Volume of Solids of Revolution

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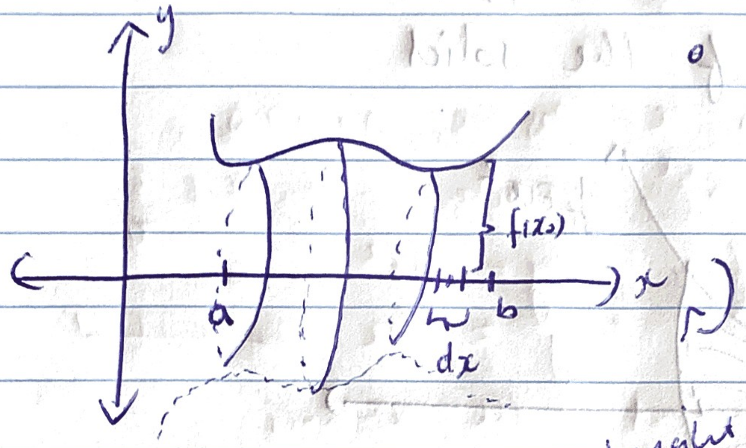
Contents

Volumes of Solids of Revolution

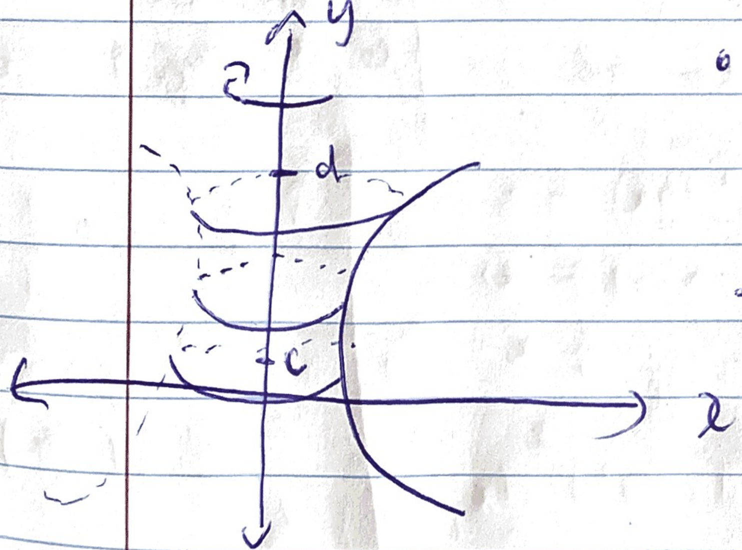
- Rotation around x-axis



$$dV = \underbrace{\pi (f(x_0))^2}_{\text{radius}} \cdot \overbrace{dx}^{\text{height}}$$

$$V = \int_a^b dV = \int_a^b \pi (f(x))^2 \cdot dx$$

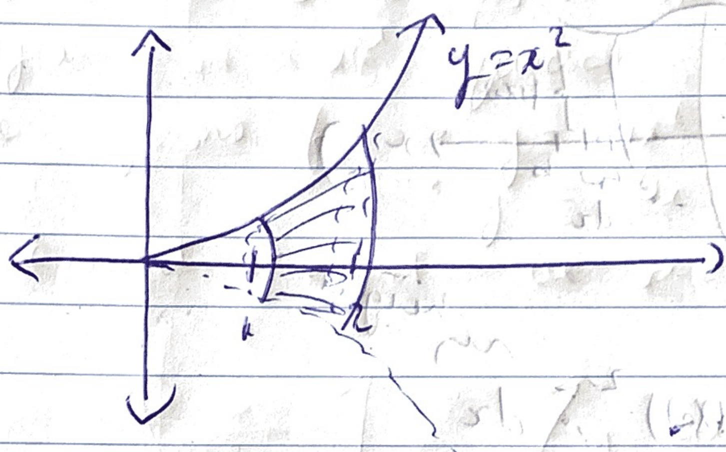
- Rotation around y-axis.



