Proposition 11.5. *Let $K \subseteq I\!R^n$ and $f : I\!R^n \to I\!R$ be differentiable at $\bar{x} \in K$. If $\bar{x}$ is a local minimizer of f in K, then*

$$\langle \nabla f(\bar{x}), v \rangle \geq 0, \ \forall \ v \in T(K; \bar{x}).$$

In other words,

$$F_0 \cap T(K; \bar{x}) = \emptyset,$$

where $F_0 = \{v \in I\!R^n : \ \langle \nabla f(\bar{x}), v \rangle < 0\}$.

Proof. Let $v \in T(K; \bar{x})$. Then, there exist $\lambda_k > 0$, $x_k \in K$, such that $x_k \to \bar{x}$ and $\lambda_k(x_k - \bar{x}) \to v$. By Taylor series

$$f(x_k) - f(\bar{x}) = \langle \nabla f(\bar{x}), x_k - \bar{x} \rangle + \|x_k - \bar{x}\| o(\|x_k - \bar{x}\|),$$

with $o(t) \to 0$ as $t \to 0$. As $\bar{x}$ is a local minimizer, $f(\bar{x}) \leq f(x_k)$ for all k sufficiently large. This along with the previous equality after multiplying by λ_k, and letting $k \to +\infty$, we get $\langle \nabla f(\bar{x}), v \rangle \geq 0$. □

Exercise 11.4. Provide an instance where the converse implication in Proposition 11.5 does not hold.

Remark 11.2. When K is convex, then by the preceding proposition we have

$$T(K; \bar{x}) = R(K; \bar{x}) = \overline{\bigcup_{t>0} t(K - \bar{x})} \ \forall \ \bar{x} \in K.$$

Thus, the standard optimality condition is recovered: $\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0$ for all $x \in K$. In such a case $[T(K; \bar{x})]^* = -N(K; \bar{x})$ as one can check it directly.

In what follows, we use the equality:

$$I\!R^m_+ + \text{int } I\!R^m_+ = \text{int } I\!R^m_+,$$

which follows from

$$I\!R^m_+ + \text{int } I\!R^m_+ \subseteq I\!R^m_+ + I\!R^m_+ = I\!R^m_+,$$

and so

$$I\!R^m_+ + \text{int } I\!R^m_+ \subseteq \text{int } I\!R^m_+ \subseteq I\!R^m_+ + \text{int } I\!R^m_+.$$

More generally, one can check that

$$P + \text{int } P = \text{int } P,$$

provided P is a convex cone with nonempty interior.

Theorem 11.6. (*An alternative Gordan-type theorem*) *Let $K \subseteq I\!R^n$ be convex; $f_i : K \to I\!R$, $i = 1, \ldots, m$ be convex functions. Prove that exactly one of the following two systems admits a solution:*
(I) $f_i(x) < 0$, $i = 1, \ldots, m$, $x \in K$;
(II) $p \in I\!R^m_+ \setminus \{0\}$, $\sum_{i=1}^{m} p_i f_i(x) \geq 0$, $\forall \ x \in K$.

Proof. Obviously both systems cannot have solutions simultaneously.
Let $F = (f_1, \ldots, f_m)$ be the function defined by $F(x) = (f_1(x), \ldots, f_m(x))$. Suppose that (I) has no solution. Then, $F(x) \notin -\text{int } I\!R^m_+$, for all $x \in K$, that is, $F(K) \cap (-\text{int } I\!R^m_+) = \emptyset$. This is equivalent to $(F(K) + I\!R^m_+) \cap (-\text{int } I\!R^m_+) = \emptyset$. Due to convexity of each f_i, we obtain the convexity of $F(K) + I\!R^m_+$. By applying a standard theorem on separation of convex sets, we get the existence of $p \in I\!R^n \setminus \{0\}$ and $\alpha \in I\!R$ such that

$$\langle p, z\rangle \geq \alpha, \ \forall \ z \in F(K) + I\!R^m_+. \tag{11.3}$$

$$\langle p, w\rangle \leq \alpha, \ \forall \ w \in -\text{int } I\!R^m_+. \tag{11.4}$$

