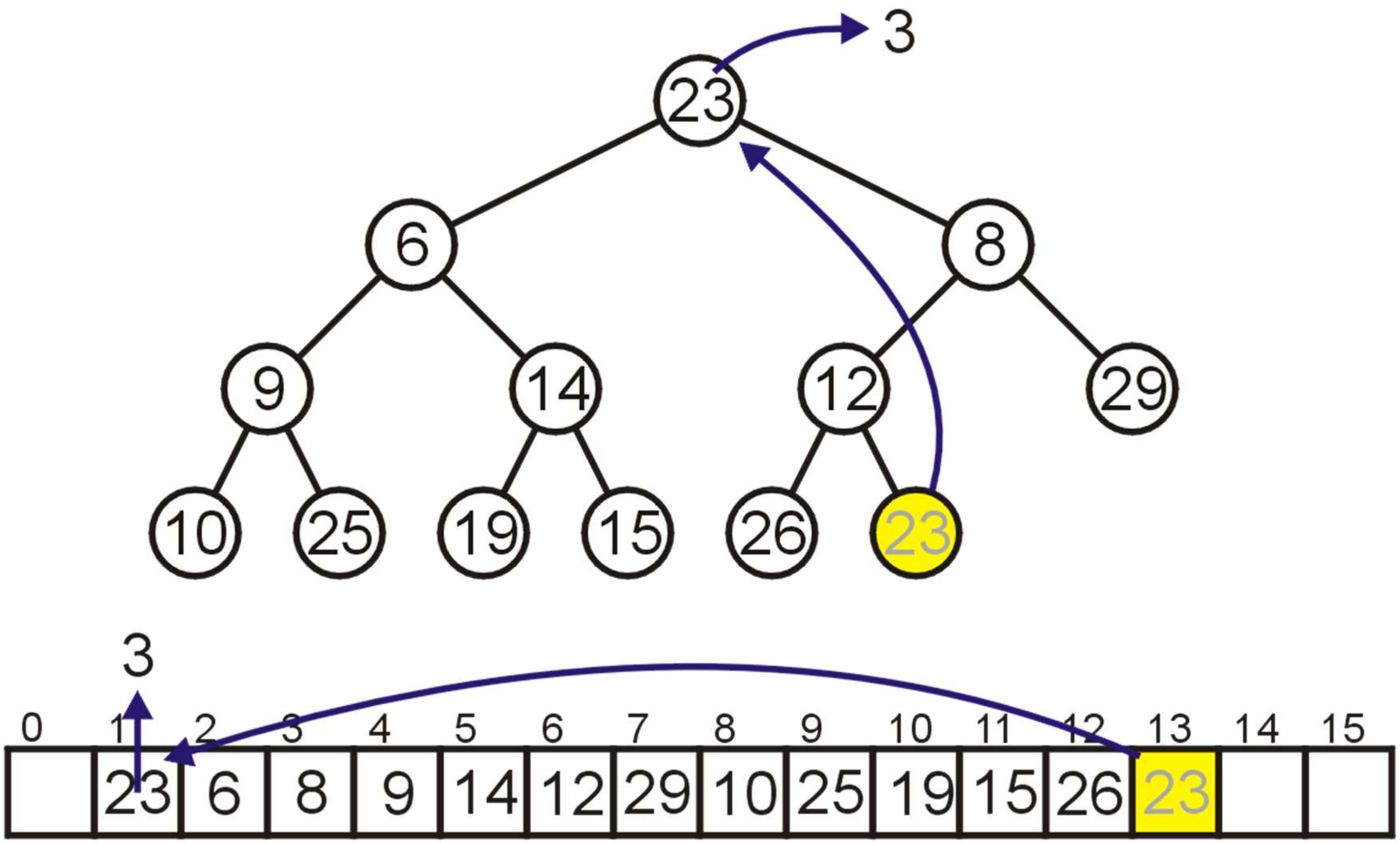
Array-Based Implementation – `insert`

- At this point, 8 is greater than its parent, so we are finished



Array-Based Implementation – `deleteMin`

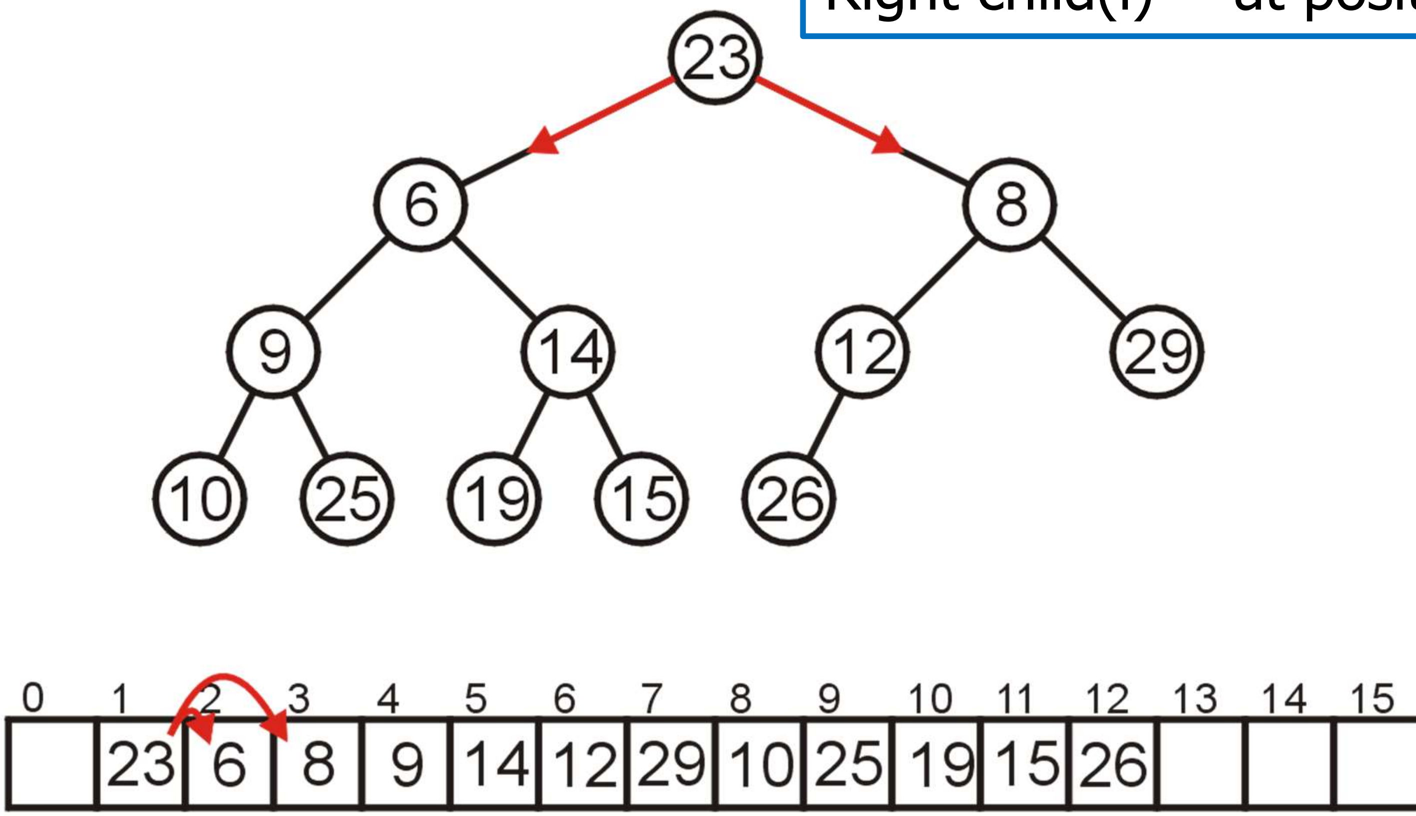
- Removing the top require copy of the last element to the top



