Transformada de Laplace

Introducción a la transformada de Laplace directa e inversa

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Transformada de Laplace de $f : [0, +\infty[\to \mathbb{R}$

$$f \xmapsto{\mathcal{L}} \widehat{f}(s) := \int_0^{+\infty} e^{-s\,t} f(t)\, dt$$

Ejemplo: $f(t) \equiv 1$

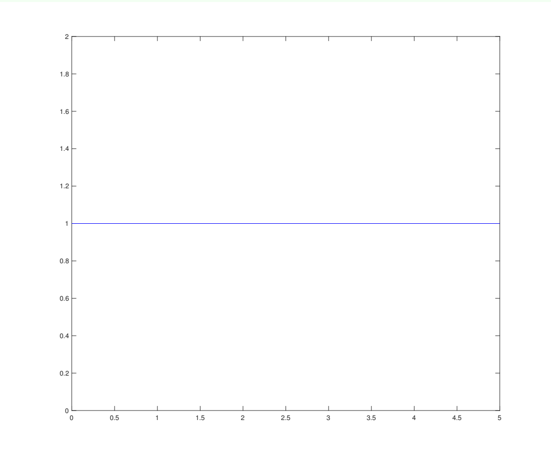
$$\mathcal{L}(\mathbf{1})(s) = \int_0^{+\infty} e^{-s\,t} \cdot 1\, dt = \int_0^{+\infty} e^{-s\,t}\, dt = \begin{cases} \frac{1}{s} & \text{si } s > 0 \\ +\infty & \text{si } s \leq 0 \end{cases}$$

Transformada inversa de Laplace

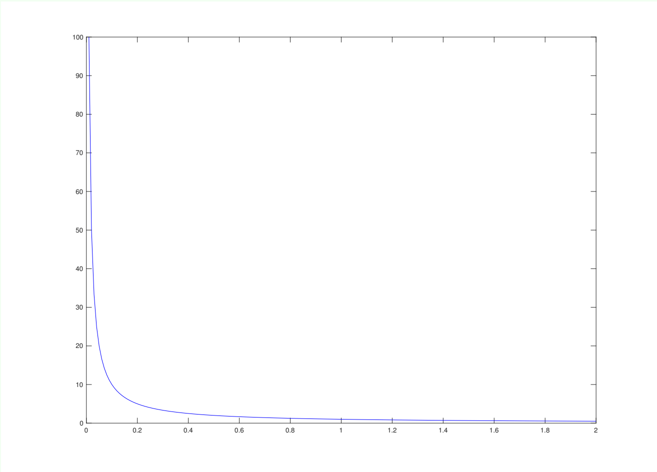
$$\widehat{f}(s) := \int_0^{+\infty} e^{-s\,t} f(t)\, dt \xmapsto{\mathcal{L}^{-1}} f$$

Ejemplo: $\widehat{f}(s) = 1/s$

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right)(t) = 1$$



$$\xmapsto{\mathcal{L}}$$



Transformada de Laplace de $f : [0, +\infty[\to \mathbb{R}$

$$f \xmapsto{\mathcal{L}} \widehat{f}(s) := \int_0^{+\infty} e^{-s\,t} f(t)\, dt$$

$$Dom\left(\widehat{f}\right) = \left\{ s \in \mathbb{R} : \text{ existe } lim_{R \to +\infty} \int_0^R e^{-s\,t} f(t)\, dt \right\}$$

Ejemplo: $f(t) = e^{\alpha\, t}$

$$\mathcal{L}\left(e^{\alpha\, t}\right)(s) = \int_0^{+\infty} e^{-s\,t} e^{\alpha\, t}\, dt = \int_0^{+\infty} e^{-(s-\alpha)t} dt = \begin{cases} \frac{1}{s-\alpha} & \text{si } s > \alpha \\ +\infty & \text{si } s \leq \alpha \end{cases}$$

