

# It Takes Two: A Peer-Prediction Solution for Blockchain Verifier’s Dilemma

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The security of blockchain systems is fundamentally based on the decentralized consensus in which the majority of parties behave honestly, and the content verification process is essential to maintaining the robustness of blockchain systems. However, the phenomenon that a secure blockchain system with few or no cheaters could not provide sufficient incentive for verifiers to honestly perform the costly verification, referred to as the Verifier’s Dilemma, could incentivize lazy reporting and severely undermine the fundamental security of blockchain systems. While existing works have attempted to insert deliberate errors to disincentivize lazy verification, the decentralized environment renders it impossible to judge the correctness of verification or detect malicious verifiers directly without additional layers of procedures, e.g., reputation systems or additional committee voting.

In this paper, we initiate the research with the development of a *Byzantine-robust peer prediction* framework towards the design of one-phase Bayesian truthful mechanisms for the *decentralized verification games* among multiple verifiers, incentivizing all verifiers to perform honest verification without access to the ground truth even in the presence of noisy observations in the verification process. Furthermore, we optimize our mechanism to realize provable robustness against collusions and other malicious behavior from the verifiers, and also show its resilience to inaccurate priors and beliefs. With the theoretically guaranteed robust incentive properties of our mechanism, our study provides a framework of incentive design for decentralized verification protocols that enhances the security and robustness of the blockchain and potentially other decentralized systems.

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## 1 INTRODUCTION

Blockchain, with prevailing examples as Bitcoin [Nakamoto, 2008] and Ethereum [Buterin et al., 2014], is an emerging technology that maintains decentralized consensus via a distributed ledger that utilizes cryptographic techniques to achieve trust and security. To ensure that the blockchain runs correctly, certain contents on the blockchain need to be “proven” and verified by other parties. The “proofs” can either be developed cryptographically, e.g., via zero-knowledge proofs [Goldreich and Oren, 1994], or in a game-theoretical way, e.g., via *verification games*. While the method of zero-knowledge proofs achieves cryptographic security, its high computational overhead limits its application in large-scale tasks, e.g. machine learning [Chen et al., 2024].

In several previous studies on blockchain verification (e.g. Ileri et al. [2016], Jia et al. [2021]), it is typical that the system prevents the provers from cheating while assuming the credibility of verifiers. However, this is not necessarily true in a fully decentralized and permissionless blockchain system, especially in verification games in which incentive compatibility is not theoretically guaranteed.

When considering game-theoretic ways to incentivize verifiers to verify honestly, a straightforward way is to introduce a penalty (“slash”) when a verifier is detected to be dishonest, as in Ethereum [Cassez et al., 2022], opML [Conway et al., 2024], etc. Nevertheless, a simple “slashing” mechanism, in which a verifier is penalized for a constant amount when the verifier is detected to have passed the verification of an invalid proof, has a two-fold challenge as follows. The first one is that in a decentralized environment, there is not a trusted “root” that makes the decision, so the decision needs to be performed in a voting protocol, but it is hard to ensure the honesty of voters with theoretical guarantees; the second one is the Verifier’s Dilemma [Luu et al., 2015]:

### **Verifier’s Dilemma:**

- If a mechanism is (incentive-)secure against strategic provers, then no (rational) provers would cheat.
- If no prover would cheat and the verification has a non-zero computational cost, then the verifier’s optimal strategy is to lazily report “Success” without actual verification.
- If all verifiers are rational and would not actually verify, then the security properties no longer hold.

Formally, the Verifier’s Dilemma can be formulated as the following theorem. The proof is deferred to Appendix D.1.

**THEOREM 1.1 (VERIFIER’S DILEMMA).** *In a verification game in which*

- *A verifier’s report is binary, e.g. “Success” or “Fail”;*
- *The verification result of an honest proof is always “Success”;*
- *Honest verification has a strictly positive cost,*

*It is impossible to design a verification mechanism with a pure-strategy Nash equilibrium that the prover and verifier(s) simultaneously act honestly.*

The Verifier’s Dilemma occurs to incentivize lazy strategies in the scenario of  $\epsilon$ , the probability of cheating, tends to zero, so that any bounded reward for catching a cheat cannot uniformly compensate for the verification cost in expectation. Hence, a conventional penalty-based approach can only lower, but not eliminate, the rate of cheating in the pool of proofs. For example, the study of [Conway et al., 2024] shows that the opML mechanism achieves a mixed-strategy Nash equilibrium that the prover has a constant probability to cheat and the verifier has a constant probability to honestly verify.

To achieve an honest pure-strategy Nash equilibrium, a natural idea is to introduce “attention challenges” that contains extra information or deliberate errors, e.g. insert deliberate objects (which can be valid or invalid), as so-called “flags” or to incentivize verifiers to find and report, as in the works of [Luu et al., 2015, Reynouard et al., 2024, Teutsch and Reitwießner, 2024, Zhao et al., 2024]. Besides, the flags may also appear from the internal uncertainties of the verification process, e.g., in ML inference [Nodehi et al., 2024]. The simple flag-insertion mechanism works well for “honest-but-lazy” verifiers, as long as the expected reward of finding the flags can cover the verification cost. However, in the scenario where the verifiers may also be *malicious*, the validity of verification results are also costly to verify, e.g., in Proof-of-Learning [Jia et al., 2021, Zhao et al., 2024], we generally need a sort of *authentication* or higher-level verifiers, or need additional phases of “committee voting” [Zhang et al., 2024] to decide on their validity and prevent dishonest verifiers from harming the system. As long as we want to design a fully decentralized system with no trusted “root” parties, there is no way to access the “ground truth” and the validity and payments can only be decided by consensus among the voting parties.

In the field of computational economics, the concept of *peer prediction* [Miller et al., 2005] refers to a wide scope of mechanisms to elicit honest information when direct verification is unavailable, which is widely adopted in the applications of dataset acquisition [Chen et al., 2020], peer grading [Dasgupta and Ghosh, 2013], etc. Particularly, a recent work [Wang et al., 2023] also utilized the peer prediction approach for the voting consensus on blockchain platforms. A widely used practice for peer prediction is to design a mutual-information-based payment rule [Kong and Schoenebeck, 2019] and leverage the data processing inequality to make sure that any manipulation cannot increase the expected utility for an agent. While these types of scoring rules are convenient for usage in scenarios with known prior, this family of peer prediction mechanisms is still unable to directly resolve the Verifier’s Dilemma because the infinitesimal cheating probability will make the logarithmic scoring rule diverge. In the practice of blockchain systems, as the Verifier’s Dilemma leads to an arbitrarily low cheating probability, its empirical value becomes difficult to estimate and the scoring rule’s sensitivity to small probabilities will severely undermine the robustness of the verification mechanism.

In contrast to most peer prediction mechanisms that guarantee Bayesian Nash equilibria requiring a known prior, a recent work [Kong, 2023] utilizes determinant-based mutual information (DMI) to design a dominantly-truthful finite-task peer prediction mechanism which has the potential to tackle with the scenario of unbalanced and unknown priors. However, although it does not need prior estimations of  $\epsilon$ , its need and assumption of multiple homogeneous tasks, i.e. at least 4 tasks sharing a common  $\epsilon$ , may still lead to complication in the analysis as it has assumptions on and is still sensitive to the latent value of  $\epsilon$ . Furthermore, the lack of permutation-proofness also renders the DMI mechanism impractical for the verification games: a verifier who flips all her reports, i.e., reporting “Dishonest” when observing “Honest” and vice versa, is genuinely malicious but still gets optimal utility. While we acknowledge the potential of the DMI mechanism to be adapted as a solution to the problem, possibly after well-designed modifications, we would still be motivated to design a conceptually “more robust” mechanism that is inherently permutation-proof and insensitive to  $\epsilon$  as long as it is small enough.

In a game-theoretic interpretation, it is worth noting that the longest-chain rule proposed by Nakamoto [2008] essentially implements an *implicit* peer prediction mechanism for decentralized consensus. In the presence of disagreement (“forks”), miners follow and *vote* on the branch that they believe would be the main chain, and get block rewards if and only if they are agreed by the most resource (work, stake, etc.) While the protocol is designed for incentivizing honest actions via consensus, a miner’s reward is indeed solely depending on other miners’ votes without reference to the “ground truth” that a previous block is honest or not. Through the testimony of history,

the success of longest-chain rule for blockchain has shown the practical value of peer prediction mechanisms in decentralized consensus. Besides, several current studies on incentivized consensus relying on votes to decide on the results and rewards for voters (e.g., Nassirzadeh et al. [2024], Zhang et al. [2024]), also represent the high-level principles of peer prediction; while they are likely to work in practice, these studies do not delve into rigorous economic analyses, or rely on ad-hoc assumptions (e.g., “the majority is the ground truth”), which may lack theoretical guarantees especially for the scenario when the verification process is stochastic, e.g., for ML training and inference, and/or the prior distributions of types are heavily unbalanced. Hence, the study on the framework of peer prediction mechanisms, beyond the current paradigm of simple majority voting, is indeed essential to the theoretical reliability of incentivized consensus mechanisms for decentralized systems. On the other hand, the closed-form incentive guarantees can in turn back up the canonical assumption on honest majority via the rationality of agents.

Particularly, while the longest-chain rule works as promised for both traditional Proof-of-Work (PoW) and Proof-of-Stake (PoS) mechanisms, as in Bitcoin and Ethereum respectively, it is based on the fact of low observation costs in the verification of PoW and PoS. In the scenarios in which verification is costly, especially for the current demand for verification of machine learning models [Conway et al., 2024, Nodehi et al., 2024, Zhao et al., 2024], the Verifier’s Dilemma motivates us to design a more robust **explicit** peer prediction mechanism for the design of reliable decentralized consensus mechanisms.

Compared to traditional peer prediction mechanisms which mainly consider incentive compatibility (IC), in our setting, we are additionally required to explicitly consider individual rationality (IR) so that an honest verifier would get an expected reward no less than the verification cost, and the verification cost may be dependent on the verification result. On the other hand, due to the permissionless property of the blockchain system, we also need a no-free-lunch (NFL) property, which is a stronger form of *informed truthfulness* [Shnayder et al., 2016], that any free-riding verifier who does not actually perform the verification cannot get a positive expected reward from the mechanism.

### 1.1 Our Contribution

In this research, we develop a theoretical framework with modeling of *decentralized verification game (DVG)*, and initiate the study that combines the ideas of *flags* and peer prediction into our proposed mechanism, named capture-the-flag peer prediction (CTF-PP), which only needs one phase in its procedure and simultaneously satisfies the following properties:

- Interim *unique* incentive compatibility (interim UniIC): Given all other verifiers act honestly, a verifier, after performing the verification, maximizes her expected utility when she reports honestly. Furthermore, if she reports a different type from her observation, her expected utility is non-positive.
- Interim individual rationality (interim IR): Given all other verifiers act honestly, a verifier, after performing the verification, gets a non-negative expected net utility, when she acts honestly.
- Interim no-free-lunch (interim NFL): Given all other verifiers act honestly, a verifier cannot get a positive expected utility via any *uninformed strategy* [Shnayder et al., 2016], i.e., without doing the verification.

Combining all the desirable properties, we characterize the notion of *incentive alignment* ( $\delta$ -IA) as a general guideline for peer prediction mechanisms in decentralized verification games, which depicts the property that any pure strategy gains a positive interim utility if *and only if* it is honest, with a margin of  $\delta$  (details in Section 3.3). With this stronger incentive guarantee, we can

ensure that the peer prediction mechanism works as desired for the tricky setting of decentralized environments in blockchains, with additional guarantee to *disincentivize free-riding behavior* in blockchain systems.

Technically, we first consider a simple case of 2-verifier DVG, modeling the design of a  $\delta$ -IA mechanism as a linear program, and show the general feasibility of this problem for all non-degenerate cases (Section 4). Furthermore, we extend our methodology for general  $n$ -verifier DVG (Section 5), and derive the Byzantine robustness (e.g., collusion-proofness) properties of our methods, developing a general guideline of  $(\delta, N)$ -compactness for robust peer prediction mechanisms, showing the *compactness* criteria that a scoring rule with *wide incentive margins and low rewards/penalties* yields good Byzantine robustness (See in Section 6). Then, we leverage the *Byzantine reduction* technique to connect the resilience against inaccurate prior distributions and beliefs with the resilience against malicious users, showing the generality of our Byzantine robustness notion for peer prediction mechanisms (See in Section 7).

Additionally, by computing a numerical solution to a typical set of parameters that applies to the model in the Incentive-Secure PoL [Zhao et al., 2024], we show the practical applicability of our approach as a theoretically guaranteed solution that breaks the critical barrier of Verifier’s Dilemma in the design of blockchain (and more generally, decentralized) verification mechanisms (Appendix C).

## 2 BACKGROUND AND RELATED WORK

Since the emergence of Bitcoin [Nakamoto, 2008], the concept of blockchain is inherently designed as an unalterable distributed ledger that maintains trustworthiness via decentralized consensus. The blockchain can be modeled as a growing linked list stored by decentralized nodes, in which each *block* contains its contents that consists of *transactions*, a hash reference of its previous block, and a certificate (e.g., PoW, PoS and etc.) that controls the access to the block. Conceptually, when a block producer, also called a *miner*, would like to pack and propose new contents on the blockchain, she needs to attach the block to a previous block, and pay certain efforts to create the certificate that gains her the access to produce the block. When a miner attach a new block to an existing block, she is supposed to have also *verified* the validity of the previous block. This process also makes the previous block unalterable, because the new block would be stored and witnessed by all the nodes of the network.

Nevertheless, in real-world blockchain ecosystems, the agents that are responsible to verify the contents, usually called as *verifiers* or validators, may be economically rational or selfish. In this context, Verifier’s Dilemma [Luu et al., 2015] occurs as a phenomenon that in a decentralized verification protocol, the verifiers may behave lazily when there are no sufficient rewards to incentivize honest verification, especially when all or overwhelming majority of the contents are valid. For example, Cao et al. [2023] propose an attack that leverages the Verifier’s Dilemma to double spend in Bitcoin. Besides, the studies of Alharby et al. [2020], Smuseva et al. [2022] make extensive analyses on Ethereum and the results show that Ethereum verifiers are frequently incentivized not to verify the contents while they are supposed to, rendering the Ethereum protocol economically vulnerable.

That said, one may argue that in the original design of Bitcoin or Ethereum, the verification of a block has negligible costs as it only needs the miner to check if all the transactions and the Proof-of-Work (PoW) or Proof-of-Stake (PoS) is valid. Since invalid blocks can be detected easily, miners might practically decide to behave honestly even if it is (slightly) irrational. Nevertheless, the development of the blockchain technology generalized the usage of blockchain system from an unalterable ledger of monetary transactions to a general decentralized platform that guarantees the integrity of diverse contents, e.g., smart contracts [Ante, 2021, Khan et al., 2021, Wang et al., 2019].

