MATH GRE PREP: WEEK 3

UCHICAGO REU 2019

(1) Evaluate
$$\oint_C y^3\,dx - x^3\,dy,$$
where C is the boundary of the positively oriented annulus with inner radius 1 and outer radius 2 centered at the origin.

(A) $-\frac{45\pi}{2}$

(B) $\frac{45\pi}{2}$

(C) 14π

(D) 36π

(E) 0

(2) Let X be $\mathbb{R}$ with the topology given by letting the cocountable sets be open.
Let $Y = (X \times [0,1])/((x,t) \sim (x',t') \iff t = t' = 1)$. Which of the following is false?

(A) Y is connected.

(B) Y is locally connected.

(C) Y is path-connected.

(D) Y is hyperconnected (all non-trivial open sets intersect).

(E) X is hyperconnected.

Date: July 22, 2019.

(3) Let X, Y, Z be vectors spaces of dimension 7. Let A_1, A_2 be subspaces of X of dimension 4. Let $B_i = f(A_i)$. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be linear maps such that $g \circ f$ is not bijective. If h is a linear map and C is a subspace of the domain, denote by $h|_C$ the restriction of h to C. Which of the following cannot happen?

(A) $f|_{A_1+A_2}$ is injective

(B) $f|_{A_1+A_2}$ is surjective

(C) $g|_{B_1+B_2}$ is injective

(D) $g|_{B_1+B_2}$ is surjective

(E) $(g \circ f)|_{A_1+A_2}$ is injective

(4) For what values of a does the system $y = x^3 - 6ax^2 + 33$ and $y = a$ have three solutions?

(A) None

(B) $a > 0$

(C) $0 < a < 33$

(D) $1 < a < 33$

(E) All $a \neq 0$ in $\mathbb{R}$.

(5) Consider the simultaneous system of differential equations:

$$\begin{aligned} x'(t) &= y(t) - x(t)/2 \\ y'(t) &= x(t)/4 - y(t)/2. \end{aligned}$$

If $x(0) = 2$ and $y(0) = 3$, then what is $\lim_{t\to\infty}(x(t) + y(t))$?

(A) The limit does not converge, or is not unique.

(B) 6

(C) 8

(D) 10

(E) 12

(6) Let B be the unit ball $\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 < 1\}$. Evaluate the integral

$$\iiint_B 3x^2 + y^2 + z^2 + 2 \ dxdydz.$$

(A) 1

(B) π

(C) 2π

(D) 4π

(E) π^2

(7) Evaluate the sum:

$$\sum_{n=1}^{\infty} \log\left(1 + \frac{8}{n^2 + 9n}\right).$$

(A) The sum does not converge.

(B) 1

(C) $\log 2$

(D) $\log 8$

(E) $\log 9$

(8) The following algorithm failed to be commented:

```
C := 1
B := 2
input(A)
    while B =< A
        if A % B == 0:
           C := B
           A := A / B
        else:
           B += 1
return C
```

If the number $368,039$ is inputted, what is the output?

(A) 1

(B) 7

(C) 29

(D) 37

(E) 7511

(9) We say $f\colon \mathbb{R} \to \mathbb{R}$ is lower semi-continuous provided $f^{-1}(a,\infty)$ is open for every $a \in \mathbb{R}$.

(I) The characteristic function $\chi_{(-1,1)}$ is lower semi-continuous.

(II) If $(f_\alpha)_{\alpha\in A}$ is a family of lower semi-continuous functions, then $f(x) := \sup_{\alpha\in A} f_\alpha(x)$ is also lower semi-continuous.

(III) If f is lower semi-continuous and $K \subseteq \mathbb{R}$ is compact, then f attains a minimum on K.

Which of the above statements are true?

(A) I only

(B) I and II

(C) I and III

(D) II and III

(E) I, II, and III

(10) Which of the following sets has the largest cardinality?

(A) The set of topologies on the real line.

(B) The set of functions (not necessarily continuous) $f : \mathbb{R} \to \mathbb{R}$.

(C) The set of all continuous functions $f : \mathbb{R}^n \to \mathbb{R}^n$ (for any/all n).

(D) The set of all functions $f : \mathbb{Z} \to \mathbb{R}^{|\mathbb{R}|}$.

(E) The set of all subsets of planes that pass through the origin of $\mathbb{R}^7$.

(11) Let $z = x+iy$, $x, y \in \mathbb{R}$, and consider a function $f(z) = g(x, y)+i\cdot h(x, y)$ with $g, h \colon \mathbb{R}^2 \to \mathbb{R}$. Suppose f is holomorphic, $g(x, y) = x^5 - 10x^3y^2 + 5xy^4$, and $f(0) = i$. What is $f(1 + 2i)$?

(A) $41 - 37i$

(B) $41 - 40i$

(C) $41 - 41i$

(D) $41 - 42i$

(E) $41 - 45i$

(12) How many similarity classes of 2×2 complex matrices are there such that $A^n = I$?