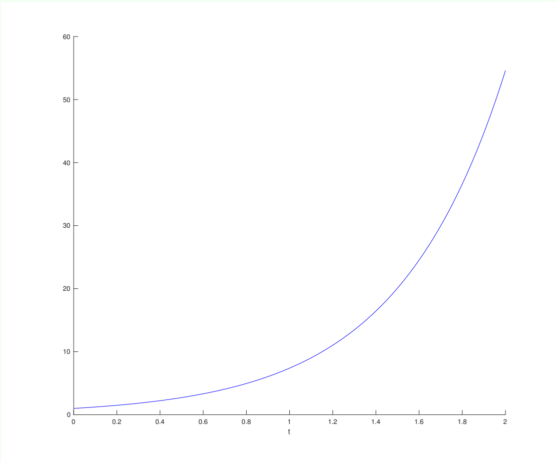
$$Dom\left(\mathcal{L}\left(e^{\alpha\, t}\right)\right) =]\alpha, +\infty[$$

Transformada inversa de Laplace

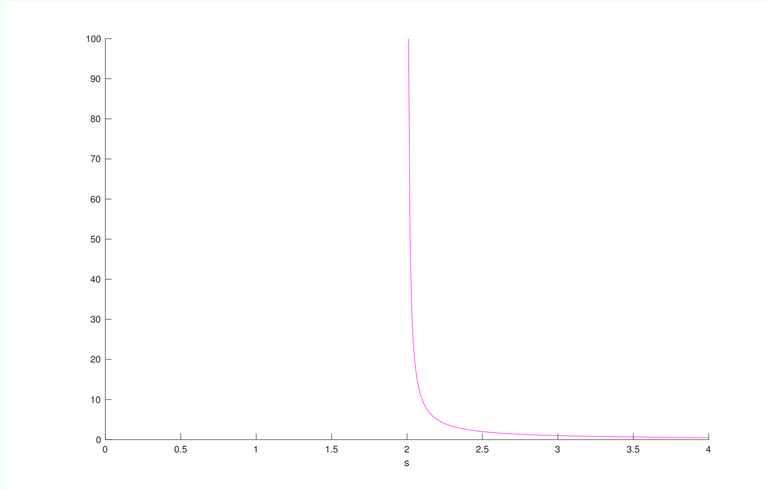
$$\widehat{f}(s) := \int_0^{+\infty} e^{-s\,t} f(t)\,dt \xmapsto{\mathcal{L}^{-1}} f$$

Ejemplo: $\widehat{f}(s) = \frac{1}{s-\alpha}$ para $s > \alpha$

$$\mathcal{L}^{-1}\left(\frac{1}{s-\alpha}\right)(t) = e^{\alpha\,t}$$



$$\xmapsto{\mathcal{L}}$$



Función de orden exponencial

Una función $f : [0, +\infty[\to \mathbb{R}$ es de orden exponencial α si:

- f es continua por partes (f es continua en $[0, +\infty[$ salvo un número finito de puntos donde f tiene limites por la derecha e izquierda)
- Existe $K > 0$ tal que

$$|f(t)| \leq K\, e^{\alpha\, t} \qquad \forall t \geq 0.$$

Propiedad

Suponga que $f : [0, +\infty[\to \mathbb{R}$ es de orden exponencial α.
Entonces, $\mathcal{L}(f)(s)$ existe para todo $s > \alpha$ y

$$lim_{s \to +\infty} \mathcal{L}(f)(s) = 0.$$

$$\int_0^{+\infty} \left| e^{-s\,t}\, f(t) \right| dt \leq K \int_0^{+\infty} e^{-s\,t}\, e^{\alpha\, t}\, dt = \frac{1}{s-\alpha}$$

para $s > \alpha$. Luego, $\mathcal{L}(f)(s)$ existe para todo $s > \alpha$.

