Good Gradings of Generalized Incidence Rings

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Abstract

This inquiry is based on both the construction of generalized incidence rings due to Gene Abrams and the construction of good group gradings of incidence algebras due to Molli Jones. We provide conditions for a generalized incidence ring to be graded isomorphic to a subring of an incidence ring over a preorder. We also extend Jones's construction to good group gradings for incidence algebras over preorders with crosscuts of length one or two.

1 Overview

Unless otherwise stated we use multiplicative notation for all semigroup, monoid, or group operations and the identity is denoted by 1.

Suppose G is a semigroup and S is a ring which does not necessarily contain a multiplicative identity. We say S is a G-graded ring if there is a direct sum

$S = \bigoplus_{a \in G} S_a$, as a group under the addition of S, such that $S_b S_c \subseteq S_{bc}$ for all $b, c \in G$. The subgroups S_a, $a \in G$, are called the homogeneous components, the elements of $\cup_{a \in G} S_a$ are called the homogeneous elements, and every element is a sum of finitely many homogeneous elements. We let ∂s be the unique element of G such that $s \in S_{\partial s}$ for any nonzero homogeneous $s \in S$. The support of S is the set $\operatorname{Supp}_G S = \{a \in G : S_a \neq 0\}$. The grading is called finite if $\operatorname{Supp}_G S$ is a finite set.

An important type of matrix algebra grading is a good grading (for example, see references [2], [3], and [4]). This definition extends easily to incidence algebras (see [4], [6], and [7]). In section 2 we state the definition of balanced relation introduced by Abrams (see [1]) and go over the construction of generalized incidence rings. Good semigroup gradings of generalized incidence rings are defined in section 3. Theorems 3.2 and 3.4 are fundamental for our constructions since they categorize good gradings of generalized incidence rings in terms of homomorphisms from the relations to the semigroup. Theorem 3.8 shows how to construct good semigroup gradings of incidence algebras over minimally connected partial orders.

Suppose $S = \bigoplus_{a \in G} S_a$ and $T = \bigoplus_{a \in G} T_a$ are G-graded rings. A homomorphism of G-graded rings is a ring homomorphism $h : S \to T$ such that $h(S_a) \subseteq T_a$ for all $a \in \operatorname{Supp}_G S$. An isomorphism which is a homomorphism of G-graded rings is called an isomorphism of G-graded rings. In the case of matrix algebras there are gradings which are not good gradings but are isomorphic to good gradings (see [3, Example 1.3]). Isomorphic gradings for good group gradings of incidence algebras over partial orders have been studied by Miller and Spiegel (see [6]).

In section 4 we state the definitions of compression maps and stable relations (see [8]). If G is a cancellative monoid then Theorem 4.3 shows compression maps provide a correspondence between good G-gradings. Stable relations are used to describe a class of generalized incidence rings which are isomorphic to subrings of incidence rings over preorders by Theorem 4.5. The isomorphism, which is described in Lemma 4.2, is an isomorphism of G-graded rings.

Good group gradings are considered in section 5. The main result of this section is Theorem 5.1, which extends [4, Theorem 4] to good group gradings for incidence algebras over preorders with crosscuts of length one or two. The conclusion of our theorem is modified to account for finite gradings. We show our result for preorders is related to generalized incidence rings in Corollary 5.2 and Example 5.3. We finish with example 5.4, which describes a partial order with a minimal element whose incidence algebra does not have the free-extension property. Unfortunately this is a counterexample to [4, Theorem 4].

2 Generalized Incidence Rings

By definition a relation ρ on a set X is a subset of $X \times X$. We adopt the usual convention of writing $x\rho y$ if $x, y \in X$ satisfy $(x, y) \in \rho$. The notation $x\rho y$ is often shorter and more convenient, but the notation $(x, y) \in \rho$ will be used when it is helpful to describe the relation as a set of ordered pairs. The directed graphs shown in Figure 1 represent reflexive relations. We omit loops in all diagrams so that the arrows match up with elements of the off-diagonal subset of relations.

The construction of generalized incidence rings does not require the relation to

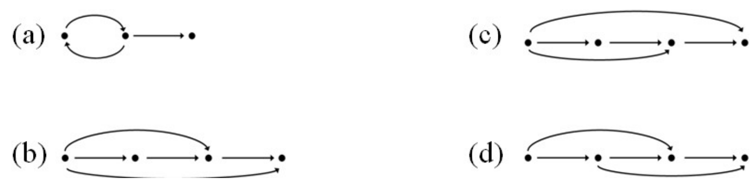

Figure 1: Directed graphs determining reflexive relations.

be a preorder. To define multiplication we will assume X is locally finite, that is, every interval of X is a finite set. (Recall an interval is a subset of the form $[x, y] = \{z \in X : x\rho z \text{ and } z\rho y\}$ for some $x, y \in X$.)

Throughout the rest of this section ρ is a locally finite relation on a set X and R is an associative ring with unity. Let $I(X, \rho, R)$ denote the set of all functions $f : X \times X \to R$ such that $f(x, y) \neq 0$ implies $x\rho y$. Componentwise operations determine an R-module such that given $r \in R$ and $f, g \in I(X, \rho, R)$ the functions $rf, f + g \in I(X, \rho, R)$ satisfy $(rf)(x, y) = r(f(x, y))$ and $(f + g)(x, y) = f(x, y) + g(x, y)$ for all $x, y \in X$.

