Solución de EDOs usando la Transformada de Laplace

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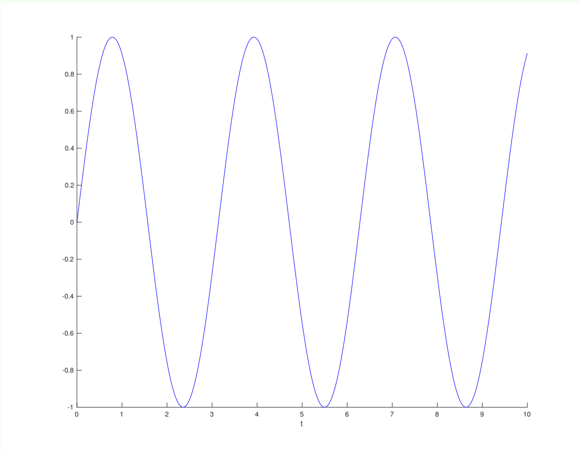
Transformada de Laplace de $f : [0, +\infty[\to \mathbb{R}$

$$f \xmapsto{\mathcal{L}} \widehat{f}(s) := \int_0^{+\infty} e^{-s\,t} f(t)\, dt$$

$$Dom\left(\widehat{f}\right) = \left\{ s \in \mathbb{R} :\ \text{existe } lim_{R \to +\infty} \int_0^{R} e^{-s\,t} f(t)\, dt \right\}$$

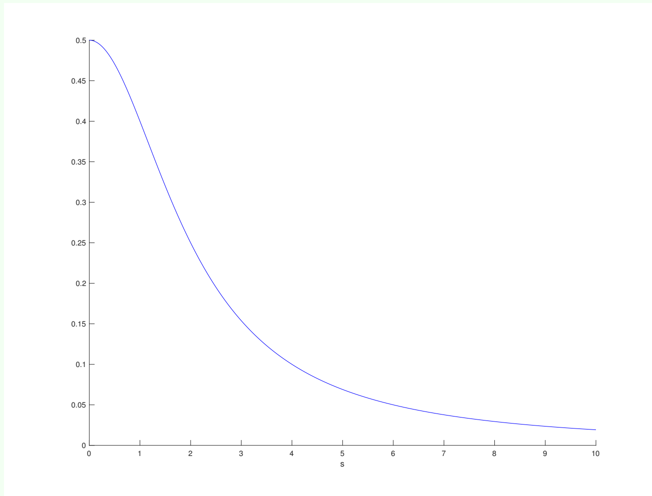
Ejemplo:

$$\mathcal{L}\left(sen\left(\beta\, t\right)\right)(s) = \frac{\beta}{s^2 + \beta^2} \qquad\qquad \forall s > 0$$



$$\xmapsto{\mathcal{L}}$$

$$\xleftarrow[\mathcal{L}^{-1}]{}$$



Función de orden exponencial

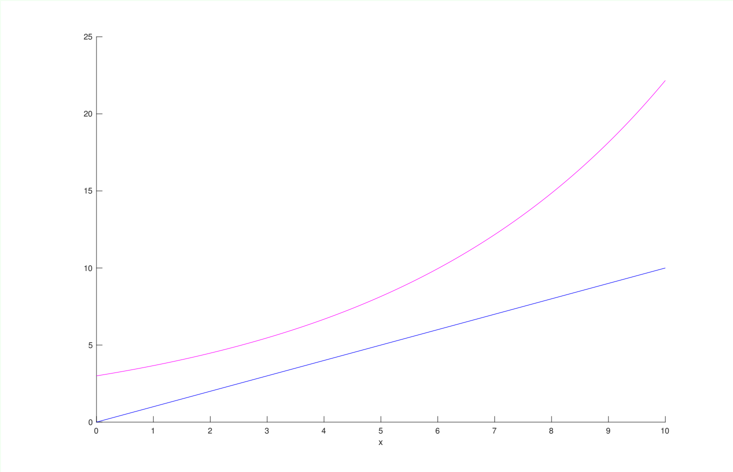
Una función $f : [0, +\infty[\to \mathbb{R}$ es de orden exponencial α si:

- f es continua por partes (*f es continua en $[0, +\infty[$ salvo un número finito de puntos donde f tiene limites por la derecha e izquierda*)
- Existe $K > 0$ tal que

$$|f(t)| \leq K e^{\alpha t} \qquad \forall t \geq 0.$$

Propiedad

Suponga que $f : [0, +\infty[\to \mathbb{R}$ es de orden exponencial α. Entonces $\mathcal{L}(f)(s)$ existe para todo $s > \alpha$.



$$f(x) = x$$

es de orden exponencial α para todo $\alpha > 0$.

