MATH GRE PREP: WEEK 4

UCHICAGO REU 2019

(1) Which points $(x, y) \in \mathbb{R}^2$ are limit points of the set $X = \{(a, b) \in \mathbb{R}^2 \ : \ b = a^{-1}\sec(a^{-1})\}$?

(A) The set has no limit points.

(B) The points $y = x^{-1}\sec(x^{-1})$ (where this is defined), along with the points $(0, y)$ with $|y| \geq 1$.

(C) Only the points $y = x^{-1}\sec(x^{-1})$ (where this is defined).

(D) Only the points $(0, y)$.

(E) Only the points $(0, y)$ with $|y| \geq 1$.

(2) Let A be a 3×3 real matrix with zero trace, and such that the trace of A^2 is one. If A is not invertible, then what is the largest eigenvalue of A?

(A) 0

(B) $\sqrt{2}/2$

(C) $\sqrt{3}/2$

(D) 1

(E) $\sqrt{3}$

(3) How many invertible 3×3 matrices are there with entires in $\mathbb{F}_2$ (the field with 2 elements)?

(A) 128

(B) 150

(C) 168

(D) 256

(E) 300

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(4) Let $ABCD$ be a quadrilateral and let AB be parallel to CD. If $AB = 10$, $BC = 13$, $CD = 14$, and $AD = 15$, what is the area of $ABCD$?

(A) 126

(B) 132

(C) 138

(D) 144

(E) 156

(5) Define a logical symbol by the following table:

A	B	$A\#B$
0	0	0
0	1	1
1	0	1
1	1	0

Which of the following is true?

(A) $(\neg B) \wedge A = (\neg((\neg A)\#B))\#(A\#B)$

(B) $(\neg B) \wedge A = (\neg(A\#B))\#(A\#(\neg B))$

(C) $(\neg B) \wedge A = ((\neg A)\#B)\#(\neg B)$

(D) $(\neg B) \wedge A = ((\neg A)\#A)\#B$

(E) It is not possible to obtain a logical formula equivalent to $(\neg B) \wedge A$ using only $\neg, \#$ and parentheses.

(6) How many integer solutions exist to the equation $x^2 + 1 = y^3 - 1$?

(A) 0

(B) 1

(C) 2

(D) 3

(E) 5

(7) A politician is heading for a meeting via limo. Running late, the driver wants to take the fastest path. Naturally, the roads are set up as a Cartesian coordinate plane with road lying on every point with one of the coordinates an integer. The driver must go 7 blocks east and 5 blocks north. However, there is an accident 4 blocks east and 3 blocks north. How many different shortest length paths are there from the starting position to the meeting that avoid the accident?

(A) 442

(B) 792

(C) 552

(D) 672

(E) 592

(8) Solve the following limit:

$$\lim_{n\to\infty} \prod_{1\le k\le n} \left(1 + \frac{k}{n}\right)^{\frac{1}{k}}.$$

(A) $e^{\pi^2/12}$

(B) $e^{\pi/6}$

(C) $e^{\pi^2/6}$

(D) $e^{\pi^3/12}$

(E) The product does not converge.

(9) Suppose that $X_1, ..., X_n$ are independent and identically distributed random variables with expectation θ and standard deviation ρ. Which of the following is an unbiased estimator of θ for all $n > 0$ and θ?

(A) $\sqrt[n]{\sum_{i=1}^n X_i}$

(B) $\sqrt{\frac{1}{n-1} \sum_{i=1}^n X_i^2}$

(C) $\sqrt{\frac{1}{n} \sum_{i=1}^n X_i^2}$

(D) $\sum_{i=1}^n (-1)^i X_n$

(E) X_1

(10) Compute the following integral:

$$\int_2^4 \frac{\sqrt{\log(9-x)}}{\sqrt{\log(9-x)} + \sqrt{\log(x+3)}} dx.$$

(A) e

(B) $\log 2$

(C) 1

(D) e^{-1}

(E) $1/2$

(11) Evaluate:

$$\int_0^\infty \frac{\cos(x)dx}{(x^2+4)^2}.$$

(A) The integral does not converge.

(B) $\frac{3\pi}{64e^2}$

(C) $\frac{3\pi}{32e^2}$

(D) $\frac{\pi}{64e^2}$

(E) $\frac{\pi}{32e^2}$

(12) Bee-lated Rates. Bees are moving honey to a conical container at a rate of 10 cm^3/min. The cone points downward, and has a height of 30 cm, and a base radius of 10 cm. At time $t = 0$, the cone is filled up to the halfway point in height, and a hole develops at the bottom point. Honey flows out of the container through this hole at a rate of $V/2$ cm^3/min, where V is the volume of honey in the container. What is the height of the honey at time $t = 2 \cdot \ln(2)$ minutes?

(A) $3\sqrt[3]{\frac{5^3}{2} + \frac{5}{\pi}}$

(B) $3\sqrt[3]{\frac{5^3}{4} + \frac{5}{\pi}}$

(C) $3\sqrt[3]{\frac{5^3}{4} + \frac{10}{\pi}}$

(D) $3\sqrt[3]{\frac{5^3}{2} + \frac{10}{\pi}}$

(E) $3\sqrt[3]{\frac{5^3}{4} + \frac{20}{\pi}}$

(13) Calculate the flux of $F = x(z^3 - y)\widehat{\mathbf{i}} + yz(2x - z^2)\widehat{\mathbf{j}} + (yz - 2xy - xz^2)\widehat{\mathbf{k}}$ through the ellipsoid determined by $9x^2 + 9y^2 + z^2 = 9$ on the region $z > 0$, with the standard normal.

