Centralizing a Monopsonistic Labor Market*

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This paper asks whether centralized matching can make monopsonistic labor markets more efficient. We construct a job matching model in which each firm views workers as interchangeable and must pay all its workers the same salary. This requirement generates monopsonistic distortions: While an efficient stable outcome will always exist, inefficient outcomes can be stable as well. There exists an efficient stable outcome that workers prefer over any other stable outcome. Firms prefer inefficient stable outcomes in which they pay lower salaries and thus extract greater profits. When production technologies are public information, a strategyproof mechanism can elicit how workers value employment, and thus implement an efficient stable outcome. However, no strategyproof mechanism can elicit firms' production technologies. Thus, centralized matching can improve monopsonistic labor markets when the market designer observes production.

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1 Introduction

When a worker chooses between jobs, she rarely has a lot of options. Most labor markets are dominated by only a few firms (Azar, Marinescu, & Steinbaum, 2020). The worker will prefer one firm over others – perhaps because it is located near her home, because it has fewer occupational hazards, or because it has an unusually active fantasy football pool. Other workers will have different preferences. Firms thus face upward-sloping labor supply curves: If a firm wants to employ more workers, it will have to increase the salary that it pays. Its optimal salary will be less than its marginal product of labor, and it will employ fewer workers than would be efficient (Robinson, 1933; Boal & Ransom, 1997; Manning, 2011). Economists are increasingly blaming this 'monopsonistic distortion' for stagnant salaries and the distorted allocation of labor across the economy (Bivens, Mishel, & Schmitt, 2018; Berger, Herkenhoff, & Mongey, 2019).

In this paper, we ask whether monopsonistic distortions can be addressed through a centralized matching mechanism. To do so, we unify the job matching and labor monopsony literatures. Following the job matching literature, we study stable outcomes: matchings of workers to firms, along with a salary schedule, from which no worker-firm coalition can profitably deviate. Canonical job matching models lack monopsonistic distortions because they assume that each worker's salary can be set independently of her colleagues' salaries (Kelso & Crawford, 1982; Hatfield & Milgrom, 2005). We instead follow the labor monopsony literature by requiring that each firm pays all its workers the same salary. This requirement introduces distortions that mimic the monopsonistic distortions found in existing labor monopsony models. We study whether centralized matching can ameliorate these distortions.

We make two high-level contributions. First, we demonstrate when and how centralized matching can address monopsonistic distortions. Inefficient outcomes can be stable. Thus, a decentralized labor market need not function efficiently. However, at least one efficient outcome will always be stable. If a market designer could stipulate that outcome, no coalition of workers and firms could profitably destabilize it. Moreover, there is always an efficient stable outcome that workers prefer over any other stable outcome. Thus, the market designer's desire for efficiency is compatible with her solidarity with the workers.

Our model contains two potential sources of private information: workers' preferences for employment at each firm, and the production technologies with which firms produce their outputs. When firms' production technologies are private information, no strategyproof mechanism can implement an efficient outcome. That is because firms will want to shade their reports, reporting that they are less productive so that they pay a lower salary. Firms can find such misreporting profitable, even if it means that the mechanism matches them to inefficiently few workers. However, when firms' production technologies are public information, a strategyproof mechanism can elicit workers' preferences and implement an efficient outcome. We thus conclude that a centralized matching mechanism can always address monopsonistic distortions when production technologies are observable.

Our second high-level contribution is to characterize how, in the presence of two-sided heterogeneity, restricted transfers generate monopsonistic distortions. The canonical Kelso and Crawford (1982) job matching model predicts that outcomes are always efficient. This prediction contrasts with labor monopsony models, which predict inefficient outcomes. It is not prima facie obvious why these literatures disagree. Our results imply that the disagreement cannot be explained by the different solution concepts that they adopt. By showing that our model produces inefficient stable outcomes, we demonstrate that monopsonistic inefficiencies are inherent to market incentives with restricted transfers. In other words, monopsonistic inefficiencies are caused by the requirement that each firm pays all its workers the same salary and are not an artifact of a particular extensive form.

Such inefficiencies are not caused by the restriction on transfers alone. We show that when workers have homogeneous preferences, or when firms are sufficiently homogenous, every stable outcome is efficient. Restricted transfers only generate monopsonistic distortions in the presence of two-sided heterogeneity.

Two restrictions differentiate our model from the canonical Kelso-Crawford job matching model. The first is that *a firm cannot pay different salaries to different workers*. The labor monopsony literature typically makes two justifications for that assumption. First, the firm may not know workers' idiosyncratic amenities and so would lack the information required to set worker-specific salaries (Card, Cardoso, Heining, & Kline, 2018). Second, workers shirk when they perceive that they are paid less than their colleagues, especially when pay differentials are not based on easily-observed productivity differences (Breza, Kaur, & Shamdasani, 2017; Akerlof & Yellen, 1990).

A second difference between our model and the Kelso-Crawford model is that we assume *workers are interchangeable in production*: a firm's output depends only on the number of workers it employs, not on the workers' identities. (Kelso and Crawford impose this restriction briefly when exploring their gross substitutes condition.) Our results are thus most applicable to labor markets in which workers can be easily substituted for one another. Such labor markets include both high-education occupations like pharmacists (Goldin & Katz, 2016) and low-education occupations like textile manufacturing. Our results are less relevant to markets in which workers' productivities are heterogeneous. We show in Appendix C that when workers are not interchangeable in production, but salaries are still constant within each firm, a stable outcome may not exist.

We ask how monopsonistic labor markets can be better designed. This question is particularly relevant in the modern labor market, where workers often find jobs through online platforms (Smith, 2016; Delgado, 2019). Search frictions are one source of distortionary employer power (Burdett & Mortensen, 1998; Manning, 2011). By eliminating search frictions, online platforms might hope to better match workers and firms. However, firms using online platforms appear to face inelastic labor supply schedules, probably because workers have differing preferences over different jobs (Dube, Jacobs, Naidu, & Suri, 2020). We show how two-sided heterogeneity can distort the worker-firm match, even in the absence of search frictions, and we advise online platforms how these distortions can be eliminated.

1.1 A summary of our results

In Section 2, we present our model. Our model comprises workers and firms. Each worker can be employed by at most one firm, while each firm can hire any number of workers. A worker's utility depends quasi-linearly on her salary and on a firm-specific idiosyncratic amenity. The magnitudes, correlations, and signs of the amenities are unrestricted. In particular they might be positive (perhaps reflecting an exciting office environment) or negative (perhaps reflecting commuting costs). We model each firm's technology with a production function. Workers are interchangeable in production, in the sense that production functions depend only on the number of workers that a firm employs.

Following the job matching literature, we require that firms treat workers as gross substitutes. In our framework, this assumption means that, if a firm is willing to employ N workers at some salary, it must be willing to employ $N-1$ workers at that salary. Under our assumption that workers are interchangeable in production, requiring that firms treat workers as gross substitutes is equivalent to requiring that firms' production functions having decreasing differences (Kelso & Crawford, 1982).

An outcome comprises a matching of workers to firms and a salary schedule associating each firm with a salary. Our solution concept is stability. An outcome is stable if no firm and set of workers can deviate from the outcome and be no worse off, with some worker or firm strictly better off.

Worker and firm preferences are both quasi-linear in salaries. Quasi-linear preferences facilitate a simple definition of match value: the sum of firm production and worker amenities. A matching is efficient if it has maximal value. A matching is hedonic efficient if it maximizes value conditional on firm sizes. (An outcome is efficient if its matching is efficient and is hedonic efficient if its matching is hedonic efficient.)

Section 2 concludes with a simple example comprising one firm and two workers. The example demonstrates that multiple matchings can be stable, and that the value of these matchings can differ: in the efficient stable outcomes both workers are employed while in the inefficient stable outcomes only one worker is. In the efficient stable outcomes, salaries are higher and firm profits are lower. The inefficient outcomes are stable because the firm cannot pay its workers two different salaries; it prefers to employ one worker at a lower salary than employ two at a higher salary.

Our model thus exhibits monopsonistic distortions. This motivates the study of how centralized matching can ameliorate these distortions, our first high-level contribution. As we discuss in subsection 1.2, these distortions are not present in models in which a firm can pay different salaries to different workers. The example thus also relates to our second high-level contribution, which is to characterize how monopsonistic distortions arise from a restriction on transfers.

In Section 3, we characterize stable outcomes. An outcome has No Envy if, given the prevailing salaries, each worker prefers her firm to any other firm. An outcome has No Firing if, given its salary, no firm would be better off matched to one fewer worker. An outcome has No Poaching if no firm can increase its salary and make at least as much profit by attracting more workers. We show that an outcome is stable if and only if it has No Envy, No Firing, and No Poaching. This characterization provides a transparent interpretation of our solution concept. We regularly use it in the proofs of our later results.

Given a discrete production function, a firm's marginal product can be defined either as the increase in output from hiring an additional worker or as the decrease in output from firing a single worker. Given that production functions have decreasing differences, the former definition will be no larger than the latter. We say an outcome has Marginal Product Salaries if every firm's salary lies within those two bounds. Section 3 shows that if an outcome has Marginal Product Salaries, then it will have No Firing and No Poaching. As a corollary, if an outcome has No Envy, then it is stable.

In Section 4, we introduce a piece of mathematical machinery: A replacement chain moves a sequence of workers from firm to firm such that each successive worker replaces the next. Thus, a replacement chain changes each firm's size by at most one worker. It follows from our gross substitutes condition that, if some matching is inefficient, its value can be increased by a replacement chain. Moreover, if the matching was in a stable outcome, this value-increasing replacement chain is acyclic: it begins and ends at different firms.[1]

The replacement chain machinery yields two immediate results. First, we use it to confirm the conventional wisdom that paying workers their marginal product will lead to efficiency. Specifically, we show that

[1]Replacement chains are similar to the vacancy chains studied by Blum, Roth, and Rothblum (1997). Blum et al. are interested in how an initial vacancy resulting, for example, from the entry of a new firm can mutate a labor market from one stable outcome to another: as each vacancy is filled, another is created. Unlike vacancy chains, a replacement chain need not leave each worker better off.

every stable outcome with Marginal Product Salaries will be efficient. Second, we show that, though not every stable outcome is efficient, every stable outcome will be *hedonic* efficient.

In Section 5 we show that every efficient matching is in a stable outcome. Our proof is constructive: given some efficient matching, we construct a salary schedule which has No Envy and Marginal Product Salaries. These two properties imply that the outcome is stable. This result is an important component of our first high-level contribution, showing how centralized matching can ameliorate monopsonistic distortions: there is an efficient outcome that, if stipulated by a market designer, no coalition of workers and firms could profitably destabilize.

In Section 6, we discuss worker and firm welfare across stable outcomes. We first show that the efficient stable outcome constructed in the previous section has greater salaries than any other stable outcome. We then show that all workers will prefer one stable outcome to another if and only if the former outcome has greater salaries than the latter. In combination, these results show that there exists an efficient stable outcome preferred by workers over any other stable outcome. We next show that if one stable outcome is preferred by all workers to another, all firms prefer the latter outcome to the former. Thus firms prefer inefficient stable outcomes over the worker-optimal efficient stable outcome. This result suggests that firms create monopsonistic distortions to increase their profits. In doing so, they harm workers and shrink social surplus.

The results in Section 6 have practical implications. Among stable outcome, there is no trade-off between worker welfare and market efficiency: the worker-optimal stable outcome provides both. As such, it is a promising target for the centralized matching mechanism considered in the following section.

In Section 7, we ask whether an efficient stable outcome can be implemented through a strategyproof mechanism. When firms' production functions are private information, it cannot be. Firms can claim that they are less productive than they actually are. This deceit results in them paying lower salaries, and thus can be profitable even if it means that they are matched to inefficiently few workers. However, when firms' production functions are public information, a strategyproof mechanism can elicit workers' preferences and implement the worker-optimal stable outcome, which we know from the previous section is efficient. This result demonstrates when centralized matching can ameliorate monopsonistic distortions: when firms' production functions are observed.

In Section 8, we study the sources of monopsonistic inefficiencies. We begin by highlighting two conditions under which the market *will* be efficient. The first such condition is that firms have 'common value amenities': workers have homogeneous preferences over the amenities provided by firms. If every firm has common value amenities, then every stable outcome is efficient. Monopsonistic inefficiencies arise when firms exclude 'expensive' workers by paying low salaries. When firms have common value amenities, no worker is relatively more expensive than any other.

We next consider the effect of duplicating firms. Two firms are duplicates if they have the same production function and provide the same amenities. When every firm has a duplicate, every stable outcome is efficient. Given that stable outcomes have No Envy, two duplicate firms must both pay the same salary in any stable outcome. This means that each could poach the other's workers by paying an infinitesimally higher salary; the temptation to do so would erode any monopsonistic rents.

The final result in Section 8 identifies the precise coalition that could destabilize an inefficient stable outcome, were transfers unrestricted. In every inefficient stable outcome, some worker is willing to work at some new firm for a salary less than what that firm would gain from hiring her. However, that new firm does not hire her, because doing so would require that it pay its existing workers more.

These final results show how monopsonistic distortions are caused by the combination of two-sided heterogeneity together with the requirement that each firm pays the same salary to all its workers. Firms exclude expensive workers, even when the worker could be paid less than what the firm would gain from hiring her. When marginal workers are inexpensive – because all workers value firms' amenities equally, or because the firm can poach workers from a duplicate – monopsonistic distortions are eliminated.

Appendix A proves the results in the main body of the text. Latter appendices contain auxiliary results. In Appendix B, we relate some definitions and results to those in other matching papers. In Appendix C, we prove that a stable outcome may not exist when workers are not interchangeable in production. In Appendix D, we motivate our model as the limiting case of a model in which workers shirk when they are paid less than their colleagues. In Appendix E, we study how the set of stable outcomes changes when each worker is duplicated. In Appendix F, we show that many plausible statistics fail to indicate which of two stable outcomes is the more efficient.

1.2 How our results relate to the existing literature

Our first high-level contribution is to assess how centralized matching can address monopsonistic distortions. To make that assessment, we connect the job matching literature to the labor monopsony literature.

The intellectual antecedent of job matching models is the Gale and Shapley (1962) college admissions model, in which students' preferences over colleges are combined with colleges' preferences over students to construct a stable matching: that is, a matching from which no set of students and colleges can profitably defect. Job matching models extend the college admissions model by pairing each matching with transfers from one side of the market to the other (Shapley & Shubik, 1971; Crawford & Knoer, 1981; Kelso & Crawford, 1982; Fleiner, 2003; Hatfield & Milgrom, 2005).

The canonical job matching model is that of Kelso and Crawford (1982). Kelso and Crawford show that, if firms treat workers as 'gross substitutes' and transfers are unrestricted, then a stable outcome will always exist. When workers' utilities are quasi-linear in salaries, every stable outcome of the Kelso and Crawford model is efficient. Kelso and Crawford assume that each worker's salary can be set independently of the salaries paid to that worker's colleagues. Thus, a blocking coalition consisting of one worker and one firm will not affect the transfers paid to other workers whom that firm employs. Such coalitions block any inefficient outcome.

The labor monopsony literature descends from Robinson's (1933) study of imperfect competition. Modern monopsony models adopt functional form restrictions more frequently than the job matching literature. For example, some models postulate a representative worker with CES labor disutility (Berger et al., 2019). Others postulate a continuum of workers with Gumbel-distributed firm amenities (Card et al., 2018; Azar, Berry, & Marinescu, 2019; Lamadon, Mogstad, & Setzler, 2019; Kroft, Luo, Mogstad, & Setzler, 2020). Firms interact in Bertrand or Cournot competition, and a firm either pays an identical salary to all its workers or discriminates solely on the basis of productivity. A recurring theme is that firms' strategic behavior distorts the labor market: unemployment is too high, productive firms are too small and unproductive firms are too large (Boal & Ransom, 1997; Manning, 2011; Berger et al., 2019; Lamadon et al., 2019).