Propiedad

Considere la función continua $f : [0, +\infty[\to \mathbb{R}$ de orden exponencial α tal que f' es continua por partes. Entonces para todo $s > \alpha$,

$$\mathcal{L}(f')(s) = s\,\mathcal{L}(f)(s) - f(0)$$

$$\mathcal{L}(f')(s) = lim_{R\to+\infty} \int_0^R e^{-s\,t}\, f'(t)\, dt = lim_{R\to+\infty} \left(e^{-s\,t}\, f(t) \Big|_{t=0}^{t=R} - \int_0^R -s\, e^{-s\,t}\, f(t)\, dt \right)$$

$$\begin{aligned} \mathcal{L}(f')(s) &= lim_{R\to+\infty} \left(e^{-s\,R}\, f(R) - e^{-s\,0}\, f(0) - \int_0^R -s\, e^{-s\,t}\, f(t)\, dt \right) \\ &= 0 - f(0) + s\,\mathcal{L}(f)(s) \end{aligned}$$

Ejemplo de uso de $\mathcal{L}(f')(s) = s\,\mathcal{L}(f)(s) - f(0)$

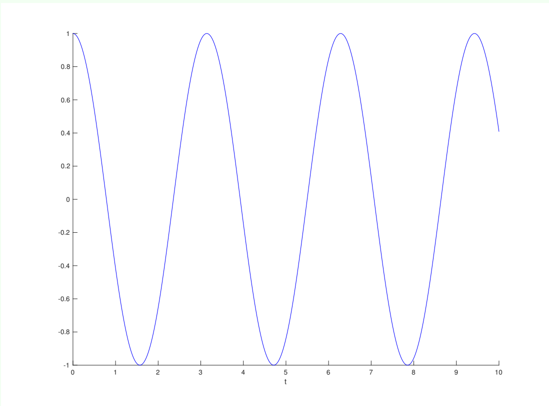
Calcular la transformada de Laplace de $\cos(\beta\, t)$

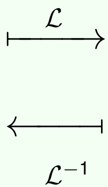
Para todo $s > 0$,

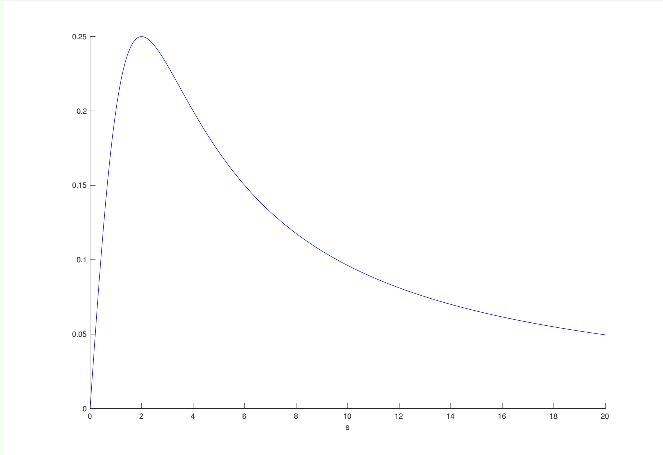
$$\mathcal{L}\left(\frac{d}{dt} sen(\beta\, t)\right)(s) = s\,\mathcal{L}(sen(\beta\, t))(s) - sen(\beta \cdot 0) = s\,\frac{\beta}{s^2+\beta^2}$$

$$\beta\,\mathcal{L}(\cos(\beta\, t))(s) = \frac{\beta\, s}{s^2+\beta^2} \qquad \Rightarrow \qquad \mathcal{L}(\cos(\beta\, t))(s) = \frac{s}{s^2+\beta^2}.$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+\beta^2}\right)(t) = \cos(\beta\, t)$$







$$\mathcal{L}\left(f'\right)(s) = s\,\mathcal{L}\left(f\right)(s) - f\left(0\right)$$

Ejemplo

Calcular la transformada de Laplace de $f\left(x\right) = x$.

