A Fine Calculus for Static Delimited Control

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11 kwietnia 2025

Flavors of delimited control

$\text{shift0}\, k.\, e$	$\langle \square \rangle$
$\text{op}\ v$	$\text{handle}\ \square\ \text{with}\ x\ \ k.\, e$

Flavors of delimited control

$\mathsf{shift0}\ k.\, e$	$\langle \square \rangle$
$\mathsf{op}\ v$	$\mathsf{handle}\ \square\ \mathsf{with}\ x\ k.\, e$
$\mathcal{S}\kappa.\, e$	$\langle \square / v \rangle$

shift0 generalized to cont vars

$$\langle \mathrm{let}\ x = \mathcal{S}\kappa.\overbrace{\ldots\ \kappa[e]\ \ldots}^{\text{body}}\ \mathrm{in}\ t\rangle \mapsto \overbrace{\ldots\ \langle \mathrm{let}\ x = e\ \mathrm{in}\ t\rangle\ \ldots}^{\text{body after subst}}$$

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Structural substitution, known from the $\lambda\mu$-calculus.
Formal evaluation rule:

$$D[\mathcal{S}\kappa.\, e] \mapsto e\{\kappa{:=}D\}$$

Data in delimiters: dynamic binding

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Can express effect handlers:

$$\begin{aligned} \mathsf{op}\ v &\equiv \mathcal{S}\kappa.\, \mathsf{ask}(\kappa)\ v\ (\lambda x.\, \kappa[x]) \\ \mathsf{handle}\ e\ \mathsf{with}\ x\, k.\, t &\equiv \langle e / \lambda x\, k.\, t\rangle \end{aligned}$$

(Previous encodings of handlers using shift0/reset unwittingly encoded dynamic binding.)

Grammar

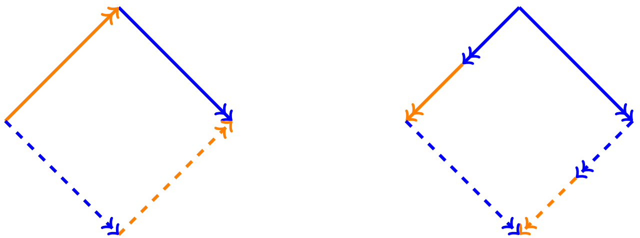
$$
\begin{aligned}
v, u &::= x \mid \lambda x.\, e \mid \mathsf{ask}(\kappa) \\
e, t &::= v \mid v\ v \mid \mathsf{let}\ x = e\ \mathsf{in}\ t \mid \mathcal{S}\kappa.\, e \mid \langle e / v \rangle \mid \kappa[e]
\end{aligned}
$$

Fine-grained reduction

$$
\begin{aligned}
(\lambda x.\, e)\, v &\to e\{x{:=}v\} && (\lambda.v)\\
\text{let } x = v \text{ in } t &\to t\{x{:=}v\} && (\text{let}.v)\\
\langle v / u\rangle &\to v && (d.v)\\
\text{let } x = \mathcal{S}\kappa.\, e \text{ in } t &\to \mathcal{S}\kappa.\, e\{\kappa{:=}\kappa[\text{let } x = \Box \text{ in } t]\} && (\text{let}.\mathcal{S})\\
\langle \mathcal{S}\kappa.\, e / v\rangle &\to e\{\kappa{:=}\langle \Box / v\rangle\} && (d.\mathcal{S})\\
\kappa[\mathcal{S}\kappa'.\, e] &\to e\{\kappa'{:=}\kappa\} && (k.\mathcal{S})\\
\langle L[\mathcal{S}\kappa.\, e] / v\rangle &\to \langle L[\mathcal{A}\, e\{\kappa{:=}\langle \Box / v\rangle\}] / v\rangle && (dL.\mathcal{S})\\
\langle L[\mathcal{A}\langle e / v\rangle] / v\rangle &\to \langle L[e] / v\rangle && (\mathcal{A}.d)\\
\langle L[\mathcal{A}\, u] / v\rangle &\to \langle L[u] / v\rangle && (\mathcal{A}.v)\\
\text{let } x = \text{let } y = e \text{ in } t_1 \text{ in } t_2 &\to \text{let } y = e \text{ in let } x = t_1 \text{ in } t_2 && (\text{let}.\text{let})
\end{aligned}
$$

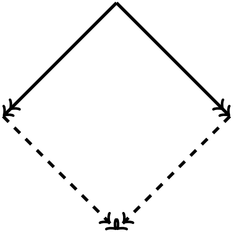
$$L ::= \Box \mid \text{let } x = e \text{ in } L$$

Correctness of reduction



Postponement and commutation of evaluation and nonevaluation.

Confluence



Now considering any pairs of reductions: much harder.

Abella formalization

Higher-order encoding of continuation variables:

```
Type shift0 ((expr -> expr) -> expr) -> expr.
```

First mechanization of parallel reductions and rewriting properties for $\lambda\mu$-calculus.

Explain and improve upon distributing delimiters

$$
\begin{aligned}
\langle \mathsf{let}\ x = e\ \mathsf{in}\ t \mid y.\, t_r \rangle &\equiv \langle \mathsf{let}\ y = \mathsf{let}\ x = e\ \mathsf{in}\ t\ \mathsf{in}\ \mathcal{A}\, t_r \rangle \\
&\to \langle \mathsf{let}\ x = e\ \mathsf{in}\ \mathsf{let}\ y = t\ \mathsf{in}\ \mathcal{A}\, t_r \rangle \\
&\leftarrow \langle \mathsf{let}\ x = e\ \mathsf{in}\ \mathcal{A} \langle \mathsf{let}\ y = t\ \mathsf{in}\ \mathcal{A}\, t_r \rangle \rangle \\
&\equiv \langle e \mid x.\, \langle t \mid y.\, t_r \rangle \rangle
\end{aligned}
$$