Polymorphic algebraic effects: theoretical properties and implementation

Wiktor Kuchta

Chapter 1

Introduction

Chapter 2

The language

We will study the deep handler calculus and type-and-effect system formulated in [2]. It is a refreshingly minimal language—the call-by-value lambda calculus with a few extensions to be able to express the essence of algebraic effects. There is only one unnamed universal operation, performed $\mathsf{do}\, v$. To be able to simulate calculi with named effects (and more), the *lift* operator, written $[e]$, is introduced. When operations are performed inside the expression e enclosed by lift, the nearest handler will be skipped and the operation will be handled by the next one instead. Naturally, the operator composes, so multiple enclosing lifts means multiple handlers skipped. In contrast to most work on algebraic effects, the effect-tracking system here is structural, we do not have any concept of predefined or user-defined (named) signatures of effects. Finally, the language features polymorphic expressions and polymorphic operations.

$$
\begin{aligned}
\mathrm{TVar} \ni{}& \alpha, \beta, \dots \\
\mathrm{Var} \ni{}& f, r, x, y, \dots \\
\mathrm{Kind} \ni \kappa ::={}& \mathsf{T} \mid \mathsf{E} \mid \mathsf{R} \\
\mathrm{Typelike} \ni \sigma, \tau, \varepsilon, \rho ::={}& \alpha \mid \tau \to_\rho \tau \mid \forall \alpha :: \kappa.\, \tau \mid \iota \mid \Delta.\, \tau \Rightarrow \tau \mid \varepsilon \cdot \rho \\
\mathrm{Val} \ni v, u ::={}& x \mid \lambda x.\, e \\
\mathrm{Exp} \ni e ::={}& v \mid e\ e \mid [e] \mid \mathsf{do}\ v \mid \mathsf{handle}\ e\ \{x, r.\, e;\ x.\, e\} \\
\mathrm{ECont} \ni K ::={}& \square \mid K\ e \mid v\ K \mid [K] \mid \mathsf{handle}\ K\ \{x, r.\, e;\ x.\, e\}
\end{aligned}
$$

Figure 2.1: Syntax.

$$\frac{}{0\text{-free}(\square)} \qquad \frac{n\text{-free}(K)}{n\text{-free}(K\ e)} \qquad \frac{n\text{-free}(K)}{n\text{-free}(v\ K)} \qquad \frac{n\text{-free}(K)}{n+1\text{-free}([K])}$$

$$\frac{n+1\text{-free}(K)}{n\text{-free}(\mathsf{handle}\ K\ \{x, r.\, e_h;\ x.\, e_r\})}$$

Figure 2.2: Evaluation context freeness.

$$\frac{e_1 \mapsto e_2}{K[e_1] \to K[e_2]} \qquad (\lambda x.\, e)\ v \mapsto e\{v/x\} \qquad [v] \mapsto v$$

$$\frac{0\text{-free}(K) \qquad v_c = \lambda z.\, \mathsf{handle}\ K[z]\ \{x, r.\, e_h;\ x.\, e_r\}}{\mathsf{handle}\ K[\mathsf{do}\ v]\ \{x, r.\, e_h;\ x.\, e_r\} \mapsto e_h\{v/x\}\{v_c/r\}}$$

$$\mathsf{handle}\ v\ \{x, r.\, e_h;\ x.\, e_r\} \mapsto e_r\{v/x\}$$

Figure 2.3: Single-step reduction.

$$\frac{\alpha :: \kappa \in \Delta}{\Delta \vdash \alpha :: \kappa} \qquad \frac{\Delta \vdash \tau_1 :: \mathsf{T} \qquad \Delta \vdash \rho :: \mathsf{R} \qquad \Delta \vdash \tau_2 :: \mathsf{T}}{\Delta \vdash \tau_1 \to_\rho \tau_2 :: \mathsf{T}} \qquad \frac{\Delta, \alpha :: \kappa \vdash \tau :: \mathsf{T}}{\Delta \vdash \forall \alpha :: \kappa.\, \tau :: \mathsf{T}}$$

$$\frac{}{\Delta \vdash \iota :: \mathsf{R}} \qquad \frac{\Delta \vdash \varepsilon :: \mathsf{E} \qquad \Delta \vdash \rho :: \mathsf{R}}{\Delta \vdash \varepsilon \cdot \rho :: \mathsf{R}} \qquad \frac{\Delta, \Delta' \vdash \tau_1 :: \mathsf{T} \qquad \Delta, \Delta' \vdash \tau_2 :: \mathsf{T}}{\Delta \vdash \Delta'.\, \tau_1 \Rightarrow \tau_2 :: \mathsf{E}}$$

$$\Delta \vdash \delta :: \Delta' \iff \mathrm{dom}(\delta) = \mathrm{dom}(\Delta') \land \forall \alpha \in \mathrm{dom}(\delta).\, \Delta \vdash \delta(\alpha) :: \Delta'(\alpha)$$