- $x = g(y)$
- $V = \int_c^d \pi (g(y))^2 \cdot dy$

Example 2.21 : What is the volume of the solid

(a)



$$V = \int_1^2 \pi (f(x))^2 \, dx$$

$$= \int_1^2 \pi (x^2)^2 \, dx$$

$$= \int_1^2 \pi x^4 \, dx$$

$$= \pi \left(x^5/5 \right) \Big|_1^2$$

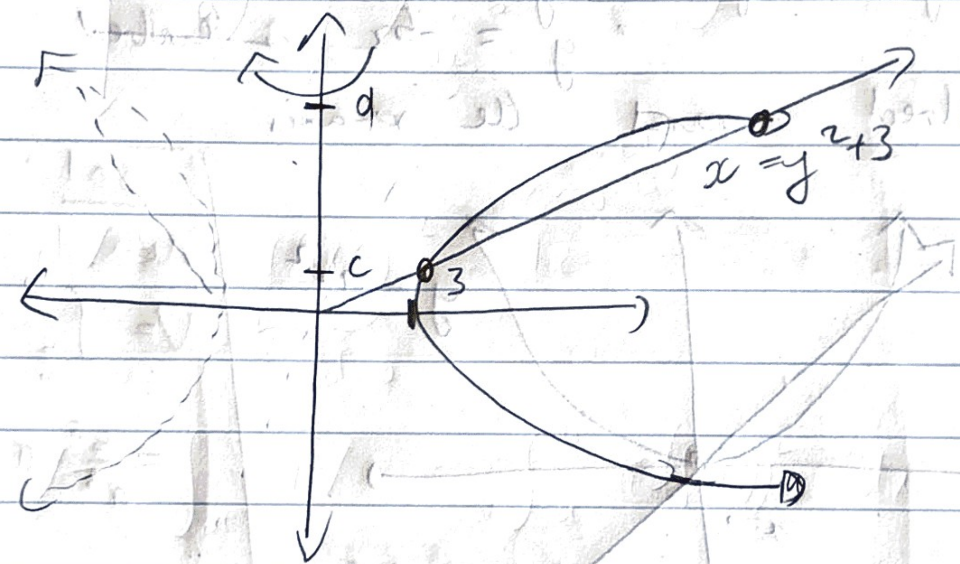
$$= \frac{\pi}{5} \left(2^5 - 1^5 \right)$$

$$= \frac{\pi}{5} (32 - 1)$$

$$\boxed{V = \frac{31}{5} \pi}$$

Example 2.21 (b): Find the volume of the solid of revolution generated by:

- The region bounded by the curve $x=y^2+3$, $x=4y$ revolved around y-axis.



$$\begin{array}{r}72\\ \times 15\\ \hline 360\\ 720\\ \hline 1080\end{array} \qquad \begin{array}{r}243\\ \times 3\\ \hline 729\end{array}$$

$$\begin{array}{r}243\\ \times 3\\ \hline 729\end{array}$$

We need points c and d.

So $y^2+3 = 4y \Rightarrow y^2-4y+3=0$

$\Rightarrow (y-3)(y-1)=0$

$\Rightarrow y=1, \; y=3.$

$\boxed{c=1,\ d=3}$

$= 270-27$
$= 250-7 = 243$

$$V = \int_1^3 \pi (4y)^2\, dy - \int_1^3 \pi (y^2+3)^2\, dy$$

$$= \pi\left[\int_1^3 16y^2\, dy - \int_1^3 y^4+6y^2+9\, dy\right]$$

$$= \pi\left[\int_1^3 -y^4+10y^2-9\, dy\right]$$

$$= \pi\left[-\frac{y^5}{5}+\frac{10}{3}y^3-9y\right]\Big|_1^3$$

$$= \pi\left[-\frac{3^5}{5}+\frac{10}{3}\cdot 3^3-9\cdot 3-\left(-\frac{1}{5}+\frac{10}{3}-9\right)\right]$$

