$$h(x,y) = \arctan\left[\frac{x+y}{x-y}\right]$$

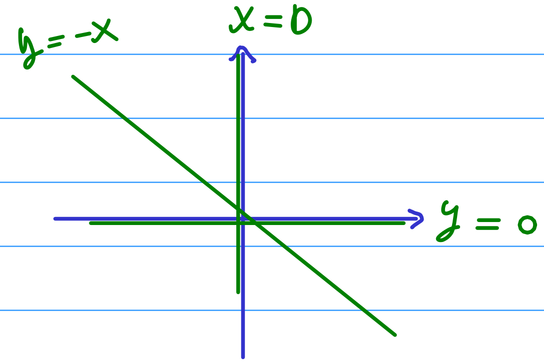
• Curvas de nivel

$$z = h(x,y) \iff z = \arctan\left[\frac{x+y}{x-y}\right]$$

$$\iff \tan(z) = \frac{x+y}{x-y}$$

$$\iff z = \frac{x+y}{x-y}$$

$z=0 \Rightarrow x=-y$



$z=1 \Rightarrow x-y = x+y \Rightarrow y=0$

$z=-1 \Rightarrow y-x = x+y \Rightarrow x=0$

$z=2 \Rightarrow 2x-2y = x+y \Rightarrow x = 3y \Rightarrow y = \frac{1}{3}x$

$z=-2 \Rightarrow -2x+2y = x+y \Rightarrow y = 3x$

Vamos como se comporta $h(x,y)$ en las siguientes trayectorias

$y=-x$

$$h(x,-x) = \arctan\left(\frac{0}{2x}\right) = \pi$$

$y=0$

$$h(x,0) = \arctan(1)$$

$x = 0$

$$h(0,y) = \arctan(-1)$$

$y = \frac{1}{3}x$

$$h\left(x, \frac{1}{3}x\right) = \arctan\left(\frac{x + \frac{1}{3}x}{x - \frac{1}{3}x}\right) = \arctan\left(\frac{4x}{2x}\right)$$

$$= \arctan(2)$$

$y = 3x$

$$h(x, 3x) = \arctan\left(\frac{x+3x}{x-3x}\right) = \arctan\left(\frac{4x}{-2x}\right) = \arctan(-2)$$

$$\lim_{(x,y)\to(-1,1)} h(x,y) = \lim_{(x,y)\to(-1,1)} \arctan\left(\frac{x+y}{x-y}\right)$$

Si $\quad x = -1$

$$g(u,v) = f(u+v,\ u-v)$$

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.$$

$$\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}$$

- $$\frac{\partial^2 g}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial x\cdot\partial y}\frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial y\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial u}$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x\partial y} + \frac{\partial^2 f}{\partial y\partial x} + \frac{\partial^2 f}{\partial y^2}$$

$$= \frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x\partial y} + \frac{\partial^2 f}{\partial y^2}$$

- $$\frac{\partial^2 g}{\partial v^2} = \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial x\partial y}\cdot\frac{\partial y}{\partial v} - \frac{\partial^2 f}{\partial y\partial x}\frac{\partial x}{\partial v} - \frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial^2 f}{\partial x^2} - 2\frac{\partial^2 f}{\partial x\partial y} + \frac{\partial^2 f}{\partial y^2}$$

$$g(u,v) = f(x(u,v),\ y(u,v)) \qquad x = u+v,\quad y = u-v$$

$$\Rightarrow u = \frac{x+y}{2},\quad v = \frac{x-y}{2}$$

$$f(x,y) = g\left(\frac{x+y}{2}, \frac{x-y}{2}\right)$$

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial g}{\partial u} \cdot \frac{1}{2} + \frac{1}{2} \frac{\partial g}{\partial v}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 g}{\partial u^2} \cdot \frac{1}{4} + \frac{\partial^2 g}{\partial u \partial v} \cdot \frac{1}{4} + \frac{1}{4} \frac{\partial^2 g}{\partial v \partial u} + \frac{1}{4} \frac{\partial^2 g}{\partial v^2}$$

$$= \frac{1}{4} \frac{\partial^2 g}{\partial u^2} + \frac{1}{2} \frac{\partial^2 g}{\partial u \partial v} + \frac{1}{4} \frac{\partial^2 g}{\partial v^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2} \frac{\partial g}{\partial u} - \frac{1}{2} \frac{\partial g}{\partial v}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{1}{4} \frac{\partial^2 g}{\partial u^2} - \frac{1}{4} \frac{\partial^2 g}{\partial u \partial v} - \frac{1}{4} \frac{\partial^2 g}{\partial v \partial u} + \frac{1}{4} \frac{\partial^2 g}{\partial v^2}$$

luego $\quad \dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial^2 f}{\partial y^2}$