These distortions are not found in Kelso and Crawford's model. Given the different modelling assumptions made by the job matching and monopsony literatures, it is not prima facie obvious why that is: Is it because of the functional forms they impose? Because of the solution concepts they employ? Or is it only because the job matching literature allows within-firm salary discrimination? By unifying the two literatures, this paper demonstrates how monopsonistic distortions arise from restrictions on transfers – the second of our two high-level contributions.

By studying market power in job matching, we follow Bulow and Levin (2006); Kojima (2007); and Azevedo (2014). Bulow and Levin study market power in centralized labor markets like those matching hospitals to doctors. They consider a stylized context in which each hospital sets an anonymous salary and is then matched to a single doctor. They assume that the efficient match is assortative: 'better' hospitals should be matched to 'better' doctors. Hospitals set salaries in mixed strategy equilibrium. Ex ante, salaries are lower than the competitive equilibrium. Ex post, the resultant match can be inefficient and unstable because better hospitals may happen to set lower salaries than worse hospitals. Bulow and Levin consider only one-to-one matching. Their model thus lacks the monopsonistic mechanism which stabilizes inefficient matchings in our model.

Kojima (2007) comments on the Bulow and Levin model. Kojima argues that Bulow and Levin's results need not extend to contexts in which each hospital is matched to many doctors. In particular, Kojima points out that strategic salary setting by firms can benefit inframarginal workers, as firms increase salaries to compete for marginal workers. Kojima limits his comparisons to the firm-optimal competitive equilibrium. Our results in Section 6 and Appendix B suggest that this perspective is limiting: no worker benefits from firms' strategic salary setting when it constitutes a departure from the worker-optimal competitive equilibrium.

Azevedo (2014) constructs a market with a finite set of firms and a continuum of workers. Firms choose quantities in Cournot competition. Azevedo first considers exogenous salaries. Exogenous salaries mean that Azevedo's model lacks the monopsonistic mechanism emphasised by our model. Nonetheless, Azevedo's model does produce inefficiencies. A firm might avoid hiring a relatively unproductive worker. The unproductive worker may then replace a worker matched to another firm. The ensuing 'rejection chain' can eventually result in the original firm being matched to a more productive worker. This can benefit the initial firm while hampering efficient employment. Our model lacks this mechanism because it assumes that workers are interchangeable in production.

Azevedo also considers endogenous salaries. When doing so, he lets salaries vary between the workers employed by a given firm. As in Kelso and Crawford's model, personalized salaries foreclose monopsonistic distortions.

2 A Model of a Monopsonistic Labor Market

A labor market $(\mathbf{F}, \mathbf{W})$ comprises a finite set of firms $\mathbf{F}$ and a finite set of workers $\mathbf{W}$. Each worker can be employed by at most one firm while each firm can hire any number of workers.

Each firm F is endowed with a production technology, which we represent with a non-decreasing function $y_F : \mathbb{N} \to \mathbb{R}^+$. Note that production depends only on the number of workers employed and not on their identity. We normalize each $y_F(0) = 0$. Each firm pays the same salary to all its workers: there is no salary discrimination within any firm's workforce. Firms face a competitive product market in which their good has price normalized to one. Thus, if firm F employs N workers at salary s, its profit will be

$$\pi_F(N, s) = y_F(N) - sN.$$

Each worker $w \in \mathbf{W}$ has quasi-linear preferences

$$u_w(F,s) = \alpha_w(F) + s,$$

where $F \in \mathbf{F} \cup \{\emptyset\}$ is the firm at which she is employed, s is the salary she is paid, and $\alpha_w(F)$ is the amenity that she receives from working at firm F. The amenity $\alpha_w(F)$ may be positive, negative, or zero, and can vary idiosyncratically across workers. It encompasses any fixed benefit or cost the worker incurs from working at a given firm. Being employed at the empty set denotes unemployment, and we normalize $\alpha_w(\emptyset) = 0$.

2.1 Matchings and outcomes

A **matching** is a function $\mu : \mathbf{F} \cup \mathbf{W} \to \mathscr{P}(\mathbf{F} \cup \mathbf{W})$ such that:

- For all workers $w \in \mathbf{W}$: $|\mu(w)| \leq 1$ and $\mu(w) \subseteq \mathbf{F}$.
- For all firms $F \in \mathbf{F}$: $\mu(F) \subseteq \mathbf{W}$.
- For all workers $w \in \mathbf{W}$ and all firms $F \in \mathbf{F}$: $w \in \mu(F)$ if and only if $\mu(w) = \{F\}$.

We use the matching to represent employment: a worker w is employed at firm F if and only if $\mu(w) = \{F\}$. Since workers are matched to at most one firm, we abuse notation and write $\mu(w) = F$ rather than $\mu(w) = \{F\}$.

An **outcome** (μ, s) comprises a matching μ and a salary function $s : \mathbf{F} \cup \emptyset \to \mathbb{R}^+$, associating each firm with a salary. We require all salaries to be non-negative, and we normalize $s(\emptyset) = 0$. To simplify our results we require that for any outcome (μ, s), for any firm F, if $\mu(F) = \emptyset$, then $s(F) = y_F(1)$. This requirement is without loss of generality because the salary paid by an unmatched firm does not affect its profit.

An outcome (μ, s) is **individually rational** if

- for all workers $w \in \mathbf{W}$: $u_w(\mu(w), s(\mu(w))) \geq 0$, and
- for all firms $F \in \mathbf{F}$: $\pi_F(|\mu(F)|, s(F)) \geq 0$.

This familiar condition requires that no worker receive less utility than she would from unemployment, and that no firm make negative profits.

A coalition (F, C, s^*), with $F \in \mathbf{F}$, $C \subseteq \mathbf{W}$, and $s^* \in \mathbb{R}^+$, **blocks** outcome (μ, s) if

- $\pi_F(|C|, s^*) \geq \pi_F(|\mu(F)|, s(F))$, and
- for all workers $w \in C$: $u_w(F, s^*) \geq u_w(\mu(w), s(\mu(w)))$,

where the inequality is strict for the firm or one of the workers. It is without loss of generality to consider only blocking coalitions comprised of a single firm: any coalition containing more than one firm could be be broken into multiple coalitions, each with only one firm. In words, a firm and some workers can destabilize an outcome if there is a salary at which, if the firm employs only those workers, all parties are better off.

Our solution concept is stability. An outcome (μ, s) is **stable** if it is individually rational and not blocked by any coalition.

We follow the matching literature in using a cooperative solution concept. Decentralized labor markets will produce stable outcomes, provided that firms and workers can freely sever existing relationships and form new relationships. Using stability as a solution concept does not imply that all stable outcomes are equally realistic as market equilibria. Equilibrium selection will depend upon the institutions of the market in question. For example, one implication of stability is that firms cannot decrease their salaries without the consent of their current workers; the plausibility of some stable outcomes will thus depend on whether contracts or regulations enforce this feature. In contrast to our approach, the labor literature often adopts non-cooperative solution concepts like Bertrand competition. In Appendix B, we show that Bertrand equilibria are generically stable.

2.2 Production functions

We required above that firms' production functions are non-decreasing and normalized such that for all firms F: $y_F(0) = 0$. Again, following the matching literature (Kelso & Crawford, 1982; Hatfield & Milgrom, 2005) we additionally impose the following gross substitutes restriction.

Firm F treats workers as **gross substitutes** if for any salary $s \in \mathbb{R}^+$ and number of employees N:

$$N \in \operatorname*{argmax}_{M \le N} \pi_F(M, s) \implies N - 1 \in \operatorname*{argmax}_{M \le N-1} \pi_F(M, s).$$

In other words, if, at some salary, a firm prefers to hire N workers over any fewer number, it must also prefer to hire $N-1$ workers over any fewer number.

Assumption 1. *Every firm treats workers as gross substitutes.*

A function $f : \mathbb{N} \to \mathbb{R}$ has **decreasing differences** if $N > M$ implies that $f(N+1) - f(N) \le f(M+1) - f(M)$. Theorem 6 in Kelso and Crawford shows that, when workers are interchangeable in production, gross substitutes is equivalent to the production functions y_F having decreasing differences. Given that our notion of gross substitutes is slightly different from theirs, we provide our own proof. (Our proofs are in Appendix A.)

Lemma 1. *Every firm's production function has decreasing differences.*

Typically, the gross substitutes condition is used to guarantee the existence of stable outcomes (Kelso & Crawford, 1982; Hatfield & Milgrom, 2005). This condition is often compared to the concavity of utility or production functions. In our setting, this connection becomes even more clear: when workers are interchangeable in production and are treated as gross substitutes, production functions exhibit diminishing returns to labor. In this way, the gross substitutes assumption is the discrete analogue of assuming concave production.

With a discrete production function, a firm's marginal product has two possible definitions. Given some matching μ, we let $\Delta^+_\mu(F)$ denote the increase in firm F's output from employing one worker *more* than the firm is employing at μ, and we let $\Delta^-_\mu(F)$ denote the decrease in firm F's output from employing one worker *fewer* than the firm is employing at μ:

$$\Delta^+_\mu(F) \equiv y_F\left(\left|\mu(F)\right| + 1\right) - y_F\left(\left|\mu(F)\right|\right);$$

$$\Delta^-_\mu(F) \equiv \begin{cases} y_F\left(\left|\mu(F)\right|\right) - y_F\left(\left|\mu(F)\right| - 1\right) & \text{if } \mu(F) \neq \emptyset; \\ \infty & \text{if } \mu(F) = \emptyset. \end{cases}$$

By Lemma 1, for any firm F and matching μ: $\Delta^+_\mu(F) \le \Delta^-_\mu(F)$.

2.3 Definitions of efficiency

Given the quasi-linear setup, we can define the **value** of a matching μ as the sum of worker amenities and firm outputs:

$$\text{value}\left(\mu\right) \equiv \sum_{F \in \mathbf{F}} y_F\left(\left|\mu(F)\right|\right) + \sum_{w \in \mathbf{W}} \alpha_w\left(\mu(w)\right).$$

A matching μ^* is **efficient** if it has maximal value:

$$\mu^* \in \operatorname*{argmax}_{\mu} \text{value}\left(\mu\right).$$

This notion of efficiency is sometimes referred to as utilitarian efficiency. We also define a more limited notion of efficiency. A matching μ^* is **hedonic efficient** if it maximizes the sum of amenities, given firm sizes:

$$\mu^* \in \underset{\mu \text{ s.t.} \forall F: |\mu(F)| = |\mu^*(F)|}{\arg\max} \sum_{w \in \mathbf{W}} \alpha_w \left(\mu(w)\right).$$

By holding firm sizes fixed, hedonic efficiency speaks only to inefficiencies caused by a mismatch of workers to firms, rather than allocative inefficiencies in production. Note that hedonic efficiency is a strictly weaker requirement than efficiency: if μ^* is efficient, then μ^* is hedonic efficient.

An outcome (μ, s) is efficient if its matching μ is efficient, and is hedonic efficient if μ is hedonic efficient.

2.4 An illustrative example

To elucidate our model, we present the following example of a monopsonistic labor market.

Example 1 (a simple monopsony). $\mathbf{F} = \{F\}$. $y_F(N) = 6N$. $\mathbf{W} = \{w_1, w_2\}$. $\alpha_{w_1}(F) = 0$. $\alpha_{w_2}(F) = -4$.

The stable outcomes of Example 1 are presented in Figure 1. The stable outcomes have one of these two matchings:

$$\mu^1 = \begin{pmatrix} w_1 & w_2 \\ F & \emptyset \end{pmatrix}; \quad \mu^2 = \begin{pmatrix} w_1 & w_2 \\ F & F \end{pmatrix}.$$

The matching μ^1 will be a stable outcome when composed with a salary $s^1(F) \in [0,2)$. The firm makes profit $\pi_F\left(1, s^1(F)\right) > 6 - 2 > 0$, worker w_1 has utility $u_{w_1}\left(F, s^1(F)\right) = s^1(F) \geq 0$, and worker w_2 has utility $u_{w_2}(\emptyset, 0) = 0$. Thus, the outcome is individually rational.

We now verify that $\left(\mu^1, s^1\right)$ is not blocked by any coalition $\left(F, C, s'\right)$. If $C = \{w_1\}$, either both F and w are indifferent between the coalition and μ^1, or one is strictly better off while the other is strictly worse off. So a blocking coalition must include w_2. It is not individually rational for w_2 to work at any salary strictly less than 4, so $w_2 \in C$ requires $s' \geq 4$. The firm would be strictly worse off paying $s' > 2$ and only employing w_2, so we must have $C = \{w_1, w_2\}$. Thus, the firm must be weakly better off employing 2 workers at s'. But $\pi_F\left(2, s'\right) \leq 2 \times 6 - 2 \times 4 = 4 < \pi_F\left(\left|\mu^1(F)\right|, s^1(F)\right)$. Therefore, there exists no blocking coalition, and $\left(\mu^1, s^1\right)$ is indeed stable.

The matching μ^2 will be a stable outcome when composed with a salary $s^2(F) \in [4,6]$. The firm makes profit $\pi_F\left(2, s^2(F)\right) = 2 \times 6 - 2 \times s^2(F) \geq 0$, worker w_1 has utility $u_{w_1}\left(F, s^2(F)\right) = s^2(F) \geq 4 \geq 0$, and worker w_2 has utility $u_{w_2}\left(F, s^2(F)\right) \geq s^2(F) - 4 \geq 0$. Thus, the outcome is individually rational.

Again, we verify that there exists no blocking coalition $\left(F, C, s'\right)$. Given both workers are employed by F at μ^2, there can be no blocking coalition $\left(F, C, s'\right)$ in which $s' < s^2(F)$. If $C = \{w_1, w_2\}$ and $s' > s^2(F)$, the firm will be strictly worse off. If $|C| = 1$ and $s' = s^2(F)$, then the firm will be weakly worse off, and either worker will be

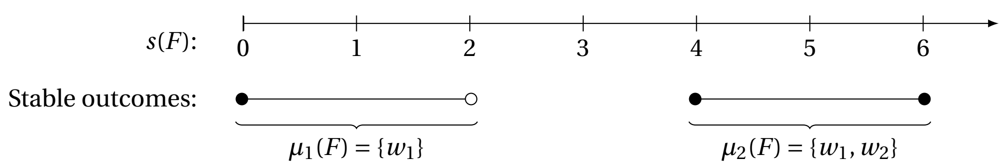


Figure 1: Stable outcomes in Example 1

indifferent. Finally, if $|C| = 1$ and $s' > s^2(F)$, then the firm will be strictly worse off. Therefore, there exists no blocking coalition, and (μ^2, s^2) is indeed a stable outcome.

Note that value$(\mu^2) = 8 >$ value$(\mu^1) = 6$. Thus, this example suffices to prove the following two results.

Proposition 1. *There can exist multiple stable outcomes, with different matchings of different value.*

Corollary 1. *There can exist an inefficient stable outcome.*

Proposition 1 demonstrates that our model behaves very differently from the Kelso and Crawford model, which always predicts efficient matchings. Note that the inefficient matching is stable directly because of the restriction on transfers. If the firm could pay different salaries to each worker, there would exist a blocking coalition to any outcome (μ^1, s^1), where $s^1(F) \in [0,2)$: the firm could continue to pay salary $s^1(F)$ to worker w_1 and offer w_2 her reservation salary, profitably employing both workers. Thus, it is exactly the restriction on transfers that creates the monopsonistic distortion: the firm prefers employing inefficiently few workers over paying all its workers a higher salary.

Yet the restriction on transfers does not exclude the efficient matching from being stable. Although the firm prefers (μ^1, s^1) to (μ^2, s^2), a firm which found itself in outcome (μ^2, s^2) could not unilaterally decrease its salary to below w_2's reservation salary. An efficient outcome can thus be stable. Whether the efficient stable outcome actually occurs will depend on whether the firm has the power to unilaterally decrease its salary at its current workers' expense. Labor market efficiency thus depends directly on whether labor market institutions empower workers or firms.

Example 1 exhibits other key features of our model that we will show hold more generally. Though μ^1 is inefficient, μ^1 is hedonic efficient: given that only one worker is employed, it is more efficient for that worker to be worker w_1. When $s(F) = 6$, a higher salary than any other stable outcome, the outcome is efficient, is better for all workers than any other stable outcome, and yields lower firm profit than any other stable outcome.