The *balance property* is satisfied by $w, x, y, z \in X$ if any of the relations $w\rho x$, $x\rho y$, $y\rho z$, and $w\rho z$ do not hold, or all four relations hold and $w\rho y$ if and only if $x\rho z$. The relation is *balanced* if it is reflexive and the balance property is satisfied by all $w, x, y, z \in X$. The relations determined by directed graphs (a), (b), and (c) in Figure 1 are not balanced because they are all missing arrows. Note that a reflexive and transitive relation is balanced but there are balanced relations, such as the one determined by (d) in Figure 1, which are not transitive.

In case ρ is balanced we can combine the proof of [1, Proposition 1.2] with the assumption that X is locally finite to construct a ring multiplication with identity

$e : X \times X \to R$ such that $e(x,x) = 1$ and $e(x,y) = 0$ for all $x, y \in X$ such that $x \neq y$. In this case we call $I(X, \rho, R)$ the *generalized incidence ring of* X *with coefficients in* R. If ρ is a partial order then $I(X, \rho, R)$ is the incidence ring over R.

The multiplication is defined so that the product of $f, g \in I(X, \rho, R)$ is the function given by equation 1 for all $x, y \in X$. This is called *convolution*.

$$(fg)(x,y) = \begin{cases} \displaystyle\sum_{z \in [x,y]} f(x,z)\, g(z,y) & \text{if } x \rho y \text{ in } X \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Obviously if the sum in equation 1 is nonzero then $x\rho y$ in X. In this case $[x,y]$ is finite so convolution uniquely determines an element of R. Consider $f, g, h \in I(X, \rho, R)$ and $w, z \in X$ with $w\rho z$. A nonzero term of $((fg)h)(w,z)$ is determined by $y \in [w,z]$ and $x \in [w,y]$. Thus there are relations $w\rho x$, $x\rho y$, $y\rho z$, $w\rho z$, and $w\rho y$. Since the relation is balanced these are equivalent to $w\rho x$, $x\rho y$, $y\rho z$, $w\rho z$, and $x\rho z$, or $x \in [w,z]$ and $y \in [x,z]$. This gives an identical term of $(f(gh))(w,z)$. Since the nonzero terms match up over the relations we have $(fg)h = f(gh)$.

For each $x, y \in X$ such that $x\rho y$ there exists $e_{xy} \in I(X, \rho, R)$ such that $e_{xy}(i,j)$ is given by equation 2 for all $i, j \in X$.

$$e_{xy}(i,j) = \begin{cases} 1 & \text{if } x = i \text{ and } y = j \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

If $w \in X$ then $w\rho w$ since a balanced relation is reflexive. It is easy to show $(e_{ww})^2 = e_{ww}$ for all $w \in X$ directly from the definition in 1. Equations 3 and 4 also hold for all $f \in I(X, \rho, R)$ and $x, y, z, w \in X$ such that $x\rho y$ and $z\rho w$.

$$f(x, y)\, e_{xy} = e_{xx} f e_{yy} \tag{3}$$

$$e_{xy}e_{zw} = \begin{cases} e_{x,w} & \text{if } y = z \text{ and } x\rho w \text{ in } X \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

In any situation where we refer to a generalized incidence ring we mean an associative ring with unity formed on the R-module of functions $I(X, \rho, R)$ where R is a ring with unity and ρ is a locally finite balanced relation on X. The operation of R on $I(X, \rho, R)$ does not play a significant role in our investigation. We reserve the term *generalized incidence algebra* for $I(X, \rho, R)$ where R is a commutative ring with unity and ρ is a locally finite balanced relation on X. If, additionally, ρ is a partial order then $I(X, \rho, R)$ is the usual incidence algebra over R (see [9]).

3 Good Gradings

DEFINITION 3.1 *Assume G is a semigroup and ρ is a relation on X.*

1. *Set* $\mathrm{Trans}(X) = \{(x, y, z) : x\rho y,\ y\rho z,\ x\rho z, \text{ and } x, y, z \in X\}$*. A* transitive triple in X *is an ordered triple in* $\mathrm{Trans}(X)$*.*

2. *We say $\Phi : \rho \to G$ is a* homomorphism *if $\Phi(x, y)\,\Phi(y, z) = \Phi(x, z)$ holds for any $x, y, z \in X$ such that $(x, y, z) \in \mathrm{Trans}(X)$.*

3. *If $I(X, \rho, R)$ is a G-graded generalized incidence ring then the grading is* good *if e_{xy} is homogeneous for all $x, y \in X$ such that $x\rho y$.*

THEOREM 3.2 *Assume G is a semigroup, $I(X,\rho,R)$ is a generalized incidence ring, and $\Phi : \rho \to G$ is a homomorphism. Let S_a be given by equation 5 for each $a \in G$. Then $I(X,\rho,R) = \bigoplus_{a\in G} S_a$ is a G-graded ring if and only if $\operatorname{Im}\Phi$ is finite.*

$$S_a = \{f \in I(X,\rho,R) : f(r) \neq 0 \text{ implies } \Phi(r) = a \text{ for all } r \in \rho\} \tag{5}$$

Proof. It is easy to see S_a is an R-submodule for all $a \in G$ and $S_a \cap S_b = \{0\}$ if $b \in G$ and $b \neq a$. We show $S_a S_b \subseteq S_{ab}$ for all $a, b \in G$. Suppose $f \in S_a$, $g \in S_b$ and $(fg)(x,y) \neq 0$ for some $x, y \in X$ with $x\rho y$. By equation 1 there exists $z \in [x,y]$ such that $x\rho z$, $z\rho y$, and $f(x,z)\,g(z,y) \neq 0$. Thus $f(x,z) \neq 0$ and $g(z,y) \neq 0$ which implies $\Phi(x,z) = a$ and $\Phi(z,y) = b$. Moreover, $ab = \Phi(x,y)$ since (x,z,y) is a transitive triple and Φ is a homomorphism. This proves $fg \in S_{ab}$ as desired.