Ejemplo: $f(t) = sen(\beta t)$

La función $t \mapsto sen(\beta t)$ es de orden exponencial 0.
Luego, $\mathcal{L}(sen(\beta t))(s)$ existe para todo $s > 0$.

Para $R > 0$ integrando por partes dos veces obtenemos

$$\left(1 + \frac{\beta^2}{s^2}\right) \int_0^R e^{-s\,t} sen(\beta t)\, dt = -\frac{1}{s} e^{-s\,R} sen(\beta R) - \frac{\beta}{s^2} e^{-s\,R} cos(\beta R) + \frac{\beta}{s^2}.$$

Así que para $s > 0$,

$$\left(1 + \frac{\beta^2}{s^2}\right) \int_0^{+\infty} e^{-s\,t} sen(\beta t)\, dt = \frac{\beta}{s^2}.$$

Luego,

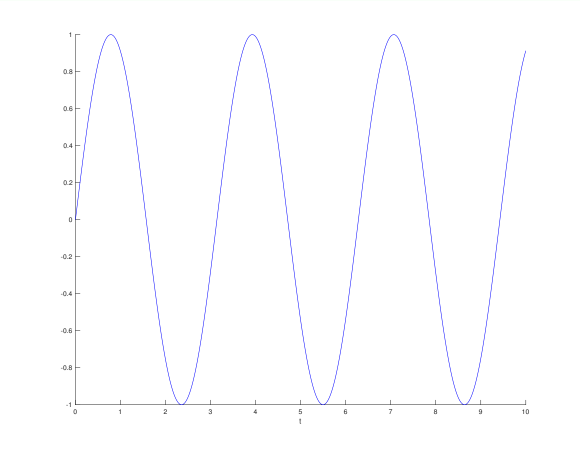
$$\mathcal{L}(sen(\beta t))(s) = \frac{\beta}{s^2 + \beta^2} \qquad \forall s > 0$$

Transformada inversa de Laplace

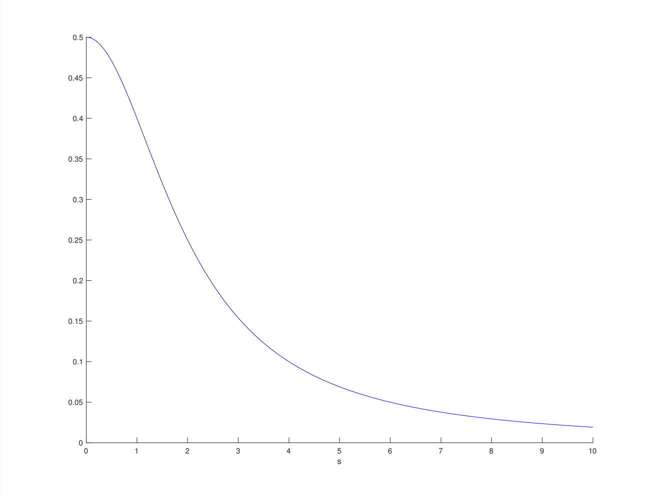
$$\widehat{f}(s) := \int_0^{+\infty} e^{-s\,t} f(t)\,dt \xmapsto{\mathcal{L}^{-1}} f$$

Ejemplo: $\widehat{f}(s) = \frac{\beta}{s^2+\beta^2}$ para $s > 0$

$$\mathcal{L}^{-1}\left(\frac{\beta}{s^2+\beta^2}\right)(t) = sen(\beta\, t)$$



$$\xmapsto{\mathcal{L}}$$



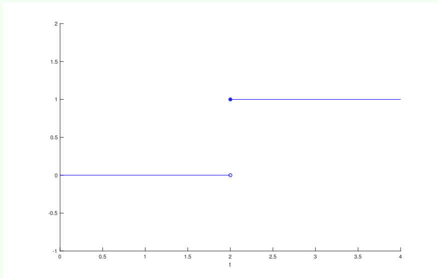
Función de Heaviside

Considere $a \geq 0$. Para todo $t \geq 0$ se define

$$\mathcal{U}_a(t) := \begin{cases} 0 & \text{si } 0 \leq t < a \\ 1 & \text{si } t \geq a \end{cases}.$$

