¿Cómo se soluciona una ecuación diferencial ordinaria no lineal de primer orden?

Cambio de variables

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$$=: z(x)$$

$$Y'(x) = f\left(\frac{Y(x)}{x}\right)$$

donde f es una función dada

$$x\, z(x) = y(x)$$

Cambio de variable

$$Z(x) = \frac{Y(x)}{x}$$

Solución

$$Y'(x) = \frac{d}{dx}(x\,Z(x)) = Z(x) + x\,Z'(x)$$

$$x\,Z'(x) + Z(x) = f(Z(x))$$

$$z'(x) = \left(f(z(x)) - z(x)\right)/x$$

$$Y'(x) = \frac{\frac{x+Y(x)}{x}}{\frac{x-Y(x)}{x}} = \frac{1+Y(x)/x}{1-Y(x)/x} = f\left(\frac{Y(x)}{x}\right)$$

$$f(z) = (1+z)/(1-z)$$

Ejemplo:

$$Y'(x) = \frac{x+Y(x)}{x-Y(x)}$$

con $Y(-1) = 1$

$$z(x) = \frac{Y(x)}{x} \Rightarrow Y(x) = x\, z(x) \Rightarrow \frac{d}{dx}(x\, z(x)) = \frac{1+z(x)}{1-z(x)}$$

$$Y'(x) = \frac{1+Y(x)/x}{1-Y(x)/x}$$

$$Z(x) = \frac{Y(x)}{x} \Rightarrow x\,Z'(x) + Z(x) = \frac{1+Z(x)}{1-Z(x)}$$

$$\boxed{Z'(x) = \frac{1}{x}\frac{1+Z(x)^2}{1-Z(x)}}$$

$$x\, z'(x) = \frac{1+z(x)}{1-z(x)} - z(x) = \frac{1+\cancel{z(x)} - \cancel{z(x)} + z(x)^2}{1-z(x)}$$

Soluciones constantes $z(x) \equiv z \in \mathbb{R}$ $\forall x$

$$0 = \frac{1}{x}\,\frac{1+z^2}{1-z} \qquad \forall x$$

$$\Leftrightarrow \quad 1+z^2 = 0 \quad \Leftrightarrow \quad z^2 = -1$$

no hay solución

$$\frac{dz}{dx} = \frac{1}{x}\,\frac{1+z^2}{1-z}$$

$$\frac{1-z}{1+z^2}\,dz = \frac{1}{x}\,dx$$

$$\int \frac{1-z}{1+z^2}\,dz = \int \frac{1}{x}\,dx = \ln(|x|) + C$$
$$= \ln(-x) + C$$

Como $y(\underline{-1}) = 1$, luego buscamos la solución para valores $x<0$

$$\int \frac{1-z}{1+z^2}\,dz = \int \frac{dz}{1+z^2} - \int \frac{z}{1+z^2}\,dz$$

$$= \arctan(z) - \frac{1}{2}\int \frac{du}{u} + \tilde{C}$$

$u = 1+z^2$
$du = 2z\,dz$

$$= \arctan(z) - \frac{1}{2}\ln(|u|) + \tilde{C}$$

$$= \arctan(z) - \frac{1}{2}\ln(1+z^2) + \tilde{C}$$

$$\frac{1-Z(x)}{1+Z(x)^2}Z'(x)=\frac{1}{x}\Rightarrow\int\frac{1-Z(x)}{1+Z(x)^2}Z'(x)\,dx=\int\frac{1}{x}dx=ln(-x)+K$$

$z=Z(x)$

$dz = z'(x)dx$

$$\int\frac{1-z}{1+z^2}dz=ln(-x)+K$$

$$\int\frac{1-z}{1+z^2}dz=\int\frac{1}{1+z^2}dz-\int\frac{z}{1+z^2}dz$$

$$=arctan(z)-\frac{1}{2}ln\left(1+z^2\right)$$

Solución dada en forma implícita

$z(x)=\frac{Y(x)}{x}$

$$arctan(Z(x))-\frac{1}{2}ln\left(1+Z(x)^2\right)=ln(-x)+K$$

$$arctan\left(\frac{Y(x)}{x}\right)-\frac{1}{2}ln\left(1+\left(\frac{Y(x)}{x}\right)^2\right)=ln(-x)+K$$

