MATH GRE PREP: WEEK 5

UCHICAGO REU 2019

(1) Consider the following multiplication table.

$\cdot$	a	b	c	d	e
a	d	c	e	a	b
b	e	d	a	b	c
c	b	e	d	c	a
d	a	b	c	d	e
e	c	a	b	e	d

Which of the following properties are true?

(I) The binary operation $\cdot$ is associative.

(II) The binary operation $\cdot$ is both left and right cancellative (e.g., for all x, y, there exist unique p, q such that $x \cdot p = y$ and $q \cdot x = y$).

(III) The binary operation $\cdot$ has an identity element.

(A) None of the above are true (the object is a magma).

(B) II only (the object is a quasigroup).

(C) I and III (the object is a monoid).

(D) II and III (the object is a loop).

(E) I, II, and III (the object is a group).

Date: August 6, 2019.

(2) Let $X = \mathbb{Z}_{>0}$. Define the Hjalmar Ekdal topology $\mathcal{T}$ on X by $Y \in \mathcal{T}$ if the successor of every odd integer in Y is also in Y. Which of the following are properties of this topology?

I. It is compact.

II. It is locally path connected. Namely, for every $x \in X$ and every neighborhood N of x, there is a subneighborhood that is path connected.

III. It is totally disconnected, i.e. all connected components are points.

(A) None

(B) I and II only

(C) I and III only

(D) II only

(E) III only

(3) Given the following system of equations, what is x?

$$\begin{aligned} x + y + z &= 10 \\ w + y + z &= 7 \\ w + x + z &= -3 \\ w + x + y &= 4 \end{aligned}$$

(A) -2

(B) -1

(C) 0

(D) 1

(E) 2

(4) Let A, B, C be sets with $|A| = 15, |B| = 10$, which of the following ensure that $|C| \geq 25$?

I. There are surjections $C \to A$ and $C \to B$ and A and B are disjoint.

II. $A \cup B \subseteq C$.

III. $C \setminus B \subseteq A$.

IV. $\mathcal{P}(A \setminus B) \setminus C = \emptyset$.

(A) I only

(B) II only

(C) IV only

(D) II and III

(E) III and IV

(5) Let $A \subset \mathbb{R}$ be a set that contains the rationals.

I. If A has positive measure (e.g., length), then $A = \mathbb{R}$.

II. If A is open, then $A = \mathbb{R}$.

III. If A is connected, then $A = \mathbb{R}$.

Which of the above must be true?

(A) All of these are true.

(B) II only

(C) I and III only

(D) III only

(E) I and II only

(6) What is

$$\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots?$$

(A) $\log 3$

(B) $\log 2$

(C) $2\log 2$

(D) e

(E) $-\frac{1}{2}+\log 2$

(7) Let $\alpha, \beta \in \mathbb{R}$ be such that

$$\lim_{x\to 0}\frac{x^2\sin(\beta x)}{\alpha x-\sin x}=1.$$

What is $6(\alpha+\beta)$?