También se denota

$$\mathcal{U}_a(t) = \mathcal{H}_a(t) = \mathcal{U}(t-a)$$



Para s > 0,

$$\mathcal{L}(\mathcal{U}_a(t))(s) = \int_0^{+\infty} e^{-s\,t}\mathcal{U}_a(t)\, dt = \int_a^{+\infty} e^{-s\,t}\, dt = e^{-s\,a}\int_0^{+\infty} e^{-s\,x}\, dx = \frac{e^{-s\,a}}{s}$$

Para $a \geq 0$,

$$\mathcal{L}^{-1}\left(\frac{e^{-s\,a}}{s}\right)(t) = \mathcal{U}_a(t)$$

Función de Heaviside

Considere $a \geq 0$. Para todo $t \geq 0$ se define $\mathcal{U}_a(t) := \begin{cases} 0 & \text{si } 0 \leq t < a \\ 1 & \text{si } t \geq a \end{cases}$.

Propiedad

Considere $f : \mathbb{R} \to \mathbb{R}$ tal que $\mathcal{L}(f)(s)$ existe. Fije $a \geq 0$. Entonces

$$\mathcal{L}(f(t-a)\mathcal{U}_a(t))(s) = e^{-as}\mathcal{L}(f)(s)$$

$$\mathcal{L}(f(t-a)\mathcal{U}_a(t))(s) = \int_0^{+\infty} e^{-st} f(t-a)\mathcal{U}_a(t)\, dt = \int_a^{+\infty} e^{-st} f(t-a)\, dt = e^{-sa}\int_0^{+\infty} e^{-sx} f(x)\, dx$$

Observación

$$\mathcal{L}(g(t)\mathcal{U}_a(t))(s) = e^{-as}\mathcal{L}(g(t+a))(s)$$

Ejemplo

Calcular la transformada de Laplace inversa de

$$\frac{1}{s^2+4}e^{-\pi s}$$

$$\mathcal{L}\left(f\left(t-a\right)\mathcal{U}_a\left(t\right)\right)\left(s\right)=e^{-a\,s}\mathcal{L}\left(f\right)\left(s\right)$$

$$\mathcal{L}\left(sen\left(\beta\,t\right)\right)\left(s\right)=\frac{\beta}{s^2+\beta^2}\qquad\qquad \forall s>0$$

$$\frac{1}{s^2+4}e^{-\pi\,s}=\frac{1}{2}\mathcal{L}\left(sen\left(2\,t\right)\right)\left(s\right)e^{-\pi\,s}=\mathcal{L}\left(\frac{1}{2}sen\left(2\,t\right)\right)\left(s\right)e^{-\pi\,s}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+4}e^{-\pi\,s}\right)\left(t\right)=\mathcal{L}^{-1}\left(e^{-\pi\,s}\mathcal{L}\left(\frac{1}{2}sen\left(2\,t\right)\right)\left(s\right)\right)\left(t\right)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+4}e^{-\pi\,s}\right)=\mathcal{U}_\pi\left(t\right)\frac{1}{2}sen\left(2\left(t-\pi\right)\right)$$