Figure 2.4: Well-formedness of types, rows, and type substitution.

$$\frac{}{\Delta \vdash \sigma <: \sigma} \qquad \frac{\Delta \vdash \tau_2^1 <: \tau_1^1 \qquad \Delta \vdash \rho_1 <: \rho_2 \qquad \Delta \vdash \tau_1^2 <: \tau_2^2}{\Delta \vdash \tau_1^1 \to_{\rho_1} \tau_1^2 <: \tau_2^1 \to_{\rho_2} \tau_2^2}$$

$$\frac{\Delta, \alpha :: \kappa \vdash \tau_1 <: \tau_2}{\Delta \vdash \forall \alpha :: \kappa.\, \tau_1 <: \forall \alpha :: \kappa.\, \tau_2} \qquad \frac{\Delta \vdash \rho :: \mathsf{R}}{\Delta \vdash \iota <: \rho} \qquad \frac{\Delta \vdash \rho_1 <: \rho_2}{\Delta \vdash \varepsilon \cdot \rho_1 <: \varepsilon \cdot \rho_2}$$

Figure 2.5: Subtyping.

$$\frac{x : \tau \in \Gamma}{\Delta; \Gamma \vdash x : \tau \,/\, \iota} \qquad \frac{\Delta \vdash \tau_1 :: \mathsf{T} \qquad \Delta; \Gamma, x : \tau_1 \vdash e : \tau_2 \,/\, \rho}{\Delta; \Gamma \vdash \lambda x.\, e : \tau_1 \to_\rho \tau_2 \,/\, \iota}$$

$$\frac{\Delta; \Gamma \vdash e_1 : \tau_1 \to_\rho \tau_2 \,/\, \rho \qquad \Delta; \Gamma \vdash e_2 : \tau_1 \,/\, \rho}{\Delta; \Gamma \vdash e_1\, e_2 : \tau_2 \,/\, \rho} \qquad \frac{\Delta \vdash \varepsilon :: \mathsf{E} \qquad \Delta; \Gamma \vdash e : \tau \,/\, \rho}{\Delta; \Gamma \vdash [e] : \tau \,/\, \varepsilon \cdot \rho}$$

$$\frac{\Delta, \alpha :: \kappa; \Gamma \vdash e : \tau \,/\, \iota}{\Delta; \Gamma \vdash e : \forall \alpha :: \kappa.\, \tau \,/\, \iota} \qquad \frac{\Delta \vdash \sigma :: \kappa \qquad \Delta; \Gamma \vdash e : \forall \alpha :: \kappa.\, \tau \,/\, \rho}{\Delta; \Gamma \vdash e : \tau\{\sigma/\alpha\} \,/\, \rho}$$

$$\frac{\Delta \vdash \tau_1 <: \tau_2 \qquad \Delta \vdash \rho_1 <: \rho_2 \qquad \Delta; \Gamma \vdash e : \tau_1 \,/\, \rho_1}{\Delta; \Gamma \vdash e : \tau_2 \,/\, \rho_2}$$

$$\frac{\Delta; \Gamma \vdash v : \delta(\tau_1) \,/\, \iota \qquad \Delta \vdash \delta :: \Delta' \qquad \Delta \vdash \Delta'.\, \tau_1 \Rightarrow \tau_2 :: \mathsf{E}}{\Delta; \Gamma \vdash \mathsf{do}\ v : \delta(\tau_2) \,/\, (\Delta'.\, \tau_1 \Rightarrow \tau_2)}$$

$$\frac{\begin{array}{c}\Delta; \Gamma \vdash e : \tau \,/\, (\Delta'.\, \tau_1 \Rightarrow \tau_2) \cdot \rho \\ \Delta, \Delta'; \Gamma, x : \tau_1, r : \tau_2 \to_\rho \tau_r \vdash e_h : \tau_r \,/\, \rho \qquad \Delta; \Gamma, x : \tau \vdash e_r : \tau_r \,/\, \rho\end{array}}{\Delta; \Gamma \vdash \mathsf{handle}\ e\ \{x, r.\, e_h; x.\, e_r\} : \tau_r \,/\, \rho}$$

Figure 2.6: Type system.

