(5) Let $h(x) = \dfrac{a^T x + \alpha}{b^T x + \beta}$ and $K = \{x \in \mathbb{R}^n : b^T x + \beta > 0\}$.

Obviously $\nabla h(x) = \dfrac{(b^T x + \beta) a - (a^T x + \alpha) b}{(b^T x + \beta)^2}$

thus

$$\nabla h(x)^T (y-x) \geq 0 \iff (b^T x + \beta) a^T (y-x) \geq (a^T x + \alpha) b^T (y-x)$$

If $x \in K$,

$$a^T (y-x) \geq \frac{a^T x + \alpha}{b^T x + \beta} b^T (y-x), \text{ or equivalently}$$

$$a^T y + \alpha - a^T x - \alpha \geq \frac{a^T x + \alpha}{b^T x + \beta} (b^T y + \beta - b^T x - \beta), \text{ or}$$

$$a^T y + \alpha \geq \frac{a^T x + \alpha}{b^T x + \beta} (b^T y + \beta - \cancel{b^T x} - \beta + \cancel{b^T x} + \beta),$$

$$h(y) = \frac{a^T y + \alpha}{b^T y + \beta} \geq \frac{a^T x + \alpha}{b^T x + \beta} = h(x).$$ Hence h is pseudoconvex.

This implies also that $\dfrac{(-a)^T x + (-\alpha)}{b^T x + \beta}$ is pseudo convex, which asserts that $-h$ is pseudoconvex.

F. Flores

(6) Evidently $\nabla f(x_1,x_2) = 2\left(x_1 - \frac{3}{2}, x_2 - 5\right)$ and so $\nabla f(1,3) = (-1,-4)$. This yields

$$\nabla f(1,3)^T((x_1,x_2)-(1,3)) = -x_1 - 4x_2 + 13 \quad \ldots (1)$$

On the other hand,

$$(x_1,x_2) \in K \Rightarrow -x_1 + x_2 \leq 2 \text{ and } 2x_1 + 3x_2 \leq 11$$

$$\Rightarrow \text{(summing up) } x_1 + 4x_2 \leq 13 \quad \ldots (2)$$

(1) and (2) give

$$\nabla f(\bar{x})(x-\bar{x}) \geq 0 \quad \forall x \in K,$$

which means that $\bar{x}$ is minimum of f on K in view of the convexity of f.

(2) (a) ✓ (b) ✓

(c) Clearly $C_1 \cup C_2$ is a cone.

$$\left.\begin{array}{l} C_1 \subseteq C_1 + C_2 \\ C_2 \subseteq C_1 + C_2 \end{array}\right\} \Rightarrow C_1 \cup C_2 \subseteq C_1 + C_2 \Rightarrow co(C_1 \cup C_2) \subseteq C_1 + C_2$$

since $C_1 + C_2$ is convex (by (b)).

Let $x_1 \in C_1$, $x_2 \in C_2$. Then

$$x_1 + x_2 = \frac{1}{2}(2x_1) + \frac{1}{2}(2x_2) \in \frac{1}{2}(C_1 \cup C_2) + \frac{1}{2}(C_1 \cup C_2)$$

$$\Rightarrow x_1 + x_2 \in co(C_1 \cup C_2). \therefore C_1 + C_2 \subseteq co(C_1 \cup C_2)$$

F. Flores

(7) (a) $f(ty_1+(1-t)y_2) = \sup_{x\in K} (ty_1+(1-t)y_2)^T x \quad (0<t<1)$

$= \sup_{x\in K}(t y_1^T x + (1-t)y_2^T x) \leq \sup_{x\in K} t\, y_1^T x + \sup_{x\in K} (1-t)\, y_2^T x$

$= t \sup_{x\in K} y_1^T x + (1-t) \sup_{x\in K} y_2^T x = t f(y_1) + (1-t) f(y_2).$

(b) We need to prove

$$f(z) \geq f(y) + \bar{x}^T (z-y) \quad \forall z \in \mathbb{R}^n.$$

Clearly, for $z \in \mathbb{R}^n$, $f(z) = \sup_{x\in K} z^T x \geq z^T x \quad \forall x \in K.$

In particular,

$f(z) \geq z^T \bar{x} = z^T \bar{x} + y^T \bar{x} - y^T \bar{x} = y^T \bar{x} + \bar{x}^T (z-y)$

$\Rightarrow \quad f(z) \geq f(y) + \bar{x}^T (z-y)$, i.e., $\bar{x} \in \partial f(y)$.

(3) $n=1$, $A = \mathbb{Q}$ (los números racionales). $L = \mathbb{R}$ y $(a\in A)$

(b) se cumple claramente, y

(a) no!

(1) Since M is a convex cone and every $x \in M, x \neq 0$, verifies $x = \frac{1}{2}(0) + \frac{1}{2}(2x)$, x cannot be extremal. thus the only possible extremal point of M should be the origin. For example, $n=2$, $A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$.

F. Flores