MATH GRE PREP: WEEK 6

UCHICAGO REU 2019

(1) Order the following real numbers, from smallest to largest.

(I) π

(II) $\frac{1+\sqrt{2}}{\sqrt{3}-1}$

(III) $e \cdot \sqrt[3]{2}$

(IV) $\alpha \in \mathbb{R} \ : \ \alpha^5 - \alpha - 101 = 0$

(A) (I) < (II) < (III) < (IV)

(B) (IV) < (II) < (I) < (III)

(C) (II) < (III) < (IV) < (I)

(D) (IV) < (II) < (III) < (I)

(E) (IV) < (I) < (II) < (III)

(2) Let $y(x)$ be a solution to the differential equation $x^{-1}(y' - 2x^{-1}y) = x^2\cos(x^2)$. Let $y(\sqrt{\pi/2}) = 1$. Then what is $y(\sqrt{2\pi})$?

(A) $2 - \pi/2$

(B) $2 - \pi$

(C) $4 - 2\pi$

(D) $4 - \pi$

(E) $8 - 2\pi$

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(3) Order the following series in terms of size of their radius of convergence about 0.

(I) $\sum_{n\geq 1} \frac{n!}{n^n} z^n$

(II) $\sum_{n\geq 1} \frac{(n!)^3}{(3n)!} z^n$

(III) $\sum_{n\geq 1} (2^n + (3/2)^n) z^n$

(IV) $\sum_{n\geq 1} (\log n)^2 z^n$

(V) $\sum_{n\geq 1} (1 + 1/n)^{n^2} z^n$

(A) III < V < IV < I < II

(B) V < III < IV < I < II

(C) V < III < IV < II < I

(D) V < IV < III < II < I

(E) II < IV < III < V < I

(4) Which of the following functions are holomorphic?

(I) $f(x, y) = x^4 - 3x^3 - 6x^2y^2 + 9xy^2 + y^4 + (9x^2y + 4xy^3 - 4x^3y - 3y^3)i$

(II) $g(x, y) = i \cdot \overline{f(x, y)}$

(III) $h(x, y) = \overline{\exp(f(x, y))}$

(A) I only

(B) II only

(C) III only

(D) II and III

(E) I, II, and III

(5) For $a, c \in \mathbb{R}_{>0}$ define $f(x)$ by

$$f(x) = \begin{cases} x^a \sin(|x|^{-c}) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Which range of a, c is equivalent to $f''(0)$ existing?

(A) $a > c$

(B) $a \geq 1 + c$

(C) $a \geq 1 + 3c$

(D) $a > 2 + c$

(E) $a > 2 + 2c$

(6) Which of the following is the weakest claim that ensures that a sequence $x_n \in X \subseteq \mathbb{R}$ has a unique accumulation point?

(A) X is closed and bounded.

(B) There exists a point $x \in X$ such that $x \in \overline{X \setminus \{x\}}$.

(C) $\exists x \in X \ : \ \forall \varepsilon > 0, \ \exists n \in \mathbb{N} \ : \ x_n \neq x, \ |x - x_n| < \varepsilon$.

(D) $\forall \varepsilon > 0, \ \exists N \in \mathbb{N} \ : \ \forall n, m \geq N, \ |x_n - x_m| < \varepsilon$.

(E) We have $|x_n - x_{n+1}| < cn^{-2}$ for some $c > 0$.

(7) What is:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{2n} \frac{n^2}{4k^2 + n^2}?$$

(A) $\frac{\pi}{2}$

(B) $\frac{1}{2} \arctan(4)$

(C) $\arctan(4)$

(D) $\frac{1}{4} \arctan(2)$

(E) $\arctan(2)$

(8) What is

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{3n+2}\right)?$$

(A) The product diverges properly to $-\infty$.

(B) $e^3/2$

(C) $e^{-3}/2$

(D) 0

(E) $e^{-3/2}$

(9) What is 7^{17} mod 60?

