MATH GRE PREP: WEEK 2

UCHICAGO REU 2019

(1) What is the coefficient of y^3x^6 in $(1+x+y)^5(1+x)^7$?

(A) 70

(B) 840

(C) 350

(D) 420

(E) 270

(2) Which of the following is closest to the value of this integral:

$$\int_0^1 \sqrt{1+\frac{1}{3x}}dx.$$

(A) 1

(B) 1.2

(C) 1.6

(D) 2

(E) The integral doesn't converge.

Date: July 15, 2019.

(3) Consider the following algorithm:

```
s = 0
x = -1000
while x < 1000
   y = -1000
   while y < 1000
      if (x * x) + (y * y) < 1 000 000:
         s = s + (x * x)
      y = y + 1
    x = x + 1
output s
```

Which of the following is closest to the output of this program?

(A) 10^6

(B) 10^9

(C) 10^{12}

(D) 10^{15}

(E) 10^{18}

(4) Consider the following complex functions, $z = x + iy$ with $x, y \in \mathbb{R}$:

I. $f(z) = x + \sin x \cosh y + i(y + \cos x \sinh y)$

II. $g(z) = x + \cos x \cosh y + i(y + \sin x \sinh y)$

III. $h(z) = y - 2xy + i(x^2 - y^2 - x) + z^2$

Which of these functions are holomorphic?

(A) I only

(B) II only

(C) III only

(D) I and III only

(E) All of them are holomorphic.

(5) Suppose that R is an integral domain. Moreover, suppose that the ideal generated by the element $r^2 - r$ for all $r \in R$ is equal to the ideal generated by 0. Which of the following hold?

I. R is abelian

II. R has characteristic 2

III. R is a field.

(A) I only

(B) I and II only

(C) II only

(D) I and III only

(E) I, II, and III

(6) Consider, for $\alpha \in \mathbb{R}$, the following set:

$$S_\alpha = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = \alpha\}.$$

Which of the following statements are true?

I. S_α has two components for $\alpha \geq 0$.

II. S_α is symmetric about the y-axis.

III. For $\alpha < 0$ the line $y = 0$ is disjoint from S_α.

(A) I only

(B) I and II only

(C) II only

(D) II and III only

(E) I, II, III

(7) Consider the group given by the presentation:

$$\langle x, y \ : \ xyx^{-1}y^{-2} = x^{-2}y^{-1}xy = 1\rangle.$$

What group is it?

(A) The trivial group

(B) $\mathbb{Z}/2\mathbb{Z}$

(C) D_4 (the group of symmetries of the square)

(D) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

(E) $\mathbb{Z}^2$

(8) Suppose you have 2 books on accounting, 2 books on beekeeping, 3 books on chandlering. Up to rotation of the circle, how many ways can they be arranged in a circle so that all of the books on each topic are together?

(A) 24

(B) 12

(C) 3

(D) 48

(E) 36

(9) How many roots of $x^4 - 4x^2 - 8x + 12$ lie in the range $[-2, 2]$?

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

(10) Which of the following integrals has the greatest value?

(A) $\int_0^{\pi/4} \sin t\, dt$

(B) $\int_0^{\pi/4} \cos t\, dt$

(C) $\int_0^{\pi/4} \cos^2 t\, dt$

(D) $\int_0^{\pi/4} \cos 2t\, dt$

(E) $\int_0^{\pi/4} \sin t \cos t\, dt$

(11) Suppose that $P(x)$ is a polynomial with non-negative coefficients and even exponents. If $a, b \in \mathbb{R}$, what is true about the function

$$f(x) = P(x) + ax + b?$$

(A) For $b \in [-|a|, |a|]$, $f(x)$ is concave.

(B) For $a \neq 0$, f is strictly convex.

(C) For all $a, b \in \mathbb{R}$, f is convex.

(D) For all $a \neq 0$, f is strictly concave.

(E) For all $a, b \in \mathbb{R}$, f is concave.

(12) Consider the function:

$$f(x, y) = (x^2 + y^2 - 5/8)^2,$$

subject to the constraint that $x + y = 1$, $x, y \geq 0$. Which of the following are true?

I. f achieves a local maximum at $x = 1/2$, $y = 1/2$.

II. f achives a global minimum at $x = 1/2$, $y = 1/2$.

III. The minimum of f is positive.

IV. f is strictly less than $1/8$.

(A) I only

(B) II only

(C) II and III only

(D) I and III only

(E) III and IV only

(13) Let C be the cylinder bounded by $x^2 + y^2 = 9$, and $z = 0, z = 5$. If $F(x, y, z) = (3x, y^3, -2z^2)$, then calculate the flux of F through C, i.e. integrate the normal vector dotted with the field over the cylinder.

(A) $-\frac{45}{2}\pi$

(B) $-\frac{45}{4}\pi$

(C) 0

(D) $-\frac{36}{2}\pi$

(E) $\frac{45}{4}\pi$

(14) The hypercube graph Q_n is the undirected graph with vertex set $V = \{0, 1\}^n$ and edge set

$$E = \{(v, v') : \text{ exactly one bit of } v \text{ and } v' \text{ is distinct}\}.$$

If $n \geq 1$, then which of the following statements is false?

(A) Q_n is connected.