Para todo $s > 0$,

$$\mathcal{L}\left(\mathbf{1}\right)(s) = \mathcal{L}\left(\frac{d}{dx}\left(x\right)\right)(s) = s\,\mathcal{L}\left(x\right)(s) - 0$$

$$\frac{1}{s} = s\,\mathcal{L}\left(x\right)(s) \qquad \Rightarrow \qquad \mathcal{L}\left(x\right)(s) = \frac{1}{s^2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)(x) = x$$

La relación

$$\mathcal{L}(f')(s) = s\,\mathcal{L}(f)(s) - f(0)$$

transforma ecuaciones diferenciales ordinarias en ecuaciones algebraicas.

Ejemplo

$$y''(x) + 4\,y(x) = x, \qquad y(0) = 0, \qquad y'(0) = 0$$

$$\mathcal{L}(y''(x) + 4\,y(x))(s) = \mathcal{L}(x)(s)$$

$$\mathcal{L}(y''(x))(s) + 4\,\mathcal{L}(y(x))(s) = \frac{1}{s^2}$$

$$\mathcal{L}(y'(x))(s) = s\,\mathcal{L}(y(x))(s) - y(0) = s\,\mathcal{L}(y(x))(s)$$

$$\mathcal{L}(y''(x))(s) = s\,\mathcal{L}(y'(x))(s) - y'(0) = s^2\,\mathcal{L}(y(x))(s)$$

$$\left(s^2 + 4\right)\mathcal{L}(y(x))(s) = \frac{1}{s^2} \qquad \Rightarrow \qquad \mathcal{L}(y(x))(s) = \frac{1}{s^2\,(s^2+4)}$$

Después de solucionar la ecuaciones algebraica,
aplicamos la transformada inversa de Laplace para obtener la solución

Ejemplo

$$y''(x) + 4\,y(x) = x, \qquad y(0) = 0, \qquad y'(0) = 0$$

$$\mathcal{L}(y(x))(s) = \frac{1}{s^2(s^2+4)}$$

Como $\frac{1}{z(z+4)} = \frac{1}{4}\cdot\frac{1}{z} - \frac{1}{4}\cdot\frac{1}{z+4}$,

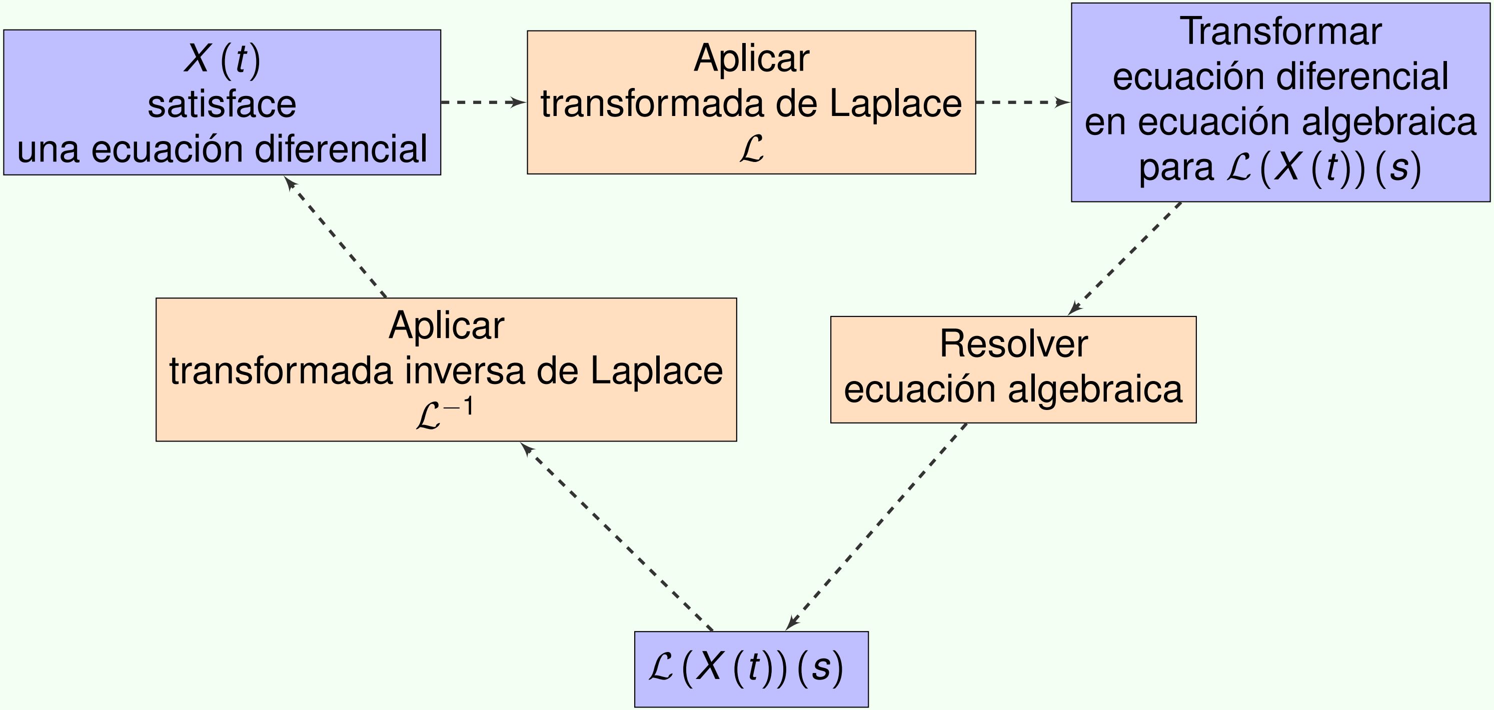
$$\mathcal{L}(y(x))(s) = \frac{1}{4}\cdot\frac{1}{s^2} - \frac{1}{4}\cdot\frac{1}{s^2+4}$$