To complete the proof we show $\operatorname{Im}\Phi$ is finite if and only if $I(X,\rho,R) = \bigoplus_{a\in G} S_a$. First assume $\operatorname{Im}\Phi$ is a finite subset of G. Then there is a positive integer m and $a_1, \ldots, a_m \in G$ such that $\operatorname{Im}\Phi = \{a_1, \ldots, a_m\}$. We must prove an arbitrarily chosen $f \in S$ is a sum of finitely many homogeneous elements.

For each $i = 1, \ldots, m$ let $f_i \in I(X,\rho,R)$ be the the function satisfying equation 6 for all $(x,y) \in X$.

$$f_i(x,y) = \begin{cases} f(x,y) & \text{if } x\rho y \text{ in } X \text{ and } \Phi(x,y) = a_i \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

By construction $f_i \in S_{a_i}$ for all $i \leq m$. It is easy to prove f is the sum of finitely many homogeneous elements, $f_1, \ldots, f_m$, as desired.

To prove the other direction assume $S = \bigoplus_{a\in G} S_a$. We choose $h \in I(X,\rho,R)$ so that for all $x, y \in X$ we have $h(x,y) = 1$ if $x\rho y$ and otherwise $h(x,y) = 0$. If

$S = \bigoplus_{a \in G} S_a$ then we may write h as the sum of homogeneous elements. Thus there is a positive integer p and distinct $b_1, \ldots, b_p \in G$ such that $h = h_1 + \cdots + h_p$ where $\partial h_i = b_i$ for each $i = 1, \ldots, p$. We will show $\operatorname{Im} \Phi \subseteq \{b_1, \ldots, b_p\}$.

For all $x, y \in X$ if $x \rho y$ we have $h(x, y) = 1$ hence $h_1(x, y) + \cdots + h_p(x, y) = 1$. Then $h_i(x, y) \neq 0$ for some $i \leq p$ so, by construction, we have $\Phi(x, y) = b_i$. Since $x, y \in X$ with $x \rho y$ were arbitrarily chosen we can conclude $\operatorname{Im} \Phi \subseteq \{b_1, \ldots, b_p\}$. This gives the desired result, $\operatorname{Im} \Phi$ is finite. ∎

Definition 3.3 *We say a* grading is induced by a homomorphism Φ *if it can be constructed in the setting of Theorem 3.2 with* $\operatorname{Im} \Phi$ *finite.*

Theorem 3.4 *Let $S = I(X, \rho, R)$ be a generalized incidence ring and let G be a monoid. Suppose $S = \bigoplus_{a \in G} S_a$ is a G-graded ring.*

1. *If the grading is good then there is a homomorphism $\Phi : \rho \to G$ given by $\Phi(x, y) = \partial e_{x,y}$ for all $x, y \in X$ with $x \rho y$.*

2. *If the grading is good and $e_{xx} \in S_1$ for all $x \in X$ then the grading is induced by Φ. Moreover, the grading is finite.*

Proof. Part 1 can be proved directly from equation 4. To prove part 2 we let $a \in \operatorname{Supp}_G S$ be arbitrarily chosen. We will show S_a is given by equation 5 so the grading is induced by Φ. Then the grading is finite since $\operatorname{Supp}_G S = \operatorname{Im} \Phi$.

Suppose $f \in I(X, \rho, R)$ and for all $x, y \in X$ with $x \rho y$ if $f(x, y) \neq 0$ then $\Phi(x, y) = a$. We must prove such an f is contained in S_a. Since S is graded there is a positive integer m and nonzero homogeneous $f_1, \ldots, f_m \in S$ such that

$f = f_1 + \cdots + f_m$ and $\partial f_i \neq \partial f_j$ if $i \neq j$. Fix $i \leq m$. Since f_i is nonzero there exist $x, y \in X$ such that $x \rho y$ and $f_i(x, y) \neq 0$. By equation 3 we have $f_j(x, y) e_{xy} = e_{xx} f_j e_{yy}$ for $j = 1, \ldots, m$. If $f_j(x, y) \neq 0$ then $\partial e_{xy} = \partial f_j$ since $\partial e_{xx} = \partial e_{yy} = 1$. When $j = i$ we conclude $\Phi(x, y) = \partial f_i$ since $\Phi(x, y) = \partial e_{xy}$ by construction and $f_i(x, y) \neq 0$ by our choice of $x, y \in X$. But if $j \neq i$ and $f_j(x, y) \neq 0$ then $\partial e_{xy} = \partial f_j$ so $\partial f_i = \partial f_j$, which is a contradiction. We are left with $f_j(x, y) = 0$ for all $j \neq i$. Therefore $f(x, y) = f_i(x, y) \neq 0$ and $\Phi(x, y) = a$ by the assumption on f. We already proved $\Phi(x, y) = \partial f_i$ so $\partial f_i = a$. Since i was arbitrarily chosen we have $\partial f_i = a$ for all $i = 1, \ldots, m$. Since $\partial f_1, \ldots, \partial f_m$ are distinct, we must have $m = 1$. Thus $f = f_1 \in S_a$, as desired.