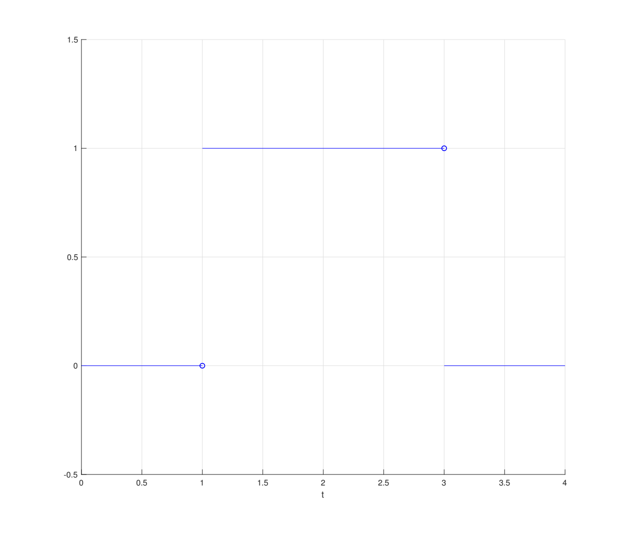
Función de Heaviside

Considere $a \geq 0$. Para todo $t \geq 0$ se define $\mathcal{U}_a(t) := \begin{cases} 0 & \text{si } 0 \leq t < a \\ 1 & \text{si } t \geq a \end{cases}$.

Propiedad

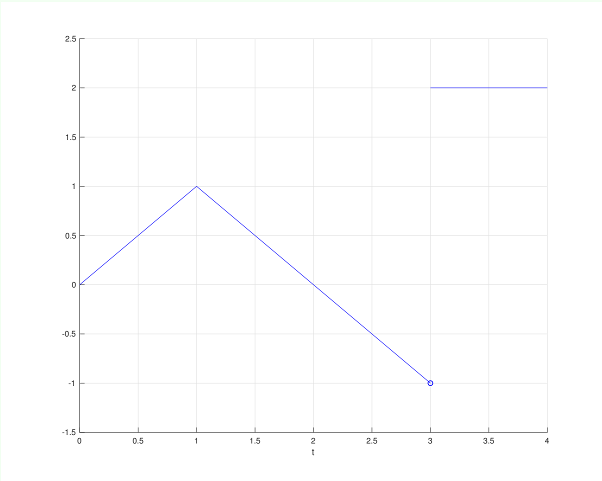
Fije $0 \leq a < b$. Entonces

$$\mathcal{U}_a(t) - \mathcal{U}_b(t) = \begin{cases} 0 & \text{si } 0 \leq t < a \\ 1 & \text{si } a \leq t < b \\ 0 & \text{si } t \geq b \end{cases}$$

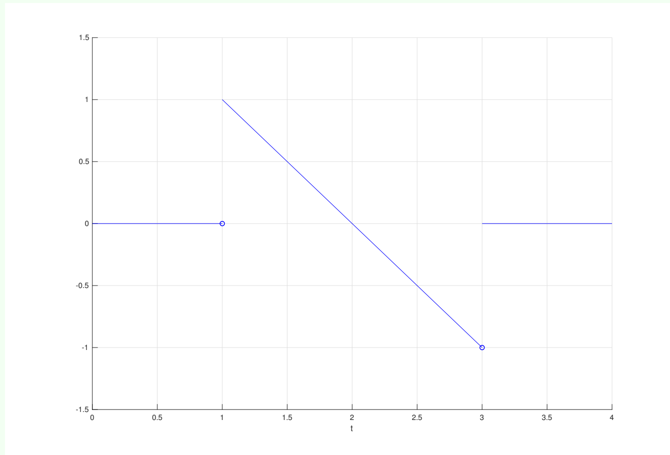


Ejemplo: Calcular la transformada de Laplace de:

$$f(t) = \begin{cases} t & \text{si } 0 \leq t < 1 \\ 2 - t & \text{si } 1 \leq t < 3 \\ 2 & \text{si } t \geq 3 \end{cases}$$



$$\xmapsto{(\mathcal{U}_1 - \mathcal{U}_3) f}$$



$$f(t) = t\left(\mathcal{U}_0(t) - \mathcal{U}_1(t)\right) + (2 - t)\left(\mathcal{U}_1(t) - \mathcal{U}_3(t)\right) + 2\,\mathcal{U}_3(t)$$

$$
\begin{aligned}
f(t) &= t\left(\mathcal{U}_0(t) - \mathcal{U}_1(t)\right) + (2-t)\left(\mathcal{U}_1(t) - \mathcal{U}_3(t)\right) + 2\,\mathcal{U}_3(t) \\
&= t\,\mathcal{U}_0(t) + (2 - 2\,t)\,\mathcal{U}_1(t) + t\,\mathcal{U}_3(t)
\end{aligned}
$$