$$= \pi\left[-\frac{243}{5}+90-27+\frac{1}{5}-\frac{10}{3}+9\right]$$

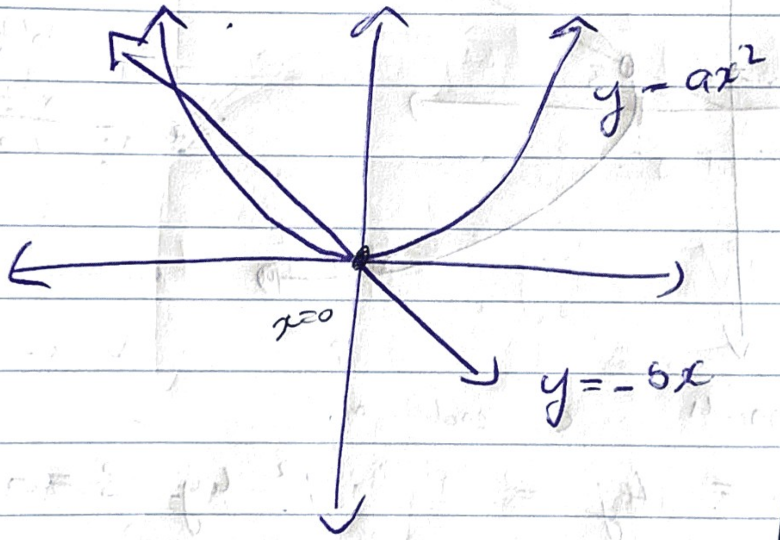
$$= \pi\left[-\frac{729}{15}+72-\frac{47}{15}\right] = \pi\left[-\frac{776}{15}+\frac{1080}{15}\right] = \boxed{\frac{304}{15}\pi}$$

Example 33

(pg 100 GRE MATH)

- let a and b be positive numbers.
- Region in 2^{nd} quadrant bounded by the graphs of:
  - $y = ax^2$
  - $y = -bx$ ~~is revolved~~

is revolved around the x-axis



Sol:

New

$$V = \int_{-\frac{b}{a}}^{0} \pi\left((-bx)^2 - (ax^2)^2\right)$$

$$= \frac{b^5}{a^3} \cdot \pi\left(\frac{2}{15}\right)$$

But for $b^5 = 2a^3$ we have $\frac{4}{15}\pi$, const

$$ax^2 = -bx \Rightarrow ax^2 + bx = 0$$

$$\Rightarrow (x)(ax+b) = 0$$

So $x = 0$ or $ax + b = 0 \Rightarrow x = -\frac{b}{a}$

Then

$$\int_{-\frac{b}{a}}^{0} \pi(-bx)^2 - \pi(ax^2)^2 \, dx$$

$$= \int_{-\frac{b}{a}}^{0} \pi \cdot b^2 \cdot x^2 - \pi a^2 \cdot x^4 \, dx$$

bla

$$= \frac{\pi b^2}{3} \cdot x^3 - \frac{\pi a^2}{5} \cdot x^5 \Big|_{-\frac{b}{a}}^{0}$$

$$= 0 - \left(\frac{\pi b^2}{3} \cdot \left(\frac{-b}{a}\right)^3 - \frac{\pi a^2}{5}\left(\frac{-b}{a}\right)^5\right) = \frac{\pi}{3} \cdot \frac{b^5}{a^3} - \frac{\pi}{5} \cdot \frac{b^5}{a^3}$$