Array-Based Implementation – `deleteMin`

- Percolate down
  - Compare Node 1 with its children: Nodes 2 and 3
  - Swap 23 and 6

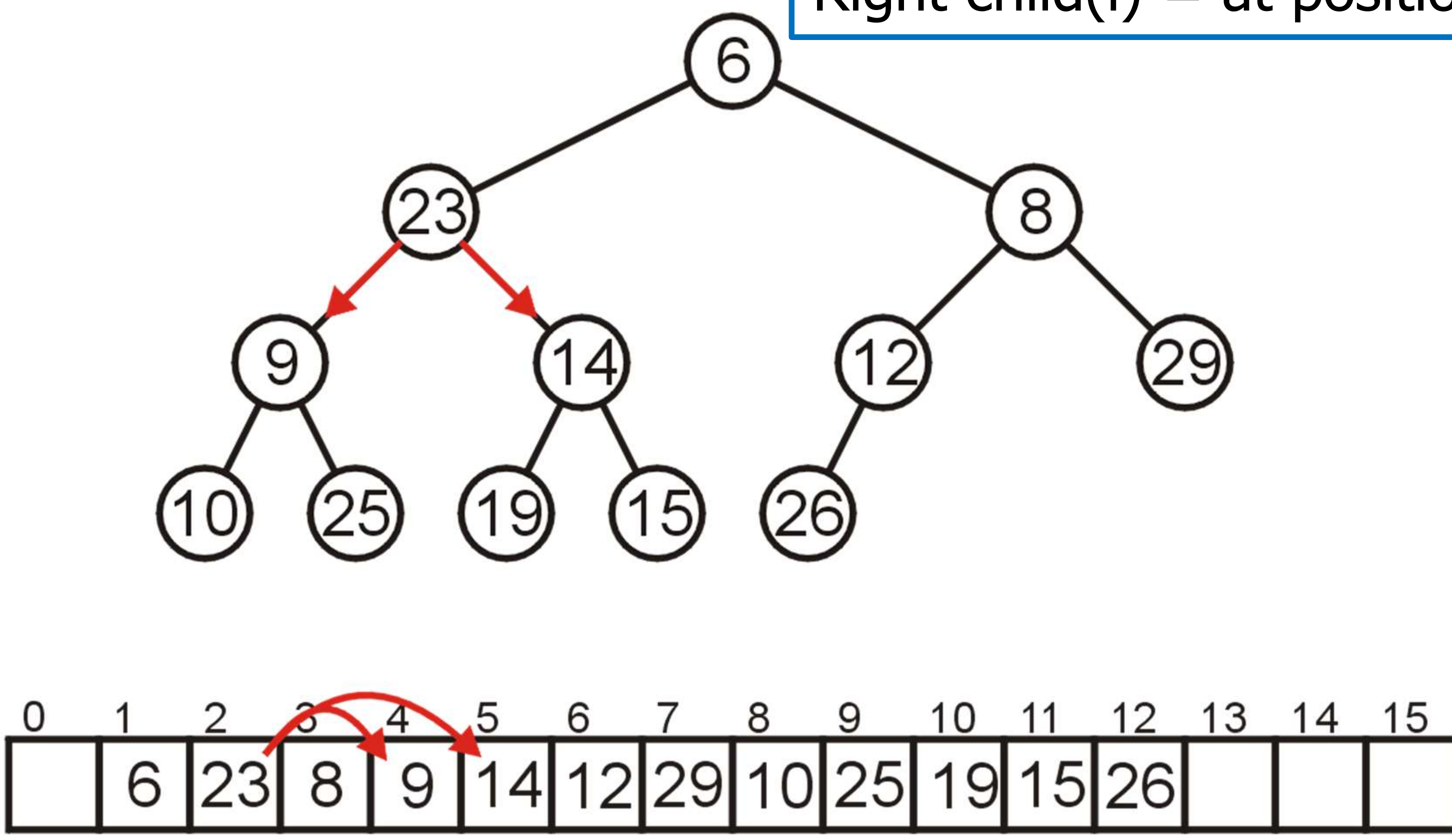
Left child(i)   = at position `2i`
Right child(i) = at position `2i + 1`