Chapter 3

The logical relation

[todo introduction to logical relations]

The logical relation is inspired by [1]. Some changes are due to language differences: we have only one universal operation which simplifies the treatment of effects, polymorphism does not manifest at the expression level—we do not have type lambdas, and our operations can be polymorphic. Instead of a binary step-indexed relation, our goal is to build a unary relation without step-indexing.

3.1 Definition

First, we define the interpretations of kinds. We call them the spaces of *semantic types* and *semantic effects*.

$$\begin{aligned}
[\![\mathsf{T}]\!] &= \mathcal{P}(\mathrm{Val}) = \mathsf{Type}, \\
[\![\mathsf{E}]\!] &= \mathcal{P}(\mathrm{Exp} \times \{0\} \times \mathsf{Type}) \\
[\![\mathsf{R}]\!] &= \mathcal{P}(\mathrm{Exp} \times \mathbb{N} \times \mathsf{Type})
\end{aligned}$$

Semantic types are simply sets of closed values. Semantic effects are sets of triples, which aim to describe a situation in which an expression being evaluated performs an effect. The components of such a triple are: the argument of the operation, the freeness of the enclosing context beyond which the operation can be handled, and the semantic type of values we can call the resumption with.

Using structural induction we simultaneously define the interpretations of types and effects, as well as relations $\mathcal{E}$ on expressions and $\mathcal{S}$ on stuck terms. We parameterize the definitions by a mapping η from type variables to semantic types or effects.

$$\llbracket \alpha \rrbracket_\eta = \eta(\alpha)$$

$$\begin{aligned}
\llbracket \tau_1 \to_\rho \tau_2 \rrbracket_\eta &= \{\lambda x.\, e \mid \forall v \in \llbracket \tau_1 \rrbracket_\eta.\, e\{v/x\} \in \mathcal{E}\llbracket \tau_2/\rho \rrbracket_\eta\} \\
\llbracket \forall \alpha :: \kappa.\, \tau \rrbracket_\eta &= \{v \mid \forall \mu \in \llbracket \kappa \rrbracket.\, v \in \llbracket \tau \rrbracket_{[\alpha \mapsto \mu]\eta}\}
\end{aligned}$$

$$\begin{aligned}
\llbracket \Delta.\, \tau_1 \Rightarrow \tau_2 \rrbracket_\eta &= \{(v, 0, \llbracket \tau_2 \rrbracket_{\eta\eta'}) \mid \eta' \in \llbracket \Delta \rrbracket \wedge v \in \llbracket \tau_1 \rrbracket_{\eta\eta'}\} \\
\llbracket \varepsilon \cdot \rho \rrbracket_\eta &= \llbracket \varepsilon \rrbracket_\eta \cup \{(v, n+1, \mu) \mid (v, n, \mu) \in \llbracket \rho \rrbracket_\eta\} \\
\llbracket \iota \rrbracket_\eta &= \emptyset
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}\llbracket \tau/\rho \rrbracket_\eta(X) &= \{e \mid \exists v \in \llbracket \tau \rrbracket_\eta.\, e \to^* v \vee \exists e' \in \mathcal{S}\llbracket \tau/\rho \rrbracket_\eta(X).\, e \to^* e'\} \\
\mathcal{S}\llbracket \tau/\rho \rrbracket_\eta(X) &= \{K[\mathsf{do}\ v] \mid \exists n, \mu.\, (v, n, \mu) \in \llbracket \rho \rrbracket_\eta \wedge n\text{-free}(K) \wedge \forall u \in \mu.\, K[u] \in X\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}\llbracket \tau/\rho \rrbracket_\eta &= \bigcap \{X \mid \mathcal{E}\llbracket \tau/\rho \rrbracket_\eta(X) \subseteq X\} \\
\mathcal{S}\llbracket \tau/\rho \rrbracket_\eta &= \mathcal{S}\llbracket \tau/\rho \rrbracket_\eta(\mathcal{E}\llbracket \tau/\rho \rrbracket_\eta)
\end{aligned}$$

We define the set $\mathcal{E}\llbracket \tau/\rho \rrbracket_\eta$ as a solution to the fixed-point equation $\mathcal{E}\llbracket \tau/\rho \rrbracket_\eta(X) = X$. The function $\mathcal{E}\llbracket \tau/\rho \rrbracket_\eta$ is monotone, so by the Knaster-Tarski theorem it has a least fixed point. From its characterization as the intersection of all $\mathcal{E}\llbracket \tau/\rho \rrbracket_\eta$-closed sets, we immediately obtain the induction principle.