3 Characterizing Stable Outcomes

In this section, we first show that stability is equivalent to three simple conditions. This result provides an easily interpretable characterization of our solution concept. We then define a condition on salary schedules that we call 'Marginal Product Salaries', which we relate to stability. These concepts will help prove and interpret later results.

3.1 Three conditions equivalent to stability

An outcome (μ, s) has **No Envy** if for every worker $w \in \mathbf{W}$ and firm $F \in \mathbf{F} \cup \{\emptyset\}$:

$$u_w\left(\mu(w), s(\mu(w))\right) \geq u_w\left(F, s(F)\right).$$

The No Envy condition states that no worker would prefer to be matched to another firm, or to be unemployed, given the prevailing salaries.

An outcome (μ, s) has **No Firing** if for every firm $F \in \mathbf{F}$ such that $|\mu(F)| > 0$:

$$\pi_F\left(\left|\mu(F)\right|, s(F)\right) \geq \pi_F\left(\left|\mu(F)\right| - 1, s(F)\right).$$

The No Firing condition states that no firm would be better off being matched to one fewer worker while paying the same salary. Using the definition of the marginal product $\Delta_\mu^-(F)$ presented in the previous section, No Firing is equivalent to the requirement that for each matched firm, $s(F) \leq \Delta_\mu^-(F)$.

Consider an outcome in which firm F pays salary s_F and each other firm F' pays $s\left(F'\right)$, i.e., the corresponding element of the salary schedule s. The maximal labor-supply available for firm F is

$$L_F(s_F, s) \equiv \left|\left\{w : u_w(F, s_F) \geq \max_{G \in \mathbf{F} \cup \{\emptyset\}} u_w\left(G, s(G)\right)\right\}\right|.$$

Note that the maximal labor-supply functions allocate a worker to multiple firms when that worker is indifferent between them. An outcome (μ, s) has **No Poaching** if for every firm $F \in \mathbf{F}$, there exists no salary $s_F > s(F)$ and employment level L such that $\left|\mu(F)\right| < L \leq L_F(s_F, s)$, and that

$$\pi_F(L, s_F) \geq \pi_F\left(\left|\mu(F)\right|, s(F)\right).$$

The No Poaching condition states that no firm can increase its salary and, by attracting additional workers, make at least as much profit as it did previously.

Proposition 2. *An outcome is stable if and only if it has No Envy, No Firing, and No Poaching.*

If the No Envy condition fails, some 'envious' worker would prefer employment at some other firm, at that firm's existing salary. The envious worker could form a blocking coalition with that firm and all-but-one of that firm's existing workers, at the existing salary. The envious worker would be better off, while the firm and the existing workers would be no worse off. The fact that the firm is no worse off follows from our assumption that production depends only on the number of workers employed and not on their identity. As we will show, No Envy can be used to impose a great deal of structure on the set of stable outcomes.

If the No Firing condition fails, some firm would be losing money on its marginal worker. The outcome would be blocked by a coalition comprising that firm and all-but-one of the firm's existing workers, at the existing salary. That coalition would leave the still employed workers no worse off and the firm would make strictly more profit.

If the No Poaching condition fails, some firm could form a blocking coalition with its existing workers and some new workers at a higher salary than it currently pays. The firm and the new workers would be no worse off, and the existing workers would be strictly better off.

The above summarizes why a stable outcome must have No Envy, No Firing, and No Poaching. To prove the converse, we first note that No Envy immediately implies individual rationality for the workers. Given Lemma 1, No Firing implies individually rational for the firms. We conclude the proof by showing that, if any blocking coalition exists, then the original outcome must not satisfy the No Envy, No Firing, or No Poaching conditions.

These three conditions encompass the ways in which a labor market outcome can be destabilized. Workers can quit or switch jobs (which is precluded by No Envy), firms can fire workers (which is precluded by No Firing), and firms can poach workers (which is precluded by No Poaching).

3.2 Marginal Product Salaries and stability

One situation in which the No Firing and No Poaching conditions will be satisfied is when firms pay workers their marginal products. Recall from Section 2 that a firm F's marginal product can be defined either as the increase in output from being matched to one worker more ($\Delta^+_\mu(F)$) or as the reduction in output from being matched to one worker fewer ($\Delta^-_\mu(F)$). By Lemma 1, the firms' production functions have decreasing differences, and so $\Delta^+_\mu(F) \le \Delta^-_\mu(F)$. We say that an outcome (μ, s) has **Marginal Product Salaries** if every firm salary is between these two quantities:

$$\forall F : s(F) \in \left[\Delta^+_\mu(F), \Delta^-_\mu(F)\right].$$

Lemma 2. *An outcome with Marginal Product Salaries will have both No Firing and No Poaching.*

If firm F's salary is at least $\Delta^+_\mu(F)$, then any higher salary will be strictly greater than $\Delta^+_\mu(F)$, and thus at such a salary the firm would be worse off hiring an additional worker. Thus, $s(F) \ge \Delta^+_\mu(F)$ implies (μ, s) will have No Poaching. The No Firing condition is equivalent to the requirement that each firm F's salaries be less than $\Delta^-_\mu(F)$.

Proposition 2 tells us that an outcome with No Envy, No Firing, and No Poaching is stable, and Lemma 2 tells us that an outcome with Marginal Product Salaries also has No Firing and No Poaching. Combining these results yields the following corollary:

Corollary 2. *If an outcome has No Envy and Marginal Product Salaries, then it is a stable outcome.*

Note that Example 1 showed us that Marginal Product Salaries are not necessary for an outcome to be stable. That is because a firm need not lose its workers when it pays less than its marginal product.

Marginal Product Salaries incentivize firms to maintain their current employment. In Subsection 4.2, we will show that stable outcomes with Marginal Product Salaries will be efficient. We will use this result to construct an efficient stable outcome that a market designer may want to implement.

4 Replacement Chains and Their Economic Implications

This section develops a piece of mathematical machinery – a 'replacement chain' – which decomposes the change from one matching to another. A replacement chain represents moving a sequence of workers from their current firm to the following worker's firm. Having developed replacement chains, we deploy them to prove two substantive results: one relating Marginal Product Salaries to the efficiency of a stable outcome, and another characterizing the inefficiency that a stable outcome might exhibit.

4.1 Replacement chains

A **replacement chain** comprises a sequence of workers $(w_k)_{k=0}^{N-1} \subseteq \mathbf{W}$ and a sequence of firms $(F_k)_{k=0}^{N} \subseteq \mathbf{F} \cup \{\emptyset\}$ such that no worker is repeated:

$$\forall k \neq j : w_k \neq w_j,$$

and no adjacent firms are the same:

$$\forall k : F_k \neq F_{k+1}.$$

We use replacement chains to describe changing a given matching by a sequence of worker moves in which each worker replaces the subsequent worker in the chain at the subsequent worker's firm. Since workers are interchangeable in production, replacement chains allow us to consider large changes to matchings in which only the production of the first and last firm is affected.

We say $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ is a replacement chain from matching μ to μ' if

$$\forall k \in 0, \ldots, N-1 : w_k \in \mu(F_k) \cap \mu'(F_{k+1}).$$

In other words, we can think of χ as representing a sequence of worker moves from the firms they are matched to under μ to the firms they are matched to under μ'. Note that these moves do not necessarily capture all the differences between μ and μ'.

For a matching μ, if χ is such that

$$\forall k \in 0, \ldots, N-1 : w_k \in \mu(F_k),$$

then we define $\mu + \chi$ to be the matching constructed by moving each worker w_k from F_k to F_{k+1}:

$$\left(\mu + \chi\right)(w) = \begin{cases} F_{k+1} & \text{if } w = w_k; \\ \mu(w) & \text{if } w \notin (w_k)_{k=0}^{N-1}. \end{cases}$$

Similarly, for a matching μ',if χ is such that

$$\forall k \in 0, \ldots, N-1 : w_k \in \mu'(F_{k+1}),$$

then we define $\mu' - \chi$ to be the matching constructed by moving each worker w_k from F_{k+1} to F_k:

$$\left(\mu' - \chi\right)(w) = \begin{cases} F_k & \text{if } w = w_k; \\ \mu'(w) & \text{if } w \notin (w_k)_{k=0}^{N-1}. \end{cases}$$

Note that if χ is a replacement chain from μ to μ', then both $\mu + \chi$ and $\mu' - \chi$ are well-defined.

If $\mu \neq \mu'$, there will exist some replacement chain from μ to μ'. For example, if there exists a worker w such that $\mu(w) \neq \mu'(w)$, then the trivial replacement chain $\left((w), \left(\mu(w), \mu'(w)\right)\right)$ is a replacement chain from μ to μ'. Note that if χ is a replacement chain from μ to μ', it need not be the case that $\mu + \chi = \mu'$. Rather, it is necessarily the case that there will exist a sequence of replacement chains $\chi_1, \chi_2, \ldots, \chi_k$, all from μ to μ', such that $\mu + \chi_1 + \chi_2 + \ldots + \chi_k = \mu'$.

Consider some replacement chain $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$. χ is **cyclic** if $F_0 = F_N$. χ is **acyclic** if $F_0 \neq F_N$.

The notion of a replacement chain is depicted in Figure 2. The replacement chain in Panel (a) moves worker w_0 from firm F_0 to firm F_1, moves worker w_1 from firm F_1 to firm F_2, moves worker w_2 from firm F_2 to firm F_3, and moves worker w_3 from firm F_3 to firm F_4. It is acyclic because it starts and ends at different firms. The replacement chain in Panel (b) is identical, except that it additionally moves worker w_4 from firm F_4 to firm F_0. It is cyclic because it starts and ends at the same firm.

Our first result regarding replacement chains claims that, from any inefficient matching, there exists a replacement chain that increases value. This result is a consequence of our gross substitutes assumption (Assumption 1). For example consider two matchings μ, μ^* such that value$\left(\mu^*\right) >$ value$\left(\mu\right)$, and μ and μ^* only differ for two workers w_1, w_2, who are both matched to firm F_1 in μ and F_2 in μ^*:

$$\{w_1, w_2\} = \mu(F_1) \cap \mu^*(F_2);\ F_1 \neq F_2.$$

There is no replacement chain χ such that $\mu^* = \mu + \chi$. However, given Assumption 1, if moving *both* w_1 and w_2 from F_1 to F_2 increases value, then either moving w_1 from F_1 to F_2 increases value, or moving w_2 from F_1 to F_2

increases value (or both). It is this underlying connection to gross substitutes that makes replacement chains so useful for formally comparing the efficiency of different outcomes. Replacement chains change firm sizes by at most one, and thus the way they change the value of a given matching is very simple to describe.

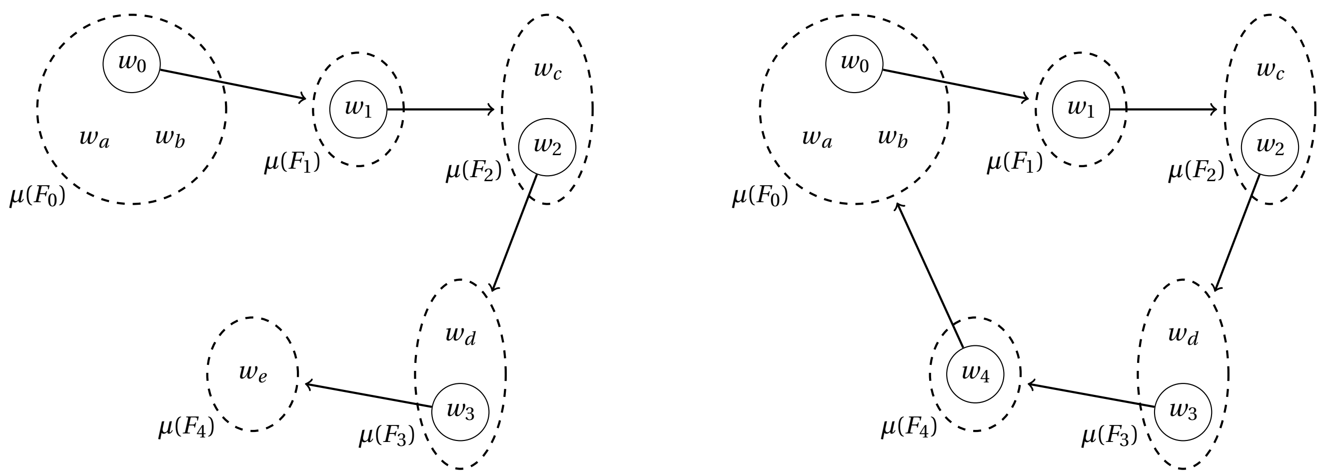
Lemma 3. *Let μ and μ^* be matchings such that value$(\mu^*) >$ value(μ). There exists a replacement chain χ from μ to μ^* such that value$(\mu+\chi) >$ value(μ). Moreover, for each firm F: $\left|(\mu+\chi)(F)\right| \leq \max\left\{\left|\mu(F)\right|, \left|\mu^*(F)\right|\right\}$.*

The proof of Lemma 3 formalizes the above intuition. We present an algorithm which necessarily finds the required replacement chain. The fact that, for each firm F, $\left|(\mu+\chi)(F)\right| \leq \max\left\{\left|\mu(F)\right|, \left|\mu^*(F)\right|\right\}$ means that the value-improving replacement chain need not grow any firm to be bigger than its size in μ^*.

Our next result demonstrates which kinds of replacement chains can increase value from a stable outcome.

Lemma 4. *Let (μ, s) be a stable outcome. There exists no cyclic replacement chain χ such that value$(\mu+\chi) >$ value(μ).*

Given that firm production depends only on the number of workers each firm employs, cyclic replacement chains do not change production. Thus, if a cyclic replacement chain increases an outcome's value, it must do so by rearranging workers to increase their total amenities. The rearrangement preserves workers' total income, and so by increasing workers' total amenities, it increases their total utility. Thus, there must be some worker who would have higher utility after the rearrangement. Some worker having higher utility after the rearrangement implies that that worker would prefer to be matched to a different firm at the prevailing salaries – i.e., the outcome does not have No Envy. By Proposition 2, if the outcome does not have No Envy then it cannot be stable. By the contrapositive, no cyclic replacement chain can increase the value of a stable outcome.



(a) An acyclic replacement chain

(b) A cyclic replacement chain

Figure 2: Two replacement chains

4.2 Marginal Product Salaries and efficiency

We will now use replacement chains to show that *any* stable outcome with Marginal Product Salaries will be efficient. Consider first an outcome (μ, s) such that the value of μ can be increased by moving one worker w from F to F':

$$\Delta^+_\mu(F') - \Delta^-_\mu(F) + \alpha_w(F') - \alpha_w(F) > 0.$$

If (μ, s) has Marginal Product Salaries, then $s(F') \geq \Delta^+_\mu(F')$ and $s(F) \leq \Delta^-_\mu(F)$, and thus:

$$s(F') - s(F) + \alpha_w(F') - \alpha_w(F) > 0,$$

which shows that (μ, s) does not have No Envy, and so by Proposition 2 is not a stable outcome.

An inefficient outcome cannot necessarily be improved by moving only one worker. However, Lemmas 3 and 4 guarantee the existence of an acyclic replacement chain with which the above argument can be extended to any inefficient outcome. This logic is formalized in the proof of the following proposition.

Proposition 3. *If a stable outcome has Marginal Product Salaries, then it is efficient.*

Proposition 3 tells us that, if we can stipulate a stable outcome in which workers are paid their marginal products, we can be assured that the outcome is efficient. Of course, knowing whether workers are paid their marginal products would require us to know firms' production functions, which may be the firms' private information. We will ask whether these production functions can be elicited in Section 7.

4.3 Stable outcomes and hedonic efficiency

Lemma 4 told us that stable outcomes never have a value-improving cyclic replacement chain. Proposition 4 captures the economic meaning of this result: the inefficiency of a stable outcome arises only through inefficient firm sizes rather than through a mismatch of workers to firms.