$$\Rightarrow \cancel{\frac{1}{4} \frac{\partial^2 g}{\partial u^2}} + \frac{1}{2} \frac{\partial^2 g}{\partial u \partial v} + \cancel{\frac{1}{4} \frac{\partial^2 g}{\partial v^2}} = \cancel{\frac{1}{4} \frac{\partial^2 g}{\partial u^2}} - \frac{1}{4} \frac{\partial^2 g}{\partial u \partial v} - \frac{1}{4} \frac{\partial^2 g}{\partial v \partial u} + \cancel{\frac{1}{4} \frac{\partial^2 g}{\partial v^2}}$$

$$\Rightarrow \frac{1}{2} \frac{\partial^2 g}{\partial u \partial v} = -\frac{1}{2} \frac{\partial^2 g}{\partial u \partial v}$$

$$\Rightarrow \frac{\partial^2 g}{\partial u \partial v} = 0 \quad / \int \Rightarrow \int \frac{\partial^2 g}{\partial u \partial v} \cdot dv = 0$$

$$\Rightarrow \frac{\partial g}{\partial u} = F(u) \; / \int$$

$$\Rightarrow g(u,v) = F'(u) + G(v)$$

$$\Rightarrow \quad g(u(x,y), v(x,y)) = F\left(\frac{x+y}{2}\right) + G\left(\frac{x-y}{2}\right)$$

$$\Rightarrow \quad f\left(\frac{x+y}{2} + \frac{x-y}{2}, \frac{x+y}{2} - \frac{x+y}{2}\right) = f(x,y)$$

$$\Rightarrow \quad f(x,y) = F\left(\frac{x+y}{2}\right) + G\left(\frac{x-y}{2}\right)$$

$$L = \lim_{(x,y)\to(0,0)} \frac{4x^2 + 2xy + 5y^2}{3x^2 + 5y^2}$$

considere la trayectoria $T = \{(x,y) \in \mathbb{R}^2 : y = mx,\ m > 0\}$

$$\lim_{x\to 0} \frac{4x^2 + 2mx^2 + 5m^2x^2}{3x^2 + 5m^2x^2} = \lim_{x\to 0} \frac{(4 + 2m + 5m^2)\cancel{x^2}}{(3 + m^2)\cancel{x^2}}$$

$$= \frac{4 + 2m + 5m^2}{3 + m^2}$$

luego, el límite anterior depende del valor de m, por lo que L dependerá de la trayectoria $\Rightarrow$ L no existe.

- $$\lim_{(x,y)\to(0,0)} \frac{x^2 - y}{x^2 + y^2}$$

veamos que sucede con la trayectoria $T = \{(x,y) \in \mathbb{R}^2 : x = 0\}$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{-y}{y^2} = \lim_{y\to 0} \frac{-1}{y}$$, este límite no existe

$\Rightarrow$ L no existe.

$$\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^2 + y^{10}} = \lim_{(x,y)\to(0,0)} \frac{x(y^5)}{x^2 + (y^5)^2} = \lim_{(x,u)\to(0,0)} \frac{xu}{x^2 + u^2} = 0$$

sea $u = y^5$ si $y \to 0 \Rightarrow u \to 0$

$$\left| \frac{xu}{x^2 + u^2} \right| = \frac{|xu|}{|x^2 + u^2|} \leq \frac{|xu| \cdot |x|}{x^2 + u^2} \leq \frac{\cancel{x^2}|u|}{\cancel{x^2}} = |u| \longrightarrow 0$$

$$\lim_{(x,y)\to(0,0)} x \operatorname{sen}\left(\frac{1}{y}\right) = 0$$

$$\left| x \operatorname{sen}\left(\frac{1}{y}\right)\right| \leq |x|\cdot 1 = |x| \longrightarrow 0$$

- $$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$$ $x = r\cos\theta$, $y = r\operatorname{sen}\theta$

$$\left|\frac{x^2y}{x^2+y^2}\right| = \left|\frac{x\cdot xy}{x^2+y^2}\right| = \frac{|x|\,|xy|}{x^2+y^2}\cdot\frac{2}{2} = \frac{|x|\cdot 2|xy|}{(x^2+y^2)\cdot 2} \leq \frac{|x|\,(\cancel{x^2+y^2})}{2(\cancel{x^2+y^2})} \longrightarrow 0$$

$$\frac{x^2y}{x^2+y^2} = \frac{r^2\cos^2\theta\cdot r\operatorname{sen}\theta}{r^2\cos^2\theta + r^2\operatorname{sen}^2\theta} = \frac{r^3\cos^2\theta\operatorname{sen}\theta}{r^2} = r\cos^2\theta\operatorname{sen}\theta$$

luego $|\cos^2\theta| \leq 1$ y $|\operatorname{sen}\theta| \leq 1$

$$\Rightarrow \quad 0 \leq r\cos^2\theta\operatorname{sen}\theta \leq r$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = \lim_{r\to 0} r\cos^2\theta\operatorname{sen}\theta = 0$$