Furthermore, with the recent rapid development of artificial intelligence (AI) technologies and the demands of trustworthy AI models, researchers are actively exploring to establish blockchain-based platforms that verify the computation of machine learning [Chen et al., 2024, Conway et al., 2024, Jia et al., 2021, Nodehi et al., 2024, Zhao et al., 2024], which brings new motivations for blockchain studies as a novel paradigm of decentralized trustworthy AI.

Unlike hash puzzles in the Bitcoin PoW, the verification of such complicated contents can be potentially costly. Particularly, in the context of ML verification, Fang et al. [2023] show that efficient byzantine-secure verification of stochastic gradient descent (SGD) computation reduces to fundamentally hard open problems in deep learning theories. Even though the study of Zhao et al. [2024] achieves substantially lower verification overheads via the relaxation to incentive-security, the verification protocol still needs to reproduce the training process of at least  $\Theta(1)$  epochs which has non-negligible computational costs. Consequently, recent studies typically resort to weaker incentive properties for verification games. For example, the recent proposal of opML [Conway et al., 2024], a protocol that designs for trustworthy ML inference on blockchain, can only reach a mixed-strategy Nash equilibrium that a (small) constant fraction of provers and verifiers behave dishonestly.

Since the Verifier’s Dilemma, unless suitably addressed, appears as a fundamental vulnerability in the incentive structure of blockchains that may severely undermine the reliability of blockchain systems, the studies of Teutsch and Reitwießner [2024], Zhang et al. [2024] work on this issue via introducing deliberate invalid objects as *attention challenges* that incentivize verification. Nevertheless, their protocols are multi-phased as they need additional dispute processes and are potentially restricted to particular applications. In our work, we are motivated to design a one-phase general-purposed solution to the Verifier’s Dilemma with theoretical incentive guarantees, expecting to resolve the critical incentive issue in decentralized verification games in a reliable and efficient paradigm.

### 3 BASIC MODELING OF DECENTRALIZED VERIFICATION GAMES

To initiate the study, we first consider the modeling of decentralized verification games (DVG). In a  $n$ -verifier DVG, there are  $n$  homogeneous players (verifiers)  $i = 1, \dots, n$  independently verifying an on-chain *proof*, and we use the terms “player” and “verifier” interchangeably.

The proof has an underlying ground-truth type  $\theta \in S$ , which can be either “Honest” ( $\theta = 0$ ), “Flag  $j$ ” ( $\theta = F_j$ , for  $j = 1, 2, \dots, m$ ), or “Dishonest” ( $\theta = 1$ ). We define  $S_* = 0, F_1, \dots, F_m$  as the set of all non-dishonest types. The actual type  $\theta$  is unknown to both the verifiers and the system. While the observations  $\{X_i\}$  can potentially be noisy, every verifier’s observation, when they actively verify the proof, is *i.i.d.* conditioned on  $\theta$ . Since the system can insert flags to maintain a pre-set flag rate (as in [Teutsch and Reitwießner, 2024, Zhao et al., 2024], etc.) that robustly incentivizes verification when no cheater occurs, we have *principal* prior probabilities  $P(\theta = F_i) = p_{F_i}$  and  $P(\theta = 0) = p_0$  as publicly known information. Throughout the paper, the term “principal” refers exclusively to this cheater-free scenario.

However, in reality, the prior distribution slightly deviates from the principal scenario as there is a small but unknown probability  $\epsilon \in [0, \epsilon_0]$  that the proof is dishonest, i.e.,  $P(\theta = 1) = \epsilon$ , with a known upper bound  $\epsilon_0$ . We assume that the appearance of dishonest proofs may alter the probabilities of types in  $S_*$  in an arbitrary way. Hence, for any  $s \in S_*$  we have  $P(\theta = s) = p_s - \epsilon_s$ , in which  $\sum_{s \in S_*} \epsilon_s = \epsilon$  but the exact values of  $\{\epsilon_s\}$  remain unknown.

Similar to [Zhao et al., 2024], we begin our discussion with the verification process in a *lossy-channel* model as follows, and then study the general case (as shown in Theorem 4.1). When verifier  $i$  verifies the proof, the distribution of the observation  $X_i$  is dependent on  $\theta$  in this way:

- **Completeness:** A honest proof is always observed as honest, i.e.  $P(X_i = 0|\theta = 0) = 1$ .
- **Probabilistic soundness:** A dishonest proof can be observed as any type, but the probabilities are known to the public, and the probability of correct detection at least  $\kappa > 0$ , i.e.  $P(X_i = 1|\theta = 1) \geq \kappa$ .
- **Benign flags:** A flag  $F_j$  can be detected with known probability  $\mu_j > 0$  or missed and observed as honest, but will never be observed as dishonest or other flags, i.e.  $P(X_i = F_j|\theta = F_j) = \mu_j$  and  $P(X_i = 0|\theta = F_j) = 1 - \mu_j$ .

For each verifier  $i$ , she first makes a decision to follow one of the following strategies:

- (1) Informed (active) strategy: Actively verifies the proof and gets the observation, which gains her access to  $X_i$  but incurs a publicly-known cost  $c(X_i) \geq 0$  which can depend on  $X_i$ .
- (2) Uninformed (lazy) strategy: Does not verify and has no access to  $X_i$ . For convenience of expression, we can denote  $X_i = \perp$  in this case, and  $c(\perp) = 0$ .

Hence, verifier  $i$ 's Bayesian belief  $\mathcal{B}$  on  $Z_{-i}$  is the conditional distribution of  $P(X_{-i}|X_i)$  for the informed strategy, or the marginal distribution  $P(X_{-i}|\perp) = P(X_{-i})$  for the uninformed strategy. Here,  $i$ 's belief of the cheating probability can be an arbitrary  $\epsilon^{(i)} \in [0, \epsilon_0]$  that can be different from the actual  $\epsilon$ , with arbitrary  $\{\epsilon_s^{(i)}\}$  such that  $\sum_{s \in S_s} \epsilon_s^{(i)} = \epsilon^{(i)}$ , and we desire to design a mechanism that uniformly satisfies the incentive guarantees for arbitrary  $\{\epsilon_s^{(i)}\}$ . In this context, we define the *principal belief* as the Bayesian belief given  $\forall \epsilon_s^{(i)} = 0$ , i.e.,  $\epsilon_0 = 0$ .

Then, verifier  $i$  reports a  $Z_i$  that maximizes  $\mathbb{E}_{Z_{-i} \sim \mathcal{B}}[R_i(Z_i, Z_{-i})]$  in which  $\mathcal{B}$  is her belief of  $Z_{-i}$ , and claims that  $Z_i$  is her observation. After each verifier  $i$  independently reports  $Z_i$  without seeing  $Z_{-i}$ <sup>1</sup>, the system has the information of  $Z_1, \dots, Z_n$ , but not  $\theta$ , and rewards each prover  $i$  based on a *scoring rule* from the reports, denoted as  $R_i(Z_i, Z_{-i})$ . Then, verifier  $i$ 's net utility is  $R_i(Z_i, Z_{-i}) - c(X_i)$ .

**Formal characterization of strategies and utilities.** For the action of any verifier  $i$ , she may first decide to be active (informed) or lazy (uninformed). If she chooses to be lazy, then she can choose a distribution  $D \in \Delta(S)$ , in which  $\Delta(S)$  is the set of all convex combinations of elements in  $S$ , and report  $Z_i \sim D$ . If she chooses to be active, she may observe  $X_i$  and report according to a respective distribution corresponding to each  $X_i \in S$ , so any informed strategy can be characterized by a mapping  $D(\cdot) : S \rightarrow \Delta(S)$ . Furthermore, the verifier can also randomly decide to be active or lazy. Hence, we formally characterize the verifiers' strategies as:

DEFINITION 3.1. *The strategy space of any verifier can be characterized as*

$$\Omega = \Omega(S) = \Delta(\{\Delta(S), \Delta(S)^S\}),$$

and we denote the strategy of verifier  $i$  as  $s_i$ .

For example,  $s_i \in \Delta(S)$  if  $i$  chooses an uninformed strategy, and  $s_i \in \Delta(S)^S$  if  $i$  chooses an informed strategy. Otherwise, if  $s_i$  is a random choice between informed and uninformed strategies, we can represent  $s_i$  with the 3-tuple  $(\lambda, \mu, \alpha)$ , denoted as

$$s_i \triangleq (\lambda, \mu, \alpha),$$

in which  $\lambda \in \Delta(S)$ ,  $\mu \in \Delta(S)^S$ ,  $\alpha \in (0, 1)$  and  $s_i = \alpha \cdot \lambda + (1 - \alpha) \cdot \mu$ .

For any verifier, she would maximize her expected utility based on her *belief* on the reports of other verifiers. The belief profile of verifier  $i$  can be characterized as  $\mathbb{B}_i : (S \cup \{\perp\}) \rightarrow \Delta(S^{n-1})$  which maps her observation to a joint distribution of other verifiers' reports. For *Bayesian* verifiers,

<sup>1</sup>This can be implemented via a cryptographic commitment scheme.

they always set their belief as  $\mathbb{B}_i(X_i) = P(\mathbf{X}_{-i}|X_i)$ . Then, with regard to belief  $\mathbb{B}_i$ , we can define the interim utility  $u_i(s_i; \mathbb{B}_i)$  as:

$$u_i(s_i; \mathbb{B}_i) = \begin{cases} \mathbb{E}_{Z_i \sim s_i, \mathbf{Z}_{-i} \sim \mathbb{B}_i(\perp)} [R_i(Z_i, \mathbf{Z}_{-i})], & s_i \in \Delta(S); \\ \mathbb{E}_{X_i \sim P(X_i)} [\mathbb{E}_{Z_i \sim s_i(X_i), \mathbf{Z}_{-i} \sim \mathbb{B}_i(X_i)} [R_i(Z_i, \mathbf{Z}_{-i}) - c(X_i)]], & s_i \in \Delta(S)^S; \\ \alpha \cdot u_i(\lambda; \mathbb{B}_i) + (1 - \alpha) \cdot u_i(\mu; \mathbb{B}_i), & s_i \triangleq (\lambda, \mu, \alpha). \end{cases} \quad (1)$$

In most parts of this paper (except for the discussion of Byzantine robustness and Appendix A), we always assume that all verifiers have principal Bayesian beliefs  $\mathbb{B}_i(X_i) = P(\mathbf{X}_{-i}|X_i)$  assuming  $\epsilon = 0$  (we justify using  $\epsilon = 0$  for actual small  $\epsilon$  in the robustness analysis of Section 7). Hence, we simplify the notation as  $u_i(s_i)$ . We call a strategy *honest* or *truthful* if and only if  $s_i = I$  is (induced by) the identity map on  $S$ , i.e.  $s \in \Delta(S)^S$  and  $s(X_i) \equiv X_i$ .

**Pure & mixed strategies.** In the strategy space  $\Omega$ , we define the subset  $\Omega_d = S \cup S^S$  as the space of *pure* strategies, in which the verifier deterministically decides to be informed or uninformed, and reacts deterministically to her observation. From the linearity of the utility function, we can see that the optimal utility is always realized by a pure strategy for any fixed belief, and hence we mainly consider pure strategies throughout this paper.

### 3.1 IR and UniIC Constraints for Informed Verifiers

The IR constraint requires that given the verifier  $i$  observes  $X_i$ , truthfully reporting it gains her an expected reward no less than  $c(X_i)$ . Since  $X_i$  and  $\mathbf{X}_{-i}$  are independent conditioned on  $\theta$ , define  $r_{X_i}(Z_i)$  as verifier  $i$ 's expected reward of reporting  $Z_i$  conditioned on observing  $X_i$ , then  $r_{X_i}(Z_i)$  can be computed as

$$r_{X_i}(Z_i) = \sum_{\mathbf{X}_{-i} \in S} R_i(Z_i, \mathbf{X}_{-i}) P(\mathbf{X}_{-i}|X_i) \quad (2)$$

$$= \sum_{\mathbf{X}_{-i} \in S} R_i(Z_i, \mathbf{X}_{-i}) \frac{P(X_i, \mathbf{X}_{-i})}{P(X_i)} \quad (3)$$

$$= \sum_{\mathbf{X}_{-i} \in S} R_i(Z_i, \mathbf{X}_{-i}) \frac{\sum_{\theta \in S} P(\theta) P(X_i|\theta) P(\mathbf{X}_{-i}|\theta)}{\sum_{\theta \in S} P(\theta) P(X_i|\theta)}. \quad (4)$$

While the probabilities are dependent on  $\epsilon$ , as long as  $\epsilon$  is small enough, for  $X_i \in S_*$ , the  $r_{X_i}$  is a continuous function w.r.t.  $\epsilon$ , so the IR constraints on  $X_i \in S_*$  can be implied by

$$\sum_{\mathbf{X}_{-i} \in S} R_i(X_i, \mathbf{X}_{-i}) \frac{\sum_{\theta \in S} \tilde{P}(\theta) \tilde{P}(X_i|\theta) \tilde{P}(\mathbf{X}_{-i}|\theta)}{\sum_{\theta \in S} \tilde{P}(\theta) \tilde{P}(X_i|\theta)} \geq c(X_i) + \delta, \quad \forall X_i \in S_*, \quad (5)$$

in which  $\tilde{P}$  denotes the *principal* probabilities assuming  $\epsilon = 0$ , and the margin  $\delta$  is introduced to ensure the incentive guarantees even if the actual priors slightly deviate from the principal priors, so that for any  $\delta > 0$ , the constraints hold robustly for some  $\epsilon_0 > 0$ . The condition is also necessary when  $\delta = 0$ . In the rest of this paper, we say a condition is “sufficient and almost necessary” when it is sufficient with a  $\epsilon_0 > 0$  depending on  $\delta$ , and when we set  $\delta = 0$ , it becomes a necessary condition. We defer the rigorous justification of the “sufficiency” and quantitative analysis on the relations between  $\delta$  and  $\epsilon_0$  to Section 7.

For the case of  $X_i = 1$ , from the lossy-channel model, we know that  $\theta = 1$ . Therefore, we have



$$r_1(1) = \sum_{\mathbf{X}_{-i} \in S} R_i(1, \mathbf{X}_{-i}) P(\mathbf{X}_{-i} | \theta = 1) \geq c(1) + \delta. \quad (6)$$

So Eqs. (5-6) are sufficient and almost necessary conditions that a CTF-PP mechanism is IR.