Proposition 4. *Every stable outcome is hedonic efficient.*

Proposition 4 contrasts with recent criticisms of centralized labor markets like the National Resident Matching Program. Proposed reforms to the National Resident Matching Program focus on improving the match between workers and firms, given firm sizes (Crawford, 2008). In contrast, Proposition 4 suggests that, when workers are interchangeable in production, firm sizes are the *only* problem with an inefficient stable outcome. In other words, in our setting, monopsonistic inefficiencies are chiefly rooted in production inefficiencies resulting from firm size distortions.

5 An Efficient Stable Outcome Exists

We know from Example 1 that stable outcomes need not be efficient. In this section, we show that at least one efficient outcome will be stable. This result speaks to our first main contribution, which asks how centralized matching can address monopsony power: The fact that there will always be a stable efficient outcome means that, if a market designer could stipulate it, no coalition of firms and workers could profitably destabilize it.

Theorem 1. *Every efficient matching is in a stable outcome.*

Our proof of Theorem 1 is constructive. In what follows, we will construct a salary schedule for some efficient matching. We will show that the resultant outcome has both No Envy and Marginal Product Salaries. By Corollary 2, this outcome is stable.

The No Envy condition requires that differences between firms' salaries be large enough such that no worker envies another worker's firm and salary. Below, we show that such salaries can also satisfy the bounds required by the Marginal Product Salaries condition.

Consider any matching μ. For any two firms F, G, let $d^1_\mu(F,G)$ be the minimum difference between their salaries required to ensure that none of G's workers strictly prefer working at F:

$$d^1_\mu(F,G) \equiv \max_{w \in \mu(G)} \{\alpha_w(F) - \alpha_w(G)\}.$$

For an outcome (μ, s) to have No Envy, it is necessary that no worker at firm G prefers working at firm F: i.e., $s(F) \le s(G) - d^1_\mu(F,G)$.

Let us now consider the maximal level of firm F's salary consistent with the outcome (μ, s) having No Envy, treating firm G's salary $s(G)$ as fixed. We require $s(F) \le s(G) - d^1_\mu(F,G)$. However, if there is a third firm H, we also require that $s(F) \le s(H) - d^1_\mu(F,H)$, and that $s(H) \le s(G) - d^1_\mu(H,G)$. Combining inequalities, this requires that $s(F) \le s(G) - \left(d^1_\mu(F,H) + d^1_\mu(H,G)\right)$. To capture the idea that $s(F)$ must simultaneously satisfy both the inequalities above, we define the object d^∞_μ:

$$d^\infty_\mu(F,G) \equiv \max_{N,\ (F_k)_{k=1}^N} \left\{ \sum_{k=1}^{N-1} d^1_\mu(F_k, F_{k+1}) \right\} \text{ such that } F_1 = F \text{ and } F_N = G.$$

In the salary schedule we propose below, the No Envy constraints will just bind. As such, for each pair of firms F and G, $s(F) = s(G) - d^\infty_\mu(F,G)$.

The function d^∞_μ optimizes over all lists of firms $(F_k)_{k=1}^N$; these lists can include a firm more than once and thus are arbitrarily long. Nonetheless, Lemma 5 will ensure that d^∞_μ is well-defined, provided that μ is hedonic efficient. Lemma 5 tells us that, if a candidate list contains duplicates, there must be a list without duplicates for which the sum $\sum_{k=1}^{N-1} d^1_\mu(F_k, F_{k+1})$ is at least as large. Thus, when constructing d^∞_μ, we can ignore any list with duplicates.

Lemma 5. *Let μ be hedonic efficient. For any list of firms $(F_k)_{k=1}^N$ such that $F_1 = F_N$: $\sum_{k=1}^{N-1} d^1_\mu(F_k, F_{k+1}) \le 0$.*

Let us now consider an efficient matching μ^*. Our candidate salary schedule s^* is defined as

$$s^*(F) \equiv \min_{G \in \mathbf{F}} \left\{ \Delta^-_{\mu^*}(G) - d^\infty_{\mu^*}(F,G) \right\}. \tag{1}$$

Given that μ^* is efficient, it must be hedonic efficient, and so Lemma 5 ensures that $d^\infty_{\mu^*}$ is well-defined.

The salary schedule s^* sets salaries as high as is possible while ensuring that the outcome (μ^*, s^*) has both No Envy and No Firing.

For reasons that will become obvious in the following section, we will refer to (μ^*, s^*) as a *worker-optimal stable outcome*. Note that there will generically be a unique worker-optimal stable outcome; there will be multiple only in the knife-edge case of there being multiple efficient matchings.

By Corollary 2, the following lemma implies Theorem 1.

Lemma 6. *A worker-optimal stable outcome has No Envy and Marginal Product Salaries.*

That (μ^*, s^*) has No Envy follows from the definition of $d^{\infty}_{\mu^*}$. Marginal Product Salaries requires that, for each firm F, $s^*(F) \in \left[\Delta^+_{\mu^*}(F), \Delta^-_{\mu^*}(F)\right]$. The upper bound $s^*(F) \le \Delta^-_{\mu^*}(F)$ follows immediately from the definition of the salary schedule. The lower bound $s^*(F) \ge \Delta^+_{\mu^*}(F)$ follows from the efficiency of μ^*: the proof of Lemma 6 shows how one could increase the value of μ^* were it the case that, for some firm F, $s^*(F) < \Delta^+_{\mu^*}(F)$.

Recall from Corollary 1 that inefficient outcomes can be stable. Whether market design can ameliorate these inefficiencies directly depends on the existence of efficient stable outcomes. In this section, we showed that the worker-optimal stable outcome is one such outcome and thus is a promising target for a centralized matching mechanism: it can be easily computed, it is efficient, and it is stable. In Section 7, we will show that a strategyproof mechanism can elicit workers' amenities if it implements the worker-optimal stable outcome.

The existence of an efficient stable outcome is consistent with the existing transferable utility matching literature, which typically finds that all stable outcomes are efficient (Kelso & Crawford, 1982; Shapley & Shubik, 1971). It contrasts with the labor monopsony literature, which typically predicts that workers in a labor market will allocated inefficiently. We reconcile the two literatures by showing that both efficient and inefficient outcomes can be stable when transfers are restricted.

6 Worker and Firm Welfare across Stable Outcomes

The previous sections have demonstrated that stable outcomes can have differing values: At least one will be efficient, while others may not be. In this section, we relate these results to worker and firm welfare. We explain why we gave the worker-optimal stable outcome its name: all workers prefer that outcome to any other stable outcome. We also show that if all workers prefer one stable outcome to another, all firms must prefer the latter outcome to the former. It follows that the worker-optimal stable outcome is worse for all firms than any other stable outcome.

These results depict who is harmed by monopsonistic distortions: workers. They also have practical implications. First, they characterize the tradeoffs that a centralized mechanism must navigate: if a market designer values efficiency, she must prioritize the welfare of the workers over the welfare of the firms. Second, our characterization of the worker-optimal stable outcome will be useful when we study strategyproofness in the following section.

6.1 The alignment of efficiency and worker welfare

We first characterize the worker-optimal stable outcome:

Lemma 7. *Let (μ^*, s^*) be a worker-optimal stable outcome. For each firm F_1, either $s^*(F_1) = \Delta^-_{\mu^*}(F_1)$, or there exists a list of firms $(F_k)_{k=1}^N$ and a list of workers $(w_k)_{k=2}^N$ such that each $w_k \in \mu^*(F_k)$, $s^*(F_N) = \Delta^-_{\mu^*}(F_N)$, and each worker w_k is indifferent between firm F_k and firm F_{k-1}: $\alpha_{w_k}(F_{k-1}) + s^*(F_{k-1}) = \alpha_{w_k}(F_k) + s^*(F_k)$.*

For any firm F_1, there is some reason why that firm's salary cannot be increased. One situation is that $s^*(F_1) = \Delta^-_{\mu^*}(F_1)$ – i.e., the firm cannot increase its salary without breaching the No Firing constraint. The other situation is that F_1 is the first firm in a sequence of firms with binding No Envy constraints; these constraints prevent each firm in the sequence from increasing its salary without also increasing the salary of the next firm. The sequence concludes at a firm F_N which has salary $s(F_N) = \Delta^-_{\mu^*}(F_N)$ and which thus cannot have a higher salary without breaching the No Firing constraint.

Lemma 7 suggests that the worker-optimal stable outcome has the highest salaries possible for the efficient matching. However, it is not obvious whether other, inefficient matchings could have higher salaries for at least some workers. For example, given that (by Lemma 1) firms' production functions have decreasing marginal products, one might worry that an outcome in which a firm is matched to inefficiently few workers would have higher salaries than an efficient outcome. The following result shows that is not the case.

Proposition 5. *If $\left(\mu^*, s^*\right)$ is the worker-optimal stable outcome then, for all stable outcomes $\left(\mu, s\right)$: $s^* \geqq s$.*

As intuition for Proposition 5, consider again an outcome in which a firm matched to inefficiently few workers has a higher salary than in the worker-optimal stable outcome. Such a salary would make some workers, who are not matched to that firm, envious, destabilizing the outcome. In sum, the interchangeability of workers rids them of any distortionary market power, meaning that they receive their highest salaries when the outcome is efficient.

Proposition 5 may be surprising since it claims that salaries – transfers from one side of the market to the other – are aligned with the efficiency of the entire market. Workers lack any distortionary market power, aligning efficiency with worker welfare.

Define the binary relation $\succeq_{\mathbf{W}}$ as representing workers' unanimous preferences across outcomes:

$$\left(\mu, s\right) \succeq_{\mathbf{W}} \left(\mu', s'\right) \iff \forall\, w : u_w\left(\mu(w), s(\mu(w))\right) \geq u_w\left(\mu'(w), s'(\mu'(w))\right).$$

If one outcome has greater salaries than another, a worker matched to the same firm in both outcomes will necessarily prefer the former outcome over the latter. Of course, workers who switch firms between the two outcomes could prefer either outcome. However, if both outcomes are stable, No Envy guarantees that each worker prefers the firm she is matched to over any other firm, given that firm's salaries. By combining inequalities, we can prove that if one *stable* outcome has greater salaries than another, all workers must prefer the former outcome over the latter.

Lemma 8. *For any two stable outcomes $\left(\mu, s\right), \left(\mu', s'\right) : s \geqq s' \iff \left(\mu, s\right) \succeq_{\mathbf{W}} \left(\mu', s'\right)$.*

Combining Lemma 8 and Proposition 5 yields the following result:

Corollary 3. *If $\left(\mu^*, s^*\right)$ is a worker-optimal stable outcome then, for all stable outcomes $\left(\mu, s\right)$: $\left(\mu^*, s^*\right) \succeq_w \left(\mu, s\right)$.*

Corollary 3 shows why we gave the worker-optimal stable outcome its name: across all stable outcomes, workers prefer this outcome over any other. In other words, workers' interests are globally aligned. Furthermore, we know from the previous section that this outcome is efficient. Thus, for the market designer, solidarity with workers is consistent with economic efficiency. Inefficiencies are caused by firms exploiting their monopsonistic labor market power; when they do not exploit that power, workers are better off.

6.2 Firm welfare and worker welfare

As with workers, we can define a binary relation representing firms' preferences. We define $\succeq_{\mathbf{F}}$ as:

$$\left(\mu, s\right) \succeq_{\mathbf{F}} \left(\mu', s'\right) \iff \forall F : \pi_F\left(\left|\mu(F)\right|, s(F)\right) \geq \pi_F\left(\left|\mu'(F)\right|, s'(F)\right).$$

If one stable outcome is preferred by all workers over another, then no firm could strictly prefer the former outcome to the latter: if they did, they could block the latter outcome by forming a coalition with the workers to which they are matched in the former coalition. This point is expressed in the following lemma.

Lemma 9. *For any two stable outcomes* $(\mu, s), (\mu', s')$*:* $(\mu, s) \succeq_{\mathbf{W}} (\mu', s') \implies (\mu', s') \succeq_{\mathbf{F}} (\mu, s)$.

Given Corollary 3, Lemma 9 implies the existence of a worker-optimal, firm-pessimal stable outcome, which has higher salaries than any other stable outcome and is efficient. We summarize these facts in the following theorem.

Theorem 2. *There exists a stable outcome* (μ^*, s^*) *such that, in comparison to any other stable outcome* (μ, s)*:*

1. *it is more efficient:* $value(\mu^*) \geq value(\mu)$,
2. *it has greater salaries:* $s^* \geqq s$,
3. *it is preferred by workers:* $(\mu^*, s^*) \succeq_{\mathbf{W}} (\mu, s)$, *and*
4. *it is less preferred by firms:* $(\mu, s) \succeq_{\mathbf{F}} (\mu^*, s^*)$.

Note that, as shown in Example 1, not all efficient stable outcomes are worker-optimal. Generically, the worker-optimal stable outcome will be unique.

In summary: across stable outcomes, worker interests are aligned with efficiency whereas firm interests are not. Beyond its normative power, Theorem 2 has interesting implications for market design. It suggests three indicators of market efficiency that can be targeted (provided that the labor market remains stable): (1) high salaries, (2) worker welfare, and (3) firms making minimal profits. Theorem 2 also gives us some insight into the cause of labor market inefficiency: inefficiency arises because firms prefer it – provided they cannot personalize salaries.

Theorem 2 also may explain aggregate unemployment. Generically, a worker employed at the worker-optimal stable outcome will strictly prefer that outcome over unemployment. (Given individual rationality, the only exception is the knife-edge case in which her maximal salary exactly offsets her disamenity of employment.) This preference means that, generically, a worker employed in any stable outcome will be employed in the worker-optimal efficient stable outcome. The worker-optimal efficient stable outcome will thus have the lowest unemployment level of any stable outcome. By exploiting their monopsony power, firms create aggregate unemployment.

Perhaps surprisingly, the converse of Lemma 9 does not hold: Given two stable outcomes, it is possible that one is better for all firms and some of the workers. In many-to-one matching, a worker might not be able to form a blocking coalition with only a firm: she might need the support of her fellow workers as well. We require that each firm pays all its workers the same salary, and thus the salary sufficiently generous to earn the support of her fellow workers may cost her the support of the firm. This phenomenon contrasts with other matching models, in which payoffs typically form a dual lattice (Knuth, 1976; Shapley & Shubik, 1971; Hatfield & Milgrom, 2005; Blair, 1988). Example B.1, in Appendix B, illustrates this phenomenon.

This section has shown that an efficient stable outcome is optimal for workers but not for firms. This result suggests that workers will cooperate with a mechanism which implements an efficient stable outcome, but that such a mechanism will struggle to elicit the same cooperation from firms. In the following section, we will confirm that this intuition is correct.

7 Designing a Centralized Labor Market

This paper asks how a centralized matching mechanism can make monopsonistic labor markets more efficient. Production functions and amenities might not be directly observed by the market designer. In this case, these primitives must be elicited. Having shown that an efficient stable outcome exists and characterized how efficiency and welfare vary across stable outcomes, we now ask whether a strategyproof mechanism can implement an efficient stable outcome. We show that no such mechanism can elicit firms' production functions, but it can elicit the amenities that workers receive from employment.

We first consider eliciting production functions from the firms:

Proposition 6. *When firms' production functions are private information, no dominant strategy mechanism can implement an efficient stable outcome.*

Proposition 6 can be proved with the following example.

Example 2 (a slightly more general simple monopsony). $\mathbf{F} = \{F\}$. $y_F(N) = \beta N$; $\beta \in \{1,6\}$. $\mathbf{W} = \{w_1, w_2\}$. $\alpha_{w_1}(F) = 0$. $\alpha_{w_2}(F) = -4$.

Example 2 generalizes Example 1, which was introduced in Section 2. When $\beta = 6$, Example 2 is identical to Example 1, and we showed earlier that the efficient matching $\mu^6(w_1) = \mu^6(w_2) = F$ will be supported by a salary $s^6(F) \in [4,6]$. When $\beta = 1$, the efficient matching is $\mu^1(w_1) = F; \mu^1(w_2) = \emptyset$. This matching will be supported by a salary $s^1(F) \in [0,1]$.