Now suppose $g \in S_a$ and $g(w, z) \neq 0$ for some $w, z \in X$ such that $w \rho z$. Equation 3 becomes $g(w, z) e_{wz} = e_{ww} g e_{zz}$ and we have $\partial e_{wz} = \partial g$ since $\partial e_{ww} = \partial e_{zz} = 1$ by assumption. Moreover $a = \partial g$ and $\partial e_{wz} = \Phi(w, z)$ so $\Phi(w, z) = a$. ■

Remark 3.5 *Equation 4 gives* $(e_{xx})^2 = e_{xx}$ *and so* $\Phi(x, x)^2 = \Phi(x, x)$ *for all* $x \in X$. *The condition* $e_{xx} \in S_1$ *for all* $x \in X$ *stated in part 2 of Theorem 3.4 may not hold for all monoids. But if* G *is a cancellative monoid then* $(e_{xx})^2 = e_{xx}$ *implies* $\Phi(x, x) = 1$ *for all* $x \in X$ *and the condition in part 2 is satisfied automatically.*

Definition 3.6 *Let* G *be a semigroup and let* ρ *be a relation on* X.

1. *A subset* β *of* ρ *is a* G-extendible set *for* ρ *if for every function* $\phi : \beta \to G$ *there exists a homomorphism* $\Phi : \rho \to G$ *such that* $\Phi|_\beta = \phi$. *If we may choose* Φ *so that* $\operatorname{Im}\Phi$ *is finite then we say* ϕ *is* grading admissible.

2. *A subset γ of ρ is a G-*essential set *for ρ if for all homomorphisms $\Phi_1, \Phi_2 : \rho \to G$ such that $\Phi_2 \neq \Phi_1$ there exists $c \in \gamma$ such that $\Phi_2(c) \neq \Phi_1(c)$.*

3. *A G-extendible and G-essential subset is called a G-*grading set *for ρ.*

REMARK 3.7 *σ is a G-grading set for ρ if and only if for every function $\phi : \sigma \to G$ there is a unique homomorphism $\Phi : \rho \to G$ such that $\Phi|_\sigma = \phi$. If ρ is balanced and ρ contains a G-grading set then all good G-gradings of a generalized incidence ring $S = I(X, \rho, R)$ such that $e_{xx} \in S_1$ for all $x \in X$ are uniquely determined by grading admissible functions from the G-grading set to G.*

We finish this section with a result on partial orders. Recall $(X, \leq)$ is minimally connected if $(X, \leq)$ is a connected, locally finite partial-order and $[x, y]$ is either empty or a chain for all $x, y \in X$. The Hasse diagram of X is the directed graph H with vertex set X and arrow set $\{(a, b) : a, b \in X \text{ and } b \text{ covers } a\}$.

THEOREM 3.8 *Assume G is a semigroup, R is an associative ring with unity, and $(X, \leq)$ is a minimally-connected partial order. Then the arrow set of the Hasse diagram of $(X, \leq)$ is a G-grading set for $\leq$.*

Proof. Let $\phi : \sigma \to G$ be given, where σ is the set of arrows in the Hasse diagram for $(X, \leq)$. We use ϕ to define a function $\Phi :\leq \to G$. Let $x, y \in X$ such that $x \leq y$ be given. Since $(X, \leq)$ is a minimally connected there is a unique chain $\{x_1, \ldots, x_m\}$ in σ such that $x = x_1 \leq x_2 \leq \ldots \leq x_m = y$ and $(x_i, x_{i+1}) \in \sigma$ for each $i < m$. We set $\Phi(x, y) = \phi(x_1, x_2) \cdots \phi(x_{m-1}, x_m)$. A straightforward check proves Φ is a homomorphism such that $\Phi|_\sigma = \phi$. Thus σ is a G-extendible set for $\leq$.

Suppose Φ_1, Φ_2 are homomorphism and $\Phi_1|_\sigma = \Phi_2|_\sigma$. Give $x, y \in X$ such that $x\rho y$ there is a unique chain $\{x_1, \ldots, x_m\}$ in σ such that $x = x_1 \leq x_2 \leq \ldots \leq x_m = y$. The homomorphism property gives $\Phi_i(x, y) = \phi(x_1, x_2) \cdots \phi(x_{m-1}, x_m)$ for $i = 1$ or $i = 2$. Thus $\Phi_1 = \Phi_2$ which proves σ is a G-essential set for ρ. Therefore σ is a G-grading set for ρ. ■

4 Compression Maps and Stable Relations

We fix the notation $\delta(\rho) = \{(x, x) : x \in X\}$ for the diagonal subset of a relation ρ on a set X. The off-diagonal set $\rho^* = \rho \backslash \delta(\rho)$ is an anti-reflexive relation on X.