For the IC constraint, we need and only need  $r_{X_i}(X_i) = \max_{Z_i \in S} \{r_{X_i}(Z_i)\}$ . With similar arguments, we can also develop sufficient and almost necessary conditions that a CTF-PP mechanism is IC. Actually, given that the IR is satisfied we can define a stronger notion of *Uniquely-IC* as follows:

- Uniquely IC (UniIC): In addition to the IC requirement, given all other verifiers act honestly and a verifier actively performed the verification, then she gets a negative expected utility when she reports any type different from her observation.

Besides conventional IC notions, the UniIC requirement additionally rules out the possibility that a dishonest verifier cheats the system without losing money. Assuming that the IR constraints are already satisfied, the UniIC constraints can be formulated as:

$$r_{X_i}(Z_i) \leq c(X_i) - \delta, \quad \forall X_i \in S, Z_i \neq X_i. \quad (7)$$

### 3.2 NFL Constraints for Uninformed Verifiers

We assume verifiers other than  $i$  are honest, i.e. they all decide on the informed strategy and  $\mathbf{Z}_{-i} = \mathbf{X}_{-i}$ . If verifier  $i$  performs the uninformed strategy, she has no information on  $\theta$  and her strategy can only be reporting any type in  $S = \{0, F_1, \dots, F_m, 1\}$ , or any convex combination of them. Hence,  $i$ 's utility when she lazily reports  $Z_i$  is denoted as:

$$r_{\perp}(Z_i) = \sum_{\mathbf{X}_{-i} \in S} R_i(Z_i, \mathbf{X}_{-i}) P(\mathbf{X}_{-i}). \quad (8)$$

From the NFL requirement and assuming small  $\epsilon \leq \epsilon_0$ , a sufficient and almost necessary condition is the following linear constraints

$$\sum_{\mathbf{X}_{-i} \in S} R_i(Z_i, \mathbf{X}_{-i}) P(\mathbf{X}_{-i}) \leq -\delta, \quad \forall Z_i \in S. \quad (9)$$

### 3.3 Incentive Alignment for Decentralized Verification Games

From the discussion in Section 3.1-3.2, we would like to design a mechanism for the decentralized verification game that simultaneously satisfies IR, UniIC, and NFL constraints. Combining the derivations above, we can summarize the sufficient and almost necessary conditions that satisfy all constraints above. Hence, we define the notion of *incentive alignment* ( $\delta$ -IA) as follows:

**DEFINITION 3.2 (INCENTIVE ALIGNMENT).** A CTF-PP mechanism is  $\delta$ -incentive-aligned ( $\delta$ -IA) if and only if for any verifier  $i$  and pure strategy  $s_i \in \Omega_d$ ,

$$u_i(s_i) \begin{cases} \geq \delta, & s_i = I; \\ \leq -\delta, & s_i \neq I. \end{cases}$$

Equivalently,

$$r_{X_i}(Z_i) - c(X_i) \begin{cases} \geq \delta, & Z_i = X_i; \\ \leq -\delta, & Z_i \neq X_i. \end{cases}$$

Here, the  $\delta$ -IA is a sufficient and almost necessary condition that IR, UniIC and NFL are simultaneously satisfied.

#### 4 THEORETICAL GUARANTEE FOR 2-VERIFIER DVG: LP MODELING AND FEASIBILITY

In this section, we show a basic result on the existence of incentive aligned CTF-PP mechanisms for any 2-verifier DVG that satisfies mild conditions.

Assume  $\epsilon = 0$ , and define the *principal* belief matrix  $B : S^2 \rightarrow \mathbb{R}$  as  $B_{xy} = P(X_{-i} = y | X_i = x)$ .<sup>2</sup> Besides, we define  $B_{\perp}$  as the blind-belief (row) vector as  $B_{\perp y} = P(X_{-i} = y)$  that describes the belief of verifier  $i$  when she does not verify the proof. Then, we can formulate the design of a  $\delta$ -IA CTF-PP mechanism as a linear programming (LP) problem.

We define decision variable as the scoring matrix  $T : S^2 \rightarrow \mathbb{R}$  with  $T_{xy} = R_i(x, y)$ , and denote

$$W = BT'. \quad (10)$$

Then  $W_{xy} = r_x(y)$ , which is the expected reward verifier  $i$  gets from the mechanism when she observes  $x$  and reports  $y$ . The IR and UniIC conditions are equivalent to the following:

$$W_{xx} \geq c(x) + \delta, \quad \forall x \in S; \quad (11)$$

$$W_{xy} \leq c(x) - \delta, \quad \forall x \in S, \quad y \in S - \{x\}. \quad (12)$$

Similarly, we denote

$$W_{\perp} = B_{\perp} T', \quad (13)$$

then  $W_{\perp y} = r_{\perp}(y)$  is the expected reward verifier  $i$  gets when she does not verify and lazily reports  $y$ . Then the NFL conditions are equivalent to the following:

$$W_{\perp} \leq -\delta. \quad (14)$$

Hence, we only need to find a feasible solution of the linear system (10-14), i.e., solve the following linear program:

$$\begin{aligned} LP_0 : \quad & \text{minimize} \quad 0 \\ & \text{s.t.} \quad (10-14). \end{aligned}$$

Here we propose our main theorem that shows the feasibility of  $LP_0$ , with the proof deferred to Appendix D.2:

**THEOREM 4.1 (MAIN THEOREM).** *If  $B$  is invertible, and  $P(X_i = y | \theta = 1) = 0$  for any  $y \neq 1$  (i.e., a non-cheating proof is never observed as a cheat), then for any  $\delta \geq 0$ , we can find a  $\delta$ -IA mechanism for the 2-verifier DVG as a feasible solution of  $LP_0$ .*

*For some  $\epsilon_0 > 0$ , the mechanism is IR, NFL and UniIC for any  $\epsilon \in [0, \epsilon_0]$ .*

Particularly, we show that our method always works for the DVG with the lossy-channel model as defined in Section 3 with the following proposition. The proof is deferred to Appendix D.3.

**PROPOSITION 4.2.** *In the lossy-channel model defined in Section 3, the principal belief matrix  $B$  is invertible.*

#### 5 GENERAL MECHANISM DESIGN FOR $n$ -VERIFIER DVG

In this section, we leverage the result for the 2-verifier DVG to design a CTF-PP mechanism for general  $n$ -verifier DVG, with additional robustness properties against collusion and malicious verifiers (to be shown in the next section).

Conceptually, in Section 4 we have shown that under mild assumptions there always exists an incentive-aligned mechanism for any 2-verifier DVG. In this section, we invoke the 2-verifier mechanism as a building block and construct our mechanism for the general setting of  $n$  verifiers.

<sup>2</sup>  $B_{1y}$  can still be defined even if  $P(\theta = 1) = \epsilon = 0$ , e.g.  $\theta = 1$  when a zero-measure set is drawn.

### 5.1 Warmup: Random Pairing Mechanism

Imagine a simple scenario of 4 verifiers, in which each verifier  $i \in \{1, 2, 3, 4\}$  independently verifies the block content  $\theta$  with their observations  $\{X_i\}$  as modeled in Section 3. To incentivize all verifiers to behave honestly, a simple mechanism works as follows:

- (1) Divide the verifiers into two pairs  $\{1, 2\}$  and  $\{3, 4\}$ .
- (2) Run the 2-verifier mechanism in each pair.
- (3) Each verifier  $i$ 's report  $Z_i$  is compared to another verifier  $j$ 's report  $Z_j$ , getting an ex-post reward of  $T_{Z_i Z_j}$  as defined in Section 4.

In this simple mechanism, from the perspective of each verifier  $i$ , she is playing the 2-verifier DVG with another verifier, with mechanism described as the scoring matrix  $T$ . Hence, as long as the 2-verifier mechanism is  $\delta$ -IA, the proposed 4-verifier mechanism also satisfies the desired  $\delta$ -IA property.

In actual decentralized environments, as the verifiers are anonymous and can arrive in arbitrary orders, their indices are essentially randomly shuffled. Hence, the pairing result is random and uniformly distributed among the following 3 cases, each with  $\frac{1}{3}$  probability:

$$\{1, 2\}, \{3, 4\}; \quad \{1, 3\}, \{2, 4\}; \quad \{1, 4\}, \{2, 3\}.$$

As we can observe, each verifier  $i$  may be paired with any of the remaining 3 verifiers with equal probabilities. Hence among the distribution of random pairing, the verifier  $i$  gets an expected reward of

$$R_i(Z_i, \mathbf{Z}_{-i}) = \frac{1}{3} \sum_{j \neq i} T_{Z_i Z_j}. \quad (15)$$

In fact, the random pairing mechanism can be generalized for any even  $n$ . From the symmetry of the random shuffling, the verifier  $i$  will be paired with any verifier  $j \neq i$  with probability  $\frac{1}{n-1}$ , and  $i$ 's expected reward is:

$$R_i(Z_i, \mathbf{Z}_{-i}) = \frac{1}{n-1} \sum_{j \neq i} T_{Z_i Z_j}. \quad (16)$$

From the argument above, the random pairing mechanism for any even  $n$ , formulated as Eq. (16), is  $\delta$ -IA as long as the 2-verifier mechanism with the scoring rule described as  $T$  is  $\delta$ -IA. The formal results are shown in Theorem 5.1 of the next subsection.

### 5.2 Revisit of Random Pairing: Pairwise-Scoring Mechanism

While the random pairing mechanism only works for even  $n$ , in practical usage of decentralized consensus, e.g., the common practice of majority voting, the case of odd  $n$  is also (if not more) essential. On the other hand, by looking at Eq. (16), the scoring rule is also inherently compatible to odd  $n$ . Actually, there can be an alternative interpretation to the scoring rule:

- (1) For fixed verifier  $i$ , collect her report  $Z_i$  and all other verifiers' reports  $\{Z_j : j \neq i\}$ .
- (2) For each  $j \neq i$ , compare  $Z_i$  and  $Z_j$  and get a sub-score  $T_{Z_i Z_j}$  invoking the 2-verifier mechanism.
- (3) Take the average of all sub-scores as the final reward to  $i$ .

**Vectorized notation.** In the 2-verifier game, the  $(Z_i, Z_j)$ -th entry of the matrix  $T$ , denoted as  $T_{Z_i Z_j}$ , depicts the reward of verifier  $i$  when she reports  $Z_i$  while the other verifier  $j$  reports  $Z_j$ . With a slight abuse of notation, if we regard each type in  $S$  as a unit one-hot column vector in the corresponding dimension, we can get  $T_{Z_i Z_j} = Z_i' T Z_j$ . Hence, from the inspiration of Section 5.1, our pairwise-scoring mechanism can be formulated as:

$$R_i(Z_i, \mathbf{Z}_{-i}) = Z_i' T \overline{\mathbf{Z}_{-i}}, \quad (17)$$

in which we denote

$$\overline{Z_{-i}} = \frac{1}{n-1} \sum_{j \neq i} Z_j. \quad (18)$$

Then, we can show that the pairwise-scoring mechanism as described as Eq. (17) has equivalent incentive structures as the 2-verifier mechanism characterized as  $T$ . Formally, we have:

**THEOREM 5.1.** *If the scoring matrix  $T$  satisfies the  $\delta$ -IA property for the 2-verifier DVG, then the scoring rule as Eq. (17) also satisfies  $\delta$ -IA for the general  $n$ -verifier DVG.*

*Furthermore, if the 2-verifier mechanism characterized as  $T$  is IR, NFL and UniIC for any  $\epsilon \in [0, \epsilon_0]$ , then the  $n$ -verifier mechanism in Eq. (17) also satisfies the same properties.*

The proof of Theorem 5.1 is deferred to Appendix D.4.

## 6 BYZANTINE ROBUSTNESS VIA MARGIN OPTIMIZATION

In the context of (Bayesian) Nash equilibria, we aim to design mechanisms in which no agent may benefit from *individual* deviations. In other words, we guarantee that each verifier is incentivized to be honest given that *all* others are honest. However, in decentralized ecosystems like blockchains, this assumption may be too strong as there may exist *malicious* players who would deliberately attack the system, i.e., trying to undermine the robustness of the system at the risk of losing their own utilities. Furthermore, just like the widely studied topic of blockchain *transaction fee mechanisms* (See, e.g., Chen et al. [2022], Chung and Shi [2023], Roughgarden [2020, 2021], Wu et al. [2023b]), blockchain players may also potentially collude with each other or create fake identities (aka. *Sybil attack*) to increase their utilities.

[TODO: Discussion of Sybil attack?]

While it may be too strong to assume that all other players are individually rational, in the field of decentralized systems, the notion of *Byzantine robustness* (See, e.g., Chen et al. [2012], Wu et al. [2023a], Yin et al. [2018]), also called as *Byzantine fault tolerance* or *Byzantine resilience*, is widely studied as a desired property that the system works robustly as expected even if a (small) portion of the system does not work in the correct way. In the works of Schoenebeck et al. [2021], Wang et al. [2023], the existence of colluding players is also considered for peer prediction mechanisms. Particularly, Schoenebeck et al. [2021] consider the multi-task setting and tackle with it as a *robust learning* problem, and Wang et al. [2023] focus on the specific *leader election* problem [Gharehchopogh and Arjang, 2014] for blockchain consensus. In another perspective, Frongillo and Witkowski [2017] look into the scenario of peer prediction with inaccurate prior knowledge, and develop a margin optimization methodology to maximize the tolerance to inaccurate priors.

Inspired by these studies, we are motivated to design a general-purposed solution for decentralized consensus with stronger *incentive alignment* guarantees, with a optimization framework of Byzantine robustness. Following the framework in the study of Schoenebeck et al. [2021], the types of players can be generally classified into the following categories:

- (1) Altruistic ( $\mathcal{A}$ ): Acting honestly without consideration of utilities;
- (2) Selfish ( $\mathcal{S}$ ): Acting in the way that maximizes their own utilities;
- (3) Colluding ( $\mathcal{C}$ ): Conducting collusions with other players (within  $\mathcal{C}$ ) to maximize their joint utility;
- (4) Malicious ( $\mathcal{M}$ ): Acting in arbitrarily manners that may not optimize their utilities, without access of non-malicious players' information.

While traditional game theory primarily focuses on the behavior of  $\mathcal{A}$  and  $\mathcal{S}$  players,  $\mathcal{C}$  and  $\mathcal{M}$  players typically fall outside the scope of its standard models. Therefore, we call  $\mathcal{A}$ ,  $\mathcal{S}$  players as **benign** and  $\mathcal{C}$ ,  $\mathcal{M}$  players as **rogue**.

Intuitively, we would like to design the mechanism in the following way: as long as rogue verifiers only constitute a small portion, all four types of verifiers are incentivized to act honestly, even though malicious players may actually act in different manners at the cost of their own utilities. However, we still assume that malicious players cannot access non-malicious players' information (e.g. observations and reports) as the unauthorized access of non-malicious players' information should be prevented by the system design, and also breaks the basic model of Bayesian games.