Array-Based Implementation – `deleteMin`

- Compare Node 2 with its children: Nodes 4 and 5
  - Swap 23 and 9

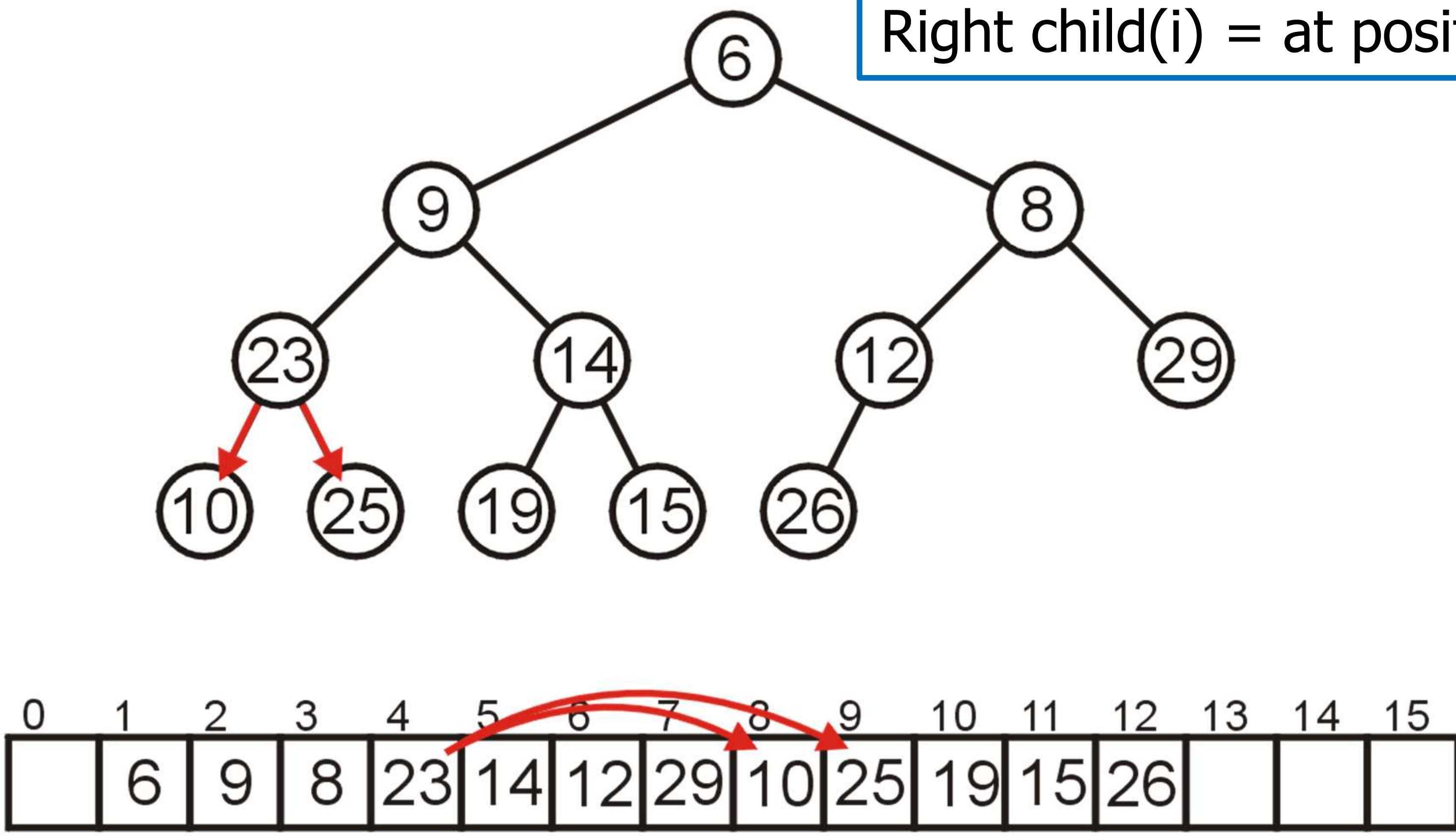
Left child(i) = at position `2i`
Right child(i) = at position `2i + 1`



Array-Based Implementation – `deleteMin`

- Compare Node 4 with its children: Nodes 8 and 9
  - Swap 23 and 10

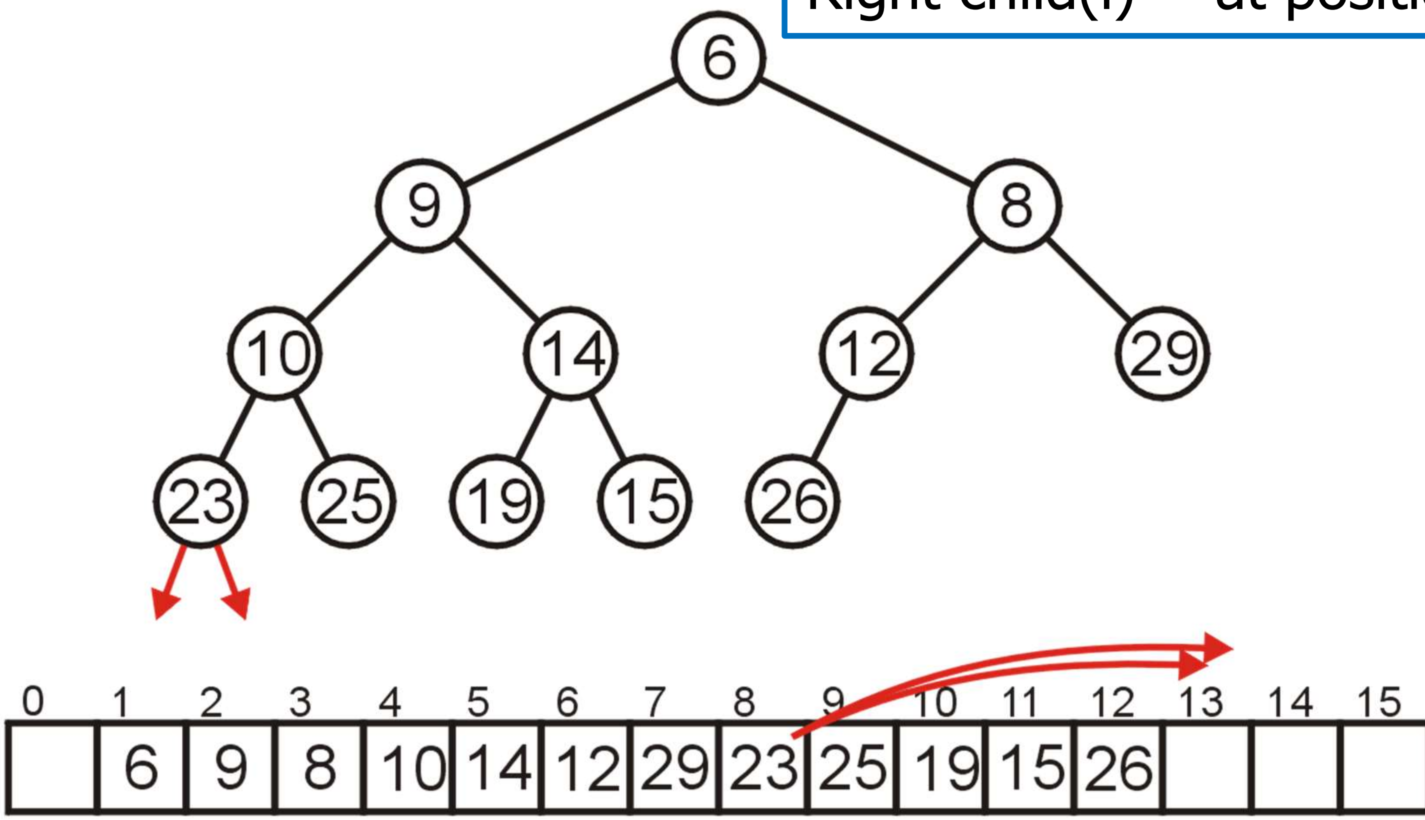
Left child(i)   = at position `2i`
Right child(i) = at position `2i + 1`



Array-Based Implementation – `deleteMin`

- The children of Node 8 are beyond the end of the array:
  - Stop

Left child(i) = at position `2i`
Right child(i) = at position `2i + 1`



Runtime Analysis

- `insert` operation
  - Worst case: Inserting an element less than the root
    - ➢ $O(\log_2 n)$
  - Best case: Inserting an element greater than any other element
    - ➢ $O(1)$
  - Average case: $O(1)$
    - ➢ Why ?

