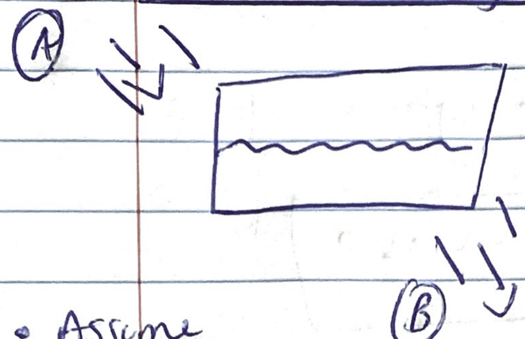
A large tank contains 20 liters of pure water. Then a process begins where water containing 50 grams per liter of salt is pumped into the tank at a rate of 3 liters per
minute. The well-mixed tank is drained at 2 liters per minute. Let y(t) be the amount of salt (in grams) in the tank at t >= 0 minutes since the pumping began.
Set up an Initial Value Problem (IVP) for y(t) that models this situation and solve the IVP.

Another mixing problem



(t=0)
Initial : 20 L water

(A) Water In : • $50 \, g/L$ (salt concentration)
• $3 \, L/min$ (Rate in)

• Assume uniform mixing

(B) Water out :
• $2 \, L/min$ (Rate out)

Let $y(t)$: Amount of salt in grams at time $t \geq 0$ minutes.

Then $y'(t)$: Change in grams of salt at time t.

Also ~~$y'(t)$ = Rate in - Rate out~~

kg salt

Also $y'(t) =$ Amount in $-$ Amount out

$$= \left(\frac{50g}{1 \text{ Liter}}\right) \cdot \left(\frac{3L}{1 \text{ min}}\right) - \left(\, ? \, \right)\left(\frac{2L}{1 \text{ min}}\right)$$

What is the concentration of salt at time t? Well $y(t)$ is kg salt.
And let $V(t)$ be the amount of water in liter at time t.

So $V(t) = 20 \text{ liter} + (\text{Rate in} - \text{Rate out})\, t$
$= 20 + (3-2)\, t$
$= 20 + t.$

So concentration salt at time t is $\dfrac{y(t)}{V(t)} = \dfrac{y(t)}{20+t}$

Thus $y' = 50 \cdot 3 - \dfrac{y}{20+t} \cdot 2$ (next page) ↓

Mixing problem (Continued)

We have $y' = 150 - \frac{2}{20+t} \cdot y$

Standard form: $\boxed{y' + \frac{2}{20+t} y = 150}$

lets use Integrating factors to solve this first order linear diff eq.

Now let μ be our integrating factor.

Then $\mu(y' + \frac{2}{20+t} y) = \mu \cdot 150$

$\Rightarrow \mu y' + \mu \cdot \frac{2}{20+t} \cdot y = \mu \cdot 150$

$\Rightarrow \frac{d}{dt}(\mu y) = \mu \cdot 150$

So $\mu' = \mu \cdot \frac{2}{20+t} \Rightarrow \int \frac{\mu'}{\mu} dt = \int \frac{2}{20+t} dt$

$\Rightarrow \ln|\mu| = 2 \ln|20+t| + c$

$\Rightarrow \ln|\mu| = \ln((20+t)^2)$

$\Rightarrow \boxed{\mu = (20+t)^2}$

Then $\int \frac{d}{dt}(\mu y) \, dt = \int 150 \cdot \mu \, dt$

$\Rightarrow \mu y = \int 150 (20+t)^2 \, dt$

$\Rightarrow (20+t)^2 y = 150 \cdot \frac{(20+t)^3}{3} + c$

$\Rightarrow (20+t)^2 y = 50 \cdot (20+t)^3 + c$

$\Rightarrow y = 50 \cdot (20+t) + \frac{c}{(20+t)^2}$

Now on next page find c using $y(0)$.

III

Mixing Problems (continued)

Since initial concentration is: • 0 grams/liter

• 20 Liter

So $y(0) = 0$

So $y(0) = 0 \Rightarrow 50(20+0) + \dfrac{c}{(20+0)^2} = 0$

$\Rightarrow 1000 + \dfrac{c}{(20+t)^2} = 0$

$\Rightarrow \dfrac{c}{20^2} = -1000$

$\Rightarrow c = -400\,000$

So $$y(t) = 50(20+t) - \frac{400\,000}{(20+t)^2}$$