Lemma 1 (Tarski induction principle)**.** *If $\mathcal{E}\llbracket \tau/\rho \rrbracket_\eta(X) \subseteq X$, then $\mathcal{E}\llbracket \tau/\rho \rrbracket_\eta \subseteq X$.*

By expanding out the definition of the function $\mathcal{E}\llbracket \tau/\rho \rrbracket_\eta$ and treating X as a predicate P, we get the more familiar principle of structural induction on $\mathcal{E}\llbracket \tau/\rho \rrbracket_\eta$.

Lemma 2 (Induction principle)**.** *Assume that $P(e)$ is implied by e evaluating to a value in $\llbracket \tau \rrbracket_\eta$ or reducing to some $K[\mathsf{do}\ v]$ such that there exist $(v, n, \mu) \in \llbracket \rho \rrbracket_\eta$ such that n-free(K) and $P(K[u])$ holds for all $u \in \mu$. Then $P(e)$ holds for all $e \in \mathcal{E}\llbracket \tau/\rho \rrbracket_\eta$.*

Lemma 3 (Value inclusion)**.** *For any τ and ρ we have $\llbracket \tau \rrbracket_\eta \subseteq \mathcal{E}\llbracket \tau/\rho \rrbracket_\eta$.*

Lemma 4 (Control-stuck inclusion)**.** *For any τ and ρ we have $\mathcal{S}\llbracket \tau/\rho \rrbracket_\eta \subseteq \mathcal{E}\llbracket \tau/\rho \rrbracket_\eta$.*

Lemma 5 (Closedness under antireduction)**.** *If $e \to^* e' \in \mathcal{E}\llbracket \tau/\rho \rrbracket_\eta$, then $e \in \mathcal{E}\llbracket \tau/\rho \rrbracket_\eta$.*

Lemma 6. *If $\llbracket \tau_1 \rrbracket_\eta \subseteq \llbracket \tau_2 \rrbracket_\eta$ and $\llbracket \rho_1 \rrbracket_\eta \subseteq \llbracket \rho_2 \rrbracket_\eta$, then $\mathcal{S}\llbracket \tau_1/\rho_1 \rrbracket_\eta \subseteq \mathcal{S}\llbracket \tau_2/\rho_2 \rrbracket_\eta$ and $\mathcal{E}\llbracket \tau_1/\rho_1 \rrbracket_\eta \subseteq \mathcal{E}\llbracket \tau_2/\rho_2 \rrbracket_\eta$.*

3.2 Compatibility lemmas

We want to establish that $\vdash e : \tau \ /\ \iota$ implies $e \in \mathcal{E}\llbracket \tau/\iota \rrbracket$.

For this purpose we will prove a semantic counterpart of each typing rule. First, we need to define a counterpart to the typing judgment. Unlike typing judgments, our relations are on closed terms only, so we get around that by using substitution. We define semantic entailment as

$$\Delta; \Gamma \models e : \tau \;/\; \rho \iff \forall \eta \in [\![\Delta]\!].\, \forall \gamma \in [\![\Gamma]\!]_\eta.\, \gamma(e) \in \mathcal{E}[\![\tau/\rho]\!]_\eta,$$

where $[\![\Delta]\!] = \{\eta \mid \forall \alpha :: \kappa \in \Delta.\, \eta(\alpha) \in [\![\kappa]\!]\}$ contains type-level mappings and $[\![\Gamma]\!]_\eta = \{\gamma \mid \forall x : \tau \in \Gamma.\, \gamma(x) \in [\![\tau]\!]_\eta\}$ contains expression-level variable substitutions.

Lemma 7 (Variable compatibility)**.**

Proof. Assume $x : \tau \in \Gamma$. We want to prove $\Delta; \Gamma \models x : \tau \;/\; \iota$. Take any $\eta \in [\![\Delta]\!]$ and $\gamma \in [\![\Gamma]\!]_\eta$. We want to show $\gamma(x) \in \mathcal{E}[\![\tau/\iota]\!]_\eta$. From the definition of $[\![\Gamma]\!]_\eta$ we know that $\gamma(x) \in [\![\tau]\!]_\eta$, so by lemma 3 we have $\gamma(x) \in \mathcal{E}[\![\tau/\iota]\!]_\eta$. $\square$