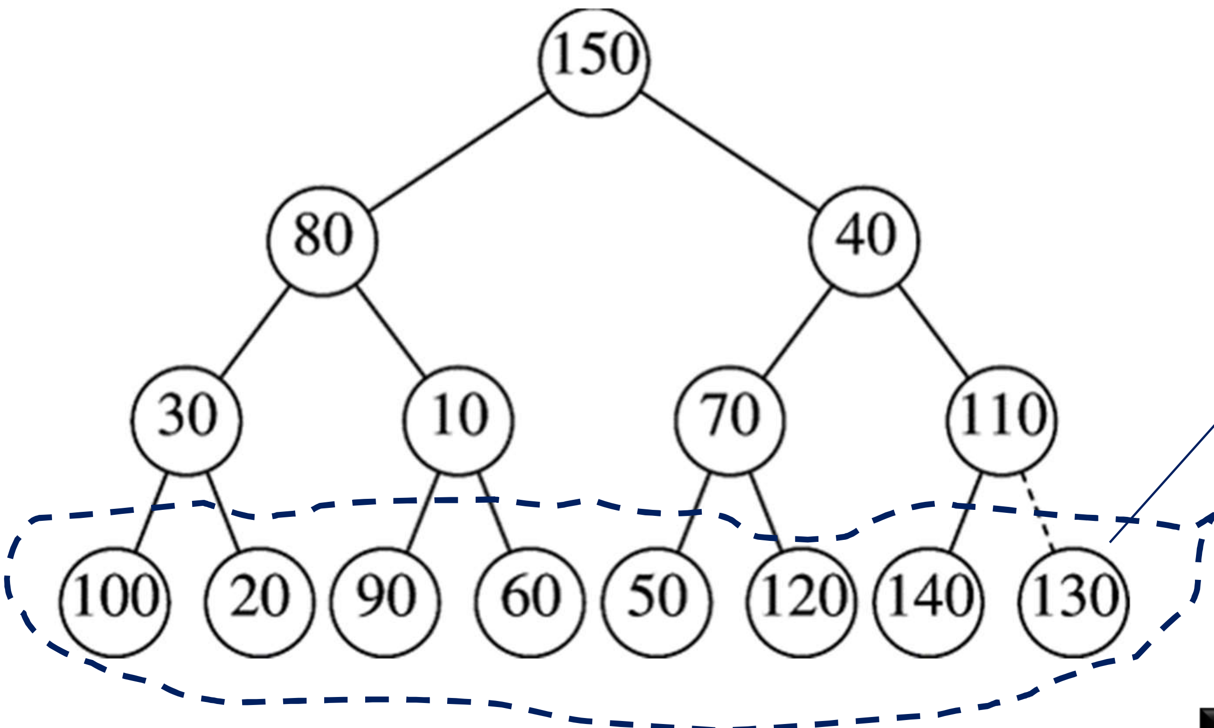
- `deleteMin` operation
  - Replacing the top element is $O(1)$
  - Percolate down the top object is $O(\log_2 n)$
  - We copy something that is already in the lowest depth
    - ➢ It will likely be moved back to the lowest depth

Building a Heap

- What if all N elements are all available upfront?
  - Construct heap from initial set of N items
- Solution 1 (insert method)
  - Perform N inserts
- Solution 2 (BuildHeap method)
  - Randomly populate initial heap with structure property
  - Perform a percolate-down from each internal node
    - ➢ To take care of heap order property

BuildHeap Example

- Input priority levels
  - { 150, 80, 40, 30, 10, 70, 110, 100, 20, 90, 60, 50, 120, 140, 130 }