From (11.4) it follows that $\alpha \geq 0$, and from (11.3), we get $\langle p, F(x) + q\rangle \geq \alpha$ for all $x \in K$ and all $q \geq 0$. Hence $\langle p, F(x)\rangle \geq 0$ for all $x \in K$ and $p \geq 0$, proving the desired result. □

Remark 11.3. It is not difficult to check that for any set $A \subseteq I\!R^m$,

$$A \cap (-\text{int } I\!R^m_+) = \emptyset \Longleftrightarrow \overline{A} \cap (-\text{int } I\!R^m_+) = \emptyset \Longleftrightarrow (A + I\!R^m_+) \cap (-\text{int } I\!R^m_+) = \emptyset$$
$$\Longleftrightarrow \overline{\text{cone}}(A + I\!R^m_+) \cap (-\text{int } I\!R^m_+) = \emptyset.$$

As a consequence, the previous theorem remains valid under the convexity of $\overline{\text{cone}}(F(K) + I\!R^m_+)$.

Exercise 11.5. Provide funtions $F = (f_1, \ldots, f_m) : K \to I\!R^m$ such that $F(K) + I\!R^m_+$ is convex without being each $f_i : K \to I\!R$. Similarly, find a function F such that $\overline{\text{cone}}(F(K) + I\!R^m_+)$ is convex but not the set $F(K) + I\!R^m_+$.

We recall the notion of pointedness of a cone: a cone P is pointed if co $P \cap (-\text{co } P) = \{0\}$. *Further, we will use the fact that*

$$\text{int } P = \{p \in P : \ \langle q, p\rangle > 0 \ \forall \ q \in P^*, \ q \neq 0\},$$

and

$$P = P^{**} = \{p \in I\!R^n : \ \langle q, p\rangle \geq 0 \ \forall \ q \in P^*\},$$

whenever P is any convex, closed cone with nonempty interior.

Theorem 11.7. *Let $P \subseteq I\!R^m$ be a convex closed cone such that* int $P \neq \emptyset$, *and $A \subseteq \mathbb{R}^m$ be any nonempty set. Then the following assertions are equivalent:*

(*a*) $\exists \ \lambda^* \in P^* \setminus \{0\}, \ \langle \lambda^*, a\rangle \geq 0 \ \forall \ a \in A$;
(*b*) cone$(A + \text{int } P)$ *is pointed;*
(*c*) co$(A) \cap (-\text{int } P) = \emptyset$.

Proof. $(a) \Longrightarrow (b)$: Suppose that $0 = \sum_{i=1}^{l} x_i$ with $x_i \in$ cone$(A + \text{int } P)$, we shall prove that $x_i = 0$ for all i. By choice, $x_i = t_i(a_i + p_i)$ with $t_i \geq 0$, $a_i \in A$, $p_i \in \text{int } P$ for $i = 1, \ldots, l$. This implies that $\sum_{i=1}^{l} t_i a_i \in -\text{int } P$. This yields a contradiction if $\sum_{i=1}^{l} t_i > 0$ under (a), since the inequality in (a) also holds for all $a \in$ co(A), and

$$\text{int } P = \{p \in P : \ \langle q, p\rangle > 0 \ \forall \ q \in P^*, \ q \neq 0\}.$$

$(b) \Longrightarrow (c)$: Assume on the contrary that co$(A) \cap (-\text{int } P) \neq \emptyset$. Then, there exist $a_i \in A$, $p_0 \in \text{int } P$, $\alpha_i \geq 0$, satisfying $\sum_{i=1}^{l} \alpha_i = 1$ and

$0 = \sum_{i=1}^{l} \alpha_i a_i + p_0$. Thus, $0 = \sum_{i=1}^{l} \alpha_i(a_i + p_0)$. By pointedness, we get $\alpha_i(a_i + p_0) = 0$ for all $i = 1, \ldots, l$. Hence, $0 = a_j + p_0 \in A + \text{int } P$ for some j, which implies that $\text{cone}_+(A + \text{int } P) = I\!R^m$, contradicting (b).
$(c) \Longrightarrow (a)$ It follows from a simple use of a standard theorem on separation of convex sets, we get the existence of $p \in I\!R^n \setminus \{0\}$ and $\alpha \in I\!R$ such that

$$\langle p, z\rangle \geq \alpha, \ \forall\ z \in \text{co } A, \text{ and } \langle p, w\rangle \leq \alpha, \ \forall\ w \in -\text{cl}(\text{int } P) = -P. \tag{11.5}$$