Lemma 8 (Abstraction compatibility)**.**

Proof. Assume $\Delta \vdash \tau_1 :: \mathsf{T}$ and $\Delta; \Gamma, x : \tau_1 \models e : \tau_2 \;/\; \rho$. We want to prove $\Delta; \Gamma \models \lambda x.\, e : \tau_1 \to_\rho \tau_2 \;/\; \iota$. Take any $\eta \in [\![\Delta]\!]$ and $\gamma \in [\![\Gamma]\!]_\eta$. By lemma 3 it suffices to show $\gamma(\lambda x.\, e) = \lambda x.\, \gamma(e) \in [\![\tau_1 \to_\rho \tau_2]\!]_\eta$. So take any $v \in [\![\tau_1]\!]_\eta$. We need to show $\gamma(e)\{v/x\} \in \mathcal{E}[\![\tau_2/\rho]\!]_\eta$. Let $\gamma' = \gamma'[x \mapsto v]$. Then $\gamma' \in [\![\Gamma, x : \tau_1]\!]_\eta$, so $\gamma(e)\{v/x\} = \gamma'(e) \in \mathcal{E}[\![\tau_2/\rho]\!]_\eta$. $\square$

For clarity of presentation, in the following we will assume Γ empty, i.e. an interpretation of Γ was already substituted.

Lemma 9 (Lift compatibility)**.**

Proof. Assume $\Delta \vdash \tau :: \mathsf{T}$, $\Delta \vdash \varepsilon :: \mathsf{E}$, and $\Delta \vdash \rho :: \mathsf{R}$. Take any $\eta \in [\![\Delta]\!]$. Let $Y = \{e \mid [e] \in \mathcal{E}[\![\tau/\varepsilon \cdot \rho]\!]_\eta\}$. We will show $\mathcal{E}[\![\tau/\rho]\!]_\eta \subseteq Y$ by showing $\mathcal{E}[\![\tau/\rho]\!]_\eta(Y) \subseteq Y$. Take any $e \in \mathcal{E}[\![\tau/\rho]\!]_\eta(Y)$.

If $e \to^* K[\mathsf{do}\, v] \in \mathcal{S}[\![\tau/\rho]\!]_\eta(Y)$, then we have $(v, n, \mu) \in [\![\rho]\!]_\eta$ such that n-free(K) and $\forall u \in \mu.\, K[u] \in Y$ We have $(v, n+1, \mu) \in [\![\varepsilon \cdot \rho]\!]_\eta$, $n+1$-free$([K])$, and $\forall u \in \mu.\, [K[u]] \in \mathcal{E}[\![\tau/\varepsilon \cdot \rho]\!]_\eta$. So $[K[\mathsf{do}\, v]] \in \mathcal{S}[\![\tau/\varepsilon \cdot \rho]\!]_\eta$ and $[e] \in \mathcal{E}[\![\tau/\varepsilon \cdot \rho]\!]_\eta$ by antireduction.

If $e \to^* v \in [\![\tau]\!]_\eta$, then $[e] \to^* [v] \to v$, so $[e] \in \mathcal{E}[\![\tau/\varepsilon \cdot \rho]\!]_\eta$. $\square$

Lemma 10 (Application to value compatibility)**.**

Proof. Fix any well-formed Δ and $\tau_1 \to_\rho \tau_2$. Take any $\eta \in [\![\Delta]\!]$ and $(\lambda x.\, e) \in [\![\tau_1 \to_\rho \tau_2/\rho]\!]_\eta$. Let $Y = \{e \mid (\lambda x.\, e)\; e \in \mathcal{E}[\![\tau_2/\rho]\!]_\eta\}$. We will show $\mathcal{E}[\![\tau_1/\rho]\!]_\eta \subseteq Y$ by showing $\mathcal{E}[\![\tau_1/\rho]\!]_\eta(Y) \subseteq Y$. Take any $e_2 \in \mathcal{E}[\![\tau_1/\rho]\!]_\eta(Y)$.

If $e_2 \to^* K[\mathsf{do}\, v] \in \mathcal{S}[\![\tau_1/\rho]\!]_\eta(Y)$, then there exists $(v, n, \mu) \in [\![\rho]\!]_\eta$ such that n-free(K) and for all $u \in \mu$ we have $K[u] \in Y$. Since n-free$((\lambda x.\, e)\; e)$ and for all $u \in \mu$ we have $((\lambda x.\, e)\; K[u] \in \mathcal{E}[\![\tau_2/\rho]\!]_\eta$, $(\lambda x.\, e)\; K[\mathsf{do}\, v] \in \mathcal{S}[\![\tau_2/\rho]\!]_\eta$ and by antireduction $(\lambda x.\, e)\; e_2 \in \mathcal{E}[\![\tau_2/\rho]\!]_\eta$.

