Ayudantía 11.

1. La sucesión de Fibonacci es: $f_0 = 1$, $f_1 = 1$

$$f_{n+1} = f_n + f_{n-1} \quad \forall n \in \mathbb{N}.$$

a) Denotando $u_n = \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix}$, determinar $A \in M_2(\mathbb{R})$ tq $u_{n+1} = A u_n$ $\forall n \in \mathbb{N}$.

Sol: Sea $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} = \begin{pmatrix} a f_n + b f_{n-1} \\ c f_n + d f_{n-1} \end{pmatrix} = \begin{pmatrix} f_n + f_{n-1} \\ f_n \end{pmatrix} \qquad \begin{matrix} a = 1 & c = 1 \\ b = 1 & d = 0. \end{matrix}$$

Por lo que la Matriz es

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad u_{n+1} = A u_n.$$

b) Demuestre que $\forall n \in \mathbb{N}$, $u_n = A^{n-1} u_1$

Solución:

Caso base: $n = 1$, $u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = I u_1 = A^0 u_1$

Hipótesis inductiva: $u_n = A^{n-1} u_1$.

Paso inductivo:
$$\begin{aligned} u_{n+1} &= A u_n && (\text{Por a)}) \\ &= A \cdot A^{n} u_1 && (\text{Por H.I.}) \\ &= A^{n} u_1 \end{aligned}$$

Por el principio de inducción matemática se concluye que $\forall n \in \mathbb{N}$, $u_n = A^{n-1} u_1$

A simétrica $\mathbb{R}$, $\Rightarrow$ A diagonalizable.

c) diagonalice A y determine una formula cerrada para F_n, (Depende de n pero no de F_{n-1}).

Solución: Como A es simétrica, es diagonalizable, por lo que $\exists P \in M_2(\mathbb{R})$ invertible tal que

$$A = P \cdot D \cdot P^{-1}$$

Donde $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\lambda_1, \lambda_2 \in \sigma(A)$.

Pero $u_{n+1} = A^n u_1 = \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$

Donde $A^n = (PDP^{-1})^n = (PD\underbrace{P^{-1}P}_{Id}D\underbrace{P^{-1}}_{Id} \cdots PDP^{-1}) = PD^nP^{-1}$

Con lo que

$$u_{n+1} = \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \overbrace{PD^nP^{-1}}^{\text{No tiene } f_{n-1}} u_1 \quad (*).$$

Calculamos $\sigma(A)$

$$P_A(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = -\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1$$

Los valores propios son ceros de P_A,

$$P_A(\lambda) = 0 \iff \lambda^2 - \lambda - 1 = 0 \iff \lambda_1 = \frac{1+\sqrt{5}}{2} = \varphi, \quad \lambda_2 = \frac{1-\sqrt{5}}{2} = \varphi^- = 1 - \varphi$$

φ es la razón áurea

$$\sigma(A) = \{\varphi, 1-\varphi\}.$$

• Ahora calculamos los vectores propios:

$$S_\varphi = \text{Ker}\,(A-\varphi I) = \left\{ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2 : \begin{pmatrix} 1-\varphi & 1 \\ 1 & -\varphi \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \right\}.$$

$$\left(\begin{array}{cc|c} 1-\varphi & 1 & 0 \\ 1 & -\varphi & 0 \end{array}\right) \quad F_2 \leftarrow F_2 - \frac{1}{1-\varphi} F_1$$

$$\left(\begin{array}{cc|c} 1-\varphi & 1 & 0 \\ 0 & \cancel{*} & 0 \end{array}\right) \quad * = \varphi - \frac{1}{1-\varphi} = 0$$

$$\varphi^2 - \varphi - 1 = 0$$
$$\varphi^2 - \varphi = 1$$
$$\varphi(\varphi - 1) = 1$$
$$\Rightarrow \varphi - 1 = 1/\varphi \quad \Rightarrow 1 - \varphi = -1/\varphi$$

$$S_\varphi = \left\{ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2 : \begin{pmatrix} 1-\varphi & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$
$$= \left\{ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2 : (1-\varphi) v_1 + v_2 = 0 \right\}.$$
$$= \left\{ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2 : v_1 = \varphi v_2 \right\}.$$
$$= \left\{ v = \begin{pmatrix} \varphi v_2 \\ v_2 \end{pmatrix} \in \mathbb{R}^2,\ v_2 \in \mathbb{R} \right\}.$$
$$= \left\langle \left\{ \begin{pmatrix} \varphi \\ 1 \end{pmatrix} \right\} \right\rangle.$$

* Tarea: $S_{\varphi^-} = \left\langle \left\{ \begin{pmatrix} 1 \\ -\varphi \end{pmatrix} \right\} \right\rangle$

