3) $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$

a) f continua en $(0,0)$ ✓

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f((0,0)+h(1,0)) - \overset{0}{f(0,0)}}{h} = \lim_{h\to 0} \frac{f(h,0)}{h}$$

$$= \lim_{h\to 0} \frac{0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f((0,0)+h(0,1)) - \overset{0}{f(0,0)}}{h} = \lim_{h\to 0} \frac{f(0,h)}{h} = 0$$

$$\nabla f(0,0) = (0,0).$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - \nabla f(0,0)\cdot(x,y) - \overset{0}{f(0,0)}}{\|(x,y)\|} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{\frac{xy(x^2-y^2)}{x^2+y^2} - \cancel{(0,0)}^{0}(x,y)}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{(x^2+y^2)\|(x,y)\|}.$$

$\forall (x,y) \neq (0,0)$:

$$0 \leq |f(x,y) - 0| = \frac{|x||y|\,|x^2-y^2|}{|x^2+y^2|\,\|(x,y)\|} \leq \frac{|x||y|\,x^2}{(x^2+y^2)\|(x,y)\|} + \frac{|x||y|\,y^2}{(x^2+y^2)\|(x,y)\|}$$

$$\leq \frac{2\|(x,y)\|^4}{\cancel{\|(x,y)\|^3}} \xrightarrow{(x,y)\to(0,0)} \underline{0}$$

y por lo tanto f es diferenciable en $(0,0)$.

$$|x+y| \leq |x|+|y| \quad /(\ )^{-1}$$

$$\frac{1}{|x+y|} \geq \frac{1}{|x|+|y|}$$

$$\left|\frac{x^2-y^2}{(x^2+y^2)}\right| = \frac{|x^2-y^2|}{x^2+y^2} = \frac{1}{x^2+y^2}|x^2-y^2| \leq$$

$$\frac{1}{x^2+y^2}|x^2| + \frac{1}{x^2+y^2}|y^2|$$

$$x^2+y^2 = \left(\sqrt{x^2+y^2}\right)^2 = \|(x,y)\|^2$$

b) $\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}((0,0)+h(0,1)) - \cancel{\frac{\partial f}{\partial x}(0,0)}^{0}}{h}$

$= \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(0,h)}{h} = \lim_{h \to 0} \frac{\frac{-h^5}{h^4}}{h} = -1.$

$\frac{\partial}{\partial y}\left(\frac{xy(x^2-y^2)}{x^2+y^2}\right) = -\frac{x(y^4+4x^2y^2-x^4)}{(y^2+x^2)^2}\Bigg|_{(h,0)} = \frac{h^5}{h^4}$

$\frac{\partial}{\partial x}\left(\frac{xy(x^2-y^2)}{(x^2+y^2)}\right) = \frac{y(x^4+4y^2x^2-y^4)}{(x^2+y^2)^2}\Bigg|_{(0,h)} = \frac{-h^5}{h^4}$

$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) = 1.$

c) el lema de Schwarz, dice que si $f \in \mathcal{C}^2 \Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

pero nos dio que $\frac{\partial^2 f}{\partial x \partial y}(0,0) = -1 \neq 1 \; \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

$\therefore f \notin \mathcal{C}^2$ en $(0,0)$.

Problema 4 $\quad f: \mathbb{R}^n \to \mathbb{R}$ campo escalar.

$F: \mathbb{R}^n \to \mathbb{R}^n$ campo vectorial.