(A) 1

(B) 2

(C) 7

(D) 49

(E) 3

(10) Evalute the following definite integral.

$$\int_0^{\frac{\pi}{2}}\left(\frac{x}{\sin x}\right)^2 dx$$

(A) π

(B) $\frac{\pi}{2}$

(C) $\frac{\pi\cdot\log(2)}{2}$

(D) $\pi\cdot\log(2)$

(E) 1

(11) Which of the following vector fields are conservative, where they are defined?

(I) $F(x, y, z) = (3x^2 z, z^2, x^3 + 2yz)$

(II) $F(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$

(III) $F(x, y, z) = (3x^2 y^2 z + 5y^3, 2x^3 yz + 15xy^2 - 7z, x^3 y^2 - 7y + 4z^3)$

(A) I only

(B) II only

(C) III only

(D) I and II

(E) I and III

(12) A square pyramid, of side length 100 cm and height 100 cm, of ice is melting at a consistent rate such that all of the ice less than y cm from the surface melts after y hours (note the *bottom* is also melting). What is the rate of change of the volume, when the height is 10 cm?

(A) $-100 \left(\frac{2\sqrt{5}}{3} \right)$

(B) $-100 \left(\sqrt{5} \right)$

(C) $-100 \left(\sqrt{5} + 1 \right)$

(D) $-100 \left(\frac{3\sqrt{5}+1}{3} \right)$

(13) Let R be a commutative ring with identity. Consider the binary operation on R: $a \star b = a + b - ab$.

(I) There exists a commutative ring R where $(R, \star)$ forms an abelian group, $|R| > 1$.

(II) For all R, $(R, \star)$ forms a group.

(III) For all R, the operation $\star$ has unique left inverses.

Which of the above are true?

(A) (I) and (II).

(B) Only (III).

(C) All three.

(D) Only (I).

(E) None of them.

(14) Let $f\colon X \to Y$ be a function, with $X, Y \subseteq \mathbb{R}$. Consider the following statements about f:

(I) $\forall \delta > 0,\ \exists \varepsilon > 0\ :\ |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$

(II) $\forall (x_n)_{n\in\mathbb{N}} \subset X,\ \lim_{n\to\infty} x_n = c \implies \lim_{n\to\infty} f(x_n) = f(c)$

(III) For every open set U, $f^{-1}(U)$ is open.

(IV) The limit $\lim_{h\to 0}(f(c+h) - f(c))/h$ is defined.

What is the negation of the following statement? The function f is differentiable wherever it is continuous.

(A) There is some point $c \in X$ that does not satisfy (IV), but satisfies (I).

(B) f satisfies (III), but has a point c not satisfying (IV).

(C) f does not satisfy (III) or (IV) for some point c.

(D) There exists a point $c \in X$ satisfying (II) such that this point does not satisfy (IV).

(E) Some point $c \in X$ does not satisfy (I) or (IV).

(15) Compute the area of a unit sphere contained between the meridians $\phi = 30°$ and $\phi = 60°$ and parallels $\theta = 45°$ and $\theta = 60°$.

(A) $\frac{\pi(\sqrt{3}-\sqrt{2})}{12}$

(B) $\frac{\pi(\sqrt{3}-\sqrt{1})}{16}$

(C) $\frac{\pi(\sqrt{2}-\sqrt{1})}{6}$

(D) $\frac{\pi(\sqrt{3})}{18}$

(E) $\frac{\pi(\sqrt{3}-1)}{12}$

(16) What is:

$$\lim_{x\to 0} \frac{\tan 2x}{\sin 5x}?$$

(A) 0

(B) 4/5

(C) ∞

(D) 1/5

(E) 2/5

(17) Evalute the integral:

$$\int_{-\infty}^{\infty} \frac{\sin(5x)dx}{1+\left(x-\frac{\pi}{2}\right)^2}.$$

(A) The integral does not converge.

