Distance in R^3

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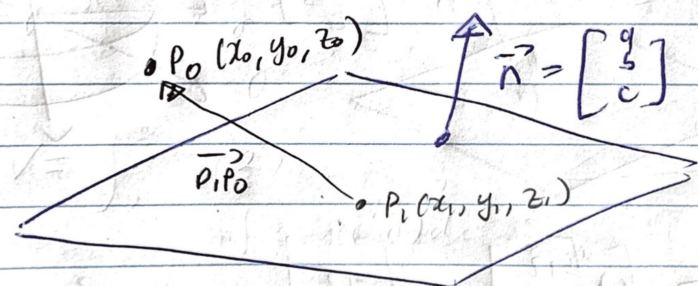
Contents

Distance in $\mathbb{R}^3$

— Stewart Chapter 12.5

Example 8 (Stewart pg 829)

Find the formula for the distance (we mean min distance) between a point $P_0 \in \mathbb{R}^3$ and the plane $P \subseteq \mathbb{R}^3$ where $ax + by + cz + d = 0$ is the cartesian equation of the plane.



- let $P_1 \in P$.

- Then the vector $\vec{P_1P_0} = \vec{P_0} - \vec{P_1} = \langle x_0 - x_1, y_0 - y_1, z_0 - z_1 \rangle$
- Now the (min) distance from P_0 to the plane P is the length of the vector obtained from projecting $\vec{P_1P_0}$ onto $\vec{n}$.

So $$\mathrm{Proj}_{\vec{P_1P_0}}(\vec{n}) = \frac{\vec{P_1P_0} \cdot \vec{n}}{\|\vec{n}\|^2} \cdot \vec{n}$$

$$= \left(\frac{a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)}{a^2 + b^2 + c^2} \right) \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Example 8 (continued)

$$\left\|\mathrm{Proj}_{\overrightarrow{P_1P_0}}(\vec{n})\right\| = \left\| \frac{a(x_0-x_1)+b(y_0-y_1)+c(z_0-z_1)}{a^2+b^2+c^2} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\|$$

$$= \left| \frac{a(x_0-x_1)+b(y_0-y_1)+c(z_0-z_1)}{a^2+b^2+c^2} \right| \cdot \left\| \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\|$$

$$= \frac{|a(x_0-x_1)+b(y_0-y_1)+c(z_0-z_1)|}{a^2+b^2+c^2} \cdot \sqrt{a^2+b^2+c^2}$$

$$= \frac{|a(x_0-x_1)+b(y_0-y_1)+c(z_0-z_1)|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|ax_0+by_0+cz_0-(ax_1+by_1+cz_1)|}{\sqrt{a^2+b^2+c^2}}$$

And since $P_1 = (x_1, y_2, z_2) \in P$ we know $ax_1+by_1+cz_1+d=0 \Rightarrow ax_1+by_1+cz_1=-d$.

So distance from P_0 to plane $ax+by+cz+d=0$

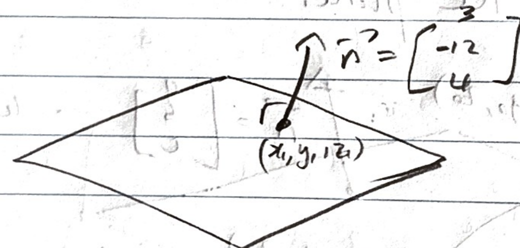
$$\boxed{\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}}$$

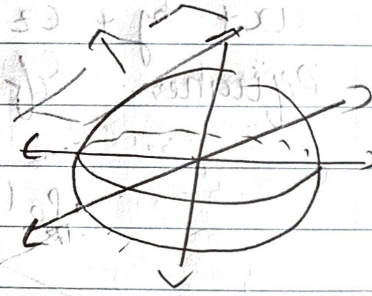
Example 39
(Rambo Practise part 2)

- Find the point on the unit ball that is the closest to the plane $3x - 12y + 4z = 9$.

- So our point $P_0(x_0, y_0, z_0)$ must satisfy
  - $x_0^2 + y_0^2 + z_0^2 = 1$ and
  - we need to minimize distance from P_0 to plane.

Note





So $\vec{L}(t) = (t)\begin{bmatrix} 3 \\ -12 \\ 4 \end{bmatrix}, \; t \in \mathbb{R}$

- Where is our sphere // to the plane?
- This will be where point of shortest distance is.

$\Rightarrow$ So our point will be somewhere on line L.