Usando la linealidad de $\mathcal{L}^{-1}$ obtenemos

$$y(x) = \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)(x) - \frac{1}{8}\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right)(x).$$

$$y(x) = \frac{1}{4}x - \frac{1}{8}sen(2\,x).$$

Método



$X(t)$ satisface una ecuación diferencial
Aplicar transformada de Laplace $\mathcal{L}$
Transformar ecuación diferencial en ecuación algebraica para $\mathcal{L}(X(t))(s)$
Resolver ecuación algebraica
$\mathcal{L}(X(t))(s)$
Aplicar transformada inversa de Laplace $\mathcal{L}^{-1}$

Propiedad

Considere $f : [0, +\infty[\to \mathbb{R}$ de orden exponencial α.
Entonces $e^{-\beta\, t} f\,(t)$ es de orden exponencial $\alpha - \beta$ y para todo $s > \alpha - \beta$ tenemos:

$$\mathcal{L}\left(e^{-\beta\, t} f\,(t)\right)(s) = \mathcal{L}\,(f)\,(s + \beta)$$

$$\mathcal{L}\left(e^{-\beta\, t} f\,(t)\right)(s) = \int_0^{+\infty} e^{-s\,t}\, e^{-\beta\, t}\, f\,(t)\, dt = \int_0^{+\infty} e^{-(s+\beta)\,t}\, f\,(t)\, dt = \mathcal{L}\,(f)\,(s+\beta)\,.$$

Ejemplo

$$\mathcal{L}\left(e^{-t} cos\,(t)\right)(s) = \mathcal{L}\,(cos\,(t))\,(s+1) = \frac{s+1}{(s+1)^2+1}$$

La relación

$$\mathcal{L}(f')(s) = s\,\mathcal{L}(f)(s) - f(0)$$

transforma ecuaciones diferenciales ordinarias en ecuaciones algebraicas.

Ejemplo

$$y''(x) + 2\,y'(x) + 2\,y(x) = 2e^{-x}\cos(x), \qquad y(0) = 2, \qquad y'(0) = -2$$

$$\mathcal{L}(y''(x) + 2\,y'(x) + 2\,y(x))(s) = \mathcal{L}\left(2e^{-x}\cos(x)\right)(s)$$

$$\mathcal{L}(y''(x))(s) + 2\,\mathcal{L}(y'(x))(s) + 2\,\mathcal{L}(y(x))(s) = 2\,\frac{s+1}{(s+1)^2+1}$$

$$\mathcal{L}(y'(x))(s) = s\,\mathcal{L}(y(x))(s) - y(0) = s\,\mathcal{L}(y(x))(s) - 2$$

$$\mathcal{L}(y''(x))(s) = s\,\mathcal{L}(y'(x))(s) - y'(0) = s^2\,\mathcal{L}(y(x))(s) - 2\,s + 2$$

$$\left(s^2 + 2\,s + 2\right)\mathcal{L}(y(x))(s) - 2\,s - 2 = 2\,\frac{s+1}{(s+1)^2+1}$$