Consecuencia:

$$P = \begin{pmatrix} \varphi & 1 \\ 1 & -\varphi \end{pmatrix} \Rightarrow \det(P) = -\varphi^2 - 1.$$

$$P^{-1} = \frac{1}{\det(P)} \cdot \overbrace{\text{Cof}(P)^T}^{\text{Adj}(P)}.$$
$$= \frac{1}{-\varphi^2-1} \cdot \begin{pmatrix} -\varphi & -1 \\ -1 & \varphi \end{pmatrix}$$
$$= \frac{1}{\varphi^2+1} \cdot \begin{pmatrix} \varphi & 1 \\ 1 & \varphi \end{pmatrix} = \frac{1}{\varphi^2+1}$$

$$u_{n+1} = \begin{pmatrix} F_{n+1} \\ P_n \end{pmatrix} = \begin{pmatrix} \varphi & 1 \\ 1 & -\varphi \end{pmatrix}\begin{pmatrix} \varphi^n & 0 \\ 0 & (1-\varphi)^n \end{pmatrix}\left(\frac{1}{\varphi^2+1}\right)\begin{pmatrix} \varphi & 1 \\ 1 & -\varphi \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
$$= \frac{1}{\varphi^2+1}\begin{pmatrix} \varphi^{n+1} & (1-\varphi)^n \\ \varphi^n & -\varphi(1-\varphi)^n \end{pmatrix}\begin{pmatrix} \varphi & 1 \\ 1 & -\varphi \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$= \frac{1}{\varphi^2+1}\begin{pmatrix} \varphi^{n+2} + (1-\varphi)^n & \varphi^{n+1} - \varphi(1-\varphi)^n \\ \varphi^{n+1} - \varphi(1-\varphi)^n & \varphi^n + \varphi^2(1-\varphi)^n \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$= \frac{1}{\varphi^2+1} \begin{pmatrix} \varphi^{n+2} + (1-\varphi)^n + \varphi^{n+1} - \varphi(1-\varphi)^n \\ \varphi^{n+1} - \varphi(1-\varphi)^n + \varphi^n + \varphi^2 + \varphi^2(1-\varphi)^n \end{pmatrix}.$$

$$= \frac{1}{\varphi^2+1} \begin{pmatrix} \varphi^{n+1}(\varphi+1) + (1-\varphi)^n(1-\varphi) \\ \varphi^n(\varphi+1) + (1-\varphi)^n(-\varphi+\varphi^2) \end{pmatrix} \qquad \varphi^2-\varphi-1=0.$$

$$= \frac{1}{\varphi^2+1} \begin{pmatrix} \varphi^{n+1}(\varphi+1) + (1-\varphi)^{n+1} \\ \varphi^n(\varphi+1) + (1-\varphi)^n \end{pmatrix}$$

$$\Rightarrow f_n = \frac{1}{\varphi^2+1}\left(\varphi^n(\varphi+1) + (1-\varphi)^n\right)$$

* Los ortogonales son Cerrados

2. Sea $(V, \langle \cdot;\cdot \rangle)$ $\mathbb{R}$-e.v. con p.i. y $S \subseteq V$ (Conjunto)

complemento ortogonal.

$S^{\perp} = \{u \in V : \forall s \in S \cdot \langle u, s\rangle = 0\}$.

a) $S^{\perp}$ es s.e.v de V.

- Veamos que $\theta \in S^{\perp}$. En efecto, $\forall s \in S$, $\langle \theta, s\rangle = 0 \Rightarrow \theta \in S^{\perp}$

- Sean $u_1, u_2 \in S^{\perp}$, $\alpha \in \mathbb{R}$. Sea $s \in S$,

$\langle u_1 + \alpha u_2, s\rangle = \cancel{\langle u_1, s\rangle}^{0} + \alpha \cancel{\langle u_2, s\rangle}^{0} = 0.$

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b) $S^{\perp} = \langle S\rangle^{\perp}$

Sea $u \in S^{\perp}$ y sea $s = \sum_{i=1}^{k} \alpha_i \overset{\in S}{s_i} \in \langle S\rangle$.

$\langle u, \sum_{i=1}^{k} \alpha_i s_i \rangle = \sum_{i=1}^{k} \alpha_i \cancel{\langle u, s_i\rangle}^{0} = 0.$

Conclusión $\forall s \in \langle S\rangle$, $\langle u, s\rangle = 0 \Leftrightarrow u \in \langle S\rangle^{\perp}$

2] Sea $u \in \langle S\rangle^{\perp} \Rightarrow \forall s \in \langle S\rangle \cdot \langle u, s\rangle = 0$

$\Rightarrow \forall s \in S, \langle u, s\rangle = 0$

$\Leftrightarrow u \in S^{\perp}.$

c) $S \subseteq S' \Rightarrow S^{\perp} \subseteq (S')^{\perp}$ (Tarea).

d) Si V es de dim finita $V = \langle S \rangle \oplus \langle S \rangle^{\perp}$.

