# Transformada de Laplace

Resumen

Carlos M. Mora

## Transformada de Laplace de $f : [0, +\infty[ \to \mathbb{R}$

$$f \xmapsto{\mathcal{L}} \widehat{f}(s) := \int_0^{+\infty} e^{-s\,t} f(t)\, dt$$

## Valores

- $\mathcal{L}(\mathbf{1})(s) = \frac{1}{s}$, $\quad\mathcal{L}^{-1}\left(\frac{1}{s}\right)(t) = 1$
- $\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}$, $\quad\mathcal{L}^{-1}\left(\frac{n!}{s^{n+1}}\right)(t) = t^n \qquad$ cuando $n = 1, 2, 3, \ldots$
- $\mathcal{L}\left(e^{\alpha\,t}\right)(s) = \frac{1}{s-\alpha}$, $\quad\mathcal{L}^{-1}\left(\frac{1}{s-\alpha}\right)(t) = e^{\alpha\,t}$
- $\mathcal{L}(sen(\beta\,t))(s) = \frac{\beta}{s^2+\beta^2}$, $\quad\mathcal{L}^{-1}\left(\frac{\beta}{s^2+\beta^2}\right)(t) = sen(\beta\,t)$
- $\mathcal{L}(senh(\beta\,t))(s) = \frac{\beta}{s^2-\beta^2}$, $\quad\mathcal{L}^{-1}\left(\frac{\beta}{s^2-\beta^2}\right)(t) = senh(\beta\,t)$
- $\mathcal{L}(\cos(\beta\,t))(s) = \frac{s}{s^2+\beta^2}$, $\quad\mathcal{L}^{-1}\left(\frac{s}{s^2+\beta^2}\right)(t) = \cos(\beta\,t)$
- $\mathcal{L}(\cosh(\beta\,t))(s) = \frac{s}{s^2-\beta^2}$, $\quad\mathcal{L}^{-1}\left(\frac{s}{s^2-\beta^2}\right)(t) = \cosh(\beta\,t)$
- $\mathcal{L}(\mathcal{U}_a(t))(s) = \frac{e^{-s\,a}}{s}$, $\quad\mathcal{L}^{-1}\left(\frac{e^{-s\,a}}{s}\right)(t) = \mathcal{U}_a(t) \qquad$ cuando $a \geq 0$
- $\mathcal{L}(\delta_a(t))(s) = e^{-s\,a}$, $\quad\mathcal{L}^{-1}\left(e^{-s\,a}\right)(t) = \delta_a(t) \qquad$ cuando $a \geq 0$
- $\mathcal{L}^{-1}\left(\frac{1}{(s^2+a^2)^2}\right)(t) = \frac{1}{2a^3}\left(sen(a\,t) - a\,t\,\cos(a\,t)\right) \qquad$ cuando $a > 0$
- $(f, g) \xmapsto{*} f * g(t) := \int_0^t f(t-s)\, g(s)\, ds \qquad \forall t \geq 0.$

## Propiedades

- $\mathcal{L}(f')(s) = s\,\mathcal{L}(f)(s) - f(0)$
- $\mathcal{L}\left(e^{-\beta x} f(x)\right)(s) = \mathcal{L}(f)(s+\beta)$, $\qquad \mathcal{L}^{-1}(F(s+\beta))(x) = e^{-\beta x}\mathcal{L}^{-1}(F)(x)$
- $\mathcal{L}(x\,f(x))(s) = -\frac{d}{ds}\mathcal{L}(f)(s)$
- $\mathcal{L}\left(t^k f(t)\right)(s) = (-1)^k \frac{d^k}{ds^k}\mathcal{L}(f)(s)$
- $\mathcal{L}(f(t-a)\mathcal{U}_a(t))(s) = e^{-a\,s}\mathcal{L}(f)(s)$ cuando $a \geq 0$
- $\mathcal{L}(g(t)\mathcal{U}_a(t))(s) = e^{-a\,s}\mathcal{L}(g(t+a))(s)$ cuando $a \geq 0$
- $\mathcal{L}(a\,g + b\,h)(s) = a\,\mathcal{L}(g)(s) + b\,\mathcal{L}(h)(s)$ para $a, b \in \mathbb{R}$.
- $\mathcal{L}(f * g)(s) = \mathcal{L}(f)(s)\,\mathcal{L}(g)(s)$
- $\mathcal{L}^{-1}(F(s)\,G(s))(t) = \mathcal{L}^{-1}(F(s))(t) * \mathcal{L}^{-1}(G(s))(t)$
- $\mathcal{L}\left(\int_0^t f(u)\,du\right)(s) = \frac{1}{s}\mathcal{L}(f)(s)$, $\qquad \mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right)(s) = \int_0^t \mathcal{L}^{-1}(F)(u)\,du$

## Función de orden exponencial

Una función $f : [0, +\infty[ \to \mathbb{R}$ es de orden exponencial $\alpha$ si:

- $f$ es continua por partes ($f$ es continua en $[0, +\infty[$ salvo un número finito de puntos donde $f$ tiene limites por la derecha e izquierda)
- Existe $K > 0$ tal que
$$|f(t)| \leq K e^{\alpha t} \qquad \forall t \geq 0.$$

Suponga que $f : [0, +\infty[ \to \mathbb{R}$ es de orden exponencial $\alpha$.
Entonces, $\mathcal{L}(f)(s)$ existe para todo $s > \alpha$ y

$$lim_{s \to +\infty} \mathcal{L}(f)(s) = 0.$$

## Función de Heaviside

Considere $a \geq 0$. Para todo $t \geq 0$ se define

$$\mathcal{U}_a(t) := \begin{cases} 0 & \text{si } 0 \leq t < a \\ 1 & \text{si } t \geq a \end{cases}.$$

También se denota $\mathcal{U}_a(t) = \mathcal{H}_a(t) = \mathcal{U}(t-a) = \mathcal{H}(t-a)$.

Fije $0 \leq a < b$. Entonces $\mathcal{U}_a(t) - \mathcal{U}_b(t) = \begin{cases} 0 & \text{si } 0 \leq t < a \\ 1 & \text{si } a \leq t < b \\ 0 & \text{si } t \geq b \end{cases}$