From the first inequality of (11.5) it follows that $\langle p, a\rangle \geq \alpha$ for all $a \in A$, and from the second inequality we get $\alpha \geq 0$. Hence $p \in P^*$, proving the desired result. □

We can go further when specializing A. We consider A to be the image of a subset $C \subseteq I\!R^n$ through a linear transformation $\mathcal{F} : I\!R^n \to I\!R^m$.

Proposition 11.8. *Let $\mathcal{F}$ be a real matrix of order $m \times n$, and write $\mathcal{F}^\top = (\mathcal{F}_1^\top, \ldots, \mathcal{F}_m^\top)$, where $\mathcal{F}_i$ is the $i-$th row of $\mathcal{F}$. Let $C \subseteq I\!R^n$ be any nonempty set. Then*

$$\mathcal{F}(C) \cap (-\text{int } I\!R^m_+) = \emptyset \Longleftrightarrow \max_{1\leq i\leq m} \langle \mathcal{F}_i^\top, v\rangle \geq 0 \ \forall\ v \in \overline{C},$$

and the following statements are equivalent:

(a) $\text{cone}(\mathcal{F}(C) + \text{int } I\!R^m_+)$ is pointed;
(b) $\mathcal{F}(\overline{\text{co}}(C)) \cap (-\text{int } I\!R^m_+) = \emptyset$;
(c) $\mathcal{F}(\text{co}(C)) \cap (-\text{int } I\!R^m_+) = \emptyset$;
(d) $\max_{1\leq i\leq m} \langle \mathcal{F}_i^\top, v\rangle \geq 0 \ \forall\ v \in \overline{\text{co}}(C)$;
(e) $\text{co}(\{\mathcal{F}_i^\top :\ i = 1, \ldots, m\}) \cap (C)^* \neq \emptyset$.

Proof. The first part is straightforward. By the previous theorem

$$\text{cone}(\mathcal{F}(C)) + \text{int } I\!R^m_+)) \text{ is pointed } \Longleftrightarrow \text{co}(\mathcal{F}(C)) \cap (-\text{int } I\!R^m_+) = \emptyset.$$

It is easy to check that $\text{co}(\mathcal{F}(C)) = \mathcal{F}(\text{co}(C))$ and

$$\begin{aligned}\mathcal{F}(\text{co}(C)) \cap (-\text{int } I\!R^m_+) = \emptyset &\Longleftrightarrow \mathcal{F}(\overline{\text{co}}(C)) \cap (-\text{int } I\!R^m_+) = \emptyset \\ &\Longleftrightarrow \overline{\mathcal{F}(\text{co}(C))} \cap (-\text{int } I\!R^m_+) = \emptyset.\end{aligned}$$

Both relations along with the fact that (a) of Theorem 11.7 amounts to writing

$$\text{co}(\{\mathcal{F}_i^\top :\ i = 1, \ldots, m\}) \cap (C)^* \neq \emptyset,$$

we get all the remaining equivalences. □

11.2. Optimization with inequality constraints

Let us consider the minimization problem with inequality constraints:

$$\begin{cases} \min f(x) \\ g_i(x) \leq 0,\ i = 1, \ldots, m \\ x \in X, \end{cases} \tag{11.6}$$

where $f, g_i : I\!R^n \to I\!R$, $i = 1, \ldots, m$, are given functions. Define the feasible set to (11.6) as

$$K \doteq \{x \in X :\ g_i(x) \leq 0,\ i = 1, \ldots, m\}.$$

For fixed $\bar{x} \in K$, we associate its active indexes set,

(11.7) $$I \doteq I(\bar{x}) \doteq \{i : \ g_i(\bar{x}) = 0\}.$$

Based on Proposition 11.8 we establish a new version, as an alternative-type result, of the Fritz John optimality conditions at points not necessarily optimal, which is new in the literature.