Definition 4.1 *Suppose ρ_1 and ρ_2 are relations on X_1 and X_2, respectively. A function $\theta : X_2 \to X_1$ is called a* compression map *if 1, 2, and 3 are satisfied. In this case we say ρ_1 is a* compression *of ρ_2.*

1. *θ is surjective and order-preserving.*

2. *For all $a_1, a_2, a_3 \in X_1$ if $(a_1, a_2, a_3) \in \mathrm{Trans}(X_1)$ then there exist $x_1, x_2, x_3 \in X_2$ such that $(x_1, x_2, x_3) \in \mathrm{Trans}(X_2)$ and $\theta(x_i) = a_i$ for $i = 1, 2, 3$.*

3. *There is a bijection $\theta^* : \rho_2^* \to \rho_1^*$ given by $\theta^*(x, y) = (\theta(x), \theta(y))$ for all $x, y \in X_2$ with $x\rho_2^* y$.*

Compression maps were introduced in [8]. Example 5.3 uses a compression map to construct a G-grading set for a non-transitive relation.

Lemma 4.2 *Suppose $I(X_1, \rho_1, R)$ and $I(X_2, \rho_2, R)$ are generalized incidence rings and $\theta : X_2 \to X_1$ is a compression map.*

1. *If G is a semigroup and $\Phi_1 : \rho_1 \to G$ is a homomorphism then a homomorphism $\Phi_2 : \rho_2 \to G$ is given by $\Phi_2(x, y) = \Phi_1(\theta(x), \theta(y))$ for all $x, y \in X$ with $x\rho_2 y$.*

2. *If $S = I(X_1, \rho_1, R)$ and $T = I(X_2, \rho_2, R)$ have gradings induced by Φ_1 and Φ_2, respectively, then there is an injective homomorphism of G-graded rings $h : S \to T$ such that equation 7 holds for all $f \in S$ and all $x, y \in X_2$.*

$$(h(f))(x, y) = \begin{cases} f(\theta(x), \theta(y)) & \text{if } x\rho_2 y \text{ in } X_2 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Proof. (1) Part 1 is trivial. (2) A routine check shows h is an R-module homomorphism. In the definition of multiplication in equation 1, it is easy to see h is a ring homomorphism if $\theta([x, y]) = [\theta(x), \theta(y)]$ for all $x, y \in X_2$ such that $x\rho_2 y$. For any $c \in [\theta(x), \theta(y)]$ there is a transitive triple $(\theta(x), c, \theta(y)) \in \mathrm{Trans}(X_1)$ so by part 2 of definition 4.1 there exists $u, v, w \in X_2$ such that $(v, v, w) \in \mathrm{Trans}(X_2)$, $\theta(u) = \theta(x)$, $\theta(v) = c$, and $\theta(w) = \theta(y)$. We find $u = x$ and $w = y$ since $\theta^*(x, y) = \theta^*(u, v)$ and θ^* is bijective. Therefore $v \in [x, y]$ and $c \in \theta([x, y])$. Since c was arbitrarily chosen we have $\theta([x, y]) \subseteq \theta([x, y])$. Moreover $\theta([x, y]) \subseteq [\theta(x), \theta(y)]$ since θ is order-preserving. Therefore $\theta([x, y]) = [\theta(x), \theta(y)]$ as desired.

Given $f \in S \setminus \{0\}$ there exist $a_1, b_1 \in X_1$ such that $f(a_1, b_1) \neq 0$. If $a_1 \neq b_1$ then there exist $x_1, y_1 \in X_2$ such that $x_1 \rho_2^* y_1$ and $\theta^*(x_1, y_1) = (a_1, b_1) \in \rho_2^*$. If $a_1 = b_1$ then there exists $x_1 \in X_2$ such that $a_1 = \theta(x_1)$ and we set $y_1 = x_1$. In either case $x_1 \rho_2 y_1$ and $h(f)(x_1, y_1) = f(a_1, b_1)$. Therefore $h(f)(x_1, y_1) \neq 0$ and $h(f)$ is nonzero. This proves h is injective.

Let $c \in \mathrm{Supp}_G S$ and $g \in S_c \setminus \{0\}$ be arbitrarily chosen. Since $h(g) \in T \setminus \{0\}$ there exist $x_2, y_2 \in X_2$ such that $x_2 \rho_2 y_2$ and $(h(g))(x_2, y_2) \neq 0$. If we set $a_2 = \theta(x_2)$ and $b_2 = \theta(y_2)$ then $a_2 \rho_1 b_2$ since θ preserves order. Moreover $(h(g))(x_2, y_2) = g(a_2, b_2)$ and $\Phi_1(a_2, b_2) = c$ since the grading on S is induced by Φ_1. We have $\Phi_2(x_2, y_2) = \Phi_1(\theta(x_2), \theta(y_2))$ so $\Phi_2(x_2, y_2) = c$ hence $h(g) \in T_c$ as desired. This proves h preserves the grading. ■

THEOREM 4.3 *Suppose ρ_1 and ρ_2 are reflexive relations on X_1 and X_2, respectively, θ is a compression map of X_2 onto X_1, and G is a cancellative monoid. There is a G-grading set of ρ_1 if and only if there is a G-grading set of ρ_2.*

Proof. We assume there is a G-grading set *of* ρ_2 and prove there is a G-grading set of ρ_1. The reverse implication can be proved using a similar argument. Suppose σ_2 is a G-grading set of ρ_2 over G. Then $\sigma_2 \subseteq \rho_2^*$ since G is cancellative. We set $\sigma_1 = \{\theta^*(a, b) : a, b \in X_1 \text{ and } (a, b) \in \sigma_2\}$.