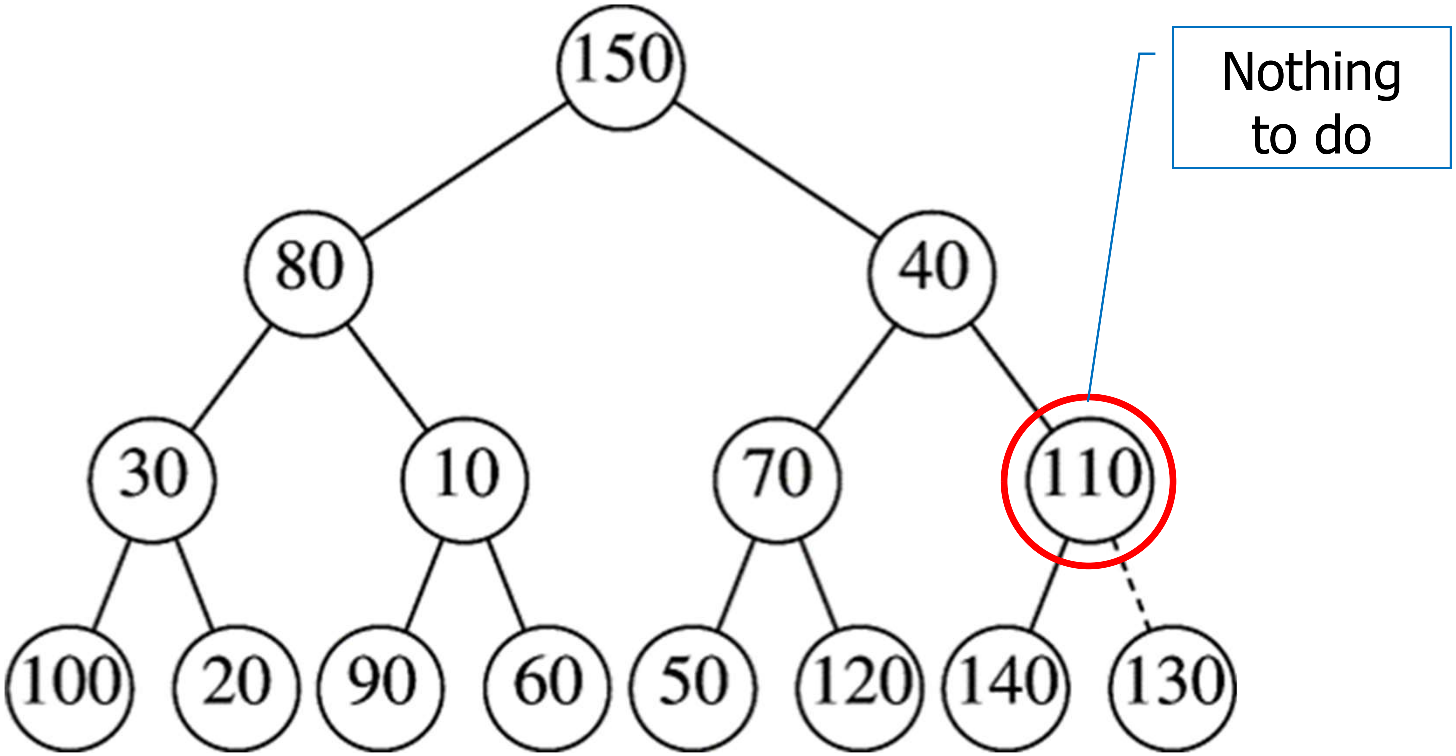


Leaves are all valid heaps (implicitly)

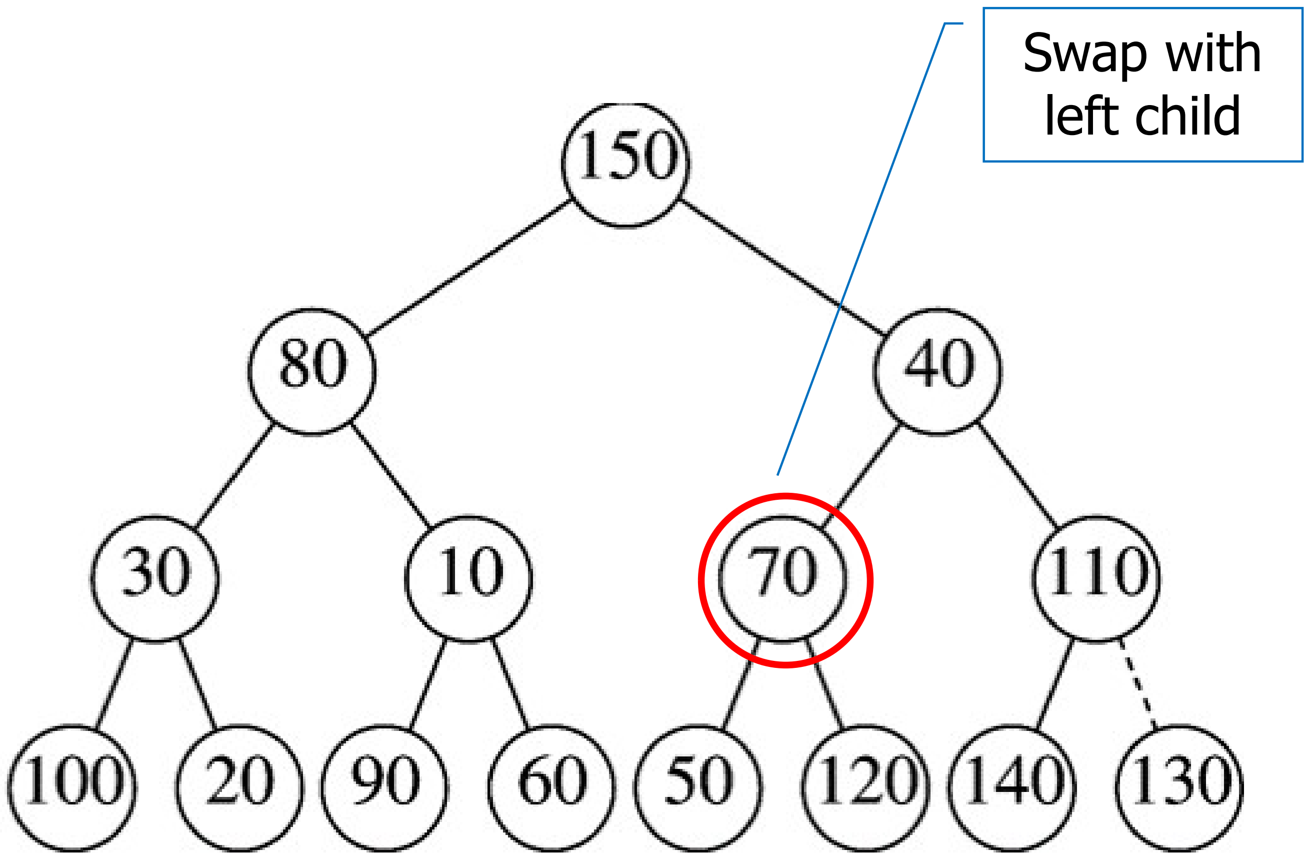
- Arbitrarily assign elements to heap nodes
- Structure property satisfied
- Heap order property violated
- Leaves are all valid heaps (implicit)