Consider the mechanism design problem of implementing (μ^6, s^6) when $\beta = 6$ and (μ^1, s^1) when $\beta = 1$; where only the firm knows the value of β. By the revelation principle, we can consider only mechanisms in which the firm reports its type. If it reports $\beta = 1$, it will be matched to one worker, pay salary $s^1(F) \in [0,1]$, and thus receive profit $\beta - s^1(F) \geq \beta - 1$. If it reports $\beta = 6$, it will be matched to two workers, pay salary $s^6(F) \in [4,6]$, and thus receive profit $2\left(\beta - s^6(F)\right) \leq 2(\beta - 4)$. In particular, when the true value of the firm's productivity is $\beta = 6$, it would receive at least profit 5 from reporting $\beta = 1$ while it would receive at most profit 4 from reporting $\beta = 6$. It will thus not report truthfully. By the revelation principle, this example constitutes a proof of Proposition 6.

In some contexts, the mechanism designer will know firms' production functions but will not know the amenities that workers receive from employment. Our next result shows that, in such cases, the mechanism designer can implement the worker-optimal stable outcome. We know from Section 5 that this outcome is efficient.

Theorem 3. *When firms' production functions are public information, there exists a strategyproof mechanism that implements the worker-optimal stable outcome, which is efficient.*

The intuition for Theorem 3 is as follows. Let (μ^*, s^*) be the worker-optimal stable outcome. Consider a worker w who would be matched to the firm $F = \mu^*(w)$ were she to report truthfully. For worker w to benefit from misreporting, she must be able either to increase the salary at firm F or to switch herself to a different firm.

By Lemma 7, either the No Firing constraint $s^*(F) \leq \Delta^-_{\mu^*}(F)$ is binding, or firm F's No Envy constraint is binding for some worker $w' \notin \mu^*(F)$: $s^*(F) + \alpha_{w'}(F) = s^*\left(\mu^*(w')\right) + \alpha_{w'}\left(\mu^*(w')\right)$. Unless worker w switches firms, she can affect neither the No Firing constraint at firm F nor the No Envy constraints for firm F. It follows that she cannot increase salaries at firm F without switching to a different firm. By misreporting, the worker *can* move herself to another firm, but we show that doing so cannot make her better off since the worker-optimal stable outcome has No Envy. The worker thus has no incentive to misreport her amenities.

This sketch suggests representing our mechanism as a VCG mechanism *a la* Green and Laffont (1977), where the firm-worker matching is the 'public good' chosen by the workers. As with our mechanism, each player in a VCG mechanism receives a transfer that depends only on the reports of the other players. This setup ensures that players lack an incentive to inflate their transfers by misreporting.

Proposition 6 and Theorem 3 echo earlier results in the two-sided matching literature: there is no mechanism that is strategyproof for both sides of the market; and implementing the worker-optimal stable outcome is strategyproof for workers but not firms (Roth & Sotomayor, 1990). Our context has the added twist that the worker-optimal stable outcome is efficient while other stable outcomes need not be.

An efficient outcome can be implemented through a dominant strategy mechanism provided that the mechanism designer observes firms' production functions. Whether the designer can observe firms' production functions will depend on context. Recall that we require that workers enter into firms' production functions interchangeably. This requirement reduces the informational complexity of a firm's production function. In contexts where workers are interchangeable, such as pharmacies, manufacturing assembly lines, and the construction trades, production functions could plausibly be inferred from engineering or accounting data.

8 The Sources of Monopsonistic Inefficiencies

We have shown in the previous sections that restrictions on transfers *can* permit inefficient stable outcomes. In this section, we explore more directly *why* these inefficiencies arise. Each of the next two subsections explores a particular condition which guarantees that all stable outcomes will be efficient. The third subsection reveals exactly how our restriction on transfers – the requirement that each firm pays all its workers the same salary – stabilizes inefficient outcomes. By characterizing distorted labor markets, these results illustrate the contexts in which the previous section's centralized matching mechanism could be useful. These final results also show how monopsonistic distortions are caused by combining worker and firm heterogeneity, together with the requirement that each firm pays the same salary to all its workers.

8.1 Common value amenities

As Example 1 made clear, the different amenities that workers receive from the same firm can cause monopsonistic inefficiencies. Proposition 7 formalizes this intuition by showing that, when there is no within-firm heterogeneity in amenities, all stable outcomes are efficient.

We say that firm F has **common value amenities** if every worker receives the same amenity from working at F:

$$\forall\, w, w' \in \mathbf{W}:\ \alpha_w(F) = \alpha_{w'}(F).$$

Proposition 7. *If every firm has common value amenities, then every stable outcome is efficient.*

By Proposition 2, stable outcomes have No Envy. When all firms have common value amenities, the No Envy condition equalizes workers' utilities. We use this structure within a replacement chain to construct a blocking coalition for any inefficient stable outcome, and thus prove Proposition 7 by contradiction.

This result exposes one source of monopsonistic inefficiency. Firms want to lower their salaries to price out 'expensive' workers, even when employing those workers is efficient. This incentive does not arise when firms have common value amenities since no worker is relatively more expensive than any other: if a firm lost one worker when it reduced its salary, it would lose them all. This intuition accords with the labor monopsony literature: differences in amenities across workers generate upward sloping labor supply curves that in turn generate monopsonistic inefficiencies.

8.2 Duplicate firms

In this subsection, we explore how firms' uniqueness relates to the efficiency of stable outcomes. As with the previous subsection, this discussion will help us better understand how firms derive their distortionary market power.

We say that two firms $F' \neq F$ are **duplicates** if

$$\forall N \in \mathbb{N} : y_F(N) = y_{F'}(N),$$
$$\text{and } \forall w \in \mathbf{W} : \alpha_w(F) = \alpha_w(F').$$

Proposition 8. *If every firm has a duplicate, then every stable outcome is efficient.*

The proof of Proposition 8 first notes that, by the No Envy condition, if two firms are duplicates they must pay the same salary. This means that, if two firms are duplicates, either could poach any worker from the other by paying an infinitesimally higher salary. No Poaching requires that doing so would be unprofitable, which implies that each firm F's salary be at least equal to their increased output from hiring another worker (i.e., $\Delta^+_\mu(F)$). In addition, the No Firing condition requires that each firm F's salary be no less than the decrease in output from firing a marginal worker (i.e., $\Delta^-_\mu(F)$). Thus, each firm will have a salary $s(F) \in \left[\Delta^+_\mu(F), \Delta^-_\mu(F)\right]$, which means that the outcome has Marginal Product Salaries. By Proposition 3, the matching must be efficient.

As is suggested by the proof, duplicate firms will compete away their monopsonistic rents. If a pair of duplicate firms were paying salaries less than their marginal products, one could profitably poach workers from the other. Stability requires that poaching workers be unprofitable, and so each firm must be paying their marginal product. Thus, duplicate firms' non-uniqueness strips them of any market power. This result shows us that restricted transfers only generate monopsonistic distortions when firms are unique.

In Appendix E, we ask what happens when each *worker* has a duplicate. In contrast to the results above, duplicating workers has no effect on the set of stable outcomes (provided that firms' production functions are 'stretched' appropriately). Theorem 2 showed that workers prefer an undistorted market, and thus it is firm power, rather than worker power, which distorts the labor market. This subsection provides additional support for this principle: Distortions are eliminated when each firm has a duplicate. When each worker has a duplicate, distortions remain.

8.3 How restricted transfers stabilize distortions

Our final result demonstrates precisely how our restriction on transfers stabilizes inefficient outcomes.

Proposition 9. *Consider an inefficient stable outcome (μ, s). There exists a salary s', a firm F, and a worker w such that $s' < \Delta^+_\mu(F)$, and w strictly prefers to work for F at salary s' than for $\mu(w)$ at salary $s(\mu(w))$.*

To prove Proposition 9, we again invoke Lemmas 3 and 4. These results tells us that, for every inefficient matching, there must be an acyclic, value-increasing replacement chain from that matching to the efficient matching. Because the replacement chain is acyclic, it increases the size of the last firm. Because the replacement chain is value-increasing, the marginal product of moving the last worker to the last firm must be greater than the salary needed to induce the worker to move.

Proposition 9 tells us that, in every inefficient stable outcome, some worker would be willing to work at some new firm for a salary less than what that firm would gain from hiring her. The firm refuses to hire her, however, because doing so would require that the firm increase the pay of its existing workers.

This section has shown how the restricted transfers combine with two-sided heterogeneity to cause monopsonistic distortions. When a firm must pay all of its workers the same salary, it is incentivized to exclude workers who would require a higher salary. This incentive is only in play when marginal workers are more expensive than inframarginal workers. This requires two-sided heterogeneity: workers having heterogeneous preferences over heterogeneous firms.

9 Conclusion

For many firms, employing more workers would require paying higher salaries. In classic job matching models, these higher salaries need only be paid to the firm's new workers; the firm can leave its existing workers' salaries unchanged. In other words, these models assume that labor markets exhibit perfect price discrimination. This paper has argued that, without price discrimination, labor markets can suffer from monopsonistic distortions. We introduced a tractable matching framework that models these monopsonistic distortions.

Why do we describe these distortions as 'monopsonistic'? As in traditional labor monopsony models, a marginal worker would be willing to work at some firm for a salary less than her marginal product. That firm, however, refuses to employ her, because doing so would require that it pay its existing workers more.

This matching framework has yielded new insights into labor market monopsony. We showed that only firm sizes are distorted: conditional on firm sizes, the matching of workers to firms is efficient. We showed that monopsonistic distortions are beneficial to firms and are harmful to workers. Further, we showed that monopsonistic distortions stem from firms exploiting two-sided heterogeneity: when each firm has a duplicate, or when each firm's amenities are equally appreciated by all workers, every stable outcome is efficient.

We have used this framework to assess a potential solution to monopsonistic distortions: a centralized matching mechanism. To be successful, such a mechanism would have to implement an outcome that is both efficient and stable. We showed that one efficient outcome is indeed always stable. To identify an efficient outcome, such a mechanism may also need to elicit the non-pecuniary amenities that workers receive from employment, or elicit the technologies with which firms produce output. We showed that a strategyproof mechanism can elicit the former but not the latter.

Ideally, a lighter-touch policy could implement an efficient outcome. For example, in a pure monopsony consisting of only one firm, a minimum salary can incentivize efficient employment. However, with multiple firms an efficient minimum salary must be firm-specific: a uniform minimum salary cannot generally allocate labor efficiently between a high amenity, low-productivity firm and a low-amenity, high-productivity firm.

While a market designer could impose firm-specific minimum salaries, doing so is not trivial. We show that stable outcomes with Marginal Product Salaries will be efficient. However, to know what the right marginal products are, the market designer must know the efficient matching. Given that these minimum salaries require that the designer know an efficient matching, it seems simpler to just impose that matching directly.

Our baseline model requires that workers be interchangeable in production. When workers are not interchangeable in production, but salaries are still required to be constant within a firm, a stable outcome may not exist. On the other hand, our results would trivially extend to cover labor markets comprising multiple occupations provided that firms could set different salaries to different occupations, that each worker could only work in one occupation, and that firms' production was additively separable over occupations. Tractable models with heterogeneous worker productivity would be a valuable goal for future work. Similarly, it would be interesting to derive the minimal restrictions on worker heterogeneity, given restricted transfers, under which a stable outcome will always exist.

Our paper applies the tools of market design to the labor market. One group of labor markets that must be explicitly designed are the online platforms now often used to match workers to firms. Whether such platforms are efficient, and whether they favor workers or firms, will depend on how they are designed. We hope that our paper can be useful to these platforms' creators.

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A Proofs of main results

Lemma 1. *Every firm's production function has decreasing differences.*

Proof. Assume towards a contradiction that some firm F's production function does not have decreasing differences:

$$\exists N \in \mathbb{N} \text{ such that } y_F(N) - y_F(N-1) > y_F(N-1) - y_F(N-2). \tag{2}$$

Without loss of generality, let N be the smallest integer such that inequality (2) holds for firm F.

Consider the salary $s_\epsilon \equiv y_F(N-1) - y_F(N-2) + \epsilon$, where $\epsilon \geq 0$. Given that $y_F(0) = 0$, firm F's profit from employing N workers at salary s_ϵ is

$$\pi_F(N, s_\epsilon) = \sum_{i=1}^{N} \left[y_F(i) - y_F(i-1) - s_\epsilon \right].$$

This implies that when $\epsilon = 0$, the marginal profit from hiring the ith worker is

$$\begin{aligned} & y_F(i) - y_F(i-1) - s_0 \\ = & \, y_F(i) - y_F(i-1) - \big(y_F(N-1) - y_F(N-2) \big). \end{aligned}$$

By the assumption that N is the smallest integer such that inequality (2) holds, the firm's marginal profit will be weakly positive for $i < N$ and strictly positive for $i = N$. Thus:

$$\forall M < N : \pi_F(N, s_0) > \pi_F(M, s_0).$$

Moreover, the continuity of the profit function with respect to the salary implies that the inequality will continue to hold for all ϵ sufficiently close to 0:

$$\exists \epsilon > 0 : \forall M < N : \pi_F(N, s_\epsilon) > \pi_F(M, s_\epsilon). \tag{3}$$

However, for any $\epsilon > 0$, the marginal profit from hiring the $(N-1)$th worker is negative. Thus:

$$\forall \epsilon > 0 : \pi_F(N-1, s_\epsilon) < \pi_F(N-2, s_\epsilon). \tag{4}$$

In combination, expressions (3) and (4) contradict Assumption 1. □

Proposition 2. *An outcome is stable if and only if it has No Envy, No Firing, and No Poaching.*

Proof. We prove Proposition 2 in six steps.

Step 1: If an outcome is stable, then it has No Firing.

Proof of Step 1: Consider a firm F such that $\mu(F) \neq \emptyset$. If the outcome lacks No Firing, then the firm is making a loss on its marginal worker. It would be better off being matched to one worker less at the same salary:

$$\forall w \in \mu(F) : \pi_F \big(\left| \mu(F) \setminus \{w\} \right|, s(F) \big) > \pi_F \big(\left| \mu(F) \right|, s(F) \big).$$

If $\left|\mu(F)\right| = 1$, the left hand size of the above inequality is 0, and thus the candidate outcome is not individually rational for the firm. If $\left|\mu(F)\right| > 1$, then for any worker $w \in \mu(F)$: $\big(F, \mu(F) \setminus \{w\}, s(F)\big)$ blocks (μ, s) because firm F would be strictly better off in the blocking coalition and every worker in $\mu(F) \setminus \{w\}$ would be indifferent. Thus, an outcome without No Firing cannot be stable.

Step 2: If an outcome is stable, then it has No Envy.

Proof of Step 2: Assume towards a contradiction that (μ, s) is stable but does not have No Envy:

$$\exists w \in \mathbf{W}, F \in \mathbf{F} \cup \{\emptyset\} : u_w\left(\mu(w), s(\mu(w))\right) < u_w(F, s(F)).$$

If $F = \emptyset$, the right hand side of that inequality is 0, and thus (μ, s) is not individually rational for w. Thus, if (μ, s) is stable but lacks No Envy, then $F \neq \emptyset$.

Consider first the case where $\mu(F) \neq \emptyset$. Since workers are interchangeable in production, the firm would be indifferent switching out any worker in its employ, keeping the same salary. Thus, for any $w' \in \mu(F)$, $\left(F, \mu(F) \cup \{w\} \setminus \{w'\}, s(F)\right)$ blocks (μ, s).

Now consider the case where $\mu(F) = \emptyset$. For such a firm, $s(F) = y_F(1)$ by definition. Thus:

$$\alpha_w(F) + y_F(1) > s\left(\mu(w)\right) + \alpha_w\left(\mu(w)\right),$$

while

$$\pi_F\left(|\{w\}|, y_F(1)\right) = \pi_F\left(\left|\mu(F)\right|, s(F)\right) = 0.$$

Thus $\left(F, \{w\}, y_F(1)\right)$ blocks (μ, s).

Step 3: If an outcome is stable, then it has No Poaching.