### 6.1 Characterization of Robust Incentive Properties

To ensure that arbitrary actions of malicious players do not affect the incentive guarantees of other players even in the worst case, we introduce the notion of *robust incentive properties* describing the scenario in which the incentive properties holding uniformly for any possible realization of malicious players' actions.

In this case, we can define  $\phi$  as the *environmental variable* that depicts the prior probabilities, the number of  $\mathcal{A}, \mathcal{S}, \mathcal{C}, \mathcal{M}$  players, and the strategies of  $\mathcal{M}$  players. While players might not have the exact information of  $\phi$ , they do have the knowledge that  $\phi$  lies in a set  $\Phi$  of *environmental assumptions*, e.g., the total number of  $\mathcal{C}, \mathcal{M}$  players does not exceed a particular threshold.

The motivation of *robust incentive properties* is to guarantee that the players will not regret their honest actions even if they learn the existence and actions of the malicious players *ex-post*, so that these malicious behavior would not affect the incentive guarantees for the majority of non-malicious players. In other words, in a mechanism with robust incentive guarantees, the desired properties hold uniformly for any  $\phi \in \Phi$ .

In this context, given different environment variables, the player  $i$  would have different beliefs of the other players' reports, and we can characterize the belief profile of player  $i$  as  $\mathbb{B}_i : (S \cup \{\perp\}) \times \Phi \rightarrow \Delta(S^{n-1})$ , which maps the tuple of her observation and the environment variable to a joint distribution of other players' reports. Different from the model in Section 3, the players do not have a prior distribution of  $\phi \in \Phi$ . Hence similar to the characterization of partial distributional knowledge in [Zheng et al., 2021], for any strategy  $s_i$ , player  $i$  would actually have a *belief interval*  $[u_i^{\Phi-}(s_i; \mathbb{B}_i), u_i^{\Phi+}(s_i; \mathbb{B}_i)]$  of its utility among all possible  $\phi$ 's, formulated as

$$u_i^{\Phi-}(s_i; \mathbb{B}_i) = \inf_{\phi \in \Phi} u_i^{\phi}(s_i; \mathbb{B}_i), \quad u_i^{\Phi+}(s_i; \mathbb{B}_i) = \sup_{\phi \in \Phi} u_i^{\phi}(s_i; \mathbb{B}_i), \quad (19)$$

in which

$$u_i^{\phi}(s_i; \mathbb{B}_i) = \begin{cases} \mathbb{E}_{Z_i \sim s_i, Z_{-i} \sim \mathbb{B}_i(\perp, \phi)} [R_i(Z_i, Z_{-i})], & s_i \in \Delta(S); \\ \mathbb{E}_{X_i \sim P(X_i)} [\mathbb{E}_{Z_i \sim s_i(X_i), Z_{-i} \sim \mathbb{B}_i(X_i, \phi)} [R_i(Z_i, Z_{-i}) - c(X_i)]], & s_i \in \Delta(S)^S; \\ \alpha \cdot u_i(\lambda; \mathbb{B}_i) + (1 - \alpha) \cdot u_i(\mu; \mathbb{B}_i), & s_i \triangleq (\lambda, \mu, \alpha). \end{cases} \quad (20)$$

In this section, we also mainly consider *Bayesian* players with  $\mathbb{B}_i(X_i, \phi) = P(\mathbf{X}_{-i} | X_i, \phi)$ , and denote  $u_i^{\phi}(s_i), u_i^{\Phi-}(s_i), u_i^{\Phi+}(s_i)$  in this case for simplicity. Then with the general guideline that the incentive properties should uniformly hold for any  $\phi \in \Phi$ , we define the notion of robust utility maximization and robust IA as follows:

**DEFINITION 6.1 (ROBUST UTILITY MAXIMIZATION).** *In a fixed mechanism, a strategy  $s_i$  robustly maximizes player  $i$ 's utility w.r.t. environmental assumption  $\Phi$ , if and only if for any strategy  $s'_i \in \Omega$ ,*

$$u_i^{\phi}(s_i) \geq u_i^{\phi}(s'_i), \quad \forall \phi \in \Phi.$$

DEFINITION 6.2 (ROBUST INCENTIVE ALIGNMENT). *A mechanism satisfies robust  $\delta$ -IA w.r.t. environmental assumption  $\Phi$ , if and only if for any player  $i$  and pure strategy  $s_i \in \Omega_d$ ,*

$$\begin{aligned} u_i^{\Phi-}(s_i) &\geq \delta, & s_i &= I; \\ u_i^{\Phi+}(s_i) &\leq -\delta, & s_i &\neq I. \end{aligned}$$

Here, we denote  $I$  as the honest informed strategy, i.e.,  $s_i \in \Delta(S)^s$  and  $s_i(X_i) \equiv X_i$ .

From the notions above, we formally define the notion of  $f(n)$ -Byzantine-robustness ( $f(n)$ -BR) as follows:

DEFINITION 6.3 (BYZANTINE ROBUSTNESS). *For a DVG with  $n$  verifiers, we call a mechanism  $f(n)$ -BR if and only if: as long as  $\Phi$  constrains that the total number of rogue ( $C$  and  $M$ ) verifiers does not exceed  $f(n)$ ,*

- *Each  $\mathcal{A}, S, C$  verifier robustly maximizes her interim utility via acting honestly with robust 0-IA guarantees, assuming that other  $\mathcal{A}, S, C$  verifiers act honestly.*
- *Each colluding party in  $C$  robustly maximizes their total interim utility via acting honestly, assuming that all  $\mathcal{A}, S$  verifiers and other colluding parties in  $C$  act honestly.*
- *Each  $M$  verifier would robustly maximize their interim utilities with robust 0-IA guarantees if she acted honestly, even though she may actually act otherwise, assuming that all  $\mathcal{A}, S, C$  verifiers act honestly.*

In the rest of this section, we show that under mild assumptions, the design in Section 5, as long as the scoring matrix  $T$  comes from a “good” solution of the linear system Eqs. (11-14), is  $\Theta(n)$ -BR, i.e., resilient against a constant portion of rogue verifiers.

## 6.2 Bang for the Buck: Criteria and Optimization of Byzantine Robustness

When we look at the pairwise-scoring mechanism Eq. (17), the reward of each verifier  $i$  is essentially based on the comparison of her report  $Z_i$  and the *average* of other verifiers’ reports  $\overline{Z}_{-i}$ . Intuitively, if only a small portion of other verifiers may act dishonestly, since their contribution to  $\overline{Z}_{-i}$  is not significant, the actual expectation of  $\overline{Z}_{-i}$  conditioned on  $X_i$  would not deviate significantly from  $\mathbb{E}[\overline{X}_{-i}|X_i]$ , and the  $\delta$  margin in our design would make the reward matrix of  $i$  still satisfy *incentive alignment* properties even with a slightly perturbed posterior distribution of  $\overline{Z}_{-i}$ .

For simplicity of discussion, in the family of rogue verifiers, we first only consider *simple malicious* ones who could not create fake identities but may act unpredictably and report in any strategy, as in the *canonical Byzantine setting* defined in Definition 6.4 below. In later sections we will show that collusions can also be reduced to the canonical Byzantine setting. For Sybil attacks, while no voting-based protocols can effectively prevent them if the attacker has unlimited resources (e.g., 51% attack [Raju et al., 2022]), our study also show that each Sybil identity can also be reduced to a (new) malicious agent and, as long as they only make up a small portion of all verifiers, our framework of Byzantine robustness can prevent them from harming the incentive structure of other verifiers. Furthermore, since gaining additional voting power in PoW or PoS protocols has additional costs, we can show that as long as the total resources (e.g., computing power for PoW or stakes for PoS) of the verifier only makes up a small portion of the network, she could not gain significant advantage via Sybil attacks compared to honest behavior. The detailed discussion on the incentives of Sybil attacks is deferred to Section [??].

DEFINITION 6.4 (CANONICAL BYZANTINE SETTING). *In a canonical Byzantine setting, each verifier acts in one of the following strategies:*

- *No-Sybil selfish ( $\mathcal{S}_*$ ): Acting in a way that maximizes their own utilities, but unable to create fake identities.*
- *No-Sybil malicious ( $\mathcal{M}_*$ ): Reporting arbitrarily, but unable to create fake identities.*

From the pairwise-scoring mechanism, we can see that conditioned on the verifier  $i$  observing  $X_i \in S \cup \{\perp\}$ , the expected utility of reporting  $Z_i$  is

$$r_{X_i}(Z_i) - c(X_i) = \frac{1}{n-1} \sum_{j \neq i} \left( \sum_{Z_j \in S} P(Z_j|X_i) T_{Z_i Z_j} - c(X_i) \right). \quad (21)$$

When verifier  $j$  is honest, we see that  $Z_j = X_j$  and  $\sum_{X_j \in S} P(Z_j|X_i) T_{Z_i Z_j} - c(X_i) = (BT')_{X_i Z_i} - c(X_i) = W_{X_i Z_i}$ , and the  $\delta$ -IA condition ensures that  $W$ 's diagonal entries are at least  $+\delta$  and other entries are at most  $-\delta$ . If  $j$  is dishonest, then her report  $Z_j$  may deviate from  $X_j$ , resulting in a different  $\sum_{X_j \in S} P(Z_j|X_i) T_{Z_i Z_j}$  and leading to a perturbation to  $r_{X_i}(Z_i)$ .

Intuitively, if the summation of all these perturbations is bounded below  $\delta$ , then  $\{r_{X_i}(Z_i)\}$  still has positive diagonal entries and negative non-diagonal entries, satisfying the robust incentive alignment property (with a smaller margin). On the other hand, we notice that  $\sum_{X_j \in S} P(Z_j|X_i) T_{Z_i Z_j}$  is a convex combination of  $\{T_{Z_i Z_j} : Z_j \in S\}$ . If we upper bound the magnitude of the scoring rule, i.e.

$$|T_{Z_i Z_j}| \leq N, \quad \forall Z_i, Z_j \in S, \quad (22)$$

then we can deduce that

$$\sum_{Z_j \in S} P(Z_j|X_i) T_{Z_i Z_j} \in [-N, N]. \quad (23)$$

Hence, each dishonest verifier  $j$  can perturb the value of  $r_{X_i}(Z_i)$  by at most  $\frac{2N}{n-1}$ , so a large incentive margin  $\delta$  with a relatively small  $N$  would achieve a good “bang for the buck” for desired Byzantine-robust guarantees. In this sense, we define  $(\delta, N)$ -compactness as:

**DEFINITION 6.5 (( $\delta, N$ )-COMPACTNESS).** For fixed observation costs  $c(\cdot)$  and principal belief matrix  $B$  (denoted as the  $(c, B)$ -environment), a scoring matrix  $T$  is called  $(\delta, N)$ -compact if and only if its entries are bounded within  $[-N, N]$  and the corresponding mechanism is  $\delta$ -IA.

For convenience, we also call a mechanism (or a scoring matrix)  $\frac{\delta}{N}$ -compact if it is  $(\delta, N)$ -compact for some  $(\delta, N)$ .

Then, we immediately derive the following lemma:

**LEMMA 6.6.** If a CTF-PP mechanism has a  $(\delta, N)$ -compact scoring matrix, then it is  $\frac{\delta}{2N}(n-1)$ -BR in the canonical Byzantine setting, as it is 0-IA even in the presence of up to  $\frac{\delta}{2N}(n-1)$  malicious players.

In the following parts we focus on the construction of  $(\delta, N)$ -compact scoring matrices with optimized  $\frac{\delta}{N}$ .

### 6.3 LP Modeling for Byzantine Robustness

In Section 4, we formulated the  $\delta$ -IA condition for the 2-verifier DVG as the linear system of (11-14), and showed that the linear system is generally feasible, so that a desirable mechanism can be found via linear programming; in Section 5 we further showed that the LP solution generalizes to the  $n$ -verifier setting, so that our proposal is a general-purposed solution for the design of DVG mechanisms.

Besides the linear constraints, the LP problem actually allows us to optimize an *objective function*, which is not specified in previous parts. Considering the motivation of Byzantine robustness, from Lemma 6.6 we would like to construct a  $(\delta, N)$ -compact scoring matrix with a large  $\frac{\delta}{N}$ . Hence, an

intuitive idea is to fix  $\delta$  and minimize  $N$ . For fixed cheat-free belief matrix  $B$ , prior distribution vector  $B_\perp$ <sup>3</sup> and observation costs  $c(\cdot)$ , we can formulate the LP problem  $LP_1(B, B_\perp, c, \delta)$  with decision variable  $(N \in \mathbb{R}, T \in \mathbb{R}^{S^2})$  as:

$$LP_1(B, B_\perp, c, \delta) : \quad \text{minimize} \quad N \quad (24)$$

$$\text{s.t.} \quad |T| \leq N, \quad (25)$$

$$(BT')_{xx} \geq c(x) + \delta, \quad \forall x \in S \quad (26)$$

$$(BT')_{xy} \leq c(x) - \delta, \quad \forall x \in S, \quad y \in S - \{x\} \quad (27)$$

$$B_\perp T' \leq -\delta. \quad (28)$$

In fact, denoting  $N_* = (B, B_\perp, c, \delta)$  as the optimal objective value of  $LP_1(B, B_\perp, c, \delta)$ , we can show an upper bound on  $N_*(B, B_\perp, c, \delta)$  as:

**THEOREM 6.7.** *Denote  $c_1 = \max_{x \in S} \{c(x)\}$  as the maximum observation cost,  $p_1 = \max\{B_\perp\} = \max_{x \in S} \{P(X_i = x)\}$  as the maximum prior probability of any observation, and  $k = |S| = m + 2$  as the number of types, then we have*

$$N_*(B, B_\perp, c, \delta) \leq \|B^{-1}\|_2 \cdot (c_1 + \delta \cdot g(k, p_1)), \quad (29)$$

in which

$$g(k, p_1) = \sqrt{\left(k + (2k - 2) \frac{p_1}{1 - p_1}\right) \left(k + \frac{2}{1 - p_1}\right)} \quad (30)$$

is only dependent on  $k, p_1$  but independent to  $n$ .

Additionally, there exists a feasible solution satisfying (29) that makes the equality hold in (26).

The proof of Theorem 6.7 is deferred to Appendix D.5. From Theorem 6.7, we immediately deduce that:

**COROLLARY 6.8.** *For any  $\xi \in [0, \xi_0]$  in which*

$$\xi_0 = \frac{1}{\|B^{-1}\|_2 \cdot \sqrt{\left(k + (2k - 2) \frac{p_1}{1 - p_1}\right) \left(k + \frac{2}{1 - p_1}\right)}}, \quad (31)$$

we can develop a  $(\xi \cdot (n - 1))$ -BR mechanism via  $LP_1(B, B_\perp, c, \delta)$  with a sufficiently large  $\delta$ .