So $(3t)^2 + (-12t)^2 + (4t)^2 = 1$

$\Rightarrow 9t^2 + 144t^2 + 16t^2 = 1$

$\Rightarrow 9t^2 + 160t^2 = 1$

$\Rightarrow t^2 = 1/169 \Rightarrow t = \pm \frac{1}{13}$

$t = \pm \frac{1}{13}$, $L(t) = \begin{bmatrix} 3 \\ -12 \\ 4 \end{bmatrix}(t)$, $t \in \mathbb{R}$

$$\boxed{\cdot\ \vec{L}\left(\tfrac{1}{13}\right) = \begin{bmatrix} \frac{3}{13} \\ \frac{-12}{13} \\ \frac{4}{13} \end{bmatrix}}\ ,\quad L\left(\tfrac{-1}{13}\right) = \begin{bmatrix} -\frac{3}{13} \\ \frac{12}{13} \\ -4/13 \end{bmatrix}$$

Plane $\cdot\ 3(3s) - 12(-12s) + 4(4s) = 9$

$\Rightarrow\ 9s + 144s + 16s = 9$

$\Rightarrow\ 169s = 9$

$\Rightarrow\ s = \frac{9}{169}$

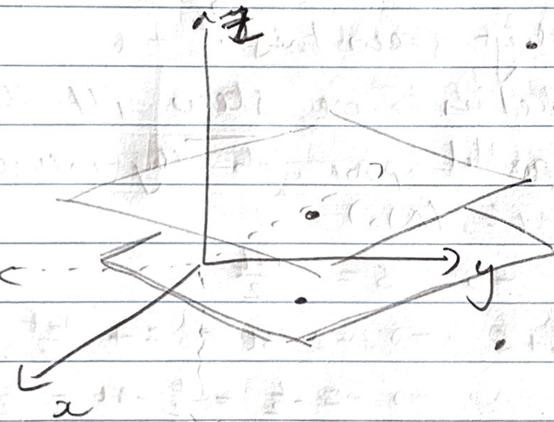
Then $d\left(L\left(\tfrac{1}{13}\right), \begin{bmatrix} 3 \\ -12 \\ 4 \end{bmatrix}\left(\tfrac{9}{169}\right)\right) < d\left(L\left(\tfrac{-1}{13}\right), \begin{bmatrix} 3 \\ -12 \\ 4 \end{bmatrix}(s)\right)$

So point is $\begin{bmatrix} \frac{3}{13} \\ \frac{-12}{13} \\ \frac{4}{13} \end{bmatrix}$ or $\boxed{\left(\tfrac{3}{13}, \tfrac{-12}{13}, \tfrac{4}{13}\right)}$.

Example 9 (Distance between parallel planes)
(Stewart pg 830)

Given two planes described by the equations
$10x+2y-2z=5$ and $5x+y-z=1$.

let $P_1 = \{(x,y,z) \in \mathbb{R}^3 \mid 10x+2y-2z=5\}$
$P_2 = \{(x,y,z) \in \mathbb{R}^3 \mid 5x+y-z=1\}$



• We know $\vec{n} = \begin{bmatrix}5\\1\\-1\end{bmatrix}$ is orthogonal to both planes.

let's choose $(\frac{1}{2},0,0)$ in P_1.

Then $L(t) = \begin{bmatrix}\frac{1}{2}\\0\\0\end{bmatrix} + \begin{bmatrix}5\\1\\-1\end{bmatrix}(t) = \begin{bmatrix}\frac{1}{2}+5t\\t\\-t\end{bmatrix}$

$5(\frac{1}{2}+5t)+(t)-(-t)=1$

$\frac{5}{2}+25t+t+t=1$

$\frac{5}{2}+27t=1$

$27t = \frac{2}{2}-\frac{5}{2} = -\frac{3}{2}$

$t = \frac{-3}{2\cdot 27} = \frac{1}{2}\cdot\frac{-1}{9} = \frac{-1}{18}$.

Find distance between $(\frac{1}{2},0,0)$ and $(\frac{4}{18},\frac{-1}{18},\frac{1}{18})$.