First we prove σ_1 is a G-essential set for ρ_1. Suppose $\Phi_1, \Phi_1' : \rho_1 \to G$ are homomorphisms such that $\Phi_1 \neq \Phi_1'$. Then there exist $a_1, a_2 \in X_1$ such that $a_1 \rho_1^* a_2$ and $\Phi_1(a_1, a_2) \neq \Phi_1'(a_1, a_2)$. We have $(a_1, a_2) = \theta^*(x_1, x_2)$ for some $x_1, x_2 \in X_2$ since θ^* is bijective. Let $\Phi_2 : \rho_2 \to G$ be the homomorphism defined as in part 1 of Lemma 4.2. Let $\Phi_2' : \rho_2 \to G$ be defined similarly, using Φ_1' in place of Φ_1. We have $\Phi_2(x_1, x_2) = \Phi_1(a_1, a_2)$ and $\Phi_2'(x_1, x_2) = \Phi_1'(a_1, a_2)$ so $\Phi_2(x_1, x_2) \neq \Phi_2'(x_1, x_2)$. Thus there exist $y, z \in X_1$ such that $(y, z) \in \sigma_2$ and $\Phi_2(y, z) \neq \Phi_2'(y, z)$ since σ_1 is a G-grading set for ϱ_1. We have $\theta^*(y, z) \in \sigma_1$, $\Phi_1(\theta^*(y, z)) = \Phi_2(y, z)$, and $\Phi_1'(\theta^*(y, z)) = \Phi_2'(y, z)$ by construction. Therefore $\Phi_1(\theta^*(y, z)) \neq \Phi_1'(\theta^*(y, z))$ and this shows σ_1 is a G-essential set for ρ_1.

Next we prove σ_1 is a G-extendible set for ρ_1. Given $\psi_1 : \sigma_1 \to G$ we let $\psi_2 : \sigma_2 \to G$ be the function given by $\psi_2 = \psi_1 \circ \theta^*$. There is also a homomorphism $\Psi_2 : \rho_2 \to G$ such that $\Psi_2|_{\sigma_2} = \psi_2$ since σ_2 is a G-extendible set for ρ_2. There is a homomorphism $\Psi_1 : \rho_1 \to G$ such that for all $a, b \in X_1$ such that $a\rho_1 b$ we have $\Psi_1(a,b) = \Psi_2\left((\theta^*)^{-1}(a,b)\right)$ if $a \neq b$ and$\Psi_1(a,b) = 1$ if $a = b$. Given $a, b \in X_1$ such that $(a,b) \in \sigma_1$ there exist $x, y \in X_1$ such that $a = \theta(x)$, $b = \theta(y)$, and $(x,y) \in \sigma_2$. We have $\theta^*(x,y) = (a,b)$ and $\Psi_2(x,y) = \Psi_1(a,b)$ by construction, $\Psi_2(x,y) = \psi_2(x,y)$ since $(x,y) \in \sigma_2$, and $\psi_2(x,y) = \psi_1(a,b)$ since $\psi_2 = \psi_1 \circ \theta^*$. This shows $\Psi_1|_{\sigma_1} = \psi_1$ and σ_1 is a G-extendible set for ρ_1. ■

Definition 4.4 *Let ρ be a reflexive relation on a set X.*

1. *(X, ρ) is* stable *if ρ is balanced and if the relations $a\rho b$, $a\rho c$, $b\rho c$, $b\rho d$, and $c\rho d$ imply $a\rho d$ for all distinct $a, b, c, d \in X$.*

2. *An element $x \in X$ is a* clasp *if there exist $w, y \in X \setminus \{x\}$ such that $w\rho x$, $x\rho y$, and $(w,y) \notin \rho$.*

3. *$x \in X$ is a* locked clasp *if there exist $u, v, w, y \in X \setminus \{x\}$ such that $(w,y) \notin \rho$ and $(u,x,y), (u,x,v), (w,x,v) \in \mathrm{Trans}(X)$.*

4. *An* unlocked clasp *is a clasp which is not locked.*

It is easy to see a preorder is stable. The balanced relation determined by (d) in Figure 1 is not stable. Neither a balanced relation which is not stable nor a stable relation which contains a locked clasp can be the compression of a preorder by [8, Theorem 2.4 and Lemma 3.4].

Theorem 4.5 *Assume ρ is a stable relation on a finite set X and every clasp in X is unlocked.*

1. *There is a preorder $\leq$ on a finite set Y and a compression map $\theta : Y \to X$.*
2. *There is an injective ring homomorphism $h : I(X, \rho, R) \to I(Y, \leq, R)$.*
3. *If G is a semigroup and $I(X, \rho, R)$ has a grading induced by $\Phi : \rho \to G$ then there is a good G-grading of $I(Y, \leq, R)$ such that $h : I(X, \rho, R) \to I(Y, \leq, R)$ is an injective homomorphism of G-graded rings.*

Proof. Part 1 is [8, Theorem 3.5]. Part 2 is a special case of part 3 when G=$\{1\}$. Part 3 follows from part 1 and Lemma 4.2. ■

5 Group Gradings

If (X, ρ) is a preorder then a subset A of X is a cross-cut if A is an antichain in X, for all $x \in X$ there exists $a \in A$ such that $x\rho a$ or $a\rho x$, and if C is a chain in X then C can be extended to a chain C' such that $A \cap C'$ is nonempty. The length of a cross-cut A of X is $|A|$. For example, the minimal elements of X form a cross-cut.

