MATH GRE PREP: EXTRA
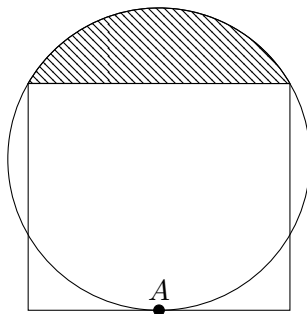
(1) In the following figure, the labeled point A is the center of the side length of a square, and the circle intersects the square in two of its vertices. What is the area of the shaded region, where s is the side length of the square?

(A) $s^2 \left(\frac{25}{64} \arcsin \left(\frac{4}{5} \right) - \frac{3}{32} \right)$

(B) $s^2 \left(\frac{25}{64} \arcsin \left(\frac{4}{5} \right) - \frac{3}{16} \right)$

(C) $s^2 \left(\frac{25}{64} \arcsin \left(\frac{3}{5} \right) - \frac{3}{32} \right)$

(D) $s^2 \left(\frac{25}{64} \arcsin \left(\frac{3}{5} \right) - \frac{3}{16} \right)$

(E) None of the above.

(2) Let W be the subset of $n \times n$ matrices over $\mathbb{R}$ such that M can be written as

$$M = AB - BA,$$

for some square matrices A and B. Which are true of W?

(I) W is a subspace.

(II) The dimension of W is $n^2 - 1$.

(III) $I \in W$

(A) None of the above are true.

(B) I

(C) III

(D) I, II

(E) I, II, III

(3) Suppose that we have k^n coins that look identical. One coin is evil and weighs less. To find the evil coin we do the following procedure: We divide our pile of coins into k identical piles. We then compare these piles using a balance to find the one with the lowest weight. Upon finding the lowest weight pile, if more than one coin remains we divide into piles again and repeat. Call this algorithm for finding the evil coin P_k. To apply on a pile that is not a power of k, we supplement the pile with other good coins.

Let $C_{i,j}$ be the number of comparisons needed to sort a pile of i coins using algorithm P_j. Sort the following numbers:

$$C_{15,2}, C_{18,2}, C_{15,3}, C_{18,3}.$$

(A) $C_{15,3} = C_{18,3} < C_{15,2} < C_{18,2}$

(B) $C_{15,3} < C_{18,3} < C_{15,2} < C_{18,2}$

(C) $C_{15,2} < C_{18,2} < C_{15,3} < C_{18,3}$

(D) $C_{15,2} < C_{18,2} < C_{15,3} = C_{18,3}$

(E) $C_{15,2} < C_{18,2} = C_{15,3} < C_{18,3}$

(4) Evaluate the following limit:

$$\lim_{x \to 0} \frac{\arctan x - x}{e^x + \cos x - x - 2}.$$

(A) 0

(B) 1

(C) -1

(D) -2

(E) The limit does not exist.

(5) Consider a graph G with vertex set $V(G) = \{(i,j) \in \mathbb{Z}^2 \ : \ 0 \le i \le 7, \ 0 \le j \le 5\}$, with edges vertical and horizontal paths between them (e.g., edges are all length 1 paths between vertices). What is the shortest possible path along the edges that visits every edge in the graph, starting at $(0,0)$ and ending at $(7,5)$?
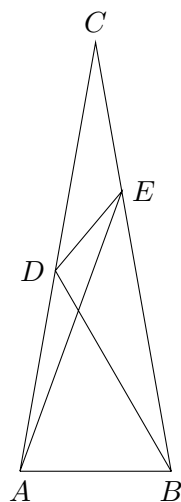Note the accident has been cleared.

(A) 82

(B) 86

(C) 90

(D) 94

(E) 98

(6) What is the probability that three randomly selected points on a circle form an acute triangle?

(A) 1/6

(B) 1/4

(C) $1/\pi$

(D) 1/3

(E) 1/2

(7)

In the figure above, if $\angle EAB = 70°$, $\angle DBA = 60°$, $\angle DAE = 10°$, and $\angle EBD = 20°$, then what is the measure of $\angle DEA$?

(A) 10°

(B) 20°

(C) 30°

(D) 45°

(E) 60°

(8) For every integer $n \geq 3$, let A_n be the area of a regular n-gon of side length n^{-1}. What is $\lim_{n\to\infty} A_n$?

(A) $4/\pi$

(B) $2/\pi$

(C) $1/\pi$

(D) $1/2\pi$

(E) $1/4\pi$

(9) Suppose we have four points on the vertices of a square of side length 1. What range contains the smallest total length of path needed to make the four points connected?

(A) (2.5,2.6)

(B) (2.6,2.7)

(C) (2.7,2.8)

(D) (2.8,2.9)

(E) (2.9,3.0)

Answers

(1) (B): This is plane geometry. Compute.

(2) (D): These are traceless matrices.

(3) (A): Doing the three-way comparison takes one use of scale.

(4) (D): Use Taylor Series to evaluate.

(5) (D): Add paths until every vertex has even valence, except start and end with odd valence.

(6) (B): Compute it.

(7) (B): Draw line parallel to AB through D; label intersection with BC as F; draw FA; label intersection with DB as G. Note CG is angle bisector of ACB. Now solve as normal.

(8) (E): Note all the perimeters are constant; compute area of circle of circumference 1.

(9) (C): The best is $1 + \sqrt{3}$.