$\begin{bmatrix}\frac{9}{18}-\frac{5}{18}\\ \frac{-1}{18}\\ \frac{1}{18}\end{bmatrix} = \begin{bmatrix}4\\-1\\1\end{bmatrix}\left(\frac{1}{18}\right)$,

$\sqrt{27} = \sqrt{3^3}$

$\sqrt{(\frac{4}{18}-\frac{1}{2})^2+(\frac{1}{18})^2+(\frac{1}{18})^2}$

$= \sqrt{(\frac{4}{18}-\frac{9}{18})^2+\frac{2}{18^2}}$

$= \frac{\sqrt{25+2}}{18}$

$= \frac{3\sqrt{3}}{18} = \boxed{\frac{\sqrt{3}}{6}}$

Example 10 (Stewart pg 830)

- Distance between 2 lines:

  - $L_1(t) = \begin{bmatrix} 1+t \\ -2+3t \\ 4-t \end{bmatrix}, t \in \mathbb{R}$

  - $L_2(s) = \begin{bmatrix} 2s \\ 3+s \\ -3+4s \end{bmatrix}, t \in \mathbb{R}$

Firstly: Are they skew lines?

- Two lines are skew iff they are not parallel, nor do they intersect.

(1) Do they intersect? (No)

- $1+t = 2s \Rightarrow s = \frac{1+t}{2}$
- $-2+3t = 3+s \Rightarrow -2+3t = 3+\frac{1+t}{2} = 3+\frac{1}{2}+\frac{t}{2}$

  $\Rightarrow -5-\frac{1}{2} = \frac{t}{2}-3t = \frac{t}{2}-\frac{6}{2}t = -\frac{5}{2}t$

  $\Rightarrow -\frac{11}{2} = -\frac{5}{2}t \Rightarrow 11 = 5t \Rightarrow \boxed{t = \frac{11}{5}}$

- $1+\frac{11}{5} = 2s \Rightarrow \frac{16}{5} = 2s \Rightarrow s = \frac{8}{5}$.
- $4-\frac{11}{5} = -3+4\left(\frac{8}{5}\right)$

$\Rightarrow \frac{9}{5} \neq -\frac{15}{5}+\frac{32}{5}$ so they can't intersect since it must be at the point where $t = \frac{11}{5}$ and $s = \frac{8}{5}$.

(2) Are they parallel?

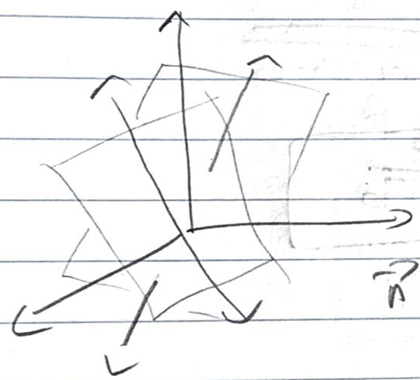
- No, since $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ are not scalar multiples.

Example 10 (continued)

So we know L_1 and L_2 are skew.

$\Rightarrow$ They lie in different planes.

$$L_1(t) = \begin{bmatrix} 1+t \\ -2+3t \\ 4-t \end{bmatrix}, \quad L_2(s) = \begin{bmatrix} 2s \\ 3+s \\ -3+6s \end{bmatrix}$$



- $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \in L_1$
- Find vector normal to both lines.

$$\vec{n} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} \end{array}$$

$$= \vec{i}\,((3)(4) - (-1)(1))$$
$$- \vec{j}\,((1)(4) - (-1)(2))$$
$$+ \vec{k}\,((1)(1) - (2)(3))$$

$$= \vec{i}\,(13) - \vec{j}\,(6) + \vec{k}\,(1-6)$$

$$= \begin{bmatrix} 13 \\ -6 \\ -5 \end{bmatrix}$$

Then $13x - 6y - 5z = d_1$

- $13(1) - 6(-2) - 5(4) = 13 + 12 - 20 = 5$

$\Rightarrow d_1 = 5$

- $13(0) - 6(3) - 5(-3) = d_2$

$\Rightarrow \quad -18 + 15 = d_2 \Rightarrow d_2 = 3.$

Example 10 (continued)

- $13x - 6y - 5z = 5$ (P_1)
- $13x - 6y - 5z = 3$ (P_2)

Then we have the distance from planes P_1 and P_2:

$$D = \frac{|5+3|}{\sqrt{13^2+6^2+5^2}} = \frac{8}{\sqrt{169+36+25}}$$

205

230

$$\boxed{D = \frac{8}{\sqrt{230}}}$$

$$\begin{array}{r} 169 \\ 36 \\ 25 \\ \hline 230 \end{array}$$

2	230
5	115
23	23

$\sqrt{230}$

$= \sqrt{2 \cdot 5 \cdot 23}$