For a partially ordered with a cross-cut of length one or two there is a subset σ of ρ such that σ is a G-grading set for any group G (see [4, Theorem 4]). However if the shortest cross-cut of a partially ordered set has length three or more then there may not be a G-grading set for a group G (see [4, Example 6]).

Theorem 5.1 *Assume (X, ρ) is a preorder and X has a cross-cut of length one or two. Then there is a subset σ of ρ such that σ is a G-grading set for any group G.*

Proof. If $r, s \in X$ satisfy $r\rho s$ and $s\rho r$ then r and s are said to be paired in X. This defines an equivalence relation on X. We set $\tilde{X} = \{[x] : x \in X\}$ where $[x] = \{y \in X : x \text{ and } y \text{ are paired}\}$ for all $x \in X$. There is a partial order $\tilde{\rho}$ on $\tilde{X}$ such that $[x]\,\tilde{\rho}\,[y]$ if and only if $x\rho y$ for all $x, y \in X$. It is easy to see $\tilde{X}$ also has a cross-cut of length one or two. A G-grading set for $\left(\tilde{X}, \tilde{\rho}\right)$ is constructed in the proof of [4, Theorem 4]. We use it to construct a G-grading set for (X, ρ). Fix $P \subseteq X$ such that $\tilde{X} = \{[y] : y \in P\}$ and $[y_1] \neq [y_2]$ for all $y_1, y_2 \in P$ such that $y_1 \neq y_2$. Let β be a subset of ρ such that $\beta \subseteq P \times P$ and $\tilde{\beta} = \{([a], [b]) : (a, b) \in \beta\}$ is a G-grading set for $\left(\tilde{X}, \tilde{\rho}\right)$. Set $\sigma = \beta \cup \gamma$ where $\gamma = \bigcup\limits_{p \in P} \{(p, x) : x \in [p] \setminus \{p\}\}$.

Let $\phi : \sigma \to G$ be given. Define $\psi : \tilde{\beta} \to G$ by $\psi([a], [b]) = \phi(a, b)$ for all $a, b \in P$ such that $(a, b) \in \beta$. Then there exists a homomorphism $\Psi : \tilde{\rho} \to G$ such that $\Psi|_{\tilde{\beta}} = \psi$. We extend ϕ to $\bar{\phi} : \sigma \cup \{(p, p) : p \in P\} \to G$ so that $\bar{\phi}|_\sigma = \phi$ and $\bar{\phi}(p, p) = 1$ for all $p \in P$. Suppose $x_1, x_2 \in X$ such that $x_1 \rho x_2$ are given. There are uniquely determined $p_1, p_2 \in P$ such that $[x_1] = [p_1]$ and $[x_2] = [p_2]$. We set $\Phi(x_1, x_2) = \bar{\phi}(p_1, x_1)^{-1} \Psi([p_1], [p_2]) \bar{\phi}(p_2, x_2)$. A routine check proves the function $\Phi : \rho \to G$ is a homomorphism.

Suppose $x_1, x_2 \in X$ such that $(x_1, x_2) \in \sigma$ are given. If $(x_1, x_2) \in \beta$ then $x_1, x_2 \in P$, $\Phi(x_1, x_2) = \bar{\phi}(x_1, x_1)^{-1} \Psi([x_1], [x_2]) \bar{\phi}(x_2, x_2)$, $\Psi([x_1], [x_2]) = \psi([x_1], [x_2])$, and $\psi([x_1], [x_2]) = \phi(x_1, x_2)$ by construction. This gives $\Phi(x_1, x_2) = \phi(x_1, x_2)$. If $(x_1, x_2) \in \gamma$ then $\Phi(x_1, x_2) = \bar{\phi}(x_1, x_1)^{-1} \Psi([x_1], [x_1]) \bar{\phi}(x_1, x_2)$ since $x_1 \in P$ and $[x_1] = [x_2]$. We have $\bar{\phi}(x_1, x_2) = \phi(x_1, x_2)$ since $(x_1, x_2) \in \gamma$ and $\Phi(x_1, x_2) = \phi(x_1, x_2)$ follows easily. We have shown $\Phi|_\sigma = \phi$ so σ is a G-extendible set for ρ.

Suppose $\Upsilon_1, \Upsilon_2 : \rho \to G$ are homomorphism such that $\Upsilon_1|_\sigma = \Upsilon_2|_\sigma$. For $i = 1, 2$

define $\tilde{\Upsilon}_i : \tilde{\rho} \to G$ by $\tilde{\Upsilon}_i([a],[b]) = \Upsilon_i(a,b)$ for all $a, b \in P$ such that $(a,b) \in \rho$. A routine check proves $\tilde{\Upsilon}_1, \tilde{\Upsilon}_2$ are homomorphisms such that $\tilde{\Upsilon}_1|_{\tilde{\beta}} = \tilde{\Upsilon}_2|_{\tilde{\beta}}$. Thus $\tilde{\Upsilon}_1 = \tilde{\Upsilon}_2$ since $\tilde{\beta}$ is a G-grading set for $\tilde{X}$. This proves $\Upsilon_1(a,b) = \Upsilon_2(a,b)$ for all $a, b \in P$ such that $(a,b) \in \rho$.