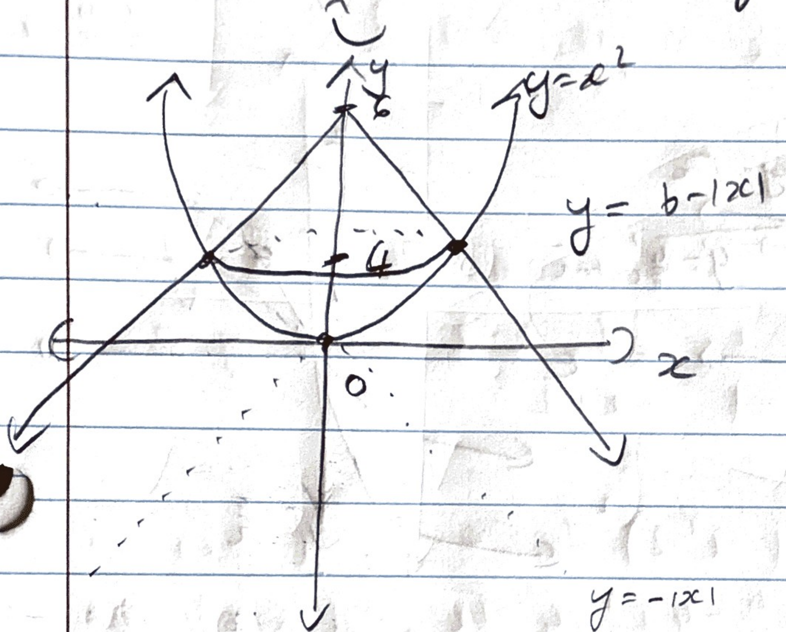
$$\frac{\pi}{3} \cdot \frac{b^5}{a^3} - \frac{\pi}{5} \cdot \frac{b^5}{a^3}$$

$$= \frac{b^5}{a^3} \cdot \pi\left(\frac{5-3}{15}\right)$$

$$= \frac{b^5}{a^3} \pi\left(\frac{2}{15}\right)$$

Example 34
(CGREa Book pg 100)

- Find the region bounded by the graph of $y = x^2$ and $y = 6 - |x|$ revolved around y-axis.
- What is the volume generated by the solid.



If $x \geq 0$, $6 - |x| = x^2$

$\Rightarrow 6 - x = x^2$

$\Rightarrow x^2 + x - 6 = 0$

$\Rightarrow (x+3)(x-2) = 0$

So $x = 2$

$\Rightarrow x^2 = 4$

$16 \times 6 = 96$

$80 + 96 = 176$

$176 - 144 = 32$

$180 - 36 = 144$, $36 \times 4 = 144$

$36 \times 6 = 216$

Separate them

$6 - x = y$

$6 - y = x$

$\sqrt[3]{216} = 6$

$\frac{64}{3}$

$96 - 64 = 32$

$$V = \int_0^6 g(y)\,dy$$

$$= \int_0^4 g_1(y)\,dy + \int_4^6 g_2(y)\,dy$$

$$= \int_0^4 \pi(\sqrt{y})^2\,dy + \int_4^6 \pi(6-y)^2\,dy$$

$$= \int_0^4 \pi \cdot y\,dy + \int_4^6 \pi(36 - 12y + y^2)\,dy$$

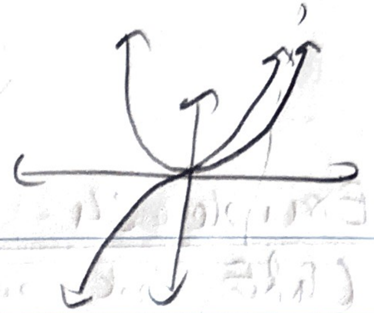
$$= \pi\left(\frac{y^2}{2}\Big|_0^4 + \left(36y - \frac{12}{2}y^2 + \frac{y^3}{3}\right)\Big|_4^6 \right)$$

$$= \pi\left(\frac{16}{2} - 0 + \left[36 \cdot 6 - 6 \cdot 6^2 + \frac{6^3}{3} - \left(36 \cdot 4 - 6 \cdot 4^2 + \frac{4^3}{3}\right)\right]\right)$$

$$= \pi\left(8 + \left[216 - 216 + 72 - \left(144 - 96 + \frac{64}{3}\right)\right)\right)$$

$$= \pi\left(8 + 72 - 144 + 96 - \frac{64}{3}\right)$$

$$= \pi\left(32 - \frac{64}{3}\right) = \pi\left(\frac{96}{3} - \frac{64}{3}\right) = \boxed{\pi\left(\frac{32}{3}\right)}$$