(A) $n^2/2$

(B) $n(n - 1)/2$

(C) $(n^2 - n + 1)/2$

(D) n^2

(E) $n(n + 1)/2$

(13) What is the set of solutions to the equation below, for $x, y \in \mathbb{R}$?

$$\begin{vmatrix} x-y & 0 & 0 \\ \cos^2 x - \sin^2 y & \cos x & \sin y \\ \sin^2 x - \cos^2 y & \sin x & \cos y \end{vmatrix} = 0.$$

(A) $\{x = y\}$

(B) $\{x = y\} \cup \{x = -y\}$

(C) $\{x = y\} \cup \{x + y = \pi/2 \bmod \pi\}$

(D) $\{x = y\} \cup \{x + y = 0 \bmod \pi\}$

(E) $\{x = -y\} \cup \{x + y = \pi \bmod 2\pi\}$

(14) Suppose f is continuously differentiable with $f(1) = 1$ and $f'(1) = 2$. Find the value of

$$\frac{d}{dx}\left(\frac{f(e^{2x-2})}{xf(x)}\right),$$

at $x = 1$.

(A) -2

(B) -1

(C) 0

(D) 1

(E) 2

(15) How many injections are there from $\{1, \ldots, 4\}$ to $\{1, \ldots, 10\}$?

(A) 0

(B) 210

(C) 2160

(D) 5040

(E) 30240

(16) Suppose that A and B are two square matrices and that $B^2A - A$ is invertible. Then which of the following is true?

(A) A is not invertible.

(B) AB is invertible.

(C) $AB - A$ is invertible.

(D) B has 1 as an eigenvalue.

(E) B has -1 as an eigenvalue.

(17) Compute the following integral:

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx.$$

(A) π

(B) $\pi/3$

(C) $\pi/2$

(D) $\pi^2/2$

(E) $\pi^2/4$

(18) Assume $f \colon \mathbb{R} \to \mathbb{R}$ is smooth. Compute the following limit:

$$\lim_{h \to 0} \frac{f(x + 4h) - 2f(x) + f(x - 4h)}{h^2}.$$

(A) 0

(B) $8f'(x)$

(C) $8f''(x)$

(D) $16f''(x)$

(E) The limit does not exist.

(19) Integrate:

$$\int_0^2 \log(1+x^2)\,dx.$$

(A) $\log 5 + 2\arctan(2) - 4$

(B) $\log 5 + \arctan(2) - 4$

(C) $2\log 5 + 2\arctan(2) - 4$

(D) $2\log 5 + \arctan(2)$

(E) $2\log 5 + 4\arctan(2)$

(20) Consider the set of integers $\mathbb{Z}$. Let $\mathscr{U}$ be the set of all subsets of $\mathbb{Z}$ that are arithmetic progressions (e.g., $U \in \mathscr{U}$ if there exist $a \in \mathbb{Z}$, $b \in \mathbb{N}_{>0}$ such that $U = \{a + bn \ : \ n \in \mathbb{Z}\}$). Let X be the integers with $\mathscr{U}$ as a base. Which of the following are true?

(I) X is metrizable.

(II) The sequence $n!$ converges in X.

(III) Addition, multiplication, and negation are continuous (e.g., X is a topological ring).

(A) I only.

(B) III only.

(C) II and III.

(D) I and II.

(E) I, II, and III.

(21) Suppose that G is a finite group and that all of its conjugacy classes are the same cardinality. Moreover, suppose that p is prime and $p^n \mid G$ but $p^{n+1} \nmid G$. Then which are true?

(I) If H and H' are subgroups of G of order p^{n-1}, then they are conjugate.

(II) G is abelian.

(III) $x \mapsto x^{-1}$ is a group automorphism.

(A) None of the above

(B) I

(C) I, II

(D) II, III

(E) I, II, III

(22) Consider the following multiplication tables.

(I)

$\cdot$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	c	b	c

(II)

$\star$	a	b	c	d
a	d	c	b	a
b	c	b	a	d
c	b	a	d	c
d	a	d	c	b

(III)

$\square$	a	b
a	a	a
b	a	a

Which represent a group?

(A) None of the above are groups.

(B) I only

(C) II only

(D) III only

(E) I and II

Answers

(1) (A): Use Stokes (with polar coordinates).

(2) (D): All the rest follow from definition. Note Y is not locally path-connected, but this is difficult.

(3) (D): If this were true, then $g \circ f$ would be surjective and hence bijective.

(4) (D): Differentiate and analyze. Alternatively, analyze answer options.

(5) (B): Solve via matrices in the standard way.

(6) (D): Note this is the only reasonable answer. Can solve via Stokes.

(7) (E): Combine terms, and get a telescoping sum.

(8) (D): Program outputs largest prime factor.

(9) (E): All of these follow quickly from the definition.

(10) (A): It should be evident all of the others are bounded by $2^{\mathbb{R}}$. Actually proving this is bigger is hard.

(11) (A): Use the Cauchy-Riemann equations to determine f.

(12) (E): The number of unordered pairs of n^{th} roots of unity.

(13) (C): Cosine addition formula.

(14) (D): Compute the derivative; should be three terms or so.

(15) (D): $10 \cdot 9 \cdot 8 \cdot 7$

(16) (C): Factor the expression.

(17) (E): Use the integration trick of flipping the bounds and adding this to original integral to kill the x.

(18) (D): This is difference quotient; or use a Taylor series.

(19) (C): Integration by parts.

(20) (E): Urysohn metrization, $n! \to 0$, and evident (but obnoxious to show).

(21) (D): The identity is its own conjugacy class, so G is abelian.

(22) (A): Fails invertibility, identity, and identity.