Propiedad

Considere $g, h : [0, +\infty[\to \mathbb{R}$ tal que $\mathcal{L}(g)(s)$ y $\mathcal{L}(h)(s)$ existen. Fije $a, b \in \mathbb{R}$. Entonces

$$
\mathcal{L}(a\,g + b\,h)(s) = a\,\mathcal{L}(g)(s) + b\,\mathcal{L}(h)(s)
$$

$$
\int_0^{+\infty} e^{-s\,t}\left(a\,g(t) + b\,h(t)\right)dt = a\int_0^{+\infty} e^{-s\,t} g(t)\,dt + b\int_0^{+\infty} e^{-s\,t} h(t)\,dt
$$

$$
\begin{aligned}
\mathcal{L}(f(t))(s) &= \mathcal{L}(t\,\mathcal{U}_0(t))(s) + \mathcal{L}((2-2\,t)\,\mathcal{U}_1(t))(s) + \mathcal{L}(t\,\mathcal{U}_3(t))(s) \\
&= \mathcal{L}(t)(s) + \mathcal{L}((2-2\,t)\,\mathcal{U}_1(t))(s) + \mathcal{L}(t\,\mathcal{U}_3(t))(s)
\end{aligned}
$$

Propiedad

Considere $f : \mathbb{R} \to \mathbb{R}$ tal que $\mathcal{L}(f)(s)$ existe. Fije $a \geq 0$. Entonces

$$\mathcal{L}(f(t-a)\mathcal{U}_a(t))(s) = e^{-as}\mathcal{L}(f)(s)$$

$$\mathcal{L}(f(t-a)\mathcal{U}_a(t))(s) = \int_0^{+\infty} e^{-st} f(t-a)\mathcal{U}_a(t)\; dt = \int_a^{+\infty} e^{-st} f(t-a)\; dt = e^{-sa}\int_0^{+\infty} e^{-sx} f(x)\; dx$$

Observación

$$\mathcal{L}(g(t)\mathcal{U}_a(t))(s) = e^{-as}\mathcal{L}(g(t+a))(s)$$

$$\begin{aligned}\mathcal{L}((2-2t)\mathcal{U}_1(t))(s) + \mathcal{L}(t\mathcal{U}_3(t))(s) &= e^{-1s}\mathcal{L}(2-2(t+1))(s) - e^{-3s}\mathcal{L}(t+3)(s)\\ &= -2e^{-s}\mathcal{L}(t)(s) - e^{-3s}\mathcal{L}(t+3)(s)\end{aligned}$$

Ejemplo: Calcular la transformada de Laplace de:

$$f(t) = \begin{cases} t & \text{si } 0 \leq t < 1 \\ 2 - t & \text{si } 1 \leq t < 3 \\ 2 & \text{si } t \geq 3 \end{cases}$$

$$\mathcal{L}(f(t))(s) = \mathcal{L}(t)(s) - 2\,e^{-s}\mathcal{L}(t)(s) - \ e^{-3\,s}\mathcal{L}(t+3)(s)$$

Como

$$\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}} \qquad\qquad \forall s > 0$$

cuando $n \in \mathbb{N} \cup \{0\}$,

$$\mathcal{L}(f(t))(s) = \frac{1}{s^2} - \frac{2}{s^2}e^{-s} - \ e^{-3\,s}\left(\frac{1}{s^2} + \frac{3}{s}\right)$$