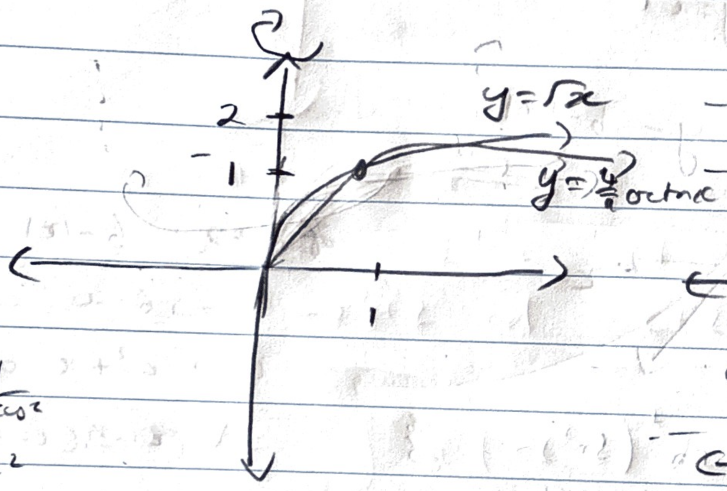


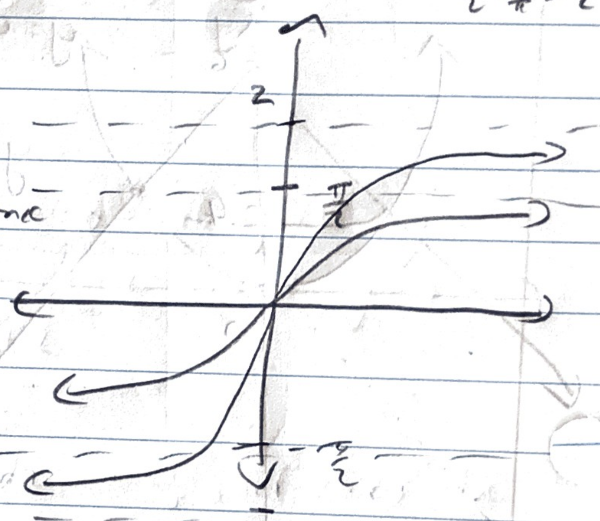
Question 23

(Rambo prac Test 2)

What is the volume of the solid of revolution bound by $y=\sqrt{x}$ and $y=\frac{4}{\pi}\arctan x$ about the y-axis?

$\frac{\pi}{2}\cdot\frac{4}{\pi}=2$





$\frac{\sin^2}{\cos^2}+\frac{\cos^2}{\cos^2}=\frac{1}{\cos^2}$

$\Rightarrow \tan^2+1=\sec^2$

$\Rightarrow \tan^2=\sec^2-1$

$x=1 \Rightarrow y=\sqrt{1}=1, \quad y=\frac{4}{\pi}\arctan(1) = \frac{4}{\pi}\cdot\frac{\pi}{4}$