- $$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$$

$x = r\cos\theta$
$y = r\operatorname{sen}\theta$

$$\frac{r^2\cos^2\theta}{r^2\cos^2\theta + r^2\operatorname{sen}^2\theta} = \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta$$

Como $|\cos^2\theta| \leq 1 \Rightarrow \cos^2\theta \leq r\cos^2\theta \leq r$

$$\Rightarrow \cos^2\theta \leq r$$

así, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{r\to 0}\cos^2\theta = \cos^2\theta$ $\Rightarrow$ el límite no existe, pues el resultado depende de θ

$$\lim_{(x,y)\to(0,0)} \frac{x^2+xy+y^2}{x^2+y^2}$$

$x = r\cos\theta$
$y = r\operatorname{sen}\theta$

$$\frac{r^2\cos^2\theta + r^2\cos\theta\operatorname{sen}\theta + r^2\operatorname{sen}^2\theta}{r^2}$$

$$= 1 + \cos\theta\operatorname{sen}\theta$$

luego $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+xy+y^2}{x^2+y^2} = \lim_{r\to 0}(1+\cos\theta\operatorname{sen}\theta) = 1+\cos\theta\operatorname{sen}\theta$

luego el limite depende de $\theta \Rightarrow$ no existe

$$\lim_{(x,y)\to(0,0)} \frac{x^5+y^4-3x^3y+2x^2+2y^2}{x^2+y^2} = 2$$

$x = r\cos\theta$
$y = r\operatorname{sen}\theta$

$$\frac{r^5\cos^5\theta + r^4\operatorname{sen}^4\theta - 3r^4\cos^3\theta\operatorname{sen}\theta + 2r^2}{r^2}$$

$$= r^3\cos^5\theta + r^2\operatorname{sen}^4\theta - 3r^2\cos^3\theta\operatorname{sen}\theta + 2$$

$|\cos^n(\theta)| \leq 1, \ \forall n \in \mathbb{N}$
$|\operatorname{sen}^n(\theta)| \leq 1, \ \forall n \in \mathbb{N}$

$$-r^3 - 3r^2 + 2 \leq r^3\cos^5\theta + r^2\operatorname{sen}^4\theta - 3r^2\cos^3\theta\operatorname{sen}\theta + 2 \leq r^3 + r^2 + 3r^2 + 2$$

$$\Leftrightarrow -r^3 - 3r^2 + 2 \leq r^3\cos^5\theta + r^2\operatorname{sen}^4\theta - 3r^2\cos^3\theta\operatorname{sen}\theta + 2 \leq r^3 + 4r^2 + 2$$

luego $\displaystyle\lim_{r\to 0}(-r^3-3r^2+2) = \lim_{r\to 0}(r^3+4r^2+2) = 2$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}\ , \quad \text{no existe}$$

$$x=0 \quad \Rightarrow \quad \lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{0-y^2}{0+y^2} = -1$$

$$y=0 \quad \Rightarrow \quad \lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^2-0}{x^2+0} = 1$$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+x^2+xy^2+y^2}{x^2+y^2}$$

$x = r\cos\theta$

$y = r\operatorname{sen}\theta$

$$\frac{r^3\cos^3\theta + r^2\cos^2\theta + r^3\cos\theta\operatorname{sen}^2\theta + r^2\operatorname{sen}^2\theta}{r^2}$$

$$= \frac{r^3(\cos^3\theta + \cos\theta\operatorname{sen}^2\theta) + r^2}{r^2}$$

$$= r(\cos^3\theta + \cos\theta\operatorname{sen}^2\theta) + 1$$

$$-r-r+1 \le r\cos^3\theta + r\cos\theta\operatorname{sen}^2\theta + 1 \le r+r+1$$

$$\lim_{r\to 0} -2r+1 = 1 = \lim_{r\to 0} 2r+1$$

$$\Rightarrow \quad \lim_{(x,y)\to(0,0)} \frac{x^3+x^2+xy^2+y^2}{x^2+y^2} = 1$$

$$\frac{x^3+x^2+xy^2+y^2}{x^2+y^2} = \frac{x^2(x+1)+(x+1)y^2}{(x^2+y^2)}$$

$$= \frac{(x+1)\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}} = (x+1)$$

luego

$$\lim_{(x,y)\to(0,0)} \frac{x^3+x^2+xy^2+y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} (x+1) = 1$$

determinar c, tal que

$$g(x,y) = \begin{cases} \dfrac{x^3+xy^2+2x^2+2y^2}{x^2+y^2} & (x,y)\neq(0,0) \\ c & (x,y)=(0,0) \end{cases}$$

sea continua

$$\lim_{(x,y)\to(0,0)} \frac{x^3+xy^2+2x^2+2y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2(x+2)+y^2(x+2)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x+2)\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}}$$

$$= \lim_{(x,y)\to(0,0)} (x+2) = 2$$

luego $c=2$