## 6.4 Reduction of Colluding Players

For the characterization of Byzantine players, it is intuitive that colluding behavior is within the scope of malicious behavior, and *the resilience against malicious players should infer the resilience against colluding players*. While this proposition is true, the reduction is actually non-trivial.

From the classification of players, we only consider the *external* effects and *individual* incentives of malicious players, i.e., their existence does not disrupt the incentive alignment guarantees of *other* players or benefit individual utilities. However, in the consideration of colluding players, we still need to prevent them from gaining *total* utility via collusion, i.e., we also need to consider *internal* effects of collusion which is not covered in previous discussion. Similar to the Side-Contract-Proofness (SCP) notion proposed by Chung and Shi [2023], we define *weak-SCP* in the scope of DVGs as follows:

<sup>3</sup>  $B'_\perp$  can be actually computed from  $B$  as the eigenvector of  $B'$  with eigenvalue 1 (See in Lemma D.1, Appendix D.5), but here we just formulate it as an extra constraint parameter.



DEFINITION 6.9 (WEAK-SIDE-CONTRACT-PROOFNESS (WEAK-SCP)). *We further define  $C_*$  players as:*

- *No-Sybil colluding ( $C_*$ ): Conducting collusions with other players (within  $C_*$ ) to maximize their joint utility, but unable to create fake identities.*

*We call a mechanism weak-Side-Contract-Proof (weak-SCP) under some conditions, if and only if as long as these conditions hold, in any collusion party, the players robustly maximize their total interim utility w.r.t. their individual (non-shared) Bayesian beliefs ( $P(X_{-i}|X_i)$ ) via acting honestly.*

We call this notion *weak-SCP* because although players collude with each other, they still update their posterior beliefs only based on their own observations, not considering other colluders' observations as they might not "fully trust each other". In turn, we call the collusion-proofness against colluders who share their beliefs based on their aggregated observations *strong-SCP*. Nevertheless, there are additional challenges in the design of strong-SCP peer prediction mechanisms and we leave it to future work. The intuition behind such challenges is that *a belief-sharing colluding party would be incentivized to report the same type even though they may have different observations*. On the bright side, from Proposition 6.10, although belief-sharing colluders may benefit from collusions, they do not disturb the incentive guarantees of other players as long as they only constitute a small portion of all players. We defer detailed discussions on strong-SCP to Appendix A.

In actual cases, there may exist multiple colluding parties, but from the argument in the player classification, for any selfish player or colluding party that intend to maximize their total utility, other colluding players outside the party do not have access to their actions or observations, and only need to be considered w.r.t. their external effects. Hence, we can deduce that

PROPOSITION 6.10. *Colluding players can be regarded as malicious players in the perspective of players outside their colluding parties.*

With Proposition 6.10, we only need to consider the existence of one colluding party of collusion players, beside a (small) number of malicious players. Formally, we have the following theorem:

THEOREM 6.11. *Assume that there are  $n$  players, among which are  $|M_*|$  no-sybil malicious players and a no-sybil colluding party of  $|C_*|$  players. If the CTF-PP mechanism has a  $(\delta, N)$ -compact scoring matrix, then the mechanism is 0-IA and weak-SCP as long as*

$$|M_*| + |C_*| \leq \frac{\delta}{2N}(n - 1).$$

The proof of Theorem 6.11 is deferred to Appendix D.6. From Theorem 6.11, we show that in the weak-SCP notion, colluding players can also be reduced to malicious players w.r.t. the weak-SCP notion for the Byzantine-robustness results of Theorem 6.7 and Corollary 6.8.

## 7 BYZANTINE REDUCTION FOR INACCURATE BELIEFS AND PRIORS

In Section 6, we discussed the construction of Byzantine-robust mechanisms for decentralized verification games in the presence of a small fraction of malicious players, assuming the prior knowledge of the distribution of  $\theta$  is accurate. However, due to the possible existence of cheating provers, as we discussed in Section 3, there is actually a "small but unknown"  $\epsilon$  probability that a proof is invalid, which is regarded as 0 in the derivation of previous sections. While we may argue that the  $\delta$  margin can indeed ensure the IA properties in the presence of "sufficiently small" perturbations of prior distributions, we still need explicit and quantitative results on the relations between  $\delta$  and  $\epsilon_0$  to show the practical robustness of our mechanism against dishonest provers.

In this section, we show that the  $(\delta, N)$ -compactness criterion is not only effective for robustness against malicious players, but also for robustness against inaccurate knowledge of prior distributions.

Formally, a  $(\delta, N)$ -compact CTF-PP mechanism maintains its IA guarantees even if the actual distribution of  $\theta$  has a  $\Theta(\delta/N)$  total variation (TV) distance from the  $P(\theta)$  we use in the construction of scoring rules, with a *Byzantine reduction* argument that reduces inaccurate beliefs to the existence of malicious players. With this technique, we not only show the general robustness of our design against malicious verifiers and inaccurate priors ( $\epsilon > 0$ , which represents the fraction of malicious provers), but also show the generality of our  $(\delta, N)$ -compactness criteria as a conceptual guideline of robust peer prediction mechanisms.

### 7.1 Byzantine Reduction: Inaccurate Beliefs $\leq$ Malicious Players

Before deriving the robustness results against inaccurate priors, we first discuss the robustness properties against inaccurate beliefs.

In the notation of Section 6.1, we suppose that the system has a perception of the environmental variable as  $\hat{\phi}$ , while the actual environmental variable is  $\phi$ . Assuming that the player  $i$  observes  $X_i \in S \cup \{\perp\}$ , her posterior belief on the distribution of  $\mathbf{X}_{-i}$  is  $\mathbb{B}_i(X_i, \hat{\phi}) = P(\mathbf{X}_{-i}|X_i, \hat{\phi})$ . However, as the actual environmental variable  $\phi$  is (slightly) different from  $\hat{\phi}$ , the actual posterior distribution of  $\mathbf{X}_{-i}$  is  $\mathbb{B}_i(X_i, \phi) = P(\mathbf{X}_{-i}|X_i, \phi)$ .

In the pairwise-scoring mechanism defined in Section 5.2, for the environmental variable  $\phi$ , the expected utility of reporting  $Z_i$  when observing  $X_i$  is

$$r_{X_i}(Z_i) - c(X_i) = \frac{1}{n-1} \sum_{j \neq i} \left( \sum_{Z_j \in S} P(Z_j|X_i, \phi) T_{Z_i Z_j} - c(X_i) \right). \quad (32)$$

Assuming that all other players are honest, i.e.,  $Z_j = X_j$ , from symmetry we have

$$r_{X_i}(Z_i) - c(X_i) = \frac{1}{n-1} \sum_{j \neq i} \left( \sum_{X_j \in S} P(X_j|X_i, \phi) T_{Z_i X_j} - c(X_i) \right) \quad (33)$$

$$= P(X_j|X_i, \phi) T_{Z_i X_j} - c(X_i), \quad (34)$$

in which the  $j$  in (34) can be an arbitrary player different from  $i$ . Intuitively, if  $P(X_j|X_i, \hat{\phi})$  is close to  $P(X_j|X_i, \phi)$ , then even if the scoring matrix  $\{T_{Z_i Z_j}\}$  is designed for the  $\delta$ -IA property according to  $\hat{\phi}$ , i.e.,

$$\sum_{X_j \in S} P(X_j|X_i, \hat{\phi}) T_{Z_i X_j} - c(X_i) \begin{cases} \geq \delta, & Z_i = X_i; \\ \leq -\delta, & Z_i \neq X_i. \end{cases} \quad (35)$$

the margin of  $\delta$  can still make the mechanism 0-IA for the actual environment of  $\phi$ , i.e.,

$$\sum_{X_j \in S} P(X_j|X_i, \phi) T_{Z_i X_j} - c(X_i) \begin{cases} \geq 0, & Z_i = X_i; \\ \leq 0, & Z_i \neq X_i. \end{cases} \quad (36)$$

Actually, if the scoring matrix  $T$  is  $(\delta, N)$ -compact, we only need that the total variation (TV) distance between  $P(X_j|X_i, \hat{\phi})$  and  $P(X_j|X_i, \phi)$  is bounded below  $\Theta(\delta/N)$ . Formally, we have:

**LEMMA 7.1.** *If (35) holds,  $\max\{|T_{Z_i Z_j}|\} \leq N$ , and  $TV(P(X_j|X_i, \hat{\phi}), P(X_j|X_i, \phi)) \leq \frac{\delta}{2N}$ , then (36) holds.*

The proof of Lemma 7.1 is deferred to Appendix D.7. From Lemma 7.1 we immediately deduce the following theorem:

**THEOREM 7.2.** Assume that there are no rogue players in environments  $\phi$  and  $\hat{\phi}$ . If a CTF-PP mechanism is  $(\delta, N)$ -compact for environment  $\hat{\phi}$ , and for a player  $j \neq i$  it holds that the total variation distance  $TV(P(X_j|X_i, \hat{\phi}), P(X_j|X_i, \phi)) \leq \frac{\delta}{2N}$  for every possible observation  $X_i \in S \cup \{\perp\}$ , then the mechanism is 0-IA for environment  $\phi$ .

Actually, we can see that if  $TV(P(X_j|X_i, \hat{\phi}), P(X_j|X_i, \phi)) = \alpha \cdot \frac{\delta}{2N}$  for  $\alpha \in [0, 1]$ , then the mechanism is  $(1 - \alpha)\delta$ -IA for environment  $\phi$ , and hence according to Lemma 6.6, it is still Byzantine-robust to at most  $(1 - \alpha) \cdot \frac{\delta}{2N} (n - 1)$  malicious players, showing that the compactness of  $\delta/N$  serves as a “reservoir” of robustness against both inaccurate beliefs and malicious players. Intuitively, it indicates that the robustness against a  $\Delta$  fraction of malicious players *implies* the robustness against a  $\Delta$  level of noise in posterior beliefs. This intuition can be interpreted via *coupling argument*: if the actual posterior distribution is different from the belief, it can be equivalently regarded as the players “within the total variation” reporting dishonestly. The detailed discussion is deferred to Appendix B.

## 7.2 Robustness for Inaccurate Priors

## 8 DISCUSSION

In this paper, we develop a theoretical framework for the decentralized verification game on blockchain proving protocols and get preliminary results that the combination of the Capture-The-Flag design (as in [Zhao et al., 2024]) and peer prediction techniques has the potential to resolve the Verifier’s Dilemma in a fully decentralized environment. On the other hand, we also explore the design of peer prediction mechanisms with broader agent strategy spaces and more general settings (e.g. observation costs dependent on observation results). In future work, we will improve and broaden the study in the following aspects:

- (1) While this paper is mainly on the elicitation of truthful verification results, we will also develop voting/aggregation mechanisms that (optimally) make decisions on whether to accept the proof.

[\[TODO: More general applications beyond blockchain...\]](#)

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## Appendix

### A DISCUSSION ON STRONG-SCP PEER PREDICTION MECHANISMS

In the scope of peer prediction mechanisms, it is assumed that players report in a way that maximizes their expected utilities w.r.t. their *beliefs* on other players' reports. Hence, different beliefs may lead to different estimations of utilities and different strategies. In the canonical setting of individual non-colluding players, their beliefs are the Bayesian posteriors conditioned on their observations, i.e.  $P(\mathbf{X}_{-i}|\mathbf{X}_i)$ .

When the collusions occur, nevertheless, the beliefs may differ in different settings as the level of information and utility sharing may vary. In the notion of side-contract-proofness (SCP) in the transaction fee mechanism design [Chung and Shi, 2023], it is assumed that the utilities are transferable via side payments, so that rather than a local Pareto improvement that weakly benefits everyone, a successful collusion only needs to improve the total utility of all players in the colluding party.

While we inherit the SCP notion that allows side payments, in the context of information elicitation, there can also be different settings on whether the players' beliefs are shared or not. In the weak-SCP notion, we consider the non-sharing-belief setting that the players estimate their utilities based on their individual observations  $P(\mathbf{X}_i|\mathbf{X}_{-i})$ , and we show in Theorem 6.11 that weak-SCP can be achieved via our design that optimizes  $(\delta, N)$ -compactness. In contrast, a "strong-SCP" notion considers the sharing-belief setting with players estimating their utilities based on the collective observations of the colluding party. Namely, their beliefs on the reports inside the party are just their true reports, and their beliefs on the reports outside the party are computed by  $P(\mathbf{X}_{-C}|\mathbf{X}_C)$ .

Nevertheless, a series of difficulties occur in the design for strong-SCP mechanisms.

#### A.1 Challenges in the Design of Strong-SCP Mechanisms

**Free-riding at low noises.** Consider the scenario when the observation noise is low, i.e., the players observe the true type  $\theta$  with high probability. Then in a colluding party, one observation is sufficient to secure a high confidence that other members would also observe that result, which is likely to be the ground-truth. Hence, it is rational for other players to lazily report that result too, saving the observation cost (which is typically high in ML verification contexts).

**Preference towards agreeing reports.** Even if the observation cost is low enough to keep the players willing to do active observation, as a wide scope of practically used peer prediction mechanisms, e.g., the correlated agreement (CA) mechanism [Shnayder et al., 2016] and mutual-information-based mechanisms [Chen et al., 2020], typically rewards agreeing reports, there can be a tendency that all colluding players report the same even if they observe differently, violating strong-SCP properties. This challenge generalizes to the family of *pairwise-scoring* mechanisms, in which the colluding party's total utility is approximately linear to their average report when  $n$  is large.<sup>4</sup> Hence, it is impossible to design a pairwise-scoring peer prediction mechanism that satisfies a "strict" strong-SCP property with sensitivity guarantees (truthful reporting yields at least  $h$  more utility than reporting the same when disagreement occurs). Formally, we have:

**THEOREM A.1.** *In an  $n$ -player decentralized verification game (DVG), for any pairwise-scoring mechanism with a pairwise scoring matrix  $T$  such that the reward of player  $i$  is given by*

$$R_i(\mathbf{Z}_i, \mathbf{Z}_{-i}) = \mathbf{Z}_i' T \overline{\mathbf{Z}_{-i}}, \quad (37)$$

<sup>4</sup>The approximate linearity property is actually not restricted to pairwise-scoring mechanisms, so the results can potentially be further generalized in practice.

and  $-N \leq T \leq N$ , it holds that:

(i) For a collusion party  $C = \{i_1, \dots, i_c\}$  with observations  $\mathbf{X}_C = \{X_{i_1}, \dots, X_{i_c}\}$ , there exists an all-same report  $\mathbf{Z}_C^* = \{X_{i^*}, \dots, X_{i^*}\}$  such that  $i^* \in C$  and

$$\mathbb{E}_{\mathbf{Z}_C=\mathbf{Z}_C^*, \mathbf{Z}_{-C} \sim P(\mathbf{X}_{-C}|\mathbf{X}_C)} \left[ \sum_{i \in C} R_i(X_i^*, \mathbf{Z}_{-i}) \right] \geq \mathbb{E}_{\mathbf{Z}_C=\mathbf{X}_C, \mathbf{Z}_{-C} \sim P(\mathbf{X}_{-C}|\mathbf{X}_C)} \left[ \sum_{i \in C} R_i(X_i, \mathbf{Z}_{-i}) \right] - h, \quad (38)$$

in which

$$h = \frac{2c(c-1)}{n-1}N. \quad (39)$$

(ii) If for any types  $s_1, s_2 \in S$ ,  $s_1 \neq s_2$ , it holds that

$$T_{s_1 s_1} \geq T_{s_1 s_2},$$

i.e., the scoring rule favors agreement, then Eq. (38) holds for  $h \leq 0$ . If it is possible to observe  $\mathbf{X}_C$  with two different observations  $X_i, X_j$  s.t.  $T_{X_i X_i} > T_{X_i X_j}$ , then Eq. (38) holds for  $h < 0$ , indicating that the mechanism is not strong-SCP.