(A) 5

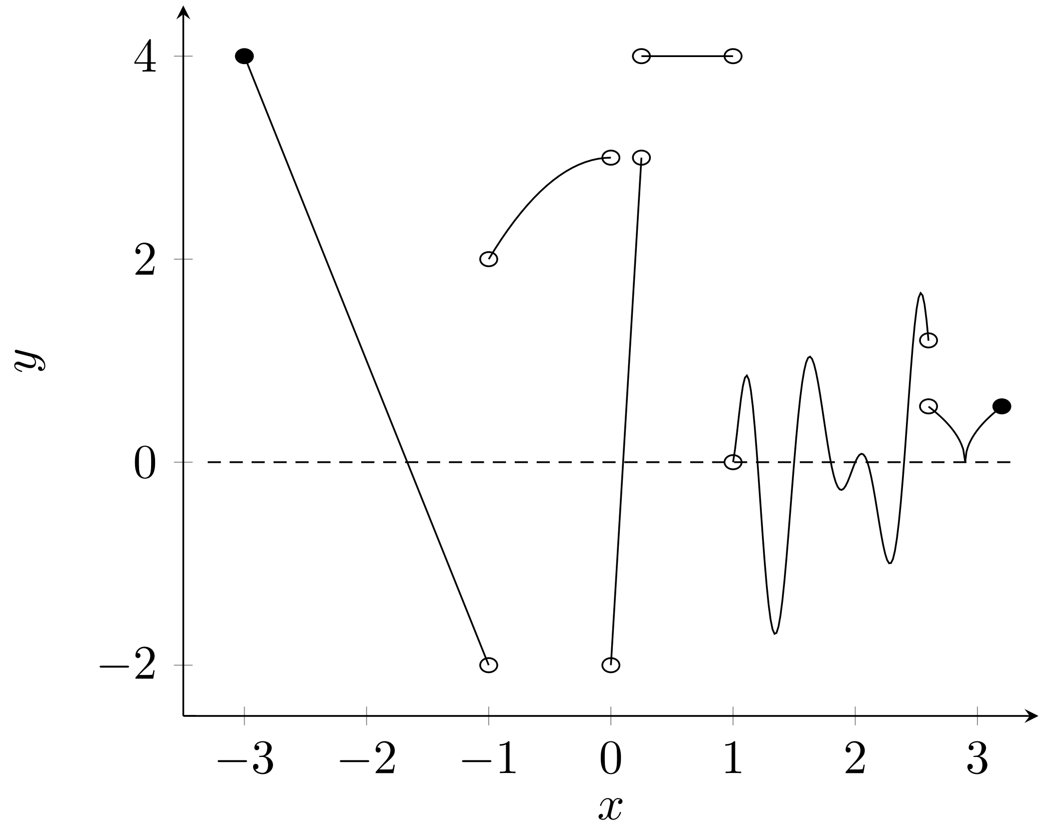
(B) 6

(C) 7

(D) 8

(E) 9

(8) Consider the following graph of $f''(x)$, for f a function defined on $[-3,3]$.



Which of the following is incorrect? (For ease of interpretation, note that every point given as an x-coordinate is a zero of $f''(x)$, e.g., the point $x = 0.1$ corresponds to the point $(0.1, 0)$ on the graph.)

(A) It is possible that f is continuously differentiable.

(B) The function f' achieves local maxima at $x = -5/3,\ x = 1.2,\ x = 1.8$, and $x = 2.1$.

(C) The function f''' has a local maximum at $x = 0.1$.

(D) If $f'(x) = \int_0^x f''(x)dx$, then f achieves its minimum in the range $[0, 0.25]$.

(E) The function f'' is differentiable wherever it is defined.

(9) Which of the following is the smallest value of n for which the following limit exists for all $r \geq n$?

$$\lim_{(x,y)\to(0,0)} \frac{x^r}{|x|^2 + |y|^2}$$

(A) 1

(B) 1.5

(C) 2

(D) 2.5

(E) 3

(10) What is the length of the curve $\langle t, t \cdot \sin t, t \cdot \cos t \rangle$, $0 \leq t \leq \pi$?

(A) $\frac{\pi}{4}\sqrt{2+\pi^2} + \frac{1}{2}\operatorname{arcsinh}(\pi/\sqrt{2})$

(B) $\frac{\pi}{2}\sqrt{2+\pi^2} + \operatorname{arcsinh}(\pi/\sqrt{2})$

(C) $\pi\sqrt{2+\pi^2} + \operatorname{arcsinh}(\pi/\sqrt{2})$

(D) $\pi\sqrt{2+\pi^2} + 2\operatorname{arcsinh}(\pi/\sqrt{2})$

(E) $2\pi\sqrt{2+\pi^2} + 4\operatorname{arcsinh}(\pi/\sqrt{2})$

(11) Solve the following differential equation.

$$y' = \cos(x - y)$$

(A) $y - \tan\frac{x-y}{2} = C$

(B) $x + \tan\frac{x-y}{2} = C$

(C) $x + \cot(x - y) = C$

(D) $y + \sin(x - y) = C$

(E) $x + \cot\frac{x-y}{2} = C$

(12) For $r \in \mathbb{R}$, consider the limit:

$$\lim_{z \to e^{i\pi/2r}} \frac{z - e^{i\pi/2r}}{z^{2r} + 1}.$$

What is the largest set (ordered by containment) where the above limit exists and is non-zero for all r in the set?

(A) $\varnothing$

(B) $r > 0$, and r is an integer

(C) $r \geq 1/2$

(D) $r > 0$

(E) $r \neq 0$

(13) Consider the matrix

$$\begin{pmatrix} 1 & 2 & x \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}.$$

For which value of x is this matrix not invertible?

(A) -1

(B) 0

(C) 2

(D) 3

(E) 5

(14) Let A be the unit 2-sphere in $\mathbb{R}^3$. Let $F = (x^3 - y^2 z^4, 2y^3, z^3 - 3zy^2)$ be a vector field. Let $\overrightarrow{n}$ be the outward-pointing normal. Evaluate:

$$\iint_A F \cdot \overrightarrow{n}\, dS.$$

(A) 3π

(B) π

(C) $\dfrac{12\pi}{5}$

(D) $\dfrac{-3\pi}{2}$

(E) 0

(15) A function f is called Hölder continuous of exponent α if:

$$\exists c \in \mathbb{R} \ : \ \forall x, y, \ |f(x) - f(y)| \leq c|x - y|^{\alpha}.$$

Which of the following is incorrect?

(A) If $f\colon [0,1] \to \mathbb{R}$ is Hölder continuous of exponent $3/2$, then f is constant.