Suppose $x_1, x_2 \in X$ such that $x_1 \rho x_2$ are given. There are uniquely determined $p_1, p_2 \in P$ such that $[x_1] = [p_1]$ and $[x_2] = [p_2]$. Transitivity and the homomorphism property gives $\Upsilon_1(x_1,x_2) = \Upsilon_1(p_1,x_1)^{-1} \Upsilon_1(p_1,p_2) \Upsilon_1(p_2,x_2)$. By the result of the previous paragraph $\Upsilon_1(p_1,p_2) = \Upsilon_2(p_1,p_2)$. Moreover $\Upsilon_1(p_2,x_2) = \Upsilon_2(p_2,x_2)$ and $\Upsilon_1(p_1,x_1) = \Upsilon_2(p_1,x_1)$ since $\Upsilon_1|_\sigma = \Upsilon_2|_\sigma$ and $(p_1,x_1), (p_2,x_2) \in \gamma$. Substitution gives $\Upsilon_1(x_1,x_2) = \Upsilon_2(p_1,x_1)^{-1} \Upsilon_2(p_1,p_2) \Upsilon_2(p_2,x_2)$ and this reduces to $\Upsilon_1(x_1,x_2) = \Upsilon_2(x_1,x_2)$. Since $x_1, x_2 \in X$ such that $(x_1,x_2) \in \rho$ were arbitrarily chosen we can conclude $\Upsilon_1 = \Upsilon_2$ and σ is a G-essential set for ρ. Therefore σ is a G-grading set for ρ. ■

Corollary 5.2 *Assume (X,ρ) is the compression of a preorder $(Y,\leq)$. If Y has a cross-cut of length one or two then there is a subset σ of ρ such that σ is a G-grading set for ρ for any group G.*

Proof. This follows from Theorems 4.3 and 5.1. ■

The directed graphs in Figure 2 represent reflexive relations. The vertices have been replaced with elements of the sets they represent.

Example 5.3 *Suppose ρ_1 is the stable relation on $X_1 = \{1,2,3,4\}$ determined by (a) in Figure 2. The only clasp in X_1 is 2, which is an unlocked clasp, so X_1 is the compression of a preordered set by part 1 of Theorem 4.5. If ρ_2 is the reflexive*

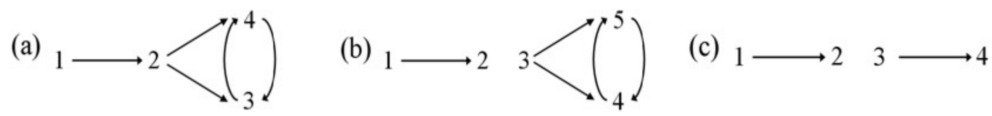

Figure 2: We use (b) and (c) to find a G-sourcing set for the relation of (a).

relation on $X_2 = \{1,2,3,4,5\}$ *determined by (b) in Figure 2 then* ρ_2 *is a preorder on* X_2 *and there is a compression map* $\theta : X_2 \to X_1$ *given by* $\theta(1) = 1$*,* $\theta(2) = 2$*,* $\theta(3) = 2$*,* $\theta(4) = 3$*, and* $\theta(5) = 4$*. By equating paired elements we may relate* (X_2, ρ_2) *to a partial order. This gives* ρ_3*, the minimally-connected partial order on* $X_3 = \{1,2,3,4\}$ *determined by (c) in Figure 2. By Theorem 3.8 a G-grading set for* ρ_3 *is* $\sigma_3 = \{(1,2),(3,4)\}$*. Following the proof of Theorem 5.1 a G-grading set for* ρ_2 *is* $\sigma_2 = \beta \cup \gamma$ *with* $\beta = \{(1,2),(3,4)\}$ *and* $\gamma = \{(4,5)\}$*. The proof of Theorem 4.3 shows a G-grading set of* ρ_1 *is* $\sigma_1 = \{(1,2),(2,3),(3,4)\}$*. Thus for any associative ring with unity R and any group G the G-gradings of* $S = I(X, \rho, R)$ *are uniquely determined by functions from* σ_1 *to G.*

Suppose σ is the Hasse diagram for a partial order $(X, \leq)$ and F is a field. The incidence algebra $A = I(X, \leq, F)$ is said to have the free-extension property (see [4, Definition 2]) if there exists $\gamma \subseteq \sigma$ such that for any group G and any function $\phi : \gamma \to G$, there is a unique grading of $I(X, \leq, F)$ such that $e_{xy} \in A_{\phi(x,y)}$ for all $x, y \in X$ such that $(x,y) \in \gamma$.

Theorem 3.4 shows a good group grading of a generalized incidence ring induced by a homomorphism must be a finite grading. This kind of result was already known for incidence algebras over partial orders (see [5, Theorem 3.3]). The free-extension property may fail for partial orders on an infinite set even if it contains a cross-cut of length one. We describe a partial order with a minimal element which does not have the free-extension property.

EXAMPLE 5.4 *Suppose F is a field, G is a group, and X is the set of natural numbers with ρ the usual ordering. The set $\sigma = \{(m, m+1) : m \in X\}$ is the arrow set of the Hasse diagram of X hence σ is a G-grading set for X by Theorem 3.8. Suppose there is an element $g \in G$ of infinite order. We let $\phi : \sigma \to G$ be the function given by $\phi(m, m+1) = g^m$ for all $m \in X$. Then there is a unique homomorphism $\Phi : \rho \to G$ such that $\Phi|_\sigma = \phi$, but this does not determine a grading for $I(X, \rho, F)$ since* $\operatorname{Im}\Phi$ *is infinite. Therefore $I(X, \rho, F)$ does not have the free-extension property even though it satisfies the requirements of [4, Theorem 4].*

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