The proof is deferred to Appendix D.8. We can interpret part (i) as that approximate linearity renders it impossible to disincentivize reporting the same type for  $c \ll n$  with significant incentive margins, as when the expected reward of an honest player is  $\Theta(N)$ , the expected reward of the party is  $\Theta(cN) \gg h$ . Furthermore, part (ii) shows that as long as the mechanism favors agreement as most existing peer prediction mechanisms typically do, this type of collusion will be strictly profitable, rendering them non-strong-SCP. While we conjecture that it may be generally impossible to design a strong-SCP single-task peer prediction mechanism under mild assumptions, Theorem A.1 shows that if it is actually possible, we need to bypass the approximate linearity and may need extremely tricky designs. On the other hand, other possible approaches to address the collusion issue in the shared-belief setting may include:

- Multi-task settings: assigning different task sets to different players and make partial-copying strategies unprofitable.
- Aggregation design: while failing to prevent collusion in the front-end elicitation phase, it may still be possible to design aggregation mechanisms to minimize its impact to the back-end decision-making (e.g., whether to accept the proof/block/etc.)

Nevertheless, we can notice that for the phenomenon that colluding players tend to copy the same report, its practical effect on the functionality of back-end decision-making could be rather “benign” as at least one of them did the honest observation, which still bypasses the Verifier’s Dilemma and can be informationally sufficient particularly in the low-noise scenario: in the back-end perspective, we can also regard such colluders as one player with more voting power in an almost unanimous voting. We leave detailed analyses and discussions on these aspects for future work.

## B A COUPLING INTERPRETATION OF BYZANTINE REDUCTION

### B.1 Coupling Argument and Total Variation Distance

In probability theory and statistics, *coupling* is a technique to compare characteristics of two distributions, especially when they are close to each other. In general, for two given distributions  $D_1, D_2$ , we can construct two dependent random variables  $V_1, V_2$  with a joint distribution  $P(V_1, V_2)$  such that the marginal distributions satisfy

$$P(V_1) = D_1, P(V_2) = D_2.$$

While the construction of the joint distribution  $P(V_1, V_2)$  is not unique, in the particular case that  $D_1$  and  $D_2$  are close to each other, we would like to make  $V_1 = V_2$  with a high probability.

As an intuitive interpretation, we can regard that in the “main world”  $\Omega_1$ , a random event  $V_1$  happens according to  $D_1$ ; in a parallel “alternative world”  $\Omega_2$ , the event is tampered to  $V_2$  which has a different distribution  $D_2$ . We can imagine that such magical manipulation is costly, so that the manipulator would like to tamper as little as possible, i.e., minimize  $P(V_1 \neq V_2)$  as long as  $V_2 \sim D_2$ . Actually, this model falls into the scope of *optimal transport*. From optimal transport theories, it holds that

$$\min_{V_1 \sim D_1, V_2 \sim D_2} P(V_1 \neq V_2) = TV(D_1, D_2). \quad (40)$$

**Example.** An academic institute has recently recruited 50 new tenure-track assistant professors, among which 7 are expected to get tenure, yielding a tenure rate of 14%. However, due to a sudden cut of funding, the tenure rate has to be lowered to 10%, hence some of the would-be decisions have to be changed. From Eq. (40), the minimum number of changed decisions is

$$50 \cdot TV(\text{Bern}(0.14), \text{Bern}(0.1)) = 50 \cdot 0.04 = 2. \quad (41)$$

In fact, to change the fewest decisions, the tenure decisions for 2 unfortunate candidates will be revoked.

## B.2 Interpretation for Robust Peer Prediction

In the context of (Byzantine)-robust peer prediction for  $n$  players, we consider a mechanism that is  $(\delta, N)$ -compact for the environment  $\hat{\phi}$  that has no rogue players. From Theorem 6.11 we know that this mechanism keeps the 0-IA incentive guarantee even if an arbitrary subset of at most  $\frac{\delta}{2N}(n-1)$  players, i.e. a  $\frac{\delta}{2N}$  fraction of other players, become malicious. For simplicity of discussion, we assume  $n \rightarrow \infty$ .

We assume that in the main world, the environment is  $\hat{\phi}$ . Conditioned on player  $i$  observing  $X_i$ , the (expected) distribution<sup>5</sup> of other players’ observations is  $P(X_j|X_i, \hat{\phi})$ . From the Byzantine robustness results, the 0-IA guarantee holds as long as at least an  $1 - \frac{\delta}{2N}$  fraction of them report honestly.

Then, we consider the alternative world in which the environment is  $\phi$ . Assuming that the player  $i$  still observes  $X_i$ , the distribution of other players’ observations is  $P(X_j|X_i, \phi)$ . From the coupling argument discussed above, a minimum of  $TV(P(X_j|X_i, \hat{\phi}), P(X_j|X_i, \phi))$  fraction of other players have different observations between two worlds.

Now we assume that all other players report honestly in the alternative world, while in the main world, they also report their observations in the alternative world. Then we see that in the main world, an exact  $TV(P(X_j|X_i, \hat{\phi}), P(X_j|X_i, \phi))$  fraction of other players are reporting dishonestly. Hence in the main world, the 0-IA guarantee holds as long as  $TV(P(X_j|X_i, \hat{\phi}), P(X_j|X_i, \phi)) \leq \frac{\delta}{2N}$ .

On the other hand, in the alternative world all players are reporting honestly, but the actual environment is  $\phi$  instead of  $\hat{\phi}$ . Since all other players report identically in two worlds, in the perspective of player  $i$ , the 0-IA guarantee still holds in the alternative world as long as  $TV(P(X_j|X_i, \hat{\phi}), P(X_j|X_i, \phi)) \leq \frac{\delta}{2N}$ . Hence we see that the Byzantine-robustness against a  $\frac{\delta}{2N}$  fraction of malicious players implies the tolerance of a  $\frac{\delta}{2N}$  error of posterior beliefs.

An intuitive interpretation is that, from the perspective of a (self-centric) player  $i$ , even if the actual posterior distribution of others’ observations differs from her belief, she can interpret this difference as arising from some players reporting dishonestly. In this interpretation, the player regards her belief (the ideal posterior distribution) as correct, while the fraction of players causing

<sup>5</sup>We consider the distribution ensemble-wise, i.e., among all possible ground-truth  $\theta$ ’s. The term “(conditional) distribution” in this part is always interpreted in this way.



the discrepancy corresponds to the error measured by the total variation distance. This ensures that as long as the error remains below the tolerance threshold  $\frac{\delta}{2N}$ , the robustness guarantee holds.

## C EXPERIMENTAL EVALUATION: VERIFICATION OF INCENTIVE-SECURE POL

In this section, we empirically demonstrate the process of designing a CTF-PP mechanism, for one set of parameters that is useful for practical interest.

### C.1 Construction of the Scoring Rule

We consider the 2-verifier DVG which captures the case of one stage in [Zhao et al., 2024]. Here, we set the distribution  $\theta$  as  $P(\theta = F_1) = P(\theta = F_2) = \frac{1}{4}$ , which means that half of all stages are flagged. Then we consider the lossy-channel model in which  $\mu_1 = \mu_2 = \frac{1}{2}$ , as each verifier independently chooses half of all stages<sup>6</sup> and each flag is detected with probability 1 when verified. According to the CTF protocol, when a cheating stage is chosen by a verifier, it has an  $\frac{1}{2}$  chance to be correctly detected, and a  $\frac{1}{4}$  chance to be observed as  $F_1$  or  $F_2$  respectively. Hence,  $P(X_i|\theta)$  is shown as in Table 1, and assuming  $\epsilon = 0$ , we compute the marginal distribution of  $X_i$  as  $B_\perp = [\frac{3}{4}, \frac{1}{8}, \frac{1}{8}, 0]$ .

Then, assuming  $\epsilon = 0$ , from  $P(X_i, X_{-i}) = \sum_\theta P(\theta)P(X_i|\theta)P(X_{-i}|\theta)$  we can compute the joint probabilities  $P(X_1, X_2)$ , as shown in Table 2. We accordingly compute the post-observation belief  $P(X_2|X_1) = \frac{P(X_1, X_2)}{P(X_1)}$  for  $X_1 \neq 1$ , and  $P(X_2|X_1 = 1) = P(X_2|\theta = 1)$ , getting the principal belief matrix  $B$  as Table 3.

	$X_i = 0$	$X_i = F_1$	$X_i = F_2$	$X_i = 1$
$\theta = 0$	1	0	0	0
$\theta = F_1$	$\frac{1}{2}$	$\frac{1}{2}$	0	0
$\theta = F_2$	$\frac{1}{2}$	0	$\frac{1}{2}$	0
$\theta = 1$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$

Table 1.  $P(X_i|\theta)$ , the observation distribution conditioned on  $\theta$ .

	$X_2 = 0$	$X_2 = F_1$	$X_2 = F_2$	$X_2 = 1$
$X_1 = 0$	$\frac{5}{8}$	$\frac{1}{16}$	$\frac{1}{16}$	0
$X_1 = F_1$	$\frac{1}{16}$	$\frac{1}{16}$	0	0
$X_1 = F_2$	$\frac{1}{16}$	0	$\frac{1}{16}$	0
$X_1 = 1$	0	0	0	0

Table 2. Joint probabilities  $P(X_1, X_2)$  ( $\epsilon = 0$ ).

	$X_2 = 0$	$X_2 = F_1$	$X_2 = F_2$	$X_2 = 1$
$X_1 = 0$	$\frac{5}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	0
$X_1 = F_1$	$\frac{1}{2}$	$\frac{1}{2}$	0	0
$X_1 = F_2$	$\frac{1}{2}$	0	$\frac{1}{2}$	0
$X_1 = 1$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$

Table 3. The principal belief matrix  $B$ .

<sup>6</sup>It is significantly more than needed, but does work.

From the nature of the CTF mechanism, in which the observation of  $F_1, F_2$  of 1 takes twice the computational cost of a stage, we set  $c(F_1) = c(F_2) = c(1) = 2c$ . On the other hand, the “observation” of a 0 has two cases: one is that the verifier has verified the stage that is not cheated or flagged, which has a  $\mu(1 - \eta) = \frac{1}{4}$  probability, and one is that the verifier does not verify the stage from the random verification protocol, which has a  $1 - \mu = \frac{1}{2}$  probability. Hence, we have an “amortized”  $c(0) = \frac{1}{3}c$ . Without loss of generality, we set  $c = 1$ .

With the knowledge of  $B, B_\perp$  and  $c$ , Eqs. (11)-(14) are the constraints that a desirable scoring rule for a CTF-PP mechanism should satisfy. For the robustness of our mechanism, we do not want the payments to have extremely large absolute values. Hence, we construct the linear program as:

$$\begin{aligned} & \text{minimize } N \\ & \text{s.t. (11)-(14), } \quad -N \leq T \leq N. \end{aligned}$$

We set a margin of  $\delta = 0.2$ , and compute a numerical solution to this LP, getting a scoring rule as shown in Table 4.

	$Z_{-i} = 0$	$Z_{-i} = F_1$	$Z_{-i} = F_2$	$Z_{-i} = 1$
$Z_i = 0$	+2.0959	-7.1643	-7.1643	-1.0764
$Z_i = F_1$	-1.5421	+6.4732	-4.4574	-1.2378
$Z_i = F_2$	-1.5421	-4.4574	+6.4732	-1.2378
$Z_i = 1$	-2.2000	+5.8000	+5.8000	+7.4000

Table 4.  $T_{Z_i Z_{-i}} = R_i(Z_i, Z_{-i})$  as a numerical solution.

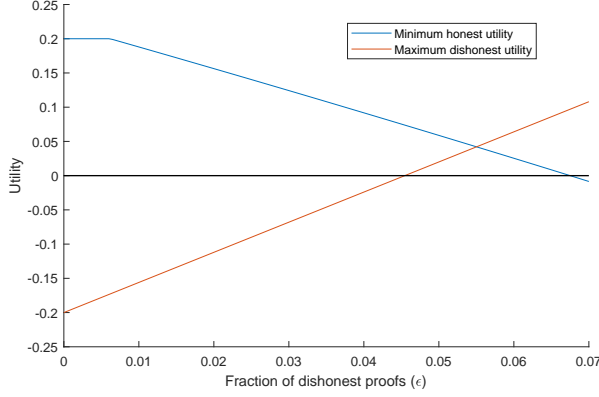
## C.2 Evaluation

With this scoring rule, given verifier  $-i$  acts honestly, we report the expected utility of verifier  $i$  is in Table 5, assuming  $\epsilon = 0$ , showing that the verifier gets a positive expected utility if and only if she verifies and reports honestly.

Furthermore, we consider the case  $\epsilon > 0$ . We plot the maximum expected utility of dishonest actions and the minimum expected utility of honest actions in Figure 1. From the plot, we show that the introduction of the margin keeps the IR, UniIC and NFL properties of our mechanism as long as  $\epsilon < 0.045$ . Hence, we demonstrate that our design of the CTF-PP mechanism for the 2-verifier DVG can incentivize honest verification even if there is no dishonest prover, thus bypassing the Verifier’s Dilemma and achieving a pure-strategy Nash equilibrium that the prover and verifiers simultaneously act honestly.

