Ayudantía 22-06.

Recordar: $W \subset \mathbb{R}^2$ es T-invariante si $T(W) \subseteq W$.

Problema 1: Hallar todos los s.e.v de $\mathbb{R}^2$ invariantes para.

a) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ ; $(x,y) \longmapsto T(x,y) = (4x+2y, -3x+11y)$.

Invariantes ~~Invariantes~~:

- de dim cero $\{\theta_{\mathbb{R}^2}\}$
- de dim 2, $\mathbb{R}^2$, $T(\mathbb{R}^2) \subseteq \mathbb{R}^2$.
- de dim 1, son los s.e.v. propios.

Sea $\mathcal{B} = \{(1,0),(0,1)\}$ base de $\mathbb{R}^2$

$$[T]_{\mathcal{B}\mathcal{B}} = \left[ [T(e_1)]_{\mathcal{B}} \mid [T(e_2)]_{\mathcal{B}} \right] = \begin{bmatrix} 4 & 2 \\ -3 & 11 \end{bmatrix}.$$

$$\sigma(T) = \sigma([T]_{\mathcal{B}\mathcal{B}}) \Rightarrow \sigma(T) = P_{[T]_{\mathcal{B}\mathcal{B}}}(\lambda) = (4-\lambda)(11-\lambda)+6.$$

Con lo que $\sigma(T) = \{\lambda_1 = 5, \lambda_2 = 10\}$.

T-invariante de dim 1

$$S_{\lambda=1} = \left\{ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : \begin{pmatrix} -1 & 2 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = \left\{ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : v_1 = 3v_2 \right\} = \left\langle \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} \right\rangle.$$

$$S_{\lambda=2} = \left\{ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : \begin{pmatrix} -3 & 6 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = \left\{ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : 3v_1 = v_2 \right\} = \left\langle \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\} \right\rangle.$$

T-invariant

b) $S(x,y) = (-y, x)$. $\quad \mathcal{B} = \{(1,0),(0,1)\}$.

$\to$ asumiendo $\mathbb{K} = \mathbb{C}$.

$[S]_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow P_{[T]_{\mathcal{B}\mathcal{B}}}(\lambda) = \lambda^2 + 1. \iff \lambda \in \{\lambda_1 = -i, \lambda_2 = i\}$.

$S_{\lambda_1} = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : \begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = \langle \{ \begin{pmatrix} 1 \\ -i \end{pmatrix} \} \rangle$.

$S_{\lambda_2} = \langle \{ \begin{matrix} 1 \\ i \end{matrix} \} \rangle$.

Problema 2: Sea $A \in M_6(\ )$ tal que su Matriz de Jordan es:

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

a) Encuentre $P_A(\lambda)$

$$J = \begin{bmatrix} J_2(0) & & & \\ & J_1(-1) & & \\ & & J_2(2) & \\ & & & J_1(1) \end{bmatrix}$$

b) $\sigma(A) = \{-1, 0, 1, 2\}$.

$\Rightarrow P_A(\lambda) = \lambda^2 (\lambda+1)(\lambda-2)^2(\lambda-1)$.

c) Multiplicidad geométrica y algebráica:

| | M.A. | Mg. |
|---|---|---|
| 0 | 2 | 1 |
| -1 | 1 | 1 |
| 2 | 2 | 1 |
| 1 | 1 | 1 |

* Mg. Contar la cantidad de cajas.
* MA. Contar dim de la caja correspondiente a $\lambda$.

4 s.e.v A-invariantes de dim 1.
(sumar Mg).

Problema 3: Determine todos las formas posibles de Jordan para una Matriz $A$ con Polinomio Característico $P_A(\lambda) = (-1-\lambda)(3-\lambda)^2$.

Sol: - Para el factor $(-1-\lambda)$ sólo podemos tener asociado $J_1(-1) = (-1)$.

- Para el factor $(3-\lambda)^2$ $J_1(3) = (3)$ o $J_2(3) = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$.

A debe tener dim $3\times 3$.