So, let us look at each internal node, from bottom to top, and fix if necessary

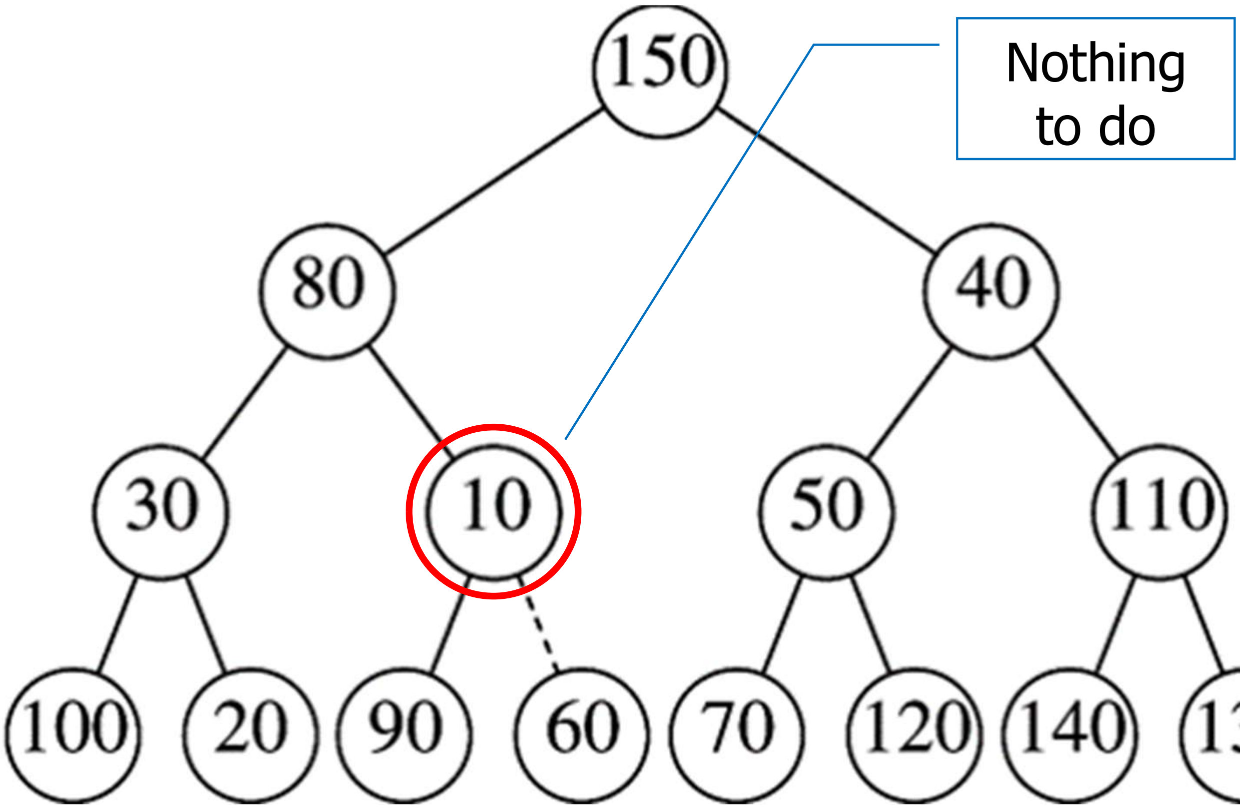
BuildHeap Example



BuildHeap Example