	Reporting 0	Reporting $F_1$	Reporting $F_2$	Reporting 1
Observing 0	<b>+0.2192</b>	-1.4504	-1.4504	-1.2000
Observing $F_1$	-4.5342	<b>+0.4655</b>	-4.9998	-0.2000
Observing $F_2$	-4.5342	-4.9998	<b>+0.4655</b>	-0.2000
Observing 1	-3.0122	-2.8285	-2.8285	<b>+0.2000</b>
Uninformed	-0.2192	-0.9046	-0.9046	-0.2000

Table 5. Verifier’s expected utility,  $\epsilon = 0$ .

Fig. 1. Verifier's expected utility,  $\epsilon > 0$ .

## D DEFERRED PROOFS

### D.1 Proof of Theorem 1.1

Assume we have such a mechanism. By the definition of Nash equilibrium, we consider a fixed verifier. Given that the prover and all other verifiers act honestly, that verifier should be incentivized to do the honest verification.

Since the prover is honest, when that verifier performs honest verification, the result should always be “Success”. However, suppose the verifier simply reports “Success” without verification. In that case, the outcome is the same but the verifier saves the verification cost, so the verifier is incentivized to deviate from the honest strategy.

That leads to a contradiction, so no such mechanism exists.

### D.2 Proof of Theorem 4.1

Notice that if  $B$  is invertible, for any given  $W : S^2 \rightarrow \mathbb{R}$ , we can compute a  $T = (B^{-1}W)'$  that satisfies  $W = BT'$ . Since we have

$$B_{\perp}y = P(X_{-i} = y) \quad (42)$$

$$= \sum_{x \in S} P(X_i = x) \cdot P(X_{-i} = y | X_i = x) \quad (43)$$

$$= \sum_{x \in S} B_{\perp x} \cdot B_{xy}, \quad (44)$$

It holds that  $B_{\perp} = B_{\perp}B$ . Hence,  $W_{\perp} = B_{\perp}T' = B_{\perp}BT' = B_{\perp}W$ . Since  $B_{\perp} \geq 0$ ,  $W_{\perp}$  is a convex combination of rows in  $W$ . Therefore, we only need to construct a  $W$  that satisfies Eq. (11) with non-diagonal entries small enough.

Here, for a constant  $M > 0$  large enough, we construct  $W$  as:

$$W_{xx} = c(x) + \delta, \quad \forall x \in S; \quad (45)$$

$$W_{xy} = -M, \quad \forall x \in S, \quad y \in S - \{x\}. \quad (46)$$

Then,

$$W_{\perp y} = \sum_{x \in S} B_{\perp x} W_{xy} \quad (47)$$

$$= B_{\perp y} W_{yy} + \sum_{x \in S - \{y\}} B_{\perp x} W_{xy} \quad (48)$$

$$= B_{\perp y} (c(y) + \delta) - (1 - B_{\perp y})M. \quad (49)$$

Since the existence of flags introduces randomness in the observation, we have  $\max\{B_{\perp}\} < 1$ . Hence, denote  $B_{\perp}^* = \max\{B_{\perp y}\}$ , we only need to let

$$M \geq \frac{B_{\perp}^* \cdot (c(y) + \delta) + \delta}{1 - B_{\perp}^*}, \quad (50)$$

Then the required constraints of Eqs. (11)-(14) are satisfied with a margin of  $\delta$ .

Then we consider the scenario that  $\epsilon > 0$  but is small enough. In this case, define  $B(\epsilon)$  and  $B_{\perp}(\epsilon)$  as the belief matrix and blind-belief matrix considering the influence of  $\epsilon$ . We can see that for any  $x \neq 1$ , since  $\tilde{P}(X_i = x) > 0$ , the influences of  $\epsilon$  on  $P(X_{-i}|X_i = x)$  and  $P(X_{-i})$  are upper bounded by  $O(\epsilon)$ , and because  $\theta \neq 1 \implies X_i \neq 1$ , Eq. (6) always holds for any  $\epsilon$ . Therefore, the margin of  $\delta$  ensures that the constraints are not violated as long as  $\epsilon_0$  is small enough.

Finally, let  $T = (B^{-1}W)'$ , then  $T$  is a scoring rule that satisfies the requirements.

### D.3 Proof of Proposition 4.2

In the 2-verifier DVG, By definition  $B_{xy} = P(X_{-i} = y|X_i = x)$ . We sort the elements of  $S$  in the order  $(0, F_1, \dots, F_m, 1)$ . For convenience, we define the observation matrix  $O$  and inference matrix  $E$  as:

$$O_{xy} = P(X_i = y|\theta = x), \quad (51)$$

$$E_{xy} = P(\theta = y|X_i = x). \quad (52)$$

From the lossy-channel model, we immediately see that  $O$  is an upper triangular matrix with non-zero diagonals, so  $O$  is invertible.

Furthermore, given  $\epsilon = 0$ , we can see that:

- If  $X_i = 0$ , then the ground truth  $\theta$  may be 0 or any flag  $F_j$ , and  $P(\theta = 0|X_i = 0) > 0$ .
- If  $X_i = F_j$ , then  $\theta = F_j$ .
- If  $X_i = 1$ , then  $\theta = 1$ .

Hence,  $E$  is a lower triangular matrix with non-zero diagonals, so  $E$  is invertible. Then, because  $X_i, X_{-i}$  are independent conditioned on  $\theta$ , we have

$$B_{X_i X_{-i}} = P(X_{-i}|X_i) \quad (53)$$

$$= \sum_{\theta \in S} P(\theta|X_i) P(X_{-i}|\theta) \quad (54)$$

$$= \sum_{\theta \in S} E_{X_i \theta} \cdot O_{\theta X_{-i}}. \quad (55)$$

Therefore,  $B = EO$ .

Since  $E, O$  are invertible, we deduce that  $B$  is invertible.

#### D.4 Proof of Theorem 5.1

In the context of Bayesian Nash equilibrium, we can assume each verifier  $j \neq i$  is honest, i.e.,  $Z_{-i} = X_{-i}$ . Hence, given that verifier  $i$  observes  $X_i \in S \cup \{\perp\}$  and reports  $Z_i \in S$ , the interim expected reward is:

$$r_{X_i}(Z_i) = \mathbb{E}\left[Z_i' T \overline{X_{-i}} \middle| X_i\right] \quad (56)$$

$$= Z_i' T \cdot \mathbb{E}\left[\overline{X_{-i}} \middle| X_i\right] \quad (57)$$

$$= Z_i' T \cdot \mathbb{E}\left[\frac{1}{n-1} \sum_{j \neq i} X_j \middle| X_i\right] \quad (58)$$

$$= \frac{1}{n-1} \sum_{j \neq i} Z_i' T \mathbb{E}\left[X_j \middle| X_i\right] \quad (59)$$

$$= \frac{1}{n-1} \sum_{j \neq i} Z_i' T \sum_{X_j \in S} P(X_j | X_i) X_j \quad (60)$$

$$= \frac{1}{n-1} \sum_{j \neq i} \sum_{X_j \in S} P(X_j | X_i) T_{Z_i X_j}. \quad (61)$$

With similar arguments as Section 4, we assume  $\epsilon = 0$ , and  $P(X_j | X_i)$  is the  $(i, j)$ -th entry of the cheat-free belief matrix  $B$  for any  $j \neq i$ . Hence, we have

$$r_{X_i}(Z_i) = \frac{1}{n-1} \sum_{j \neq i} \sum_{X_j \in S} P(X_j | X_i) T_{Z_i X_j} \quad (62)$$

$$= \sum_{X_j \in S} B_{X_i X_j} T_{Z_i X_j} \quad (63)$$

$$= (BT')_{X_i Z_i}. \quad (64)$$

Hence, the linear program of Eqs. (11-14) works equivalently for the  $n$ -verifier DVG when we use the linear average mechanism as Eq. (17) with exactly the same incentive structure. So any incentive property satisfied in the 2-verifier mechanism  $T$  is also satisfied in the linear average mechanism in Eq. (17).

#### D.5 Proof of Theorem 6.7

We first observe that any feasible solution of  $LP_1(B, B_\perp, c, \delta)$  can be constructed with feasible solutions of  $LP_1(B, B_\perp, c, 0)$  and  $LP_1(B, B_\perp, 0, 1)$ , i.e.,

**OBSERVATION 1.** *If  $(N_c, T_c)$  is a feasible solution of  $LP_1(B, B_\perp, c, 0)$  and  $(N_\delta, T_\delta)$  is a feasible solution of  $LP_1(B, B_\perp, 0, 1)$ , then  $(N_c + \delta N_\delta, T_c + \delta T_\delta)$  is a feasible solution of  $LP_1(B, B_\perp, c, \delta)$ .*

Hence, we can estimate upper bounds of optimal  $N_c$  and  $N_\delta$  separately. Here we denote  $\|\cdot\|_2$  as the matrix  $\ell_2$ -norm, and denote  $W = BT'$ . From the assumption in Theorem 4.1 that  $B$  is invertible,  $T$  can be constructed as  $(B^{-1}W)'$  and it holds that  $\forall x, y \in S, |T_{xy}| \leq \|T\|_2 = \|(B^{-1}W)'\|_2 \leq \|B^{-1}\|_2 \cdot \|W\|_2$ . Hence, we can estimate the upper bounds on entrywise maximums of  $T$  via  $\ell_2$  norms of  $W$ , respectively.

For  $LP_1(B, B_\perp, c, 0)$ , if we construct  $W = \text{diag}(c)$ , then the corresponding  $T_c = (B^{-1}W)'$  satisfies conditions (26-28). Because  $\|W\|_2 = \|\text{diag}(c)\|_2 = \max\{c\}$ , we have

$$|T_c| \leq \|B^{-1}\|_2 \cdot \max\{c\}. \quad (65)$$

Hence,  $N_c = \|B^{-1}\|_2 \cdot \max\{c\}$  is feasible for  $LP_1(B, B_\perp, c, 0)$ .

Before analysis for  $LP_1(B, B_\perp, 0, 1)$ , we prove a lemma:

LEMMA D.1.  $B'_\perp$  is an eigenvector of  $B'$  with eigenvalue 1, i.e.,  $B_\perp B = B_\perp$ .

PROOF. Let  $j$  be an arbitrary verifier other than  $i$ . From the discussion of the uninformed strategy (in Section 3), we have

$$B_{\perp y} = P(X_j = y | X_i = \perp) \quad (66)$$

$$= P(X_j = y). \quad (67)$$

On the other hand,

$$(B_\perp B)_y = \sum_{x \in S} B_{\perp x} B_{xy} \quad (68)$$

$$= \sum_{x \in S} P(X_i = x) \cdot P(X_j = y | X_i = x) \quad (69)$$

$$= \sum_{x \in S} P(X_i = x, X_j = y) \quad (70)$$

$$= P(X_j = y). \quad (71)$$

Hence we have  $B_{\perp y} = (B_\perp B)_y$  for  $\forall y \in S$ , so  $B_\perp B = B_\perp$ .  $\square$

From Lemma D.1, the  $LP_1(B, B_\perp, 0, 1)$  can be reformulated as:

$$LP_2(B, B_\perp, 0, 1) : \text{ minimize } N \quad (72)$$

$$\text{s.t.} \quad |B^{-1}W| \leq N, \quad (73)$$

$$W_{xx} \geq 1, \quad \forall x \in S \quad (74)$$

$$W_{xy} \leq -1, \quad \forall x \in S, \quad y \in S - \{x\} \quad (75)$$

$$B_\perp W \leq -1. \quad (76)$$

Here, we can construct

$$W_{xy} = \begin{cases} 1, & y = x; \\ -\frac{1+B_{\perp y}}{1-B_{\perp y}}, & y \neq x. \end{cases}$$

From the construction we immediately see that conditions (74-75) are satisfied. For condition (76), we have

$$(B_\perp W)_y = \sum_{x \in S} B_{\perp x} W_{xy} \quad (77)$$

$$= B_{\perp y} W_{yy} + \sum_{x \in S - \{y\}} B_{\perp x} W_{xy} \quad (78)$$

$$= B_{\perp y} \cdot 1 + (1 - B_{\perp y}) \cdot \left( -\frac{1 + B_{\perp y}}{1 - B_{\perp y}} \right) \quad (79)$$

$$= -1. \quad (80)$$

Hence, the  $W$  is feasible for  $LP_2(B, B_\perp, 0, 1)$ . Now we estimate an upper bound on  $\|W\|_2$ . We first show a lemma:

LEMMA D.2. For any matrix  $A$ ,

$$\|A\|_2 \leq \sqrt{\|A\|_1 \|A\|_\infty}. \quad (81)$$

PROOF. Denote  $A^*$  as the conjugate transpose of  $A$ , which is equal to  $A'$  when  $A$  is real, and denote  $\lambda_{\max}(\cdot)$  as the maximum eigenvalue. Then it holds that

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^*A)}. \quad (82)$$

Because the maximum eigenvalue is a lower bound on the  $\ell_\infty$  norm, we have

$$\lambda_{\max}(A^*A) \leq \|A^*A\|_\infty \quad (83)$$

$$\leq \|A^*\|_\infty \|A\|_\infty \quad (84)$$

$$= \|A\|_1 \|A\|_\infty. \quad (85)$$

Hence we prove  $\|A\|_2 \leq \sqrt{\|A\|_1 \|A\|_\infty}$ .

□

Now we sort  $\{B_{\perp y} : y \in S\}$  as  $p_1 \geq p_2 \geq \dots \geq p_k$ , in which  $k = |S| = m + 2$ . Then we have

$$\|W\|_1 = \max_{y \in S} \sum_{x \in S} |W_{xy}| \quad (86)$$

$$= \max_{1 \leq j \leq k} \left\{ 1 + (k-1) \frac{1+p_j}{1-p_j} \right\} \quad (87)$$

$$= 1 + (k-1) \frac{1+p_1}{1-p_1} \quad (88)$$

$$= k + (2k-2) \frac{p_1}{1-p_1}, \quad (89)$$

and

$$\|W\|_\infty = \max_{x \in S} \sum_{y \in S} |W_{xy}| \quad (90)$$

$$= \max_{1 \leq i \leq k} \left\{ 1 + \sum_{j \neq i} \frac{1+p_j}{1-p_j} \right\} \quad (91)$$

$$\leq \sum_{i=1}^k \frac{1+p_i}{1-p_i} \quad (92)$$

$$= k + 2 \sum_{i=1}^k \frac{p_i}{1-p_i} \quad (93)$$

$$\leq k + 2 \sum_{i=1}^k \frac{p_i}{1-p_1} \quad (94)$$

$$= k + \frac{2}{1-p_1}. \quad (95)$$

Therefore, we have

$$\|W_2\| \leq \sqrt{\left(k + (2k-2) \frac{p_1}{1-p_1}\right) \left(k + \frac{2}{1-p_1}\right)} \quad (96)$$

and

$$N_\delta = \|B^{-1}\|_2 \cdot \sqrt{\left(k + (2k-2) \frac{p_1}{1-p_1}\right) \left(k + \frac{2}{1-p_1}\right)} \quad (97)$$

is feasible for  $LP_2(B, B_\perp, 0, 1)$ .