If $e_2 \to^* v \in [\![\tau_1]\!]_\eta$, then $(\lambda x.\, e)\; e_2 \to^* e\{v/x\} \in \mathcal{E}[\![\tau_2/\rho]\!]_\eta$, so $(\lambda x.\, e)\; e_2 \in \mathcal{E}[\![\tau_2/\rho]\!]_\eta$ by antireduction. $\square$

Lemma 11 (Application compatibility)**.**

Proof. Fix any well-formed Δ and $\tau_1 \to_\rho \tau_2$. Take any $\eta \in [\![\Delta]\!]$ and $e_2 \in \mathcal{E}[\![\tau_1/\rho]\!]_\eta$. Let $Y = \{e_1 \mid e_1\ e_2 \in \mathcal{E}[\![\tau_2/\rho]\!]_\eta\}$. We will show $\mathcal{E}[\![\tau_1 \to_\rho \tau_2/\rho]\!] \subseteq Y$ by showing $\mathcal{E}[\![\tau_1 \to_\rho \tau_2/\rho]\!]_\eta(Y) \subseteq Y$. Take any $e_1 \in \mathcal{E}[\![\tau_1 \to_\rho \tau_2/\rho]\!]_\eta(Y)$.

If $e_1 \to^* K_1[\mathsf{do}\ v] \in \mathcal{S}[\![\tau_1 \to_\rho \tau_2/\rho]\!]_\eta(Y)$, then there exists $(v, n, \mu) \in [\![\rho]\!]_\eta$ such that n-free(K_1) and for all $u \in \mu$ we have $K_1[u] \in Y$. Since n-free$(K_1\ e_2)$ and for all $u \in \mu$ we have $K_1[u]\ e_2 \in \mathcal{E}[\![\tau_2/\rho]\!]_\eta$, $K_1[\mathsf{do}\ v]\ e_2 \in \mathcal{S}[\![\tau_2/\rho]\!]_\eta$ and by antireduction $e_1\ e_2 \in \mathcal{E}[\![\tau_2/\rho]\!]_\eta$.

Now assume $e_1 \to^* (\lambda x.\, e) \in [\![\tau_1 \to_\rho \tau_2/\rho]\!]_\eta$. By the previous lemma $(\lambda x.\, e)\ e_2 \in \mathcal{E}[\![\tau_2/\rho]\!]_\eta$, so by antireduction $e_1\ e_2 \in \mathcal{E}[\![\tau_2/\rho]\!]_\eta$. □

Lemma 12 (Handle compatibility)**.**

Proof. Assume $\Delta, \Delta'; x : \tau_1, r : \tau_2 \to_\rho \tau_r \models e_h : \tau_r \ /\ \rho$ and $\Delta; x : \tau \models e_r : \tau_r \ /\ \rho$. Let h stand for $\{x, r.\, e_h;\ x.\, e_r\}$. Take any $\eta \in [\![\Delta]\!]$. Let $Y = \{e \mid \mathsf{handle}\, e\, h \in \mathcal{E}[\![\tau_r/\rho]\!]_\eta\}$. We will show $\mathcal{E}[\![\tau/(\Delta'.\, \tau_1 \Rightarrow \tau_2) \cdot \rho]\!]_\eta \subseteq Y$ by showing $\mathcal{E}[\![\tau/(\Delta'.\, \tau_1 \Rightarrow \tau_2) \cdot \rho]\!]_\eta(Y) \subseteq Y$. Note that only τ_1 and τ_2 require Δ' to be in context. Take $e \in \mathcal{E}[\![\tau/(\Delta'.\, \tau_1 \Rightarrow \tau_2) \cdot \rho]\!]_\eta(Y)$.

If $e \to^* v \in [\![\tau]\!]_\eta$, then $\mathsf{handle}\, e\, h \to^* e_r\{v/x\} \in \mathcal{E}[\![\tau_r/\rho]\!]_\eta$, so the claim follows by antireduction.

If $e \to^* K[\mathsf{do}\ v] \in \mathcal{S}[\![\tau/(\Delta'.\, \tau_0 \Rightarrow \tau_2) \cdot \rho]\!]_\eta(Y)$, then we have $(v, n, \mu) \in [\![(\Delta'.\, \tau_1 \Rightarrow \tau_2) \cdot \rho]\!]_\eta$ such that n-free(K) and $\forall u \in \mu.\, K[u] \in Y$.