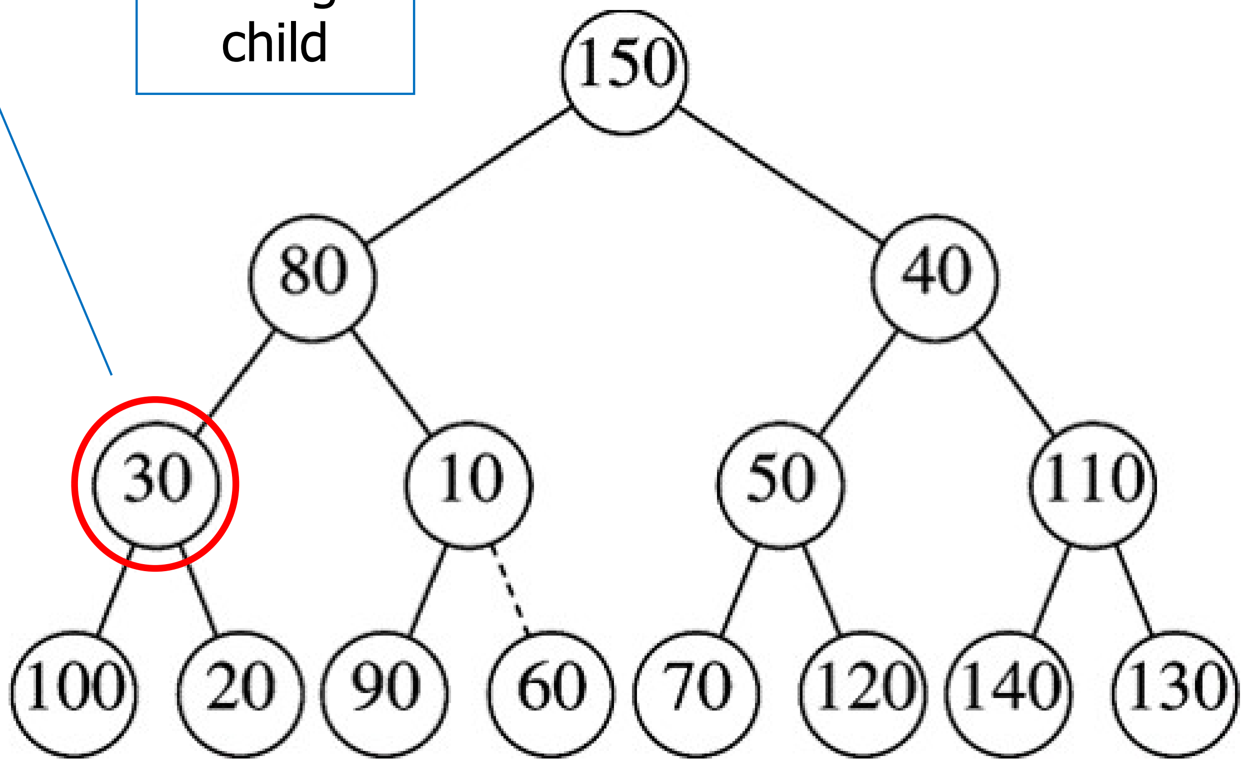


BuildHeap Example

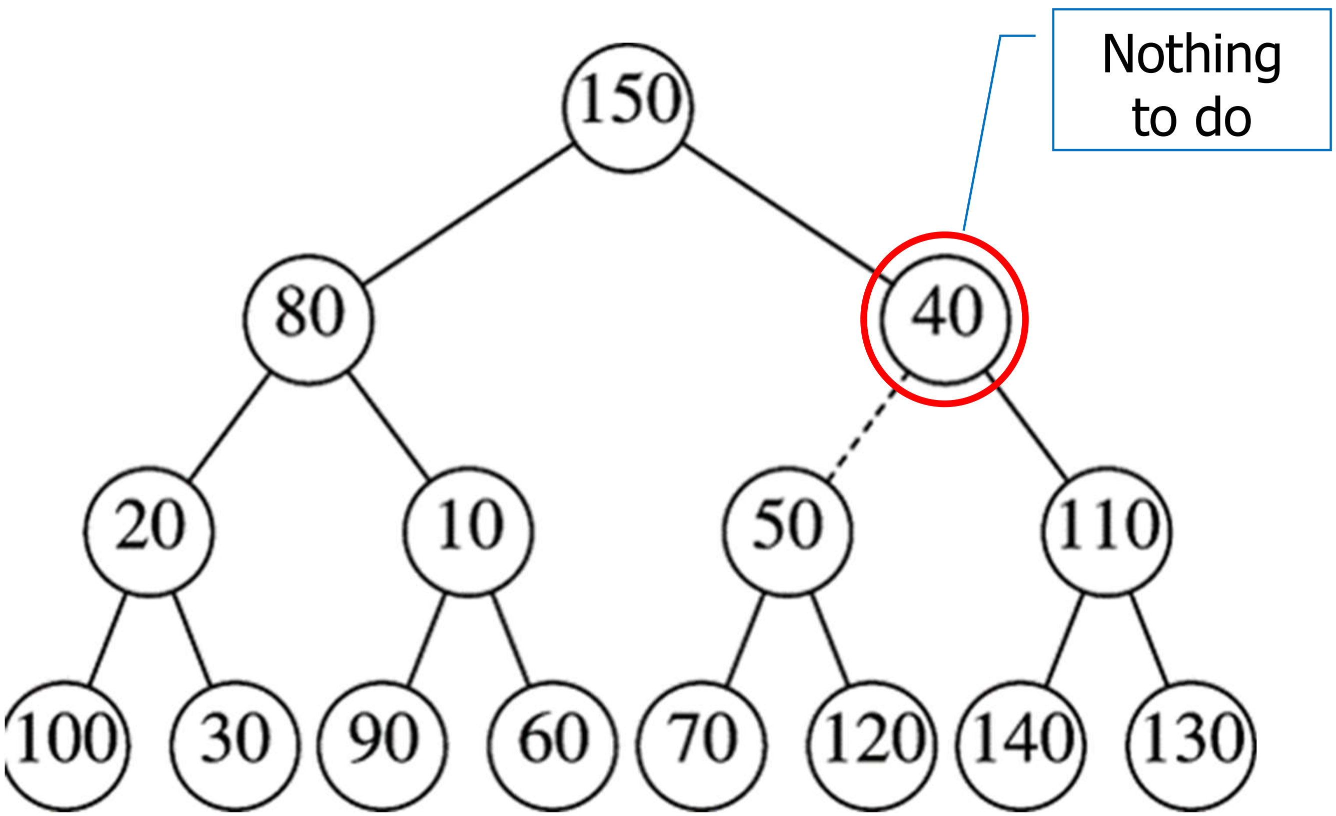


BuildHeap Example

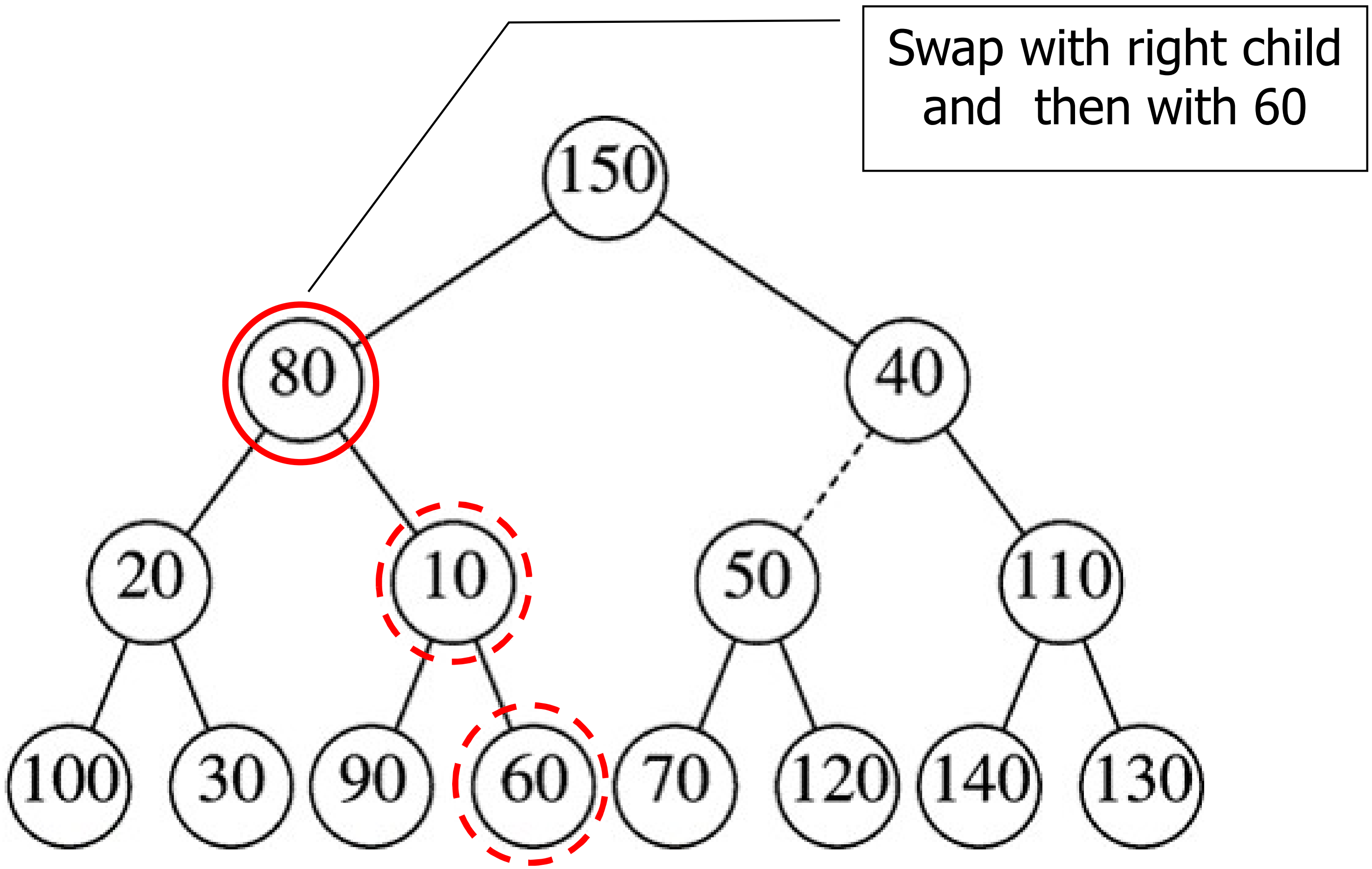
Swap with right child



