How can the van Kampen theorem for fundamental groupoids be used to compute the fundamental group of the circle?

1 Answer

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Ah, that's a sweet computation. To be fair, the standard computation of $\pi_1(S^1)$ isn't that much harder, but the van Kampen approach is very categorical and, in some ways, conceptually simpler.

I'll start with some background which you may not need: if you already know what van Kampen theorem for groupoids says, particularly the version relative to a subset of basepoints, you may hop right to the end of this answer. I'm including the setup because I think many people who know about the fundamental group may be unaware of the fundamental groupoid.

If X is any topological space, we let $\pi_\bullet(X)$ be the fundamental groupoid, which is simply the category consisting of

- Objects: the points of X
- Morphisms: continuous paths, up to homotopy relative to endpoints.

So if $a, b \in X$ are any two points, they are objects of $\pi_\bullet(X)$, and the set of morphisms $\mathrm{Mor}(a, b)$ is the set of homotopy classes $[\gamma]$ where $\gamma : [0,1] \to X$ is a continuous path with $\gamma(0) = a$ and $\gamma(1) = b$, and $\gamma \sim \gamma'$ if there's a continuous homotopy $H : [0,1] \times [0,1] \to X$ with $H(0,\cdot) = \gamma(\cdot)$ and $H(1,\cdot) = \gamma'(\cdot)$ while keeping $H(\cdot,0) = a$ and $H(\cdot,1) = b$ at all times.

This is similar to the definition of the fundamental group, except we never chose a basepoint. In fact, the fundamental group with respect to the basepoint x_0 is simply $\pi_1(X, x_0) = \mathrm{Mor}(x_0, x_0)$ in $\pi_\bullet(X)$. If you restrict the groupoid to just those morphisms from x_0 to itself, you obviously get the fundamental group.

Also, $\pi_\bullet$ is a covariant functor. This means that if $f : X \to Y$ is a continuous function, then there's a corresponding induced map $\pi_\bullet(f) : \pi_\bullet(X) \to \pi_\bullet(Y)$ defined in the natural way: any path in X can be composed with f to define a path in Y, and this naturally respects everything it needs to respect. Note that $\pi_\bullet(f)$ is itself a functor, between the groupoid categories $\pi_\bullet(X)$ and $\pi_\bullet(Y)$.

Now, consider the van Kampen scenario: $X = U \cup V$ where U and V are open sets, intersecting at $U \cap V$. In the category of topological spaces, the diagram

$$\begin{array}{ccc} U \cap V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

that for any other such diagram with Y replacing X, there's a unique map $X \to Y$ making everything commute. Simply, any such candidate Y is a space containing U and V, and the smallest such space is X itself.

The fun property of $\pi_\bullet$ is that, as a functor, it preserves pushouts. So given a van Kampen diagram as above, The diagram induced by $\pi_\bullet$ is *also* a pushout diagram, this time in the category of groupoids:

$$\begin{array}{ccc} \pi_\bullet(U \cap V) & \longrightarrow & \pi_\bullet(V) \\ \downarrow & & \downarrow \\ \pi_\bullet(U) & \longrightarrow & \pi_\bullet(X) \end{array}$$

This provides an algebraic description of the groupoid $\pi_\bullet(X)$ in terms of the other groupoids: it is a pushout, meaning it's like a "free product amalgamated along the intersection". To better understand what this means, you may want to study the pushout in the category of groups, which is an amalgamated product . We won't get too deep into this now – the case of the circle will turn out to be simple enough.

Proving that $\pi_\bullet$ preserves pushouts is actually quite easy, and this really captures the van Kampen theorem in full, for the case of groupoids. But there is one enhancement that makes things easier: instead of the entire groupoid $\pi_\bullet(X)$ we usually pick a subset $A \subset X$ and look at $\pi_\bullet(X, A)$ which is the same groupoid restricted to the objects in A. In other words, $\pi_\bullet(X, A)$ is the full subcategory of $\pi_\bullet(X)$ consisting of the objects in A and all the morphisms between them. It's like "the fundamental group with multiple basepoints".

Going back to the van Kampen scenario, you should wonder which subsets A are such that $\pi_\bullet(X, A)$ still preserves the diagram. Clearly A has to be part of all players: U, V and $U \cap V$, and in fact it should meet each of their connected components, for otherwise it is "blind" to the structure of that component, much like $\pi_1(X, x_0)$ says nothing about any connected component of X that doesn't include x_0.

It turns out that this obvious necessary condition is also sufficient. The van Kampen theorem for $\pi_\bullet(X, A)$ says that if A meets all connected components of U, V and $U \cap V$ then $\pi_\bullet(X, A)$ is again the pushout of the other groupoids.

Actual answer starts here

Now, let's see what all of this says in the case of $X = S^1$, a circle. We clearly want U and V to be two open arcs, overlapping a little bit at each of their ends. Then $U \cap V$ is two separate small arcs, which is exactly why the standard van Kampen is no use here: the intersection is disconnected. So, we let $A = \{p, q\}$ be just two points: p in one small arc and q in the other.

What is $\pi_\bullet(U, A)$? U is an arc, which is contractible. There is clearly a morphism $p \to q$, and a morphism $q \to p$, and they are inverses of each other, since any path $p \to q \to p$ can be shrunk to p alone. So $\pi_\bullet(U, A)$ is simply a category with two objects p, q and a single isomorphism $i_U : p \to q$ (with an inverse ${i_U}^{-1} : q \to p$).

The same goes for $\pi_\bullet(V, A)$. It's the same abstract groupoid, but we'll call its isomorphism $i_V : p \to q$, since in a minute we will need them to tell them apart.

always (think of a constant path at p, and another at q), but there's nothing that leads from p to q so there are no morphisms at all. We have the "discrete category" on two elements.

What is the pushout of these two groupoids, along the discrete groupoid? We're going to still have two objects p, q, and now there are two isomorphisms $i_U, i_V : p \to q$, one coming from $\pi_\bullet(U, A)$ and the other from $\pi_\bullet(V, A)$. As a result, you can start at p, use i_U to get to q, and use ${i_V}^{-1}$ to come back. If you use ${i_U}^{-1}$ you get the identity, but with ${i_V}^{-1}$ you don't since there's nothing in the pushout diagram that forces ${i_V}^{-1} \circ i_U = \mathrm{id}_p$. (This is a little hand-wavy, but it can be formalized. It's the fact that a pushout is a *free* product, except for the constraints imposed by the amalgamated part).

In fact, any morphism $p \to p$ in the pushout is a composition of various i_U and ${i_V}^{-1}$'s, or in other words it's a certain power $({i_V}^{-1} \circ i_U)^n$ for some integer n. (There's no point in ever using ${i_U}^{-1} \circ i_U$, for example, since that's the trivial morphism). Call such power ϕ_n, and observe that composing ϕ_n and ϕ_m simply yields ϕ_{n+m}, so the part of the groupoid which starts and ends at p is just the group $\mathbb{Z}$, and this is of course the fundamental group $\pi_1(S_1, p)$.

Usually, the group $\pi_1(S^1, p)$ is shown to be $\mathbb{Z}$ using a covering map from $\mathbb{R}$ and deck transformations. Here, we did not appeal to covering spaces at all, but we did need to understand the algebraic structure of the two-pointed groupoid with two distinct isomorphisms. It's a much more "algebraic" proof, which (mini-)demonstrates the power of category theory in algebraic topology.