Las Posibilidades.

1. $J^1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ 2. $J^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ 3. $J^3 = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$\shortparallel$

4. $J^4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ 5. $J^5 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ 6. $J^6 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Problema 4: Encuentre la descomposición de Jordan.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Sol:

1. Encontrar $\sigma(A)$:

Como es Triangular, Det $= \prod_{i=1}^{5} a_{ii}$

$\Rightarrow P_A(\lambda) = \underbrace{(1-\lambda)^3}_{\lambda_1} \underbrace{(3-\lambda)}_{\lambda_2} \underbrace{(2-\lambda)}_{\lambda_3}$. ; $m_{\lambda_1} = 3$, $m_{\lambda_2} = 1$, $m_{\lambda_3} = 1$.

2. Encontrar los $S_\lambda$.

$\bullet\ S_{\lambda_1} = \ker(A - Id) = \ker \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \left\{ v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} : \begin{matrix} v_5 = 0 \\ v_4 = 0 \\ v_3 = 0 \end{matrix} \right\} = \left\langle \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \right\rangle.$

3. Buscar vector cola que complete $S_{\lambda_1}$.

1) $(A - 1 \cdot Id)v = e_1$ X.

2) $(A - 1 \cdot Id)v = e_2 \iff \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$ $\Rightarrow v_5 = 0,\ v_4 = 0,\ v_3 = 1/3$

$\Rightarrow v_{22} = \begin{pmatrix} 0 \\ 0 \\ 1/3 \\ 0 \\ 0 \end{pmatrix}.$

4. Mellan $S_{\lambda 2}$, $S_{\lambda 3}$ $\qquad \lambda_2 = 3$, $\lambda_3 = 2$.

- $S_{\lambda 2 = 3} = \operatorname{Ker}(A - 3\,Id) = \operatorname{Ker}\begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 3 & 4 & 5 \\ 0 & 0 & -2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \Rightarrow \begin{array}{l} v_5 = 0 \\ v_1 = 0 \\ -2v_2 + 3v_3 + 4v_4 = 0 \\ v_3 = v_4 \end{array}$ $\qquad v_2 = \frac{7}{2} v_4$

$\Rightarrow S_{\lambda 2 = 3} = \left\langle \left\{ \begin{pmatrix} 0 \\ 7/2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\} \right\rangle$.

- $S_{\lambda 3 = 2} = \operatorname{Ker}(A - 2\,Id) = \operatorname{Ker}\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & 4 & 5 \\ 0 & 0 & -1 & 2 & 3 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$ $\quad \begin{array}{l} v_1 = 0 \\ v_4 = -3v_5 \\ v_3 = v_4 \end{array}$ $\quad v_2 = 3v_3 + 4v_4 + \frac{21}{3} v_4 - \frac{5}{3} v_4 = \frac{16}{3} v_3$

$\Rightarrow S_{\lambda 3 = 2} = \left\langle \left\{ \begin{pmatrix} 0 \\ 14/3 \\ 1 \\ 1 \\ -1/3 \end{pmatrix} \right\} \right\rangle$.

| $\lambda$ | $E_1(\lambda)$ | $E_2(\lambda) \setminus E_1(\lambda)$ | |
|---|---|---|---|
| $\lambda_1 = 1$ | $v_{11} = e_1 \longrightarrow$<br>$v_{12} = e_2 \longrightarrow$ | X<br>$(0, 0, 1/3, 0, 0)^T$ | |
| $\lambda_2 = 3$ | $u_2$ $(0, 7/2, 1, 1, 0)^T$ | | |
| $\lambda_3 = 2$ | $u_4$ $(0, 14/3, 1, 1, -1/3)^T$ | | |

$P = [e_1 \ e_2 \ v_{22} \ u \ w] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cancel{7/2} & 14/3 \\ 0 & 0 & 1/3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1/3 \end{bmatrix}$

$\Rightarrow J = \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{3} & 0 \\ 0 & 0 & 0 & 0 & \boxed{2} \end{bmatrix}$