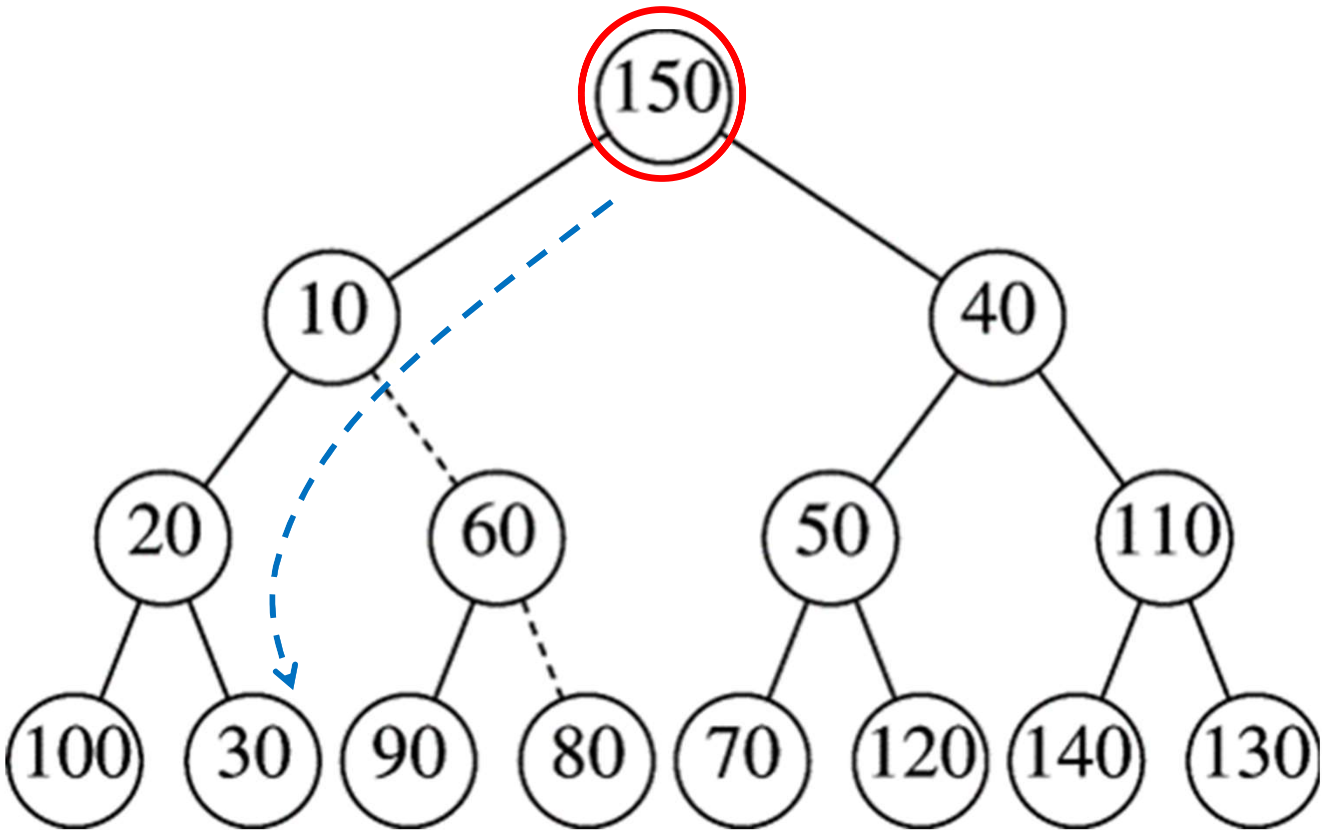
BuildHeap Example



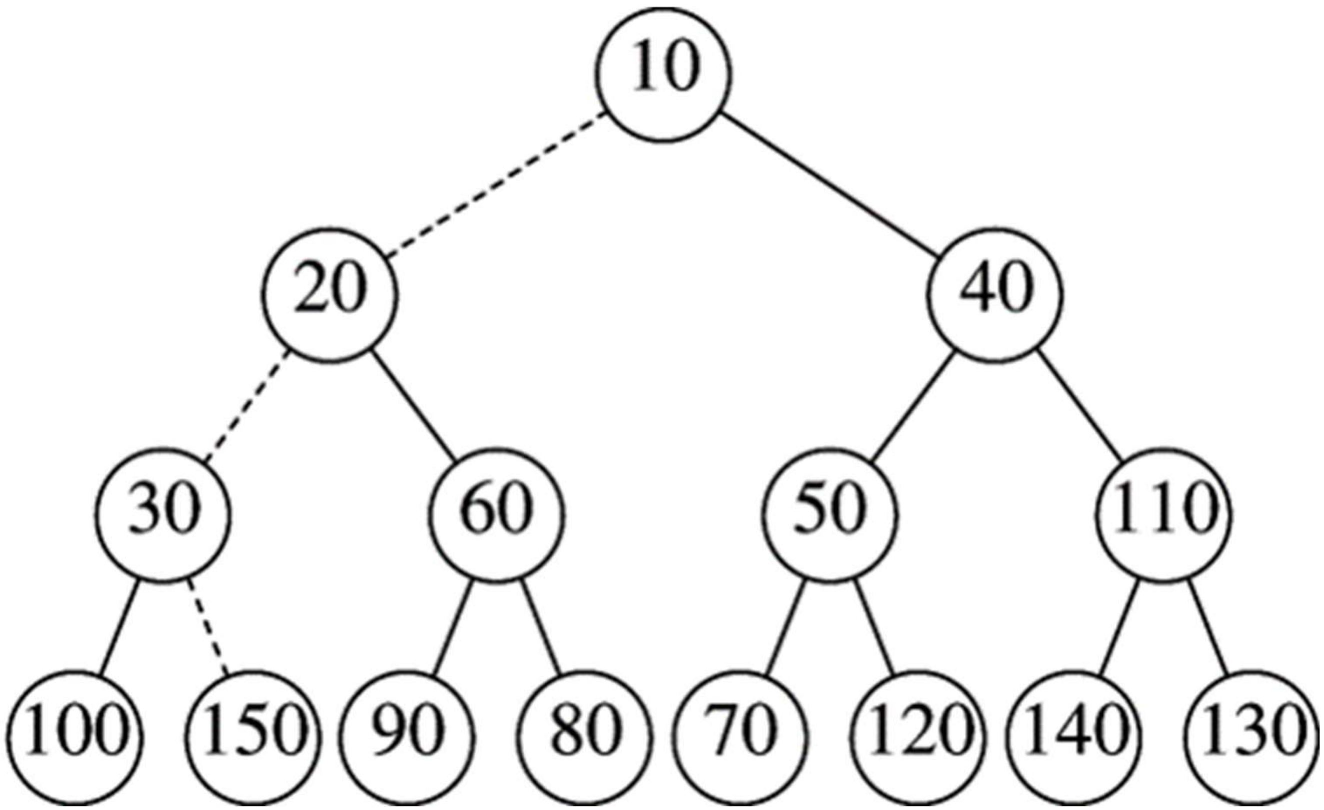
BuildHeap Example



BuildHeap Example



BuildHeap Example



Final Heap

Any Question So Far?