$y=\frac{4}{\pi}\arctan(x)$

$y\cdot\frac{\pi}{4}=\arctan(x)$

$\tan\left(y\cdot\frac{\pi}{4}\right)=x$

$$V=\int_0^1 \pi\left(\tan\left(\tfrac{\pi}{4}y\right)\right)^2 dy-\int_0^1 \pi(y^2)^2 dy$$

$$=\pi\left[\int_0^1 \sec^2\left(\tfrac{\pi}{4}y\right)-1\,dy-\int_0^1 y^4\,dy\right]$$

$$=\pi\left[\left(\tan\left(\tfrac{\pi}{4}y\right)\cdot\tfrac{4}{\pi}-y\right)\Big|_0^1-\left(\tfrac{y^5}{5}\right)\Big|_0^1\right]$$

$$=\pi\left[\left(\tan\left(\tfrac{\pi}{4}\right)\cdot\tfrac{4}{\pi}-1\right)-0-\left(\tfrac{1}{5}-0\right)\right]$$

$$=\pi\left[\tfrac{4}{\pi}-\tfrac{\pi}{\pi}-\tfrac{1}{5}\right]$$

$$=4-\pi-\pi/5$$

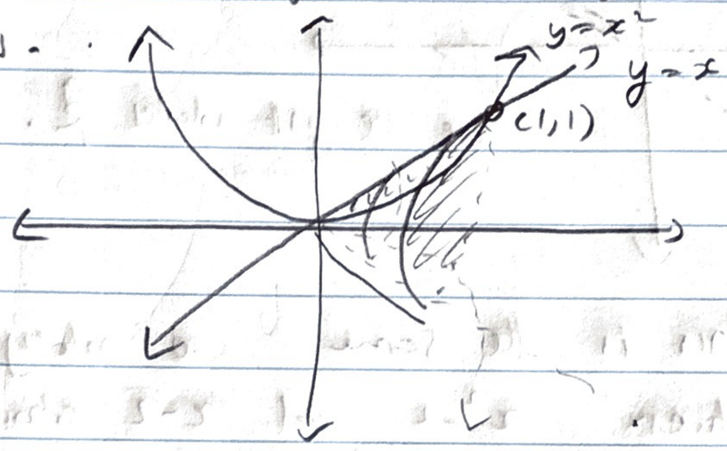
$$=\frac{20}{5}-\frac{5\pi}{5}-\pi/5$$

$$\boxed{V=\frac{20-6\pi}{5}}$$

Example 4

(Stewart Calculus pg 442)

- Region $\mathcal{R}$ is enclosed by the curves $y=x$ and $y=x^2$. Find volume of the solid of rotation about the x-axis.



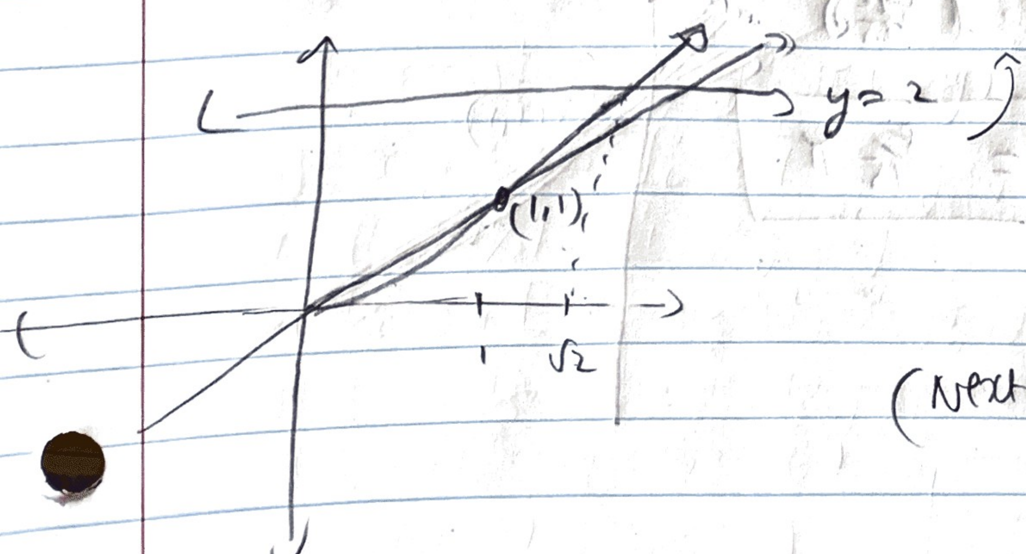
$V = \int_0^1 \pi (x)^2 \, dx - \int_0^1 \pi \cdot (x^2)^2 \, dx$

$= \int \pi x^2 \, dx - \int_0^1 \pi x^4 \, dx$

$= \pi \left(\int_0^1 x^2 - x^4 \, dx \right)$

$= \pi \left(\frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_0^1 = \pi \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{\pi}{12}$

Example 5: Now rotate the region about the line $y=2$.