Solución: Sea $\{e_1, \dots, e_k\}$ base (ortonormal) de $\langle S \rangle$ (siempre se puede).

Definimos:

$$p : V \longrightarrow \langle S \rangle$$
$$v \longmapsto p(v) = \sum_{j=1}^{k} \langle v, e_j \rangle e_j$$

Se prueba que es lineal, Además, se define:

$$p^{\perp} : V \longrightarrow$$
$$v \longmapsto p^{\perp}(v) = v - p(v)$$

$$\langle e_i, p^{\perp}(v) \rangle = \langle e_j, v - p(v) \rangle = \langle e_i, v \rangle - \langle e_i, \sum_{j=1}^{k} \underbrace{\langle v, e_j \rangle}_{\in \mathbb{R}} e_j \rangle$$
$$= \langle e_i, v \rangle - \sum_{j=1}^{k} \langle v, e_j \rangle \cancel{\langle e_i, e_j \rangle}$$

Base ortogonal.

$$= \langle e_i, v \rangle - \langle v, e_i \rangle \langle e_i, e_j \rangle$$

$$= 0$$

Se extiende $\forall s \in \langle S \rangle$

Ahora, dado $v \in V$

$$v = \underset{\in \langle S \rangle}{P(v)} + \underbrace{(v - P(v))}_{\in \langle S \rangle^{\perp}}$$

Por lo que $V = \langle S \rangle + \langle S \rangle^{\perp}$

Además, si $x \in \langle S \rangle \cap \langle S \rangle^{\perp}$

Como $x \in \langle S \rangle^{\perp}$, $\forall s \in \langle S \rangle$, $\langle s, x \rangle = 0$

$\Rightarrow$ Para $s = x$, $\langle x, x \rangle = 0$

$\|x\|^2 = 0 \quad \Rightarrow x = \Theta$

e) $\langle S \rangle = (S^{\perp})^{\perp}$

$\subseteq]$ Sea $s \in \langle S \rangle$, $\Xi := S^{\perp} = \langle S \rangle^{\perp}$

$\forall \xi \in \Xi$, $\langle s, \xi \rangle = 0$.

$s \in \Xi^{\perp} = (S^{\perp})^{\perp}$.

$\supseteq]$ Sea $u \in (S^{\perp})^{\perp} \subseteq V = \langle S \rangle \oplus S^{\perp}$

$\exists\ v_1 \in S \wedge v_2 \in S^{\perp} = \langle S \rangle^{\perp}$ tal que.
$u = v_1 + v_2$. $\Rightarrow v_2 = u - v_1$

$v_1 \in \langle S \rangle \subseteq (S^{\perp})^{\perp}$

Como $(S^{\perp})^{\perp}$ es s.e.v es cerrado para $+$ y $\odot$ $\Rightarrow u - v_1 \in (S^{\perp})^{\perp}$
Además, $u - v_1 = v_2 \in S^{\perp}$
$\Rightarrow u - v_1 \in (S^{\perp})^{\perp} \cap S^{\perp}$ (Por inciso d), $S^{\perp} \oplus (S^{\perp})^{\perp} = V$).

$\Rightarrow u = v_1 \in \langle S \rangle$

$\therefore (S^{\perp})^{\perp} \subseteq \langle S \rangle$

Problema 3:

$(V, \langle \cdot, \cdot \rangle)$ $\overbrace{\text{real con P.I.}}^{\text{Pre-Hilbert}}$; $T: V \to V$. T.L. simétrica ($\langle T(u), v\rangle = \langle u, T(v)\rangle$).

tal que $\forall u \in V$, $\langle T(u), u \rangle = 0$.

a) $\forall u, v \in V$, $\langle T(u), v \rangle = 0$.

$$
\begin{aligned}
0 = \langle T(u \cdot), u - v \rangle &= \langle T(u) - T(v), u - v \rangle \\
&= \langle T(u) - T(v), u \rangle - \langle T(u) - T(v), v \rangle \\
&= \cancel{\langle T(u), u \rangle} - \langle T(v), u \rangle - \langle T(u), v \rangle + \cancel{\langle T(v), v \rangle} \\
&= -\langle T(v), u \rangle - \langle T(u), v \rangle \\
&\overset{\text{Sim}}{=} -\langle T(v), u \rangle - \langle T(v), u \rangle \\
&= -2 \langle T(v), u \rangle \\
&\overset{\text{Sim}}{=} -2 \langle T(u), v \rangle \\
&\Rightarrow \langle T(u), v \rangle = 0.
\end{aligned}
$$

b) $T = \theta$.

Sol: Por a), $\forall u, v \in V$, $\langle T(u), v \rangle = 0$.

$v = T(u) \Rightarrow \langle T(u), T(u) \rangle = 0 = \| T(u) \|^2 \Rightarrow T(u) = \theta$, $\forall u \in V$.

$\Rightarrow T = \theta$