Solución de la ecuación algebraica

$$\left(s^2+2\,s+2\right)\mathcal{L}\left(y\left(x\right)\right)\left(s\right)-2\,s-2=2\,\frac{s+1}{\left(s+1\right)^2+1}$$

$$\left(\left(s+1\right)^2+1\right)\mathcal{L}\left(y\left(x\right)\right)\left(s\right)=2\left(s+1\right)+2\,\frac{s+1}{\left(s+1\right)^2+1}$$

$$\mathcal{L}\left(y\left(x\right)\right)\left(s\right)=2\frac{s+1}{\left(s+1\right)^2+1}+2\,\frac{s+1}{\left(\left(s+1\right)^2+1\right)^2}$$

Aplicando la transformada inversa de Laplace obtenemos

$$y\left(x\right)=2\,\mathcal{L}^{-1}\left(\frac{s+1}{\left(s+1\right)^2+1}\right)\left(x\right)+2\,\mathcal{L}^{-1}\left(\frac{s+1}{\left(\left(s+1\right)^2+1\right)^2}\right)\left(x\right)$$

$$=2\,e^{-x}cos\left(x\right)+2\,\mathcal{L}^{-1}\left(\frac{s+1}{\left(\left(s+1\right)^2+1\right)^2}\right)\left(x\right)$$

Propiedades

$$\mathcal{L}\left(e^{-\beta x} f(x)\right)(s) = \mathcal{L}(f)(s+\beta), \qquad \mathcal{L}(x\, f(x))(s) = -\frac{d}{ds}\mathcal{L}(f)(s)$$

$$\frac{d}{ds}\left(\frac{1}{(s+1)^2+1}\right) = -2\,\frac{(s+1)}{\left((s+1)^2+1\right)^2}$$

$$\mathcal{L}(sen(x))(s) = \frac{1}{s^2+1}$$

$$-\frac{d}{ds}\mathcal{L}(sen(x))(s+1) = 2\,\frac{(s+1)}{\left((s+1)^2+1\right)^2}$$

$$-\frac{d}{ds}\mathcal{L}(sen(x))(s+1) = -\frac{d}{ds}\mathcal{L}\left(e^{-x}\, sen(x)\right)(s) = \mathcal{L}\left(x\, e^{-x}\, sen(x)\right)(s)$$

$$2\,\mathcal{L}^{-1}\left(\frac{s+1}{\left((s+1)^2+1\right)^2}\right)(x) = x\, e^{-x}\, sen(x)$$