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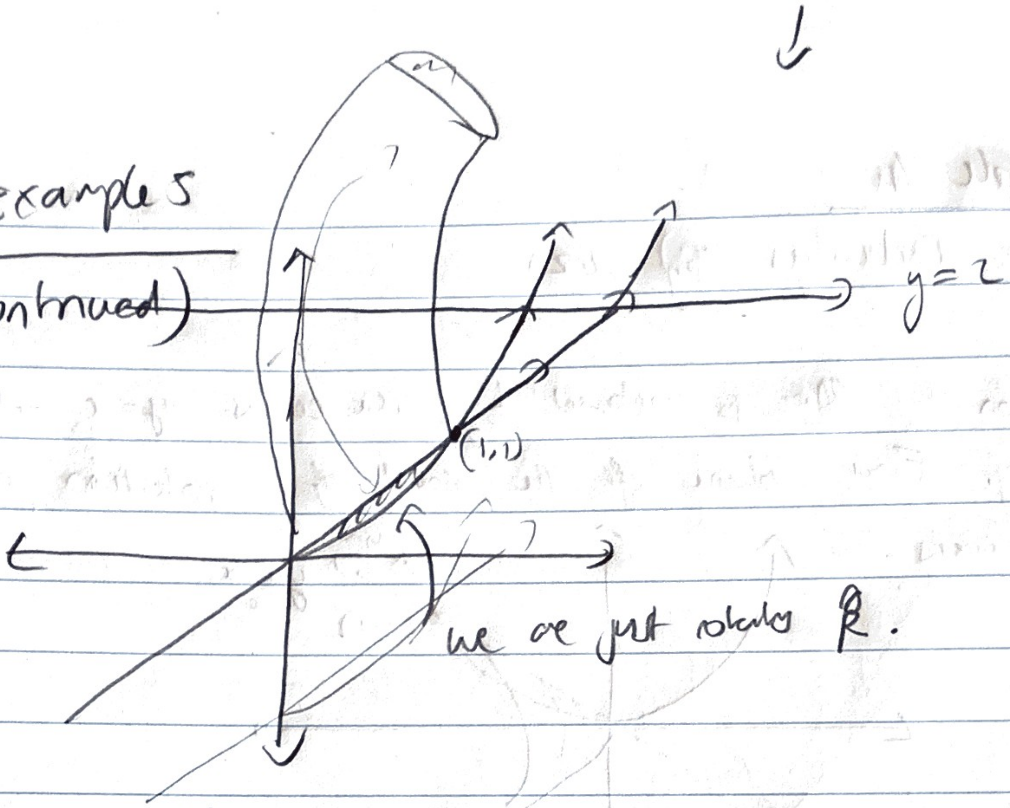
Examples 5

(continued)



* this is the same as rotating the region between x^2-2 and $x-2$ about $y=0$.