THEOREM 11.9. (*Fritz John necessary optimality conditions of alternative-type*) *Let us consider problem (11.6) and $\bar{x} \in K$, with $X \subseteq \mathbb{R}^n$. Let f, g_i, $i \in I$, be differentiable at $\bar{x}$. Then, exactly one of the following two assertions holds:*

(*a*) *there exists $v \in \mathbb{R}^n$ such that*

(11.8) $$\begin{array}{ll} \langle \nabla f(\bar{x}), v\rangle < 0, & v \in \overline{\text{co}}[T(X;\bar{x})]; \\ \langle \nabla g_i(\bar{x}), v\rangle < 0, & i \in I, \end{array}$$

(*b*) *there exist $\lambda_0 \geq 0$, $\lambda_i \geq 0$, $i \in I$, not all zero, satisfying*

(11.9) $$\lambda_0 \nabla f(\bar{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) \in [T(X;\bar{x})]^*,$$

or equivalently, $\max\limits_{i \in I} \{\langle \nabla f(\bar{x}), v\rangle, \langle \nabla g_i(\bar{x}), v\rangle\} \geq 0, \ \forall \ v \in T(X;\bar{x}).$

Furthermore, if each g_i is differentiable at $\bar{x}$, condition (11.9) *can be written as*

(11.10) $$\left\{ \begin{array}{l} \lambda_0 \nabla f(\bar{x}) + \sum\limits_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) \in [T(X;\bar{x})]^*; \\ \lambda_i g_i(\bar{x}) = 0, \quad i = 1, \ldots, m. \end{array} \right.$$

PROOF. This is a direct application of Proposition 11.8 to

$$\mathcal{F} \doteq \begin{pmatrix} \nabla f(\bar{x})^\top \\ \nabla g_I(\bar{x})^\top \end{pmatrix},$$

where $\nabla g_I(\bar{x})^\top$ is the matrix having as rows $\nabla g_i(\bar{x})^\top$ for $i \in I$. □

The first relation in (11.10) *is called the* Fritz John optimality conditions *and the real numbers λ_0, $\lambda_1, \ldots, \lambda_m$, are named the* Lagrange multipliers, *while the second relation are termed the* complementarity conditions.

EXERCISE 11.6. Provide a nonconvex set X in such a way that $T(X;\bar{x})$ is convex.

In connection to the previous theorem, we will see that a stronger system than (11.8) *is implied by optimality: To be more precise, besides differentiability of g_i at $\bar{x}$, for $i \in I$; continuity of g_i at $\bar{x}$ for $i \notin I$, local minimality of $\bar{x}$ imply that the system*

(11.11) $$\begin{array}{ll} \langle \nabla f(\bar{x}), v\rangle < 0, & v \in T(X;\bar{x}); \\ \langle \nabla g_i(\bar{x}), v\rangle < 0, & i \in I, \end{array}$$

has no solution. Thus, if $T(X;\bar{x})$ is convex, we have the impossibility of system (11.8), *and so by Theorem 11.9,* (*b*) *holds. This will be expressed in Corollary 11.11, whose formulation is very versatile as we shall show* *in Section 11.3, and it encompasses many recent results appearing in the literature. Thus, the interesting case is when $T(X;\bar{x})$ is nonconvex.*

We start by proving a result about the impossibility of system (11.11) *under local optimality.*

PROPOSITION 11.10. *Let us consider problem (11.6) and $\bar{x} \in K$. In addition, let $X \subseteq \mathbb{R}^n$ be any nonempty set; f, g_i, $i \in I$, be differentiable at $\bar{x}$; g_i, $i \notin I$, be continuous at $\bar{x}$. If $\bar{x}$ is a local solution to (11.6), then system* (11.11) *has no solution.*