Proof of Step 3: Assume towards a contradiction that (μ, s) is stable but lacks No Poaching: there exists F, $s' > s(F)$, and $L \in \mathbb{N}$ with $\left|\mu(F)\right| < L \leq L_F\left(s', s\right)$ such that

$$\pi_F\left(L, s'\right) = y_F(L) - Ls' \geq \pi_F\left(\left|\mu(F)\right|, s(F)\right) = y_F\left(\left|\mu(F)\right|\right) - \left|\mu(F)\right| s(F). \tag{5}$$

If $\mu(F) = \emptyset$, then $s(F) = y_F(1)$. Thus, $s' > y_F(1)$, meaning firm F would make negative profit if matched to one worker. By Lemma 1, its production function has decreasing differences and so firm F would make a negative profit when matched to any positive number of workers. This contradicts expression (5), which requires that there be a positive L such that $\pi_F\left(L, s'\right) \geq \pi_F\left(\left|\mu(F)\right|, s(F)\right) = 0$.

If $\mu(F) \neq \emptyset$, every worker $w \in \mu(F)$ is strictly better off being employed at salary s' rather than salary $s(F)$. By assumption, there is a size-L set of workers C who weakly prefer being matched to F at salary s' over their current match. As $L > 0$ we can require that $\mu(F) \cap C \neq \emptyset$. Thus, there is a worker in C who strictly prefers being matched to F at salary s' over their current match. By assumption, firm F weakly prefers being matched to C at salary s' over being matched to $\mu(F)$ at salary $s(F)$. Thus, $\left(F, C, s'\right)$ blocks (μ, s).

Step 4: If an outcome has No Envy, then it is individually rational for workers.

Proof of Step 4: If an outcome has No Envy, then $\forall w$:

$$u_w\left(\mu(w), s(\mu(w))\right) \geq u_w(\emptyset, 0) = 0.$$

Step 5: If an outcome has No Firing, then it is individually rational for firms.

Proof of Step 5: Note that individual rationality for unmatched firms is trivial: if $\mu(F) = \emptyset$, then $\pi_F\left(\left|\mu(F)\right|, s(F)\right) = 0$.

An outcome having No Firing requires that for all matched firms F:

$$s(F) \leq y_F\left(\left|\mu(F)\right|\right) - y_F\left(\left|\mu(F)\right| - 1\right). \tag{6}$$

By Lemma 1, each firm's production function has decreasing differences, and so

$$\pi_F\left(\left|\mu(F)\right|, s(F)\right) = \sum_{i=1}^{|\mu(F)|}\left[y_F(i) - y_F(i-1) - s(F)\right]$$
$$\geq \sum_{i=1}^{|\mu(F)|}\left[y_F\left(\left|\mu(F)\right|\right) - y_F\left(\left|\mu(F)\right|-1\right) - s(F)\right].$$

Combined with inequality (6), this implies that $\pi_F\left(\left|\mu(F)\right|, s(F)\right) \geq 0$.

Step 6: If an outcome has No Envy, No Firing and No Poaching, then there are no coalitions that block it.

Proof of Step 6: Assume towards a contradiction that the coalition (F, C, s') blocks (μ, s), an outcome with No Envy, No Firing and No Poaching, where C is a nonempty subset of $\mathbf{W}$.

First consider the case where $s' < s(F)$. Given that $s' < s(F)$, any worker previously matched to F would be strictly worse off in the coalition. Thus, $C \cap \mu(F) = \emptyset$. Consider a worker $w \in C$. Since (μ, s) has No Envy,

$$\alpha_w\left(\mu(w)\right) + s\left(\mu(w)\right) \geq \alpha_w(F) + s(F) > \alpha_w(F) + s',$$

and thus the worker prefers the original outcome over being matched to F at salary s'. This contradicts the assumption that (F, C, s') blocks (μ, s).

Next consider the case where $s' > s(F)$. As the firm is no worse off,

$$\pi_F\left(|C|, s'\right) \geq \pi_F\left(\left|\mu(F)\right|, s(F)\right). \tag{7}$$

By No Firing, firm F is not losing money on its marginal worker: $s(F) \leq \Delta_\mu^-(F)$. Together with $s' > s(F)$ and inequality (7), this implies that $|C| > |\mu(F)|$. As all workers in C are no worse off in the coalition, $|C| \leq L_F\left(s', s\right)$. With inequality (7), this contradicts the assumption that (μ, s) has No Poaching.

Finally consider the case where $s' = s(F)$. Since (μ, s) has No Envy, $\forall w \in C$:

$$\alpha_w\left(\mu(w)\right) + s\left(\mu(w)\right) \geq \alpha_w(F) + s(F) = \alpha_w(F) + s',$$

and thus workers in C are at best indifferent between the original outcome and being matched to F at salary s'. Thus, firm F must be strictly better off in the coalition. Thus, the firm would also be strictly better off being matched to C at salary slightly higher than s'. Thus, the $s' = s(F)$ case reduces to the $s' > s(F)$ considered above. □

Lemma 2. *An outcome with Marginal Product Salaries will have both No Firing and No Poaching.*

Proof. That an outcome with Marginal Product Salaries has No Firing follows from $s(F) \leq \Delta_\mu^-(F)$.

Fix a firm F and a salary $s' > s(F)$. Because $s(F) \geq \Delta_\mu^+(F)$:

$$s' > \Delta_\mu^+(F) \equiv y_F\left(\left|\mu(F)\right| + 1\right) - y_F\left(\left|\mu(F)\right|\right). \tag{8}$$

To show that the outcome has No Poaching, we must show that there exists no employment level L such that $\left|\mu(F)\right| < L \leq L_F(s_F, s)$ and that $\pi_F(L, s') \geq \pi_F\left(\left|\mu(F)\right|, s(F)\right)$. Consider any employment level $L > \left|\mu(F)\right|$:

$$\pi_F\left(L, s'\right) = \sum_{i=1}^{L}\left[y_F(i) - y_F(i-1) - s'\right] = \sum_{i=1}^{|\mu(F)|}\left[y_F(i) - y_F(i-1) - s'\right] + \sum_{i=|\mu(F)|+1}^{L}\left[y_F(i) - y_F(i-1) - s'\right]$$
$$< \sum_{i=1}^{|\mu(F)|}\left[y_F(i) - y_F(i-1) - s(F)\right] + \sum_{i=|\mu(F)|+1}^{L}\left[y_F(i) - y_F(i-1) - s'\right], \tag{9}$$

where the inequality follows from $s(F) < s'$. By Lemma 1, each firm's production function has decreasing differences, and so

$$\sum_{i=|\mu(F)|+1}^{L} \left[y_F(i) - y_F(i-1) - s'\right] \leq \sum_{i=|\mu(F)|+1}^{L} \left[y_F\left(\left|\mu(F)\right|+1\right) - y_F\left(\left|\mu(F)\right|\right) - s'\right],$$

which, in combination with expression (8), implies that

$$\sum_{i=|\mu(F)|+1}^{L} \left[y_F(i) - y_F(i-1) - s'\right] < 0.$$

With inequality (9), this implies that

$$\pi_F\left(L, s'\right) < \sum_{i=1}^{|\mu(F)|} \left[y_F(i) - y_F(i-1) - s(F)\right] = \pi_F\left(\left|\mu(F)\right|, s(F)\right).$$

□

Lemma 3. *Let μ and μ^* be matchings such that value$\left(\mu^*\right) >$ value$\left(\mu\right)$. There exists a replacement chain χ from μ to μ^* such that value$\left(\mu+\chi\right) >$ value$\left(\mu\right)$. Moreover, for each firm F: $\left|\left(\mu+\chi\right)(F)\right| \leq \max\left\{\left|\mu(F)\right|, \left|\mu^*(F)\right|\right\}$.*

Proof. Let us first define a maximal replacement chain. $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ is a **maximal** chain from μ to μ' if it cannot be extended in either direction: $\mu'(F_0) \subset \left(\mu(F_0) \cup \left\{(w_k)_{k=0}^{N-1}\right\}\right)$ and $\mu(F_N) \subset \left(\mu'(F_N) \cup \left\{(w_k)_{k=0}^{N-1}\right\}\right)$. (These subsets are strict because if $\mu'(F_0) = \mu(F_0)$ or $\mu(F_N) = \mu'(F_N)$ then no worker would be moved by χ.)

Our proof of Lemma 3 is algorithmic. The state of an algorithm is a matching μ°. The algorithm is as follows:

1. Initialize the algorithm with $\mu^\circ \leftarrow \mu^*$. Then go to 2.
2. If there exists a cyclic replacement chain from μ to μ°, go to 3. Otherwise, go to 4.
3. Let χ be a cyclic replacement chain from μ to μ°. If value$\left(\mu+\chi\right) >$ value$\left(\mu\right)$, then χ is the required replacement chain and the algorithm can terminate. If not, set $\mu^\circ \leftarrow \mu^\circ - \chi$, and go to 2.
4. If there exists a replacement chain from μ to μ°, go to 5. Otherwise, terminate the algorithm.
5. Let χ be a maximal replacement chain from μ to μ°. (Such a chain exists at this point of the algorithm because there exists no cyclic replacement chain from μ to μ°.) If value$\left(\mu+\chi\right) >$ value$\left(\mu\right)$, then χ is the required replacement chain and the algorithm can terminate. If not, set $\mu^\circ \leftarrow \mu^\circ - \chi$, and go to 2.

When the algorithm does not terminate on lines 3 or 5, the state μ° becomes more similar to μ. When $\mu^\circ = \mu$, there is no replacement chain from μ to μ°, and so the algorithm will terminate at line 4. Because the matching is discrete, this means that the algorithm must terminate eventually.

Lemma 3 will hold provided that the algorithm never terminates at line 4, and that for each replacement chain χ proposed in lines 3 and 5: χ is a replacement chain from μ to μ^*, and for each firm F: $\left|\left(\mu+\chi\right)(F)\right| \leq \max\left\{\left|\mu(F)\right|, \left|\mu^*(F)\right|\right\}$. We prove these results in turn.

Step 1: The algorithm never terminates at line 4.

Proof of Step 1: We first show that value$\left(\mu^\circ\right)$ is weakly increasing as the algorithm proceeds. The state matching

μ° is altered in lines 3 and 5. In line 3, the replacement chain $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ is cyclic, and thus it does not change the number of workers matched to any firm. Thus:

$$\text{value}\left(\mu^\circ\right) - \text{value}\left(\mu^\circ - \chi\right) = \sum_{k=0}^{N-1} \left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k)\right] = \text{value}\left(\mu + \chi\right) - \text{value}\left(\mu\right),$$

which is non-positive if the algorithm does not terminate. Thus, $\text{value}\left(\mu^\circ\right) \leq \text{value}\left(\mu^\circ - \chi\right)$.

In line 5, the replacement chain $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ is acyclic. It thus removes a worker from F_0 and adds a worker to F_N. As such:

$$\text{value}\left(\mu + \chi\right) - \text{value}\left(\mu\right) = \Delta_\mu^+(F_N) - \Delta_\mu^-(F_0) + \sum_{k=0}^{N-1} \left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k)\right]; \tag{10}$$

$$\text{value}\left(\mu^\circ\right) - \text{value}\left(\mu^\circ - \chi\right) = \Delta_{\mu^\circ}^-(F_N) - \Delta_{\mu^\circ}^+(F_0) + \sum_{k=0}^{N-1} \left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k)\right]. \tag{11}$$

Note that because χ is maximal, $\mu(F_N) \subset \mu^\circ(F_N)$ and $\mu^\circ(F_0) \subset \mu(F_0)$. (These subsets are strict.) Thus:

$$\begin{aligned}\Delta_{\mu^\circ}^-(F_N) &\equiv y_{F_N}\left(\left|\mu^\circ(F_N)\right|\right) - y_{F_N}\left(\left|\mu^\circ(F_N)\right| - 1\right)\\ &\leq y_{F_N}\left(\left|\mu(F_N)\right| + 1\right) - y_{F_N}\left(\left|\mu(F_N)\right|\right) = \Delta_\mu^+(F_N),\end{aligned}$$

where the second line follows from $\left|\mu(F_N)\right| < \left|\mu^\circ(F_N)\right|$ and the fact that, by Lemma 1, y_{F_N} has decreasing differences. Similarly, $\Delta_{\mu^\circ}^+(F_0) \geq \Delta_\mu^-(F_0)$. Combining these inequalities with equations (10) and (11) yields

$$\text{value}\left(\mu^\circ\right) - \text{value}\left(\mu^\circ - \chi\right) \leq \text{value}\left(\mu + \chi\right) - \text{value}\left(\mu\right),$$

which is non-positive if the algorithm does not terminate. Thus, $\text{value}\left(\mu^\circ\right) \leq \text{value}\left(\mu^\circ - \chi\right)$. This completes the proof that $\text{value}\left(\mu^\circ\right)$ is weakly increasing as the algorithm proceeds.

Initially, $\text{value}\left(\mu^\circ\right) = \text{value}\left(\mu^*\right) > \text{value}\left(\mu\right)$. Thus, at every stage of the algorithm: $\text{value}\left(\mu^\circ\right) > \text{value}\left(\mu\right)$. But if at line 4 there are no replacement chains from μ to μ°, then $\mu^\circ = \mu$ and thus $\text{value}\left(\mu^\circ\right) = \text{value}\left(\mu\right)$. Thus, there will always be at least one replacement chain from μ to μ°.

Step 2: Each replacement chain proposed in lines 3 and 5 is a replacement chain from μ to μ^*.

Proof of Step 2: Let $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ be a candidate replacement chain proposed in lines 3 or 5. We must show that $\forall k: w_k \in \mu(F_k) \cap \mu^*(F_{k+1})$. Given that χ is a replacement chain from μ to μ°: $w_k \in \mu(F_k) \cap \mu^\circ(F_{k+1})$. Given that $F_k \neq F_{k+1}$, that $F_k = \mu(w_k)$, and that $F_{k+1} = \mu^\circ(w_k)$, this implies that $\mu^\circ(w_k) \neq \mu(w_k)$. But when the state μ° is updated in lines 3 and 5, workers are only ever moved from their match in μ^* to their match in μ. Thus, given that $\mu^\circ(w_k) \neq \mu(w_k)$ it must be the case that $\mu^\circ(w_k) = \mu^*(w_k)$. Given that $F_{k+1} = \mu^\circ(w_k)$, this completes the proof that, for each k,

$$w_k \in \mu(F_k) \cap \mu^*(F_{k+1}).$$

Step 3: As the algorithm runs, the state matching μ° is such that $\forall F: \left|\mu^\circ(F)\right| \leq \max\left\{\left|\mu(F)\right|, \left|\mu^*(F)\right|\right\}$.

Proof of Step 3: Assume towards a contradiction that, at some point of the algorithm, this is not the case. Given that μ° is initially set equal to μ^*, this requires that there be a point in the algorithm such that

$$\forall F: \left|\mu^\circ(F)\right| \leq \max\left\{\left|\mu(F)\right|, \left|\mu^*(F)\right|\right\}; \quad \exists F_0: \left|\left(\mu^\circ - \chi\right)(F_0)\right| > \max\left\{\left|\mu(F_0)\right|, \left|\mu^*(F_0)\right|\right\}, \tag{12}$$

where $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ is a replacement chain proposed in lines 3 or 5. Replacement chains proposed in line 3 are cyclic and thus do not change the number of workers employed at any firm. Thus, it must be the case that χ is proposed in line 5.

Replacement chains proposed in line 5 are maximal from μ to μ°, and so $\mu^\circ(F_0) \subset \mu(F_0)$. Thus, $\left|\mu^\circ(F_0)\right| < \left|\mu(F_0)\right|$. Subtracting the replacement chain χ moves at most one worker to firm F_0, and so $\left|\left(\mu^\circ - \chi\right)(F_0)\right| \leq \left|\mu(F_0)\right| \leq \max\left\{\left|\mu(F_0)\right|, \left|\mu^*(F_0)\right|\right\}$, which contradicts expression (12).

Step 4: Each replacement chain χ proposed in lines 3 and 5 is such that for each firm F: $\left|\left(\mu+\chi\right)(F)\right| \leq \max\left\{\left|\mu(F)\right|, \left|\mu^*(F)\right|\right\}$.

Proof of Step 4: This proof is similar to that of Step 3. Assume towards a contradiction that some replacement chain $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ is such that

$$\left|\left(\mu+\chi\right)(F_N)\right| > \max\left\{\left|\mu(F_N)\right|, \left|\mu^*(F_N)\right|\right\}. \tag{13}$$

Replacement chains proposed in line 3 are cyclic and thus do not change the number of workers employed at any firm. Thus, it must be the case that χ is proposed in line 5.