(B) πe^{-5}

(C) $2\pi e^{-5}$

(D) $3\pi e^{-5}$

(E) $4\pi e^{-5}$

(18) Evaluate:

$$\int_0^\infty \frac{dx}{1+x^3}.$$

(A) $\frac{2\pi}{3\sqrt{3}}$

(B) 1

(C) $\frac{1}{2}$

(D) $\frac{\pi}{4}$

(E) $\frac{\pi^2}{4}$

(19) A dartist is playing darts on a dartboard equal to the unit disk in $\mathbb{R}^2$. Each throw is independent, and has (x, y) coordinates distributed according to the probability density function:

$$f(x, y) = \frac{C}{x^2 + y^2 + 1}.$$

The dartist is trying to hit the bullseye equal to the disk of radius $1/10$. What is the probability that at least one of the two next darts hits the bullseye?

(A) $\frac{\pi C}{2} \log \frac{101}{100} \cdot (2 - \pi C \log \frac{101}{100})$

(B) $\pi C \log \frac{101}{100} \cdot (2 - \pi C \log \frac{101}{100})$

(C) $2\pi C \log \frac{101}{100} \cdot (2 - \pi C \log \frac{101}{100})$

(D) $\pi C \log \frac{101}{100}$

(E) $2\pi C \log \frac{101}{100}$

(20) Let A and B be two matrices with complex entries that do not commute. Which of the following is necessarily false?

(A) A and B are similar.

(B) A and B are simultaneously triangulizable.

(C) A and B share an eigenvector.

(D) A and B have a basis of simultaneous eigenvectors.

(E) A and B have the same characteristic and minimal polynomials.

(21) (An old Chestnut) What is
$$\arctan(1) + \arctan(2) + \arctan(3)?$$

(A) $\pi/2$

(B) $4\pi/5$

(C) π

(D) 2π

(E) 3

(22) (Alcuin c. 735-804 CE) A certain bishop ordered 12 loaves of bread to be divided amongst the clergy. He stipulated that each priest should receive two loaves, each deacon should receive half a loaf and each reader should receive a quarter of a loaf. It turned out that the number of clerics and the number of loaves were the same. Assuming all loaves are distributed, how many priests must there have been?

(A) 3

(B) 4

(C) 5

(D) 6

(E) The answer is undetermined.

Answers

(1) (E): Determine $\alpha < 3$ and II,III¿π.

(2) (D): Solve by an integrating factor.

(3) (B): Compute the root test.

(4) (D): Use Cauchy-Riemann to determine f is anti-holomorphic.

(5) (D): This follows from the limit definition of the derivative.

(6) (D): Note (C) does note make it unique, and (E) is too strong. (A) and (B) are ridiculous.

(7) (B): Recognize this as a Riemann sum.

(8) (D): Use the approximation $\log(1 + x) \approx x$.

(9) (C): Note $\varphi(60) = 16$, so $x^{17} = x$ for x coprime to 60.

(10) (D): Integrate by parts twice, use double angle formula for sine. Or estimate.

(11) (E): Compute the curl.

(12) (C): Compute this, noting similarity (e.g., $h = s$). Recall the volume $V = s^2h/3$.

(13) (E): Disprove (I) and (II) by letting $b = 1$. For (III), let e be a left identity and compute.

(14) (D): The negation is there exists a point c where f is continuous and not differentiable.

(15) (A): Pass to spherical coordinates and evaluate.

(16) (E): Use Taylor series to evaluate.

(17) (B): This is the real part of an integral on the Riemann sphere; use residue theorem.

(18) (A): Use a partial fractions decomposition.

(19) (B): Convert to polar coordinates and integrate (u-substitution).

(20) (D): It should be clear this is the strongest claim.

(21) (C): This is the argument of $(1+i)(1+2i)(1+3i)$.

(22) (E): Alcuin apparently believes that there are more than zero readers, giving (C) as the answer.