PROOF. If on the contrary, v is a solution to (11.11) there exist sequences $\lambda_k > 0$, $x_k \in X$, $x_k \to \bar{x}$, satisfying $\lambda_k(x_k - \bar{x}) \to v$. By Taylor series,

$$f(x_k) - f(\bar{x}) = \langle \nabla f(\bar{x}), x_k - \bar{x} \rangle + \|x_k - \bar{x}\| o(\|x_k - \bar{x}\|)$$

with $o(t) \to 0$ as $t \to 0$. On multiplying this equality by λ_k, letting $k \to +\infty$ and using the first inequality of (11.11), we get the existence of k_1 such that

$$f(x_k) < f(\bar{x}), \quad \forall\, k \geq k_1. \tag{11.12}$$

It only remains to check that x_k is feasible for all k sufficiently large to reach a contradiction.
Let $i \in I$. We get similarly as for f

$$g_i(x_k) - g_i(\bar{x}) = \langle \nabla g_i(\bar{x}), x_k - \bar{x} \rangle + \|x_k - \bar{x}\| o(\|x_k - \bar{x}\|)$$

with $o(t) \to 0$ as $t \to 0$. On multiplying by λ_k this equality and letting $k \to +\infty$, we obtain, for some k_2,

$$g_i(x_k) < 0,\ \forall\, i \in I,\ \forall\, k \geq k_2. \tag{11.13}$$

Since g_i is continuous for $i \notin I$, there exists k_3 such that

$$g_i(x_k) < 0,\ \forall\, i \notin I,\ \forall\, k \geq k_3. \tag{11.14}$$

On combining (11.13) and (11.14), we conclude that x_k is feasible for k sufficiently large. Hence $\bar{x}$ cannot be a local solution to (11.6), and the proof is completed. □

We are now ready to establish the result which is suitable for our purposes.

COROLLARY 11.11. (*Fritz John necessary optimality conditions*) *Let us consider problem (11.6) and $\bar{x} \in K$. In addition, let $X \subseteq \mathbb{R}^n$ be a set such that $T(X;\bar{x})$ is convex; f, g_i, $i \in I$, be differentiable at $\bar{x}$; g_i, $i \notin I$, be continuous at $\bar{x}$. If $\bar{x}$ is a local solution to (11.6), then there exist $\lambda_0 \geq 0$, $\lambda_i \geq 0$, $i \in I$, not all null at the same time, satisfying*

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) \in [T(X;\bar{x})]^*.$$

PROOF. By the previous proposition, system (11.11) has no solution, but such a system is exactly that expressed in (a) of Theorem 11.9. Therefore (b) of the same theorem holds, and the corollary follows. □

11.3. Some Particular Situations

Let us show some interesting specializations appearing in applications where $T(X;\bar{x})$ is convex.

11.3.1. The set X is not convex with $T(X;\bar{x})$ being convex. *Let us consider the problem with an additional quadratic equality constraint:*

$$\min f(x) \ s.\ t.\ g_i(x) \leq 0,\ i = 1, \ldots, m;\ q(x) = 0;\ x \in \mathbb{R}^n, \tag{11.15}$$

where q is a function of the form

$$q(x) \doteq \frac{1}{2} x^\top A x + a^\top x + \alpha,$$

with A being a (real) symmetric matrix, $a \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$. Clearly the set

$$X \doteq \{x \in \mathbb{R}^n :\ q(x) = 0\ \}$$

is not necessarily convex even if q is convex. Let $\bar{x}$ feasible for Problem (11.15). *It is not difficult to find that (see for instance* [**26**, Theorem 2.1]*)*

$$T(X;\bar{x}) = \Big\{ v \in \mathbb{R}^n :\ (A\bar{x} + a)^\top v = 0\ \Big\} \text{ if } A\bar{x} + a \neq 0;$$

whereas

$$T(X;\bar{x}) = \Big\{ v \in \mathbb{R}^n :\ v^\top A v = 0\ \Big\} \text{ if } A\bar{x} + a = 0.$$

This set, en general, is nonconvex. If additionally, q is convex, that is, A is positive semidefinite, a more precise formulation may be obtained since $v^\top A v = 0 \iff Av = 0$:

$$T(X;\bar{x}) = \begin{cases} (A\bar{x} + a)^\perp & \text{if } A\bar{x} + a \neq 0; \\ \ker\ A & \text{if } A\bar{x} + a = 0. \end{cases} \tag{11.16}$$