If $n = 0$, then $(v, n, \mu) \in [\![\tau_1 \Rightarrow \tau_2]\!]_{\eta\eta'}$ for some $\eta' \in [\![\Delta']\!]$. More specifically, $v \in [\![\tau_1]\!]_{\eta\eta'}$ and $\mu = [\![\tau_2]\!]_{\eta\eta'}$. We have $\mathsf{handle}\, e\, h \to^* \mathsf{handle}\ K[\mathsf{do}\ v]\, h \to e_h\{v/x\}\{v_c/r\}$, where $v_c = \lambda z.\, \mathsf{handle}\ K[z]\, h$. To show $v_c \in [\![\tau_2 \to_\rho \tau_r]\!]_{\eta\eta'}$, take any $u \in [\![\tau_2]\!]_{\eta\eta'}$ and show $\mathsf{handle}\ K[u]\, h \in \mathcal{E}[\![\tau_r/\rho]\!]_{\eta\eta'} = \mathcal{E}[\![\tau_r/\rho]\!]_\eta$. Which holds, since $K[u] \in Y$. Therefore, $e_h\{v/x\}\{v_c/r\} \in \mathcal{E}[\![\tau_r/\rho]\!]_\eta$ and the claim follows by antireduction.

If $n > 0$, then $\mathsf{handle}\ K[\mathsf{do}\ v]\, h \in \mathcal{S}[\![\tau_r/\rho]\!]_\eta$, since $n-1$-free$(\mathsf{handle}\ K\, h)$, $(v, n-1, \mu) \in [\![\rho]\!]_\eta$, and $\forall u \in \mu.\, \mathsf{handle}\ K[u]\, h \in \mathcal{E}[\![\tau_r/\rho]\!]_\eta$. Again, the claim follows by antireduction.

□

Lemma 13. *If Δ and Δ' disjoint, $\Delta \vdash \delta :: \Delta'$, and $\Delta, \Delta' \vdash \tau :: \kappa$, $\eta \in [\![\Delta]\!]$, and η' extends η by mappings $\alpha \mapsto [\![\delta(\alpha)]\!]_\eta$, then $[\![\tau]\!]_{\eta'} = [\![\delta(\tau)]\!]_\eta$.*

Proof. By induction on the kinding rules.

If $\tau = \alpha \in \Delta$, then both sides are equal to $\eta(\alpha)$.

If $\tau = \alpha \in \Delta'$, then equality follows from the definition of η'.

If $\tau = \iota$, then both sides are empty.

If $\tau = \forall \alpha :: \kappa.\, \tau'$, then $[\![\tau]\!]_{\eta'} = \bigcap\{[\![\tau']\!]_{\eta'[\alpha \mapsto \mu]} \mid \mu \in [\![\kappa]\!]\}$ and $[\![\delta(\tau)]\!]_\eta = \bigcap\{[\![\delta(\tau')]\!]_{\eta[\alpha \mapsto \mu]} \mid \mu \in [\![\kappa]\!]\}$, which are equal by the inductive hypothesis (taking $\Delta, \alpha :: \kappa$ as Δ in the statement).

If $\tau = \Delta''.\, \tau_1 \Rightarrow \tau_2$, then $[\![\tau]\!]_{\eta'} = \{(v, 0, [\![\tau_2]\!]_{\eta'\eta''}) \mid \eta'' \in [\![\Delta'']\!] \wedge v \in [\![\tau_1]\!]_{\eta'\eta''}\}$ and $[\![\delta(\tau)]\!]_\eta = \{(v, 0, [\![\delta(\tau_2)]\!]_{\eta\eta''}) \mid \eta'' \in [\![\Delta'']\!] \wedge v \in [\![\delta(\tau_1)]\!]_{\eta\eta''}\}$, which are equal by induction (taking Δ, Δ'' as Δ in the statement).

If $\tau = \varepsilon \cdot \rho$, then $[\![\tau]\!]_{\eta'} = [\![\varepsilon]\!]_{\eta'} \cup \{(v, n+1, \mu) \mid (v, n, \mu) \in [\![\rho]\!]_{\eta'}\}$ and $[\![\delta(\tau)]\!]_\eta = [\![\delta(\varepsilon)]\!]_\eta \cup \{(v, n+1, \mu) \mid (v, n, \mu) \in [\![\delta(\rho)]\!]_\eta\}$, which are equal by the inductive hypothesis. □

Lemma 14 (Do compatibility)**.**

Proof. Assume $\Delta \vdash \delta :: \Delta'$, $\Delta \vdash \Delta'.\, \tau_1 \to \tau_2 :: \mathsf{E}$. Take any $\eta \in [\![\Delta]\!]$. Assume $v \in [\![\delta(\tau_1)]\!]_\eta$. We want to show $\mathsf{do}\ v \in \mathcal{E}[\![\delta(\tau_2)/(\Delta'.\, \tau_1 \Rightarrow \tau_2)]\!]_\eta$.