$$f_p(x,y) = \begin{cases} \dfrac{x^3 + x^2 y}{|x|^p + |y|^p} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f_p}{\partial x}(0,0) = \lim_{h\to 0} \frac{f_p(h,0)}{h} = \lim_{h\to 0} \frac{h^3}{\frac{|h|^p}{h}} = \lim_{h\to 0} \frac{h^2}{|h|^p} = \lim_{h\to 0} |h|^{2-p}$$

$$\Rightarrow 2-p>0 \Leftrightarrow p<2.$$

$$= \begin{cases} 0 & , p<2 \\ 1 & , p=2 \\ \nexists & , p>2 \end{cases}$$

$$\frac{\partial f_p}{\partial y}(0,0) = \lim_{h\to 0} \frac{f_p(0,h)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

$$\text{Si } p=2: \lim_{\bar{x}\to \theta} \frac{f_2(x,y) - \nabla f_2(0,0)\cdot(x,y) - f_2(0,0)}{\|(x,y)\|}$$

$$\lim_{\bar{x}\to\theta} \frac{\frac{x^3+x^2y}{x^2+y^2} - \frac{\cancel{(1,0)}(x,y)}{x}}{\|(x,y)\|} = \lim_{\bar{x}\to\theta} \frac{\cancel{x^3}+x^2y-\cancel{x^3}-xy^2}{(x^2+y^2)\|(x,y)\|}$$

$$= \lim_{\bar{x}\to\theta} \frac{x^2y-xy^2}{(x^2+y^2)\|(x,y)\|}, \quad \text{sea } T = \{(x,y)\in\mathbb{R}^2 : y=-x, x>0\}$$

$$\overset{y=-x}{=} \lim_{x\to 0} \frac{-\cancel{x^3}-\cancel{x^3}}{\cancel{2x^2}\sqrt{2x^2}} = \lim_{x\to 0} \frac{-x}{\sqrt{2}\,|x|} \begin{cases} \to \frac{1}{\sqrt{2}}, & x\to 0^- \\ \to -\frac{1}{\sqrt{2}}, & x\to 0^+ \end{cases}$$

$\therefore$ f no es dif en $(0,0)$ si $p=2$.

si $p<2$, $\nabla f_p(0,0) = (0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+x^2y}{(|x|^p+|y|^p)\|(x,y)\|}.$$

$$0 \leq \left| \frac{x^3 + x^2 y}{(|x|^p + |y|^p)\,\|(x,y)\|} \right| = \frac{|x|^3 + x^2|y|}{|x|^p\|(x,y)\| + \cancel{|y|^p\|(x,y)\|}} \leq \frac{|x|^3 + x^2|y|}{|x|^p\|(x,y)\|}$$

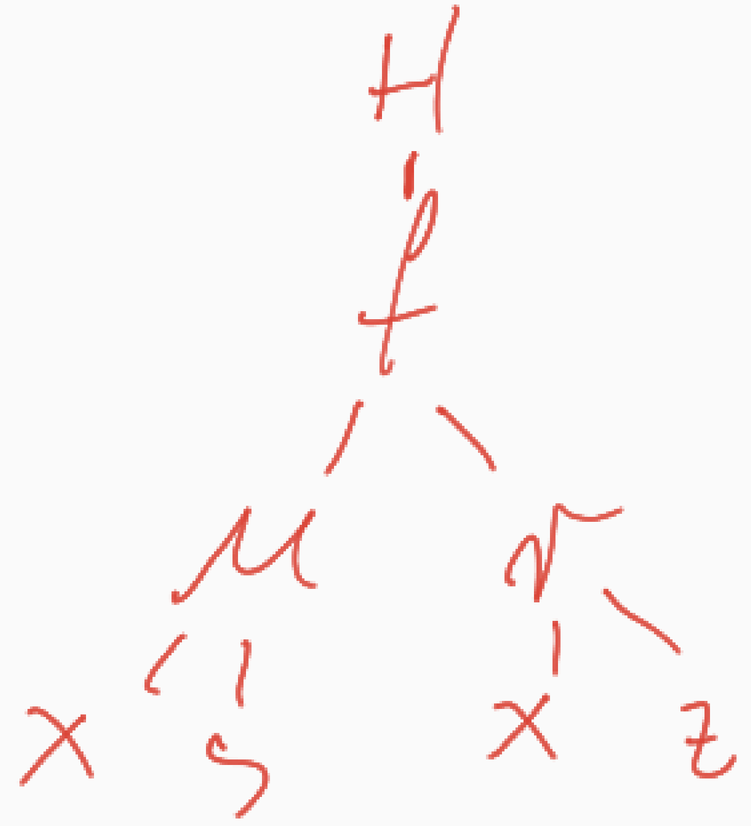
$$= \frac{|x|^3}{|x|\,\|(x,y)\|} + \frac{x^2|y|}{|x|^p\|(x,y)\|} = \frac{|x|^{3-p}}{\|(x,y)\|} + \frac{|x|^{2-p}|y|}{\|(x,y)\|} \qquad 2-p \geq 0.$$