or $\underline{-x^2+2}$ and $\underline{-x+2}$

$$* \text{So } V = \int_0^1 \pi(-x^2+2)^2\,dx - \int_0^1 \pi(-x+2)^2\,dx$$

$(-x+2)(-x+2)$

$= x^2-2x-2x+4$

$$= \pi\left(\int_0^1 x^4 - 4x^2 + 4\,dx - \int_0^1 x^2 - 4x + 4\,dx\right.$$

$$= \pi\left(\int_0^1 x^4 - 5x^2 + 4x\,dx\right)$$

$$= \pi\left(\frac{x^5}{5} - \frac{5}{3}x^3 + 2x^2\right)\Big|_0^1$$

$$= \pi\left(\tfrac{1}{5} - \tfrac{5}{3} + 2\right)$$

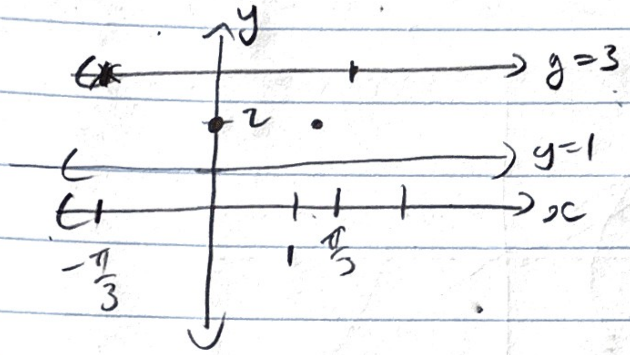
$$= \pi\left(\frac{3}{15} - \frac{25}{15} + \frac{30}{15}\right)$$

$$= \pi\left(\tfrac{8}{15}\right)$$

$$\text{So } \boxed{V = \tfrac{8}{15}\pi}$$

Exercise 13
(Stewart pg 447)

- Given region R bounded by $y = 1 + \sec x$, $y = 3$
- find Volume of solid of revolution about $y = 1$.



$*\ \sec x = \frac{1}{\cos(x)}$

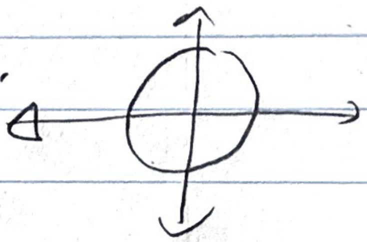
$1 + \sec(0) = 1 + \frac{1}{\cos(0)} = 2$

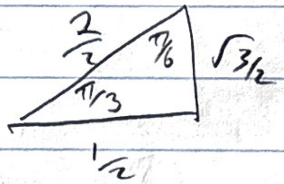
$\cdot\ 1 + \sec\theta = 3$

$\Rightarrow \sec\theta = 2$

$\Rightarrow \frac{1}{\cos\theta} = 2$

$\Rightarrow \cos\theta = 1/2.$





$\cos(\theta) = \frac{1}{2}$

$\Rightarrow \theta = \pm\frac{\pi}{3}.$

· Same as R bounded by $y = \sec x$, $y = 2$ and rotated around $y = 0$.

$$\text{So } \frac{1}{2}V = \int_0^{\frac{\pi}{3}} \pi \cdot (2)^2 dx - \int_0^{\frac{\pi}{3}} \pi(\sec x)^2 dx$$

$$\frac{1}{2}V = \pi\left[\int_0^{\frac{\pi}{3}} 4\,dx - \int_0^{\frac{\pi}{3}} \sec^2 x\,dx\right]$$

$$= \pi\left(4x\Big|_0^{\frac{\pi}{3}} - \tan x\Big|_0^{\frac{\pi}{3}}\right)$$

$$= \pi\left(4\tfrac{\pi}{3} - 0 - (\tan(\tfrac{\pi}{3}) - 0)\right)$$

$$= \pi\left(\tfrac{4\pi}{3} - \sqrt{3}\right)$$

$$= \frac{4\pi^2}{3} - \sqrt{3}\pi$$

$$= \frac{4\pi^2}{3} - \frac{3\sqrt{3}}{3}\pi$$

$\frac{\sqrt{3}}{2} \cdot \frac{2}{1} = \sqrt{3}$

$$\frac{1}{2}V = \frac{4\pi^2 - 3\sqrt{3}}{3}. \Rightarrow \boxed{V = \frac{8\pi^2 - 6\sqrt{3}}{3}}$$