Thus,

$$[T(X;\bar{x})]^* = \begin{cases} \mathbb{R}(A\bar{x} + a) & \text{if } A\bar{x} + a \neq 0; \\ (\ker\ A)^\perp = A(\mathbb{R}^n) & \text{if } A\bar{x} + a = 0. \end{cases}$$

Hence, the Fritz John conditions (11.11) *reduces to the existence of λ_0, λ_i, $i \in I$, not all zero, $\lambda \in \mathbb{R}$, $y \in \mathbb{R}^n$, such that*

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) = \begin{cases} \lambda(A\bar{x} + a) & \text{if } A\bar{x} + a \neq 0; \\ Ay & \text{if } A\bar{x} + a = 0. \end{cases}$$

11.3.2. The set X is convex. *In this case, $T(X;\bar{x})$ is convex, and since $T(X;\bar{x}) = \overline{\bigcup_{t \geq 0} t(X - \bar{x})}$, we get $[T(X;\bar{x})]^* = -N(X;\bar{x})$, and so (11.11) can be written as*

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) \in -N(X;\bar{x}).$$

Thus, Theorem 3.2.2 in Bector et al. (2005) is obtained.

11.3.3. The set X is open or $\bar{x} \in \operatorname{int} X$. *In this situation, $T(X;\bar{x}) = I\!R^n$, and therefore condition* (11.11) *reduces to*

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) = 0.$$

This is nothing else than Theorem 4.2.8 in Bazaraa et al. (2006), and Theorem 3.2.1 in Bector et al. (2005) when $X = I\!R^n$.

11.3.4. The set X is an affine subspace. *This case deals with $X = \{x \in I\!R^n : \ Hx = d\} = \bar{x} + \ker H$, with H being a real $p \times n$ matrix and $\bar{x} \in X$. Thus, we obtain $T(X;\bar{x}) = \ker H$ and $[T(X;\bar{x})]^* = (\ker H)^* = (\ker H)^{\perp}$. Hence* (11.11) *is expressed as*

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) \in (\ker H)^{\perp} = H^{\top}(I\!R^p),$$

that is, there exists $y_i \in I\!R$, $i = 1, \ldots, p$, such that

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) + H^{\top} y = 0.$$

11.3.5. The set X is polyhedral. *Let us consider (see for instance Birbil et al., 2007)*

$$(11.17) \qquad \begin{cases} \min f(x) \\ g_i(x) \leq 0, \ i = 1, \ldots, m, \\ h_j^{\top} x \leq d_j, \ j = 1, \ldots, p, \\ x \in I\!R^n. \end{cases}$$

In this case, we can take $X = \{x \in I\!R^n : \ Hx \leq d\}$ with H being a $p \times n$ matrix, and refine (11.11)*. More precisely, by denoting $h_j^{\top}$ to be the rows of H and setting $J \doteq \{j : \ h_j^{\top} \bar{x} = d_j\}$, the conclusion of Corollary 11.11 reduces to the existence of $\lambda_0 \geq 0$, $\lambda_i \geq 0$, $i \in I$, not all zero, and $u_j \geq 0$, $j \in J$, such that*

$$(11.18) \qquad \lambda_0 \nabla f(\bar{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) + \sum_{j \in J} u_j h_j = 0.$$

Indeed, by setting $C \doteq \{v \in I\!R^n : \ h_j^{\top} v \leq 0, \ j \in J\}$, which is a closed convex cone, one can easily check that $T(X;\bar{x}) = C$ (see Lemma 5.1.4 in Bazaraa et al., 2006) By applying Corollary 11.11, we get the existence of $\lambda_0 \geq 0$, $\lambda_i \geq 0$, $i \in I$, not all zero, such that

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) \in C^*.$$

The conclusion follows once we notice that $C^* = \{H_J u : \ u \leq 0\}$, with H_J is the matrix with columns h_j for $j \in J$. Thus we recovered the Fritz-John conditions as appears in Birbil et al. (2007).

We now discuss an example where $T(X;\bar{x})$ is nonconvex. Our theorem will be applicable to such an example, and no result existing in the literature is.