Hence,  $N_c + \delta N_\delta$  is feasible for  $LP_1(B, B_\perp, c, \delta)$ . Because our constructions for both parts make the equality hold in (26), the final construction makes the equality hold naturally.

### D.6 Proof of Theorem 6.11

For the 0-IA property, according to Proposition 6.10 we only need to equivalently consider the case of  $|\mathcal{M}_*| + |\mathcal{C}_*|$  malicious players in the canonical Byzantine setting. From Lemma 6.6, the mechanism is 0-IA even if up to  $\frac{\delta}{2N}(n-1)$  malicious players are considered. Since  $|\mathcal{M}_*| + |\mathcal{C}_*| \leq \frac{\delta}{2N}(n-1)$ , it is indeed 0-IA.

Then we consider the colluding party. If all players in  $\mathcal{C}_*$  act honestly, since the mechanism is  $(\delta, N)$ -compact, each of them would get an interim utility of at least  $\delta$  if there were no malicious players. As there are  $|\mathcal{M}_*|$  malicious players and each can perturb  $r_{X_i}(Z_i)$  by at most  $\frac{2N}{n-1}$ , the actual interim utility of each player is at least  $\delta - \frac{2N}{n-1}|\mathcal{M}_*|$ , so the total interim utility of the colluding party is

$$u_{\mathcal{C}_*}^{\text{honest}} \geq |\mathcal{C}_*| \cdot \left( \delta - \frac{2N}{n-1}|\mathcal{M}_*| \right). \quad (98)$$

Assuming the mechanism is not weak-SCP, then there exists a case in which  $1 \leq d \leq |\mathcal{C}_*|$  players in  $\mathcal{C}_*$  act dishonestly and increase the total interim utility of the colluding party. Hence, compared to the case that all players in  $\mathcal{C}_*$  act honestly, we can model this scenario as colluding players in  $\mathcal{C}_*$  change their actions, and consider the increment of their utilities.

As we assumed,  $d$  players in  $\mathcal{C}_*$  change their actions from honest to dishonest. Since there are now at most  $|\mathcal{M}_*| + d$  dishonest players, the interim utility of each player in  $\mathcal{C}_*$  is at most  $-\delta + \frac{2N}{n-1}(|\mathcal{M}_*| + d)$ ; on the other hand,  $d$  players changing their actions may increase the interim utility of each player in  $\mathcal{C}_* - \mathcal{C}'$  by at most  $\frac{2N}{n-1}d$ . Hence, the increment of the total interim utility in  $\mathcal{C}_*$  is

$$\Delta \leq d \cdot \left( \left( -\delta + \frac{2N}{n-1}(|\mathcal{M}_*| + d) \right) - \left( \delta - \frac{2N}{n-1}|\mathcal{M}_*| \right) \right) + (|\mathcal{C}_*| - d) \cdot \frac{2N}{n-1}d \quad (99)$$

$$= \left( -2\delta + \frac{2N}{n-1}(2|\mathcal{M}_*| + |\mathcal{C}_*|) \right) d \quad (100)$$

$$\leq \left( -2\delta + \frac{4N}{n-1}(|\mathcal{M}_*| + |\mathcal{C}_*|) \right) d \quad (101)$$

$$= \left( -2\delta + \frac{4N}{n-1} \cdot \frac{\delta}{2N}(n-1) \right) d \quad (102)$$

$$= 0. \quad (103)$$

Therefore, we show that the deviation cannot increase the colluding party's total utility, i.e. the mechanism is SCP when  $|\mathcal{M}_*| + |\mathcal{C}_*| \leq \frac{\delta}{2N}(n-1)$ .

### D.7 Proof of Lemma 7.1

We only need to prove that

$$\left| \sum_{X_j \in S} P(X_j | X_i, \hat{\phi}) T_{Z_i X_j} - \sum_{X_j \in S} P(X_j | X_i, \phi) T_{Z_i X_j} \right| \leq \delta.$$



In fact,

$$\begin{aligned} & \left| \sum_{X_j \in S} P(X_j|X_i, \hat{\phi}) T_{Z_i X_j} - \sum_{X_j \in S} P(X_j|X_i, \phi) T_{Z_i X_j} \right| \\ &= \left| \sum_{X_j \in S} T_{Z_i X_j} \left( P(X_j|X_i, \hat{\phi}) - P(X_j|X_i, \phi) \right) \right| \end{aligned} \quad (104)$$

$$\leq \sum_{X_j \in S} |T_{Z_i X_j}| \cdot \left| P(X_j|X_i, \hat{\phi}) - P(X_j|X_i, \phi) \right| \quad (105)$$

$$\leq \sum_{X_j \in S} N \cdot \left| P(X_j|X_i, \hat{\phi}) - P(X_j|X_i, \phi) \right| \quad (106)$$

$$= N \cdot \sum_{X_j \in S} \left| P(X_j|X_i, \hat{\phi}) - P(X_j|X_i, \phi) \right|. \quad (107)$$

From the definition of TV distance, we have

$$TV(P(X_j|X_i, \hat{\phi}), P(X_j|X_i, \phi)) = \frac{1}{2} \sum_{X_j \in S} \left| P(X_j|X_i, \hat{\phi}) - P(X_j|X_i, \phi) \right|. \quad (108)$$

Hence,

$$N \cdot \sum_{X_j \in S} \left| P(X_j|X_i, \hat{\phi}) - P(X_j|X_i, \phi) \right| = 2N \cdot TV(P(X_j|X_i, \hat{\phi}), P(X_j|X_i, \phi)) \quad (109)$$

$$\leq 2N \cdot \frac{\delta}{2N} \quad (110)$$

$$= \delta. \quad (111)$$

Here we prove Lemma 7.1.

## D.8 Proof of Theorem A.1

(1) From Eq. (37) we have

$$\sum_{i \in C} R_i(Z_i, \mathbf{Z}_{-i}) = \sum_{i \in C} \frac{1}{n-1} \left( \sum_{j \neq i} Z_i' T Z_j \right) \quad (112)$$

$$= \frac{1}{n-1} \sum_{i \in C} \left( \sum_{j \notin C} Z_i' T Z_j + \sum_{j \in C \setminus \{i\}} Z_i' T Z_j \right) \quad (113)$$

$$= \frac{c(n-c)}{n-1} \overline{\mathbf{Z}_C}' T \overline{\mathbf{Z}_{-C}} + \frac{1}{n-1} \sum_{i \in C} \sum_{j \in C \setminus \{i\}} Z_i' T Z_j, \quad (114)$$

in which  $\overline{\mathbf{Z}_C}, \overline{\mathbf{Z}_{-C}}$  are the average reports of players inside and outside the colluding party  $C$ , respectively. Hence, we have

$$\mathbb{E}_{\mathbf{Z}_{-C} \sim P(\mathbf{X}_{-C}|\mathbf{X}_C)} \left[ \sum_{i \in C} R_i(Z_i, \mathbf{Z}_{-i}) \right] = \overline{\mathbf{Z}_C}' \cdot \frac{c(n-c)}{n-1} \mathbb{E}_{\mathbf{Z}_{-C} \sim P(\mathbf{X}_{-C}|\mathbf{X}_C)} \left[ T \overline{\mathbf{Z}_{-C}} \right] + \frac{1}{n-1} \sum_{j \in C \setminus \{i\}} Z_i' T Z_j. \quad (115)$$

We notice that for fixed  $n, c$ , scoring matrix  $T$ , and shared belief  $P(\mathbf{X}_{-C}|\mathbf{X}_C)$ , the term

$$\frac{c(n-c)}{n-1} \mathbb{E}_{\mathbf{Z}_{-C} \sim P(\mathbf{X}_{-C}|\mathbf{X}_C)} \left[ T \overline{\mathbf{Z}_{-C}} \right]$$

is a constant and  $\overline{\mathbf{Z}_C}' \cdot \frac{c(n-c)}{n-1} \mathbb{E}_{\mathbf{Z}_{-C} \sim P(\mathbf{X}_{-C}|\mathbf{X}_C)} \left[ T \overline{\mathbf{Z}_{-C}} \right]$  is linear to  $\overline{\mathbf{Z}_C}$ , the *mean* of all players' reports in  $C$ . Hence, we have

$$\overline{\mathbf{Z}_C}' \cdot \frac{c(n-c)}{n-1} \mathbb{E}_{\mathbf{Z}_{-C} \sim P(\mathbf{X}_{-C}|\mathbf{X}_C)} \left[ T \overline{\mathbf{Z}_{-C}} \right] \leq \max_{i \in C} \left\{ \mathbf{Z}_i' \cdot \frac{c(n-c)}{n-1} \mathbb{E}_{\mathbf{Z}_{-C} \sim P(\mathbf{X}_{-C}|\mathbf{X}_C)} \left[ T \overline{\mathbf{Z}_{-C}} \right] \right\}. \quad (116)$$

Let  $i^*$  be the  $i \in C$  that yields the maximum in RHS. On the other hand, we have

$$\frac{1}{n-1} \sum_{i \in C} \sum_{j \in C \setminus \{i\}} \mathbf{Z}_i' T \mathbf{Z}_j = \frac{1}{n-1} \sum_{i \in C} \sum_{j \in C \setminus \{i\}} T_{Z_i Z_j}. \quad (117)$$

Since  $\forall T_{Z_i Z_j} \in [-N, N]$ ,

$$\frac{1}{n-1} \sum_{i \in C} \sum_{j \in C \setminus \{i\}} \mathbf{Z}_i' T \mathbf{Z}_j \in \left[ -\frac{c(c-1)}{n-1} N, \frac{c(c-1)}{n-1} N \right]. \quad (118)$$

Let  $\forall \mathbf{Z}_i = \mathbf{X}_{i^*}$  and  $h = \frac{2c(c-1)}{n-1} N$ , we can see that Eq. (38) holds.

(2) We let  $\forall \mathbf{Z}_i = \mathbf{X}_i$ . Since for any  $X_i \neq X_j$  it holds that  $T_{X_i X_i} \geq T_{X_i X_j}$ , and we first adopt the assumption that  $\exists i, j \in C$  s.t.  $T_{X_i X_i} > T_{X_i X_j}$ , we have

$$\frac{1}{n-1} \sum_{i \in C} \sum_{j \in C \setminus \{i\}} \mathbf{Z}_i' T \mathbf{Z}_j = \frac{1}{n-1} \sum_{i \in C} \sum_{j \in C \setminus \{i\}} T_{X_i X_j} \quad (119)$$

$$< \frac{1}{n-1} \sum_{i \in C} \sum_{j \in C \setminus \{i\}} T_{X_i X_i} \quad (120)$$

$$= \frac{c-1}{n-1} \sum_{i \in C} T_{X_i X_i} \quad (121)$$

$$= \frac{c-1}{n-1} \sum_{i \in C} \mathbf{X}_i' \cdot \text{diag}(T) \quad (122)$$

$$= \overline{\mathbf{X}_C}' \cdot \frac{c(c-1)}{n-1} \text{diag}(T). \quad (123)$$

Conditioned on the players in  $C$  all actively verify and observe  $\mathbf{X}_C$ , their observation costs are fixed, and their expected total reward is:

$$r(\mathbf{X}_C) = \overline{\mathbf{X}_C}' \cdot \frac{c(n-c)}{n-1} \mathbb{E}_{\mathbf{X}_{-C} \sim P(\mathbf{X}_{-C}|\mathbf{X}_C)} \left[ T \overline{\mathbf{X}_{-C}} \right] + \frac{1}{n-1} \sum_{i \in C} \sum_{j \in C \setminus \{i\}} T_{X_i X_j}. \quad (124)$$

On the other hand, we let  $k = |S|$  be the number of types and define  $f: \mathbb{R}^k \rightarrow \mathbb{R}$  s.t.

$$f(Y) = Y' \cdot \frac{c(n-c)}{n-1} \mathbb{E}_{\mathbf{X}_{-C} \sim P(\mathbf{X}_{-C}|\mathbf{X}_C)} \left[ T \overline{\mathbf{X}_{-C}} \right] + Y' \cdot \frac{c(c-1)}{n-1} \text{diag}(T). \quad (125)$$

From the arguments above, it holds that

$$r(\mathbf{X}_C) < f(\overline{\mathbf{X}_C}). \quad (126)$$

Since  $f(\cdot)$  is a linear function, we have

$$f(\overline{\mathbf{X}_C}) \leq \max_{i \in C} f(\mathbf{X}_i). \quad (127)$$

Let  $i^* = \arg \max_{i \in C} f(X_i)$ , then we deduce that

$$r(\mathbf{X}_C) < f(X_{i^*}). \quad (128)$$

Therefore, for the all-same report  $\mathbf{Z}_C^* = \{X_{i^*}, \dots, X_{i^*}\}$ , we have

$$r(\mathbf{Z}_C^*) = \overline{\mathbf{Z}}_C'^* \cdot \frac{c(n-c)}{n-1} \mathbb{E}_{\mathbf{X}_{-C} \sim P(\mathbf{X}_{-C}|\mathbf{X}_C)} \left[ T\overline{\mathbf{X}_{-C}} \right] + \frac{1}{n-1} \sum_{i \in C} \sum_{j \in C \setminus \{i\}} T_{Z_i^* Z_j^*} \quad (129)$$

$$= X_{i^*}' \cdot \frac{c(n-c)}{n-1} \mathbb{E}_{\mathbf{X}_{-C} \sim P(\mathbf{X}_{-C}|\mathbf{X}_C)} \left[ T\overline{\mathbf{X}_{-C}} \right] + \frac{1}{n-1} \sum_{i \in C} \sum_{j \in C \setminus \{i\}} T_{X_{i^*} X_{i^*}} \quad (130)$$

$$= X_{i^*}' \cdot \frac{c(n-c)}{n-1} \mathbb{E}_{\mathbf{X}_{-C} \sim P(\mathbf{X}_{-C}|\mathbf{X}_C)} \left[ T\overline{\mathbf{X}_{-C}} \right] + X_{i^*}' \cdot \frac{c(c-1)}{n-1} \text{diag}(T) \quad (131)$$

$$= f(X_{i^*}) \quad (132)$$

$$> r(\mathbf{X}_C). \quad (133)$$

Hence we show that such collusion is strictly profitable in the shared-belief setting, implying that the mechanism is not strong-SCP.

If we do not adopt the assumption that  $\exists i, j \in C$  s.t.  $T_{X_i X_i} > T_{X_i X_j}$ , then Eq. (120) still holds with “ $\leq$ ”, and all the following strict inequalities become non-strict, implying that Eq. (38) holds with  $h \leq 0$ .