$Y(-1)=1$

$$K=arctan(-1)-\frac{1}{2}ln(2)=-\pi/4-\frac{1}{2}ln(2)$$

$$y'(x) = f(\overbrace{a\,x + b\,y(x) + c}^{=\,z(x)})$$

donde f es una función dada y $a, b, c \in \mathbb{R}$ con $b \neq 0$

Cambio de variable

$$z(x) = a\,x + b\,y(x) + c$$

Solución

$$z'(x) = a + b\,\underbrace{y'(x)}_{=\,f(z(x))}$$

$$z'(x) = a + b\,f(z(x))$$

$z'(x) = 1 + y'(x) \Rightarrow y'(x) = z'(x) - 1$

$z'(x) - 1 = \cos(z(x)) \Rightarrow z'(x) = 1 + \cos(z(x))$

Ejemplo:

$=z(x)$

$$\begin{cases} y'(x) = \cos(x + y(x)) \\ y(0) = 3\pi/2 \end{cases}$$

Cambio de variable

$$z(x) = x + y(x) \qquad \Rightarrow \qquad z'(x) = 1 + y'(x)$$

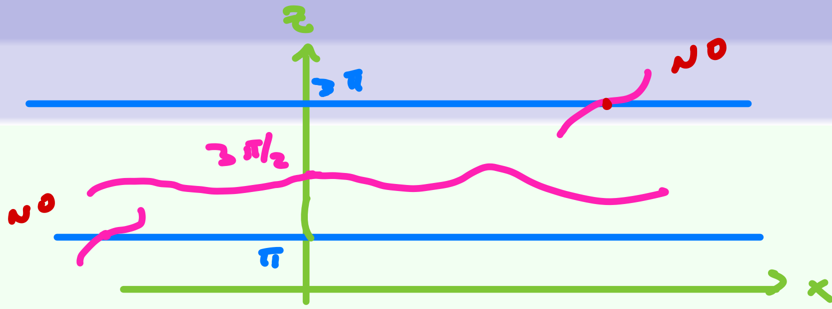
$= \cos(z(x))$

$$z'(x) = 1 + \cos(z(x))$$

$$y(0) = 3\pi/2 \qquad \Rightarrow \qquad z(0) = 0 + y(0) = 3\pi/2$$

$$\begin{cases} z'(x) = 1 + \cos(z(x)) \\ z(0) = 3\pi/2 \end{cases}$$

Soluciones constantes



$z(x) \equiv z$ $\in \mathbb{R}$

$$0 = z'(x) = 1 + \cos(\overset{= z}{z(x)}) = 1 + \cos(z)$$

$$\cos(z) = -1 \quad \Rightarrow \quad z = \pi + 2\,k\,\pi$$

con $k \in \mathbb{Z}$

Como $z(0) = 3\pi/2$, usando el teorema de existencia y unicidad obtenemos que

$$\pi < z(x) < 3\,\pi$$

$$\frac{dz}{dx} = 1 + \cos(z) \Leftrightarrow \frac{dz}{1+\cos(z)} = dx$$

Separar variables

$$z'(x) = 1 + \cos(z(x)) > 0 \quad \Rightarrow \quad \frac{1}{1+\cos(z(x))} z'(x) = 1$$

$$\int \frac{dz}{1+\cos(z)} = \int dx = x + C$$

$$= \int_0^x t\, dt$$

Integrar

$z(0) = 3\pi/2$

$$\int_0^x \frac{1}{1+\cos(z(t))} z'(t)\; dt = x$$

$$z = z(t) \Rightarrow \int_0^x \frac{1}{1+\cos(z(t))} z'(t)\; dt = \int_{3\pi/2}^{z(x)} \frac{1}{1+\cos(z)}\, dz$$

$dz = z'(t)\, dt$

$$\int_{3\pi/2}^{z(x)} \frac{1}{1+\cos(z)}\, dz = x$$

$\tan'(x) = \frac{1}{\cos^2(x)}$

Integrar

Como $1 + \cos(z) = 2\cos^2(z/2)$,

$$\int_{3\pi/2}^{z(x)} \frac{1}{1+\cos(z)}\, dz = \int_{3\pi/2}^{z(x)} \frac{1}{2}\frac{1}{\cos^2(z/2)}\, dz = \tan(z(x)/2) - \tan(3\pi/4)$$