Replacement chains proposed in line 5 are maximal from μ to μ°, and so $\mu(F_N) \subset \mu^\circ(F_N)$. Thus, $\left|\mu(F_N)\right| < \left|\mu^\circ(F_N)\right|$. The replacement chain χ moves at most one worker to firm F_N, and so $\left|\left(\mu+\chi\right)(F_N)\right| \leq \left|\mu^\circ(F_N)\right|$. By Step 3, $\left|\mu^\circ(F_N)\right| \leq \max\left\{\left|\mu(F_N)\right|, \left|\mu^*(F_N)\right|\right\}$. Combining inequalities, $\left|\left(\mu+\chi\right)(F_N)\right| \leq \max\left\{\left|\mu(F_N)\right|, \left|\mu^*(F_N)\right|\right\}$. This contradicts expression (13).

□

Lemma 4. *Let (μ, s) be a stable outcome. There exists no cyclic replacement chain χ such that value$\left(\mu+\chi\right) >$ value$\left(\mu\right)$.*

Proof. Assume towards a contradiction that there exists a cyclic replacement chain $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ such that value$\left(\mu+\chi\right) >$ value$\left(\mu\right)$. Given that χ is cyclic, it does not change the number of workers employed by any firm. Thus, the only difference between value$\left(\mu+\chi\right)$ and value$\left(\mu\right)$ is workers' amenities. It thus follows from value$\left(\mu+\chi\right) >$ value$\left(\mu\right)$ that

$$\sum_{k=0}^{N-1}\left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k)\right] > 0. \tag{14}$$

Given that χ is cyclic, $F_N = F_0$, and so inequality (14) implies that

$$\sum_{k=0}^{N-1}\left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k) + s(F_{k+1}) - s(F_k)\right] > 0.$$

As such, there must be some k such that $\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k) + s(F_{k+1}) - s(F_k) > 0$. Given that $\mu(w_k) = F_k$, this implies that

$$\alpha_{w_k}(F_{k+1}) + s(F_{k+1}) > \alpha_{w_k}\left(\mu(w_k)\right) + s\left(\mu(w_k)\right).$$

Thus, (μ, s) lacks No Envy. By Proposition 2 it cannot be stable. □

Proposition 3. *If a stable outcome has Marginal Product Salaries, then it is efficient.*

Proof. Let (μ, s) be a stable outcome with Marginal Product Salaries. By Lemma 3, if μ is not efficient, there exists a replacement chain $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ such that value$\left(\mu+\chi\right) >$ value$\left(\mu\right)$. By Lemma 4 and the fact that (μ, s) is stable, χ is acyclic. It follows that

$$\text{value}\left(\mu+\chi\right) - \text{value}\left(\mu\right) = \Delta_\mu^+(F_N) - \Delta_\mu^-(F_0) + \sum_{k=0}^{N-1}\left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k)\right]. \tag{15}$$

By Proposition 2, (μ, s) has No Envy. As such, for each $k = 0, \ldots, N-1$: $\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k) \leq s(F_k) - s(F_{k+1})$. Thus

$$\sum_{k=0}^{N-1} \left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k)\right] \leq \sum_{k=0}^{N-1} \left[s(F_k) - s(F_{k+1})\right] = s(F_0) - s(F_N). \tag{16}$$

By Marginal Product Salaries: $s(F_N) \geq \Delta^+_\mu(F_N)$ and $s(F_0) \leq \Delta^-_\mu(F_0)$. Combining these inequalities with equation (15) and inequality (16) yields

$$\text{value}(\mu + \chi) - \text{value}(\mu) \leq 0.$$

This contradicts our earlier claim that $\text{value}(\mu + \chi) > \text{value}(\mu)$. □

Proposition 4. *Every stable outcome is hedonic efficient.*

Proof. Two replacement chains $\chi_A = \left((w_k)_{k=0}^{N_A-1}, (F_k)_{k=0}^{N_A}\right)$ and $\chi_B = \left((w_l)_{l=0}^{N_B-1}, (F_l)_{l=0}^{N_B}\right)$ are **worker-disjoint** if they share no worker: $\{w_k\}_{k=0}^{N_A-1} \cap \{w_l\}_{l=0}^{N_B-1} = \emptyset$.

Step 1: If χ_A and χ_B are worker-disjoint cyclic replacement chains from μ to μ', then

$$\text{value}(\mu + \chi_A + \chi_B) = \text{value}(\mu + \chi_A) + \text{value}(\mu + \chi_B) - \text{value}(\mu).$$

Proof of Step 1: Note that $(\mu + \chi_A + \chi_B)$ and $(\mu + \chi_B + \chi_A)$ are both well-defined since each replacement chain moves a worker from μ to μ' not yet moved by the other chain.

Let $\chi_A = \left((w_k)_{k=0}^{N_A-1}, (F_k)_{k=0}^{N_A}\right)$ and let $\chi_B = \left((w_l)_{l=0}^{N_B-1}, (F_l))_{l=0}^{N_B}\right)$. Since χ_A and χ_B are both cyclic replacement chains from μ to μ':

$$\begin{aligned}
\text{value}(\mu + \chi_A) - \text{value}(\mu) &= \sum_{w \in (w_k)_{k=0}^{N_A-1}} \left[\alpha_w(\mu'(w)) - \alpha_w(\mu(w))\right]; \\
\text{value}(\mu + \chi_B) - \text{value}(\mu) &= \sum_{w \in (w_l)_{l=0}^{N_B-1}} \left[\alpha_w(\mu'(w)) - \alpha_w(\mu(w))\right]; \\
\text{value}(\mu + \chi_A + \chi_B) - \text{value}(\mu) &= \sum_{w \in (w_k)_{k=0}^{N_A-1} \cup (w_l)_{l=0}^{N_B-1}} \left[\alpha_w(\mu'(w)) - \alpha_w(\mu(w))\right].
\end{aligned}$$

Since χ_A and χ_B are worker-disjoint, $(w_k)_{k=0}^{N_A-1} \cap (w_l)_{l=0}^{N_B-1} = \emptyset$, which implies that:

$$\sum_{w \in (w_k)_{k=0}^{N_A-1} \cup (w_l)_{l=0}^{N_B-1}} \left[\alpha_w(\mu'(w)) - \alpha_w(\mu(w))\right] = \sum_{w \in (w_k)_{k=0}^{N_A-1}} \left[\alpha_w(\mu'(w)) - \alpha_w(\mu(w))\right] + \sum_{w \in (w_l)_{l=0}^{N_B-1}} \left[\alpha_w(\mu'(w)) - \alpha_w(\mu(w))\right],$$

Combining the above expressions implies that

$$\text{value}(\mu + \chi_A + \chi_B) - \text{value}(\mu) = \text{value}(\mu + \chi_A) + \text{value}(\mu + \chi_B) - 2 \cdot \text{value}(\mu).$$

Thus, $\text{value}(\mu + \chi_A + \chi_B) = \text{value}(\mu + \chi_A) + \text{value}(\mu + \chi_B) - \text{value}(\mu)$.

Step 2: Every stable outcome is hedonic efficient.

Proof of Step 2: Assume towards a contradiction that (μ°, s°) is a stable outcome and μ° is not hedonic efficient. Let

$$\mu^* \in \underset{\mu \text{ s.t.} \forall F: |\mu(F)| = |\mu^\circ(F)|}{\arg\max} \left\{\sum_{w \in \mathbf{W}} \alpha_w(\mu(w))\right\}$$

be a matching with the same firm sizes as μ° but which is hedonic efficient.

Select an arbitrary worker w_0 for whom $\mu^\circ(w_0) \neq \mu^*(w_0)$; let $F_0 = \mu^\circ(w_0)$ and let $F_1 = \mu^*(w_0)$. For every firm F: $\left|\mu^\circ(F)\right| = \left|\mu^*(F)\right|$. Thus, there must be some worker $w_1 \in \mu^\circ(F_1)$ such that $\mu^\circ(w_1) \neq \mu^*(w_1)$. We can iteratively continue to identify new worker-firm pairs w_j, F_j such that $w_j \in \mu^\circ(F_j) \cap \mu^*(F_{j+1})$. Because the number of firms is finite we must eventually find a firm F_N such that $F_N = F_i$ with $i < N$. We have constructed the cyclic replacement chain $\chi_1 = \left(\left(w_j\right)_{j=i}^{N-1}, \left(F_j\right)_{j=i}^{N}\right)$. Now repeat the above process to find a sequence of cyclic worker-disjoint replacement chains $\{\chi_m\}_{m=1}^{M}$ from μ° to μ^* such that $\left(\mu^\circ + \chi_1 + \chi_2 + \ldots + \chi_M\right) = \mu^*$. Thus

$$\text{value}\left(\mu^*\right) = \text{value}\left(\mu^\circ + \chi_1 + \chi_2 + \ldots + \chi_M\right). \tag{17}$$

Iterating Step 1 implies that

$$\text{value}\left(\mu^\circ + \chi_1 + \chi_2 + \ldots + \chi_M\right) = \text{value}\left(\mu^\circ + \chi_1\right) + \text{value}\left(\mu^\circ + \chi_2\right) + \ldots + \text{value}\left(\mu^\circ + \chi_M\right) - (M-1)\cdot\text{value}\left(\mu^\circ\right). \tag{18}$$

Each χ_m is cyclic and thus, by Lemma 4 , for all $m \in \{1, \ldots, M\}$: $\text{value}\left(\mu^\circ + \chi_m\right) \leq \text{value}\left(\mu^\circ\right)$. With equations (17) and (18), this implies that

$$\text{value}\left(\mu^*\right) \leq M \cdot \text{value}\left(\mu^\circ\right) - (M-1)\cdot\text{value}\left(\mu^\circ\right) = \text{value}\left(\mu^\circ\right).$$

This contradicts the assumption that $\text{value}\left(\mu^\circ\right) < \text{value}\left(\mu^*\right)$. □

Lemma 5. *Let μ be hedonic efficient. For any list of firms $(F_k)_{k=1}^{N}$ such that $F_1 = F_N$: $\sum_{k=1}^{N-1} d_\mu^1(F_k, F_{k+1}) \leq 0$.*

Proof. Assume towards a contradiction that there exists a list of firms $(F_k)_{k=1}^{N}$ such that $F_1 = F_N$ and that $\sum_{k=1}^{N-1} d_\mu^1(F_k, F_{k+1}) > 0$. For each $k = 1, 2, \ldots, N-1$, let $w_k \in \arg\max_{w \in \mu(F_{k+1})} \{\alpha_w(F_k) - \alpha_w(F_{k+1})\}$. By the definition of d_μ^1 it follows that $d_\mu^1(F_k, F_{k+1}) = \alpha_{w_k}(F_k) - \alpha_{w_k}(F_{k+1})$. Substituting this into the above inequality yields

$$\sum_{k=1}^{N-1} \left[\alpha_{w_k}(F_k) - \alpha_{w_k}(F_{k+1})\right] > 0. \tag{19}$$

Define the replacement chain $\chi = \left((w_k)_{k=1}^{N-1}, (F_k)_{k=1}^{N}\right)$. Recall that each $w_k \in \mu(F_{k+1})$ and thus the matching $\mu - \chi$ is the same as μ except with each worker w_k moved from firm F_{k+1} to firm F_k. Given that $F_N = F_1$, each firm has the same number of workers in $\mu - \chi$ as in μ, and so:

$$\text{value}\left(\mu - \chi\right) - \text{value}\left(\mu\right) = \sum_{k=1}^{N-1} \left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k)\right].$$

With inequality (19), this means that $\text{value}\left(\mu - \chi\right) > \text{value}\left(\mu\right)$. This contradicts the hedonic efficiency of μ. □

Lemma 6. *A worker-optimal stable outcome has No Envy and Marginal Product Salaries.*

Proof. We will prove Lemma 6 in three steps.

Step 1: The outcome (μ^*, s^*) has No Envy.

Proof of Step 1: Consider firms F, H and worker $w \in \mu^*(F)$. By definition, $s^*(F) = \min_{G\in\mathbf{F}} \left\{\Delta_{\mu^*}^-(G) - d_{\mu^*}^\infty(F, G)\right\}$. Thus, there is some firm G' such that $s^*(F) = \Delta_{\mu^*}^-(G') - d_{\mu^*}^\infty(F, G')$. Similarly, H's salary is given by

$$\begin{aligned} s^*(H) \equiv \min_{G\in\mathbf{F}} \left\{\Delta_{\mu^*}^-(G) - d_{\mu^*}^\infty(H, G)\right\} &\leq \Delta_{\mu^*}^-(G') - d_{\mu^*}^\infty(H, G') \\ &\leq \Delta_{\mu^*}^-(G') - \left(d_{\mu^*}^1(H, F) + d_{\mu^*}^\infty(F, G')\right) \\ &= s^*(F) - d_{\mu^*}^1(H, F) \\ &= s^*(F) - \max_{w' \in \mu^*(F)} \{\alpha_{w'}(H) - \alpha_{w'}(F)\} \\ &\leq s^*(F) - \alpha_w(H) + \alpha_w(F) \end{aligned}$$

Thus, $s^*(H) + \alpha_w(H) \le s^*(F) + \alpha_w(F)$, as No Envy requires.

Step 2: For each firm F, $s^*(F) \le \Delta^-_{\mu^*}(F)$.

Proof of Step 2: Consider any firm F. By Lemma 5: $d^{\infty}_{\mu^*}(F,F) = d^1_{\mu^*}(F,F) = 0$. Thus, $s^*(F) \le \Delta^-_{\mu^*}(F) - d^{\infty}_{\mu^*}(F,F) = \Delta^-_{\mu^*}(F)$.

Step 3: For each firm F, $s^*(F) \ge \Delta^+_{\mu^*}(F)$.