Ejemplo

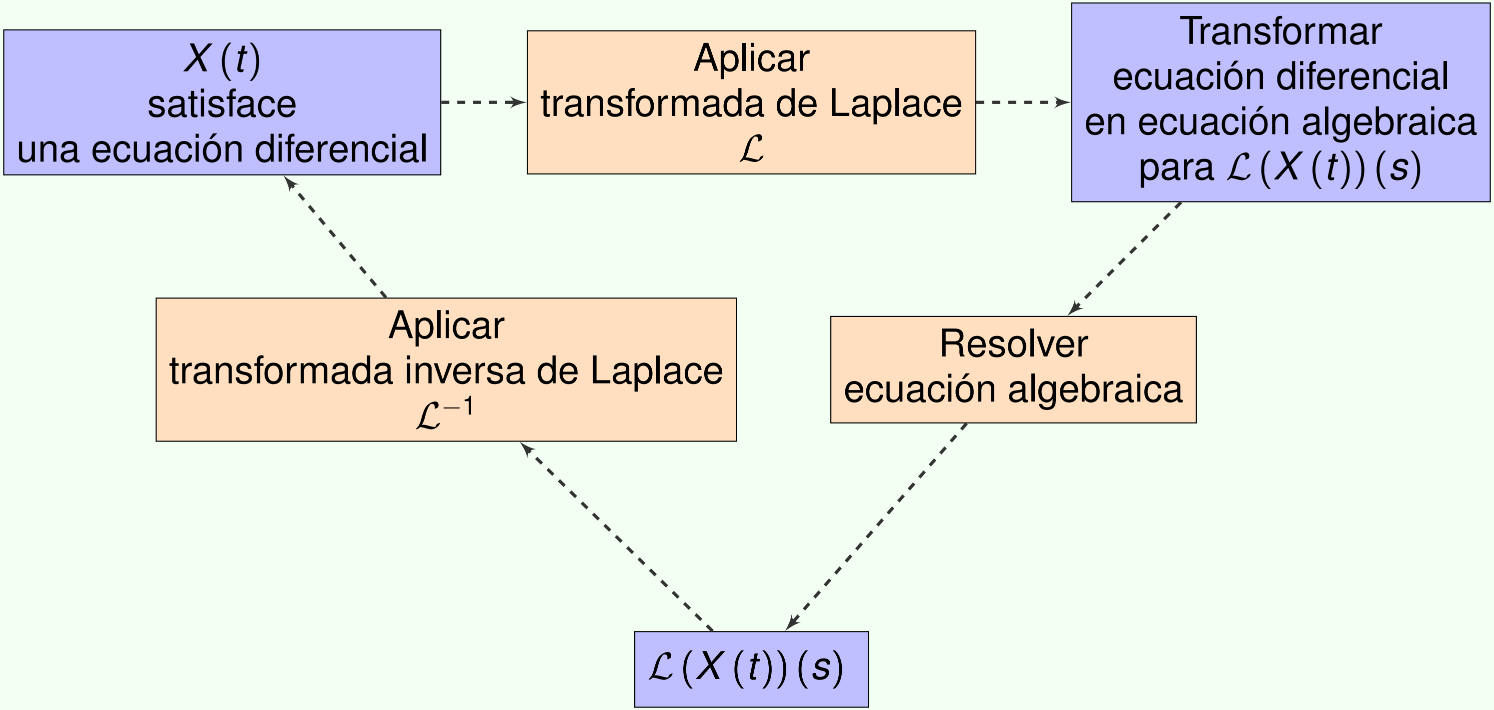
$$y''(x)+2\,y'(x)+2\,y(x)=2e^{-x}\cos(x)\,,\qquad y(0)=2,\qquad y'(0)=-2$$

Aplicando la transformada inversa de Laplace obtenemos

$$\begin{aligned} y(x) &= 2\,\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+1}\right)(x)+2\,\mathcal{L}^{-1}\left(\frac{s+1}{\left((s+1)^2+1\right)^2}\right)(x) \\ &= 2\,e^{-x}cos(x)+2\,\mathcal{L}^{-1}\left(\frac{s+1}{\left((s+1)^2+1\right)^2}\right)(x) \end{aligned}$$

$$y(x)=2\,e^{-x}cos(x)+x\,e^{-x}\,sen(x)$$

Método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Aplicar
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Propiedad

Considere $f : [0, +\infty[\to \mathbb{R}$ de orden exponencial α. Entonces $t^k f(t)$ es de orden exponencial β para todo $\beta > \alpha$ y

$$\mathcal{L}\left(t^k f(t)\right)(s) = (-1)^k \frac{d^k}{ds^k}\mathcal{L}(f)(s) \qquad \forall s > \alpha.$$

$$\begin{aligned}
\frac{d^k}{ds^k}\mathcal{L}(f)(s) &= \frac{d^k}{ds^k}\int_0^{+\infty} e^{-s\,t} f(t)\,dt \\
&= \int_0^{+\infty} \frac{d^k}{ds^k}\left(e^{-s\,t} f(t)\right) dt = \int_0^{+\infty} \frac{d^k}{ds^k}\left(e^{-s\,t}\right)\, f(t)\,dt \\
&= \int_0^{+\infty} (-t)^k\, e^{-s\,t} f(t)\,dt \\
&= (-1)^k \int_0^{+\infty} e^{-s\,t}\, t^k f(t)\,dt
\end{aligned}$$

$$\frac{d^k}{ds^k}\mathcal{L}(f)(s) = (-1)^k\, \mathcal{L}\left(t^k f(t)\right)(s)$$