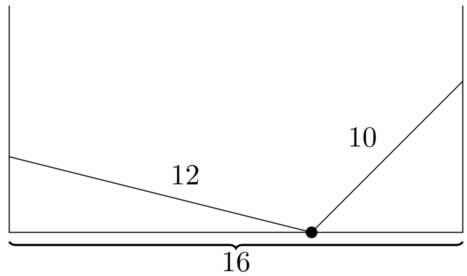
(B) Q_n has a Hamiltonian path, i.e. a path that visits each vertex exactly one.

(C) Q_n has $2^n n$ edges.

(D) Q_n has 2^n vertices.

(E) Q_n is 2-colorable.

(15) There are two ladders, of length 10 and 12 meters, in a small enclosure of total width 16 meters, connected at the bottom, in the following arrangement:



Thus, if the right ladder slides down the wall, then the left ladder slides up the wall. If the ladder of length 10 is 6 meters up the wall and sliding down the wall at a rate of 2 meters per second, at what rate is the ladder of length 12 sliding up the wall?

(A) $2/\sqrt{3}$

(B) $3/\sqrt{3}$

(C) $1/\sqrt{5}$

(D) $2/\sqrt{5}$

(E) $3/\sqrt{5}$

(16) A subset $U \subseteq \mathbb{R}^2$ is *radially open* if for every $x \in U$ and every $v \in \mathbb{R}^2$, there exists $\varepsilon > 0$ such that $x + sv \in U$ for every $s \in (-\varepsilon, \varepsilon)$. Then the collection of radially open sets defines a topology on $\mathbb{R}^2$. Let X be $\mathbb{R}^2$ equipped with the radially open topology, and let Y be $\mathbb{R}^2$ with the standard Euclidean topology. Consider the following statements about X.

I. X is Hausdorff.

II. X is second countable.

III. The identity map $Y \to X$ is continuous.

Which of the above statements are true?

(A) I only

(B) II only

(C) I and II only

(D) I and III only

(E) II and III only

(17) The group D_4 is the group of symmetries of an axis aligned square in the plane. Suppose the square is centered at $(0,0)$. How many elements of D_4 are conjugate to the reflection over the x-axis?

(A) 1

(B) 2

(C) 3

(D) 4

(E) 8

(18) Find an antiderivative of $\dfrac{3x+11}{x^2-x-6}$.

(A) $4\log|3-x| - \log|x+2|$

(B) $2\log|3-x| + \log|x+2|$

(C) $\log|x^2-x-6|$

(D) $3\log|2-x| + 4\log|x+3|$

(E) $2\log|2-x| + 2\log|x+3|$

(19) Suppose V and W are finite dimensional vector spaces, and that $f\colon V \to W$ is a linear map. Suppose $\{e_1, ..., e_n\} \subset V$ and that $\{f(e_1), \ldots, f(e_n)\}$ is a basis of W. Then which of the following are true?

I. $\{e_1, ..., e_n\}$ is a basis of V.

II. There exists a linear map $g\colon W \to V$ such that $g \circ f = Id_V$

III. There exists a linear map $g\colon W \to V$ such that $f \circ g = Id_W$.

(A) I only

(B) II only

(C) III only

(D) I and III only

(E) II and III only

(20) Find the volume of the solid obtained by rotating the region bounded by $y = x^4$, $y = 0$, and $x = 1$ about the line $x = 2$.

(A) 1

(B) $4\pi/5$

(C) $7\pi/15$

(D) 2

(E) $\pi^2/2$

(21) Which of the following are the characteristic polynomial of a matrix that is diagonalizable over the reals?

I. $x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1$

II. $x^4 + x^2 - 2$

III. $x^4 - 3x^2 + 1$

(A) I only

(B) III only

(C) I and II only

(D) II and III only

(E) I, II, and III

(22) Assume $y(x)$ solves the differential equation

$$2xyy' = \sin x - y^2$$

on $(0, \infty)$ and $y(\pi) = \sqrt{3/\pi}$. Compute the value of $y(2\pi)$.

(A) π

(B) $\sqrt{3/\pi}$

(C) $\sqrt{\pi/3}$

(D) $\sqrt{1/2\pi}$

(E) $\sqrt{2\pi}$

Answers

(1) (B): Binomial theorem.

(2) (C): Estimate from above and below (easiest).

(3) (C): Interpret the algorithm as an integral.

(4) (D): Compute the Cauchy-Riemann equations.

(5) (E): We have $r(r-1) = 0$ for all r, so $R = \mathbb{F}_2$.

(6) (D): Note I is false when $\alpha = 0$.

(7) (A): Show xy equal to two things, so then $x = y$, and then the trivial group follows.

(8) (D): $2 \cdot 2 \cdot 6$ for how the books can be arranged in section, $\cdot 2$ for arranging sections.

(9) (B): Analyze f'', and then f', and then f. Or compute a Sturm sequence.

(10) (B): It stays biggest longest. Or you could evaluate...

(11) (C): The second derivative is non-negative.

(12) (A): The rest can be disproved by plugging in points. Or solve with Lagrange multipliers.

(13) (B): Use Stokes to solve the integral.

(14) (C): It has fewer edges.

(15) (E): This is a simple related rates probem.

(16) (A): Open $\implies$ radially open, and the restriction of X to S^1 is discrete.

(17) (B): Compute.

(18) (A): Use a partial fractions decomposition.

(19) (B): III, we can find a section of f by sending $f(e_i)$ to e_i.

(20) (C): Standard computation.

(21) (B): Determine if they have complex roots. For (I), i is a root. For (II), rule of signs.

(22) (D): Note $y^2 + 2xyy' = (xy^2)'$.