$$\leq \|(x,y)\|^{\cancel{3}\,2-p-\cancel{1}} + \|(x,y)\|^{2-p+\cancel{1-1}} = 2\|(x,y)\|^{2-p} \longrightarrow 0.$$

$2-p>0 \Rightarrow p<2.$

Si $p<2$, entonces f es diferenciable en $(0,0)$.

3) $g, h : \mathbb{R} \to \mathbb{R}$ diferenciables, $u(x,y) = \dfrac{x}{y}$, $v(x,z) = h(x)\,g(z)$. Define $H(x,y,z) = f(u(x,y), v(x,z))$ donde $f(u,v) = u^2 + v^2$. $\dfrac{\partial H}{\partial x}(x,y,z)$.



$$\frac{\partial H}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= 2u \, \frac{1}{y} + 2v \, h'(x)\, g(z)$$

$$= 2\,\frac{x}{y} \cdot \frac{1}{y} + 2\, h(x) g(z)\, h'(x) g(z)$$

$$= 2\,\frac{x}{y^2} + 2 h(x)\, h'(x)\, g^2(z).$$

$$\frac{\partial H}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} = 2u \cdot \frac{-x}{y^2} = 2\frac{x}{y} \cdot \frac{-x}{y^2} = -2\frac{x^2}{y^3}$$

$$\frac{\partial H}{\partial z} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial z} = 2r \cdot h(x)\,\rho'(z) = 2h^2(x)\,\rho(z)\,\rho'(z).$$

$$f(x,y) = (2y - \sin x,\ e^{x+3y},\ xy + y^3) \qquad \mathbb{R}^2 \to \mathbb{R}^3$$

$$g(x,y,z) = (3x + y - z,\ x + yz + 1) \qquad \mathbb{R}^3 \to \mathbb{R}^2.$$

$$D(f \circ g)(0), \quad D(g \circ f)(0).$$

$$Jf = \frac{\partial(f_1, f_2, f_3)}{\partial(x,y)} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{pmatrix} = \begin{pmatrix} -\cos x & 2 \\ e^{x+3y} & 3e^{x+3y} \\ y & x+3y^2 \end{pmatrix}$$

$$Jg = \frac{\partial(g_1, g_2)}{\partial(x,y,z)} = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \end{pmatrix} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & z & y \end{pmatrix}$$

$$D(f\circ g)\theta = Df(g(\theta)) = Df(g(0,0,0))\cdot Dg(0,0,0).$$

$$= Df(0,1)\cdot Dg(0,0,0)$$

$$= \left.\begin{pmatrix} -\cos x & 2 \\ e^{x+3y} & 3e^{x+3y} \\ y & x+3y^2 \end{pmatrix}\right|_{(0,1)} \cdot \left.\begin{pmatrix} 3 & 1 & -1 \\ 1 & z & 7 \end{pmatrix}\right|_{(0,0,0)}$$

$$= \underset{3\times 2}{\begin{pmatrix} -1 & 2 \\ e^3 & 3e^3 \\ 1 & 3 \end{pmatrix}} \cdot \underset{2\times 3.}{\begin{pmatrix} 3 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}} = \begin{pmatrix} -1 & -1 & 1 \\ 6e^3 & e^3 & -e^3 \\ 6 & 1 & -1 \end{pmatrix}$$