$$\tan(z(x)/2) + 1 = x \Rightarrow z(x) = 2\arctan(x-1) \; + 2\pi$$

$z(x) \in \left]\pi, 3\pi\right[$

Solución

Ya que $z(x) = x + y(x)$,

$$x + y(x) = 2\arctan(x-1) + 2\pi \quad \Rightarrow \quad y(x) = 2\arctan(x-1) - x \; + 2\pi$$

$$\frac{z(x)}{2} \in \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[\Rightarrow \frac{z(x)}{2} - \pi \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$

$$\tan\left(\frac{z(x)}{2} - \pi\right) = -\tan\left(\pi - \frac{z(x)}{2}\right) = -\tan\left(-\frac{z(x)}{2}\right) = \tan\left(\frac{z(x)}{2}\right)$$

$\tan(-x) = -\tan(x)$ $\quad$ $\tan(x+\pi) = \tan(x)$

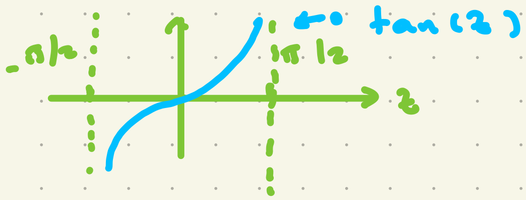
$$\tan\left(\frac{z(x)}{2}\right) + 1 = x$$

$$\tan\left(\frac{z(x)}{2}\right) = x - 1$$

$$\tan\left(\frac{z(x)}{2} - \pi\right) = x - 1$$

$$\in$$

$$\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$



$$\frac{z(x)}{2} - \pi = \arctan(x-1)$$

$$\frac{z(x)}{2} = \pi + \arctan(x-1)$$

$$z(x) = 2\pi + 2\arctan(x-1)$$

Función inversa

$$y'(x) = f(x, y(x))$$

donde f es una función dada.

Cambio de variable

$$z(x) = y^{-1}(x) \iff y^{-1} \circ y = I$$

Solución

$y^{-1}(y(x)) = x$

$y(y^{-1}(x)) = x$

$$z(y(x)) = x \quad \Rightarrow \quad z'(y(x))\, \underbrace{y'(x)}_{= f(x, y(x))} = 1$$

$$z'(y(x)) = \frac{1}{f(x, y(x))} = \frac{1}{f(z(y(x)), y(x))}$$

$$z'(y) = \frac{1}{f(z(y), y)}$$

$y'(x) = \frac{1}{x+y(x)^2} \iff \frac{dy}{dx} = \frac{1}{x+y^2}$

$x+y^2 = \frac{dx}{dy}$ $\boxed{x'(y) = x(y)+y^2}$

Ejemplo:

$$\frac{d}{dx}y(x) = \frac{1}{x+y(x)^2}$$

$y^{-1}(y(x)) = x$ $\Rightarrow$ $z(y(x)) = x$

$z(y) = y^{-1}(y)$ $\quad z'(y(x))\, y'(x) = 1$

Cambio de variable

$f(x,y) = \frac{1}{x+y^2}$

$$z(x) = y^{-1}(x)$$

$z'(y) = \frac{1}{f(z(y),y)} \Rightarrow z'(y) = \frac{1}{\frac{1}{z(y)+y^2}} \quad \Rightarrow \quad z'(y) = z(y) + y^2$

$z'(y(x)) \frac{1}{x+y(x)^2} = 1 \Rightarrow z'(y(x)) = x + y(x)^2$

$z'(y(x)) = z(y(x)) + y(x)^2$

$y(x) = y \Rightarrow z'(y) = z(y) + y^2$

$$z'(y) = z(y) + y^2$$

$$e^{-y} z'(y) - e^{-y} z(y) = y^2 \quad \Rightarrow \quad \frac{d}{dy}\left(e^{-y} z(y)\right) = e^{-y}\, y^2$$

$$e^{-y} z(y) = -e^{-y}\left(y^2 + 2y + 2\right) + 2 + C$$

$$z(y) = K\, e^{y} - y^2 - 2y - 2$$

$z(x) = y^{-1}(x)$

$$\underbrace{z(y(x))}_{=x} = K e^{y(x)} - y(x)^2 - 2y(x) - 2$$

Solución dada implícitamente

Evaluando en $y = y(x)$ se llega a

$$x = K\, e^{y(x)} - y(x)^2 - 2y(x) - 2$$

con $K \in \mathbb{R}$