By lemma 4 it suffices to show $\mathsf{do}\ v \in \mathcal{S}[\![\delta(\tau_2)/(\Delta'.\, \tau_1 \Rightarrow \tau_2)]\!]_\eta$. By taking the empty context in the definition of $\mathcal{S}$ and lemma 3, it suffices to show $(v, 0, [\![\delta(\tau_2)]\!]_\eta) \in [\![\Delta'.\, \tau_1 \Rightarrow \tau_2]\!]_\eta$. From the interpretation of polymorphic effects, it would be enough to show $(v, 0, [\![\delta(\tau_2)]\!]_\eta) \in [\![\tau_1 \Rightarrow \tau_2]\!]_{\eta'}$, where η' extends η by mappings $\alpha \mapsto [\![\delta(\alpha)]\!]_\eta$ for all $\alpha \in \Delta'$. By the interpretation of operations, it remains to show $v \in [\![\tau_1]\!]_{\eta'}$ and $[\![\delta(\tau_2)]\!]_\eta = [\![\tau_2]\!]_{\eta'}$. Which follows immediately from lemma 13. $\square$

Lemma 15 (Subtyping compatibility)**.**

Proof. By induction on subtyping rules it can be shown that if $\sigma_1 <: \sigma_2$, then $[\![\sigma_1]\!]_\eta \subseteq [\![\sigma_2]\!]_\eta$ (for σ_i of any kind).

So if $\Delta \vdash \tau_1 <: \tau_2$, $\Delta \vdash \rho_1 <: \rho_2$ and $\Delta; \Gamma \models e : \tau_1 \;/\; \rho_1$, then by lemma 6 $\Delta; \Gamma \models e : \tau_2 \;/\; \rho_2$. $\square$

Lemma 16 (Polymorphism introduction compatibility)**.**

Proof. Assume $\Delta, \alpha :: \kappa; \models e : \tau \;/\; \iota$. Take $\eta \in [\![\Delta]\!]$. We know e evaluates to a value in $[\![\tau]\!]_{\eta[\alpha \mapsto \mu]}$ for any $\mu \in [\![\kappa]\!]$. Therefore this value is in $[\![\forall \alpha :: \kappa.\, \tau]\!]_\eta$, and by antireduction $e \in \mathcal{E}[\![\forall \alpha :: \kappa.\, \tau/\iota]\!]_\eta$. $\square$

Lemma 17 (Polymorphism elimination compatibility)**.**

Proof. Assume $\Delta \vdash \sigma :: \kappa$ and $\Delta; \models e : \forall \alpha :: \kappa.\, \tau \;/\; \rho$. By lemma 13 we have $[\![\tau\{\sigma/\alpha\}]\!]_\eta = [\![\tau]\!]_{\eta[\alpha \mapsto [\![\sigma]\!]_\eta]}$, which is a superset of $[\![\forall \alpha :: \kappa.\, \tau]\!]$. So we have $\Delta; \models e : \tau\{\sigma/\alpha\} \;/\; \rho$ by lemma 6. $\square$

Theorem 1 (Termination of evaluation)**.**

Proof. If $\vdash e : \tau/\iota$, then $e \in \mathcal{E}[\![\tau/\iota]\!]$ and e has to terminate to a value, since $[\![\iota]\!]$ is empty and hence $\mathcal{S}[\![\tau/\iota]\!]$ is empty. $\square$

Bibliography

[1] Dariusz Biernacki, Maciej Piróg, Piotr Polesiuk, and Filip Sieczkowski. Handle with care: relational interpretation of algebraic effects and handlers. *Proc. ACM Program. Lang.*, 2(POPL):8:1–8:30, 2018.

[2] Maciej Piróg, Piotr Polesiuk, and Filip Sieczkowski. Typed equivalence of effect handlers and delimited control. In Herman Geuvers, editor, *4th International Conference on Formal Structures for Computation and Deduction, FSCD 2019, June 24-30, 2019, Dortmund, Germany*, volume 131 of *LIPIcs*, pages 30:1–30:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.