(A) -2π

(B) -1

(C) 0

(D) 1

(E) 2π

(14) Consider the three points $(1, 4)$, $(-2, 5)$, and $(-5, -1)$. What is the shortest distance between one of the points, and the line determined by the other two points?

(A) $\frac{21}{\sqrt{10}}$

(B) $\frac{21}{\sqrt{61}}$

(C) $\frac{7}{\sqrt{5}}$

(D) $\sqrt{7}$

(E) $\frac{21}{\sqrt{69}}$

(15) Up to isomorphism, how many groups of order 35 are there?

(A) 1

(B) 2

(C) 3

(D) 5

(E) 7

(16) Suppose that $f\colon [0,1] \to \mathbb{R}$ has the following property: for every point $y \in [0,1]$, for all $\delta > 0$, there exists $\epsilon > 0$ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. What is this property equivalent to?

(A) Continuity

(B) Boundedness

(C) Equicontinuity

(D) Uniform continuity

(E) Lower semicontinuity

(17) Evaluate:

$$\lim_{x\to\infty} x \left(\arctan\left(\frac{x+1}{x+2}\right) - \frac{\pi}{4} \right).$$

(A) $1/2$

(B) $-1/2$

(C) 1

(D) 0

(E) $-1/3$

(18) Let R be a commutative ring with 1. We say that $n \in R$ is nilpotent if there is a positive integer k such that $n^k = 0$.

(I) The set of nilpotent elements is an ideal of R.

(II) If n is nilpotent then $1 - n$ is a unit.

(III) If R has no non-zero nilpotent elements then it is an integral domain.

Which of the above statements are true?

(A) I only

(B) II only

(C) I and II

(D) I and III

(E) I, II, and III

(19) Let C_n be the boundary of a regular unit n-gon counterclockwise with its base between $(0,0)$ and $(0,1)$ in the xy-plane. What is the value of the line integral

$$\oint_{C_n} (2x - y + 2yx)\,dx + (x + 3y + x^2)\,dy?$$

(A) $\tan(2\pi/n)/2$

(B) $n\tan(2\pi/n)$

(C) $n\tan(\pi/n)$

(D) $n\cot(\pi/n)/2$

(E) $n\cot(\pi/n)$

(20) What is the length of the curve determined by $x(t) = 4\sin(t/4)$ and $y(t) = 1 - 2\cos^2(t/4)$?

(A) $4(\sqrt{2} + 2\,\mathrm{arccosh}(1))$

(B) $4(\sqrt{2} + \mathrm{arcsinh}(1))$

(C) $8(\sqrt{2} + \mathrm{arccosh}(1))$

(D) $8(\sqrt{2} + \mathrm{arcsinh}(1))$

(E) $8(\sqrt{2} + 2\,\mathrm{arcsinh}(1))$

(21) For $t > 0$, solve

$$ty' = -2y + \sin t.$$

(A) $y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{C}{t^2}$

(B) $y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{C}{t}$

(C) $y = \frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{C}{t^2}$

(D) $y = \frac{\cos t}{t} - \frac{\sin t}{t^2} + \frac{C}{t^2}$

(E) $y = -\frac{\cos t}{t} - \frac{\sin t}{t^2} + \frac{C}{t^2}$

(22) Consider the following algorithm, which is run on a computer that does integer arithmetic, i.e. it always rounds down (like a normal computer should!).

```
input(n)
set m = n + 2
set i = 0
while m > 1:
   begin
      set m = m/2
      set i = i + 1
    end
set m = n + 2
while i >= 0:
   begin
     set j = m / (2 ** i)
     print j
     set m = m - j * (2 ** i)
     set i = i - 1
   end
```

If the algorithm is run on the input `n = 101` then what sequence of digits will be the output?

(A) 1100101

(B) 1100011

(C) 01100110

(D) 01100111

(E) 1100111

Answers

(1) (C): Since $|\sec\theta| \geq 1$, we have no limit points when $x = 0$.

(2) (B): Solve the system, observing one of the eigenvalues is zero.

(3) (C): Count columns: $7 \cdot 6 \cdot 4$

(4) (D): Cut it into a paralellogram and triange, calculate area of triangle, find height.

(5) (E): Interpret everything as arithmetic modulo 2.

(6) (C): There are an even number of solutions, and then find one.

(7) (A): Calculate all paths, and subtract off ones through the accident.

(8) (A): Take the logarithm, estimate from above and below.

(9) (E): Recall that an unbiased estimator has the correct expectation.

(10) (C): Use the trick of flipping the bounds and adding this to the original integral to simplify.

(11) (C): This is half the value from $(-\infty, \infty)$; evaluate on Riemann sphere by residue theorem.

(12) (D): This is a first-order linear differential equation. Solve it.

(13) (C): Solve via Stokes.

(14) (B): It is evident what the choice should be after drawing the picture, then just compute.

(15) (A): Sylow's theorems.

(16) (B): Pick δ really big.

(17) (B): Use the sum formula for arctan, or Laurent series.

(18) (C): All reasonably clear (solve for $(1-n)^{-1}$ by the usual series; $\mathbb{F}_2 \times \mathbb{F}_2$ is a counterexample for III).

(19) (D): Use Stokes to solve.

(20) (B): Use definition of arc length. For the integral of $\sqrt{1+u^2}$, use hyperbolic substitution.

(21) (A): Use an integrating factor.

(22) (E): Convert 103 (note the $m = n + 2$) to binary.