(B) If $f\colon [0,1] \to \mathbb{R}$ is C^1, then f is Hölder continuous of exponent α, for all $0 \leq \alpha \leq 1$.

(C) If f is Hölder continuous of exponent $\alpha > 0$, then f is uniformly continuous.

(D) If $f\colon \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, then f is Hölder continuous of exponent α for all $0 < \alpha \leq 1$.

(E) The function $f(x) = \sqrt{x}$ is Hölder continuous of exponent $1/2$.

(16) Suppose that A is a real matrix with non-negative eigenvalues, and B is a real matrix with eigenvalues of absolute value less than one.

I. $I + A$ is invertible

II. $I + B$ is invertible

III. $I - A$ is inverible

IV. $I - B$ is invertible

Which of the above are true?

(A) II only

(B) II and IV only

(C) I and III only

(D) I, II, and IV only

(E) All of the above are true.

(17) What is the length of the curve $x^{2/3} + y^{2/3} = 4$?

(A) 48

(B) 52

(C) 56

(D) 60

(E) 64

(18) Suppose A is a 3×3 matrix wtih entries in $\mathbb{R}$. Assume that $\det(A) = 6$, $\mathrm{tr}(A) = 6$, and that 3 is an eigenvalue of A. Compute $\mathrm{tr}(A^2)$.

(A) -7

(B) 7

(C) 14

(D) 36

(E) 40

(19) Let M_n be the vector space of $n \times n$ matrices over $\mathbb{R}$. For a matrix $A \in M_n$, define $L_A \colon M_n \to M_n$ by
$$L_A(B) = AB.$$
Let U be the subset of M_n comprising of upper triangular matrices with diagonal entries 1 endowed with the obvious linear structure. Which of the following is false?

(A) The map $L_A \colon M_n \to M_n$ is linear.

(B) If $A \in U$ then the restriction $L_A|_U$ is a linear isomorphism of U.

(C) $\dim M_n = n^2$ and $\dim U = \frac{n(n-1)}{2}$

(D) If $A = \lambda I$, then $\det L_A = \lambda^{n^2}$.

(E) L_A is invertible if and only if A is invertible.

(20) Consider the polynomial
$$x^3 - 3x + a.$$
Which is the largest range of a for which this polynomial has three distinct real roots?

(A) $a > 0$

(B) $|a| < 2$

(C) $|a| \leq 2$

(D) $|a| < 1/2$

(E) $|a| \leq 3$

(21) Find the maximum of x^2y on the curve $x^2 + 2y^2 = 6$.

(A) 3

(B) 4

(C) 5

(D) 6

(E) 7

(22) Suppose that N is a nonzero 2 by 2 matrix over $\mathbb{C}^2$ such that $N^{2019} = 0$. Then which matrix need N be similar to, over $\mathbb{C}$, of course.

I.
$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

II.
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

III.
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

(A) I only

(B) II only

(C) I and II only

(D) I and III only

(E) I, II, and III

Answers

(1) (D): Checking associativity is a pain: note it is not $\mathbb{Z}/5\mathbb{Z}$ the only group of order 5.

(2) (D): Obviously non-compact, and $\{1,2\}$ is a connected component. Note $f(x) = \lfloor x \rfloor$ is a path from $0 \to 1$.

(3) (B): Solve it (note: fastest to sum all of the equations).

(4) (C): Consider disjointness in the other cases.

(5) (D): Enumerate the rationals, and take exponentially decreasing open sets containing each one. Consider the union.

(6) (B): Recognize the Taylor series for log.

(7) (C): Use Taylor series to evaluate, should get $\alpha = 1$ and $\beta = 1/6$.

(8) (E): It is not differentiable at about $x = 2.8$.

(9) (D): You can take two different paths when $r = 2$; else, just let $y = 0$ to bound it from above.

(10) (B): Compute the integral (hyperbolic substitution).

(11) (E): Substitute $u = x - y$, solve separable differential equation.

(12) (C): There is an issue with choice of branch, e.g., let $r = 1/4$ and note $(e^{i\pi/2r})^{2r} = 1$.

(13) (D): Set determinant equal to zero.

(14) (C): Use Stokes theorem (after converting to spherical coordinates).

(15) (D): This is only true locally. Note the identity function is Lipschitz, but not Hölder.

(16) (D): Use Jordan blocks. Or recall that $\det(A + B) \geq \det(A) + \det(B)$.

(17) (A): This is an astroid; you could parameterize, to make the integral easier.

(18) (C): Eigenvalues are $1, 2, 3$; note eigenvalues of square is square of eigenvalues.

(19) (B): U is not a linear subspace.

(20) (B): Differentiate to determine where local maximum/minimum is.

(21) (B): Use Lagrange multiplers.

(22) (E): Use Jordan Canonical form, and deduce.