Proof of Step 3: Assume towards a contradiction that, for some firm F_1: $s^*(F_1) < \Delta^+_{\mu^*}(F_1)$. By the definition of the salary schedule s^*, there exists a list of firms $(F_k)_{k=1}^N$ such that $s^*(F_1) = \Delta^-_{\mu^*}(F_N) - \sum_{k=1}^{N-1} d^1_{\mu^*}(F_k, F_{k+1})$. Combining expressions, we have that

$$\Delta^-_{\mu^*}(F_N) - \sum_{k=1}^{N-1} d^1_{\mu^*}(F_k, F_k) < \Delta^+_{\mu^*}(F_1).$$

For each $k = 1,2,\dots,N-1$, let $w_k \in \arg\max_{w\in\mu(F_{k+1})}\{\alpha_w(F_k) - \alpha_w(F_{k+1})\}$. By the definition of d^1_μ it follows that $d^1_{\mu^*}(F_k, F_{k+1}) = \alpha_{w_k}(F_k) - \alpha_{w_k}(F_{k+1})$. Substituting this into the above inequality and rearranging implies that

$$\Delta^+_{\mu^*}(F_1) - \Delta^-_{\mu^*}(F_N) + \sum_{k=1}^{N-1}\left[\alpha_{w_k}(F_k) - \alpha_{w_k}(F_{k+1})\right] > 0. \tag{20}$$

Define the replacement chain $\chi = \left((w_k)_{k=1}^{N-1}, (F_k)_{k=1}^N\right)$. By inequality (20), the value of μ^* could be improved by moving each worker w_k from firm F_{k+1} to firm F_k: $\text{value}\left(\mu^* - \chi\right) > \text{value}\left(\mu^*\right)$. This contradicts the efficiency of μ^*, completing the proof of Step 3. □

Lemma 7. *Let $\left(\mu^*, s^*\right)$ be a worker-optimal stable outcome. For each firm F_1, either $s^*(F_1) = \Delta^-_{\mu^*}(F_1)$, or there exists a list of firms $(F_k)_{k=1}^N$ and a list of workers $(w_k)_{k=2}^N$ such that each $w_k \in \mu^*(F_k)$, $s^*(F_N) = \Delta^-_{\mu^*}(F_N)$, and each worker w_k is indifferent between firm F_k and firm F_{k-1}: $\alpha_{w_k}(F_{k-1}) + s^*(F_{k-1}) = \alpha_{w_k}(F_k) + s^*(F_k)$.*

Proof. Consider some firm F_1. By equation (1), there exists a firm F_N such that

$$s^*(F_1) = \Delta^-_{\mu^*}(F_N) - d^{\infty}_{\mu^*}(F_1, F_N).$$

If $F_1 = F_N$, then, by Lemma 5, $d^{\infty}_{\mu^*}(F_1, F_N) = d^1_{\mu^*}(F_1, F_N) = 0$ and so $s^*(F_1) = \Delta^-_{\mu^*}(F_1)$. If $F_1 \ne F_N$, there exists a path of firms $(F_k)_{k=2}^N$ such that

$$s^*(F_1) = \Delta^-_{\mu^*}(F_N) - \sum_{k=1}^{N-1} d^1_\mu(F_k, F_{k+1}).$$

For each $k = 2,\dots,N$, let $w_k \in \arg\max_{w\in\mu(F_k)}\{\alpha_w(F_{k-1}) - \alpha_w(F_k)\}$. By the definition of d^1_μ it follows that

$$s^*(F_1) = \Delta^-_{\mu^*}(F_N) - \sum_{k=1}^{N-1}\left[\alpha_{w_{k+1}}(F_k) - \alpha_{w_{k+1}}(F_{k+1})\right]. \tag{21}$$

We will show that, for each j,

$$s^*(F_j) = \Delta^-_{\mu^*}(F_N) - \sum_{k=j}^{N-1}\left[\alpha_{w_{k+1}}(F_k) - \alpha_{w_{k+1}}(F_{k+1})\right]. \tag{22}$$

Let us first show that $s^*(F_j) \le \Delta^-_{\mu^*}(F_N) - \sum_{k=j}^{N-1}\left[\alpha_{w_{k+1}}(F_k) - \alpha_{w_{k+1}}(F_{k+1})\right]$. By Lemma 6, $\left(\mu^*, s^*\right)$ has No Envy. No Envy implies that, for each k, $s^*(F_{k+1}) + \alpha_{w_{k+1}}(F_{k+1}) - s^*(F_k) - \alpha_{w_{k+1}}(F_k) \ge 0$. Summing across k yields

$$\sum_{k=j}^{N-1}\left[s^*(F_{k+1}) + \alpha_{w_{k+1}}(F_{k+1}) - s^*(F_k) - \alpha_{w_{k+1}}(F_k)\right] \ge 0.$$

Simplifying the telescoping sum and rearranging yields

$$s^*(F_j) \le s^*(F_N) - \sum_{k=j}^{N-1} \left[\alpha_{w_{k+1}}(F_k) - \alpha_{w_{k+1}}(F_{k+1})\right].$$

By Lemma 6, the outcome (μ^*, s^*) has Marginal Product Salaries and thus $s^*(F_N) \le \Delta^-_{\mu^*}(F_N)$. This completes the proof that $s^*(F_j) \le \Delta^-_{\mu^*}(F_N) - \sum_{k=j}^{N-1}\left[\alpha_{w_{k+1}}(F_k) - \alpha_{w_{k+1}}(F_{k+1})\right]$.

Thus, it remains to show that, for each j, $s^*(F_j) \ge \Delta^-_{\mu^*}(F_N) - \sum_{k=j}^{N-1}\left[\alpha_{w_{k+1}}(F_k) - \alpha_{w_{k+1}}(F_{k+1})\right]$. As before, consider the No Envy requirement that $s^*(F_{k+1}) + \alpha_{w_{k+1}}(F_{k+1}) - s^*(F_k) - \alpha_{w_{k+1}}(F_k) \ge 0$, and sum across k:

$$\sum_{k=1}^{j-1} \left[s^*(F_{k+1}) + \alpha_{w_{k+1}}(F_{k+1}) - s^*(F_k) - \alpha_{w_{k+1}}(F_k)\right] \ge 0.$$

Simplifying and rearranging this inequality yields

$$s^*(F_1) \le s^*(F_j) - \sum_{k=1}^{j-1} \left[\alpha_{w_{k+1}}(F_k) - \alpha_{w_{k+1}}(F_{k+1})\right].$$

Subtracting equation (21) from both sides yields

$$0 \le s^*(F_j) - \sum_{k=1}^{j-1} \left[\alpha_{w_{k+1}}(F_k) - \alpha_{w_{k+1}}(F_{k+1})\right] - \Delta^-_{\mu^*}(F_N) + \sum_{k=1}^{N-1} \left[\alpha_{w_{k+1}}(F_k) - \alpha_{w_{k+1}}(F_{k+1})\right],$$

which can be rearranged to find that $s^*(F_j) \ge \Delta^-_{\mu^*}(F_N) - \sum_{k=j}^{N-1}\left[\alpha_{w_{k+1}}(F_k) - \alpha_{w_{k+1}}(F_{k+1})\right]$. This completes the proof of equation (22). The claim that $\alpha_{w_k}(F_{k-1}) + s^*(F_{k-1}) = \alpha_{w_k}(F_k) + s^*(F_k)$ follows from subtracting equation (22) for $j = k$ and $j = k-1$. □

Proposition 5. *If (μ^*, s^*) is the worker-optimal stable outcome then, for all stable outcomes (μ, s): $s^* \geqq s$.*

Proof. Let (μ^*, s^*) be a worker-optimal stable outcome. Consider any other stable outcome (μ, s). Let $\mathscr{I} = \{F : s(F) > s^*(F)\}$. We will show that $\mathscr{I} = \emptyset$.

Step 1: $\forall w \in \mathbf{W} : \mu^*(w) \in \mathscr{I} \implies \mu(w) \in \mathscr{I}$.

Proof of Step 1: Consider a worker w for whom $\mu^*(w) \in \mathscr{I}$ and a firm F such that $F \notin \mathscr{I}$. By Lemma 6, (μ^*, s^*) has No Envy, and so it must be the case that $\alpha_w(\mu^*(w)) + s^*(\mu^*(w)) \ge \alpha_w(F) + s^*(F)$. $F \notin \mathscr{I}$ implies that $s^*(F) \ge s(F)$, while $\mu^*(w) \in \mathscr{I}$ implies that $s(\mu^*(w)) > s^*(\mu^*(w))$. Combining these inequalities implies that

$$\alpha_w(\mu^*(w)) + s(\mu^*(w)) > \alpha_w(F) + s(F).$$

Thus, if $\mu(w) = F$, then (μ, s) would lack No Envy. By Proposition 2, this contradicts the assumption that (μ, s) is stable.

Step 2: $\sum_{F\in\mathscr{I}} |\mu^*(F)| \le \sum_{F\in\mathscr{I}} |\mu(F)|$.

Step 2 follows directly from Step 1.

Step 3: $\forall F \in \mathscr{I} : |\mu^*(F)| \ge |\mu(F)|$.

Proof of Step 3: Assume towards a contradiction that there exists a firm $F \in \mathscr{I}$ such that $|\mu(F)| > |\mu^*(F)|$. By Lemma 6, $s^*(F) \ge \Delta^+_{\mu^*}(F)$. By Lemma 1, y_F has decreasing differences, which with $|\mu(F)| > |\mu^*(F)|$ implies that $\Delta^+_{\mu^*}(F) \ge \Delta^-_{\mu}(F)$. Thus $s^*(F) \ge \Delta^-_{\mu}(F)$. Given that $F \in \mathscr{I} : s(F) > s^*(F)$. In summary:

$$s(F) > \Delta^-_{\mu}(F),$$

which means that (μ, s) lacks No Firing. By Proposition 2, this contradicts the assumption that (μ, s) is stable.

Step 4: $\forall F \in \mathscr{I}, |\mu^*(F)| = |\mu(F)|$.

Step 4 follows from Steps 2 and 3.

Step 5: $\forall w \in \mathbf{W}$: $\mu^*(w) \in \mathscr{I} \iff \mu(w) \in \mathscr{I}$.

Proof of Step 5: By Step 4: $|\{w : \mu(w) \in \mathscr{I}\}| = |\{w : \mu^*(w) \in \mathscr{I}\}|$. By Step 1: $\{w : \mu^*(w) \in \mathscr{I}\} \subseteq \{w : \mu(w) \in \mathscr{I}\}$. Thus, $\{w : \mu^*(w) \in \mathscr{I}\} = \{w : \mu(w) \in \mathscr{I}\}$.

Step 6: If for some $G \in \mathbf{F}$ there exists $F \in \mathscr{I}$ and $w \in \mu^*(G)$ such that $\alpha_w(F) + s^*(F) = \alpha_w(G) + s^*(G)$, then $G \in \mathscr{I}$.

Proof of Step 6: Given that $F \in \mathscr{I}$, $\alpha_w(F) + s(F) > \alpha_w(F) + s^*(F)$. By Proposition 2, both (μ, s) and (μ^*, s^*) have No Envy. By No Envy for (μ, s): $\alpha_w(\mu(w)) + s(\mu(w)) \geq \alpha_w(F) + s(F)$ and by No Envy for (μ^*, s^*): $\alpha_w(G) + s^*(G) \geq \alpha_w(\mu(w)) + s^*(\mu(w))$.

Combining these inequalities with the equality $\alpha_w(G) + s^*(G) = \alpha_w(F) + s^*(F)$ implies that

$$\alpha_w(\mu(w)) + s(\mu(w)) > \alpha_w(\mu(w)) + s^*(\mu(w)),$$

and thus $\mu(w) \in \mathscr{I}$. By Step 5 and the fact that $G = \mu^*(w)$: $G \in \mathscr{I}$.

Step 7: $\mathscr{I} = \emptyset$.

Proof of Step 7: Assume towards a contradiction that there exists $F_1 \in \mathscr{I}$. By Step 4, $|\mu^*(F_1)| = |\mu(F_1)|$, and so $\Delta^-_{\mu^*}(F_1) = \Delta^-_{\mu}(F_1)$. Given that $F_1 \in \mathscr{I}$, $s(F_1) > s^*(F_1)$. Given that (μ, s) is stable, Proposition 2 tells us that it has No Firing, and so $s(F_1) \leq \Delta^-_{\mu}(F_1)$. Combining these expressions implies that $s^*(F_1) < \Delta^-_{\mu^*}(F_1)$.

Given that $s^*(F_1) < \Delta^-_{\mu^*}(F_1)$, Lemma 7 tells us that there exists a list of firms $(F_k)_{k=1}^N$ and workers $(w_k)_{k=2}^N$ such that each $w_k \in \mu^*(F_k)$, $s^*(F_N) = \Delta^-_{\mu^*}(F_N)$, and, for each k, $\alpha_{w_k}(F_{k-1}) + s^*(F_{k-1}) = \alpha_{w_k}(F_k) + s^*(F_k)$. By iteratively applying Step 6, this implies that $F_N \in \mathscr{I}$. By Step 4, $|\mu^*(F_N)| = |\mu(F_N)|$, and so $\Delta^-_{\mu^*}(F_N) = \Delta^-_{\mu}(F_N)$. So, $s(F_N) > s^*(F_N) = \Delta^-_{\mu^*}(F_N) = \Delta^-_{\mu}(F_N)$. The outcome (μ, s) thus lacks No Firing. By Proposition 2, this contradicts the assumption that it is stable. □

Lemma 8. *For any two stable outcomes* $(\mu, s), (\mu', s')$: $s \geqq s' \iff (\mu, s) \succeq_{\mathbf{W}} (\mu', s')$.

Proof. We first show $s \geqq s' \implies (\mu, s) \succeq_{\mathbf{W}} (\mu', s')$. Since (μ, s) is stable, Proposition 2 tells us that it has No Envy. As such, it must be the case that for every worker w:

$$\alpha_w(\mu(w)) + s(\mu(w)) \geq \alpha_w(\mu'(w)) + s(\mu'(w)),$$

while $s \geqq s'$ implies that $\alpha_w(\mu'(w)) + s(\mu'(w)) \geq \alpha_w(\mu'(w)) + s'(\mu'(w))$.

We now show $(\mu, s) \succeq_{\mathbf{W}} (\mu', s') \implies s \geqq s'$. For every worker:

$$\alpha_w(\mu(w)) + s(\mu(w)) \geq \alpha_w(\mu'(w)) + s'(\mu'(w)) \geq \alpha_w(\mu(w)) + s'(\mu(w)),$$

where the first inequality follows from $(\mu, s) \succeq_{\mathbf{W}} (\mu', s')$, while the second follows from the fact that (μ', s') is stable and so by Proposition 2 has No Envy. This implies $s(F) \geq s'(F)$ for every firm F such that $\mu(F) \neq \emptyset$.

For firms F such that $\mu(F) = \emptyset$, $s(F) = y_F(1)$ by definition. If $\mu'(F) = \emptyset$ for such firms, then by definition $s'(F) = y_F(1) = s(F)$. If $\mu'(F) \neq \emptyset$ for such firms then $\Delta^-_{\mu'}(F) \leq y_F(1)$, because by Lemma 1 y_F has decreasing differences. By Proposition 2, (μ', s') has No Firing and so $s'(F) \leq \Delta^-_{\mu'}(F)$. Combining these inequalities with $y_F(1) = s(F)$ implies that $s'(F) \leq s(F)$. □

Lemma 9. *For any two stable outcomes* $(\mu, s), (\mu', s')$*:* $(\mu, s) \succeq_{\mathbf{W}} (\mu', s') \implies (\mu', s') \succeq_{\mathbf{F}} (\mu, s)$.

Proof. Let $(\mu, s) \succeq_{\mathbf{W}} (\mu', s')$ be two stable outcomes. Assume towards a contradiction there exists a firm F such that $\pi_F(|\mu(F)|, s(F)) > \pi_F(|\mu'(F)|, s'(F))$. Since (μ', s') is individually rational for firms, $\pi_F(|\mu'(F)|, s'(F)) \geq 0$. Thus, $\mu(F) \neq \emptyset$. By assumption, all workers in $\mu(F)$ weakly prefer (μ, s) to (μ', s'). Thus, $(F, \mu(F), s(F))$ blocks (μ', s'), contradicting the assumption that (μ', s') is a stable outcome. □

Theorem 3. *When firms' production functions are public information, there exists a strategyproof mechanism that implements the worker-optimal stable outcome, which is efficient.*

Proof. Consider the mechanism which asks each worker for her amenities and then implements a worker-optimal stable outcome. Note that regardless of the veracity of workers' reported amenities, the mechanism will use the true production functions since we assumed they are public information. We will show that, under such a mechanism, it is a weakly-dominant strategy for each worker to report her true amenities.

Let $\alpha_w \equiv (\alpha_w(F))_{F \in \mathbf{F}}$ concatenate each worker w's amenities. Let α_w° represent worker w's reported amenities. Fix a particular worker $\hat{w} \in \mathbf{W}$. Assume towards a contradiction that there exists a report $\alpha_{\hat{w}}^\circ \neq \alpha_{\hat{w}}$ such that $\hat{w}$ strictly benefits from reporting $\alpha_{\hat{w}}^\circ$, given the other workers' reports. Let (μ, s) denote the outcome produced when all workers w (including $w = \hat{w}$) report α_w°. Let (μ^*, s^*) denote the outcome produced when all workers $w \neq \hat{w}$ report α_w°, while worker $\hat{w}$ truthfully reports $\alpha_{\hat{w}}$. Let $F^* \equiv \mu^*(\hat{w})$ and let $F^\circ \equiv \mu(\hat{w})$. Our assumption that $\hat{w}$ benefits from misreporting requires:

$$\alpha_{\hat{w}}(F^*) + s^*(F^*) < \alpha_{\hat{w}}(F^\circ) + s(F^\circ). \tag{23}$$

By Lemma 6, both (μ, s) and (μ^*, s^*) have No Envy for their respective reports, though not necessarily for the true amenities. For clarity, we will say an outcome has No Reported Envy if that outcome would have No Envy if reported amenities were true.

Since (μ^*, s^*) has No Reported Envy:

$$\alpha_{\hat{w}}(F^*) + s^*(F^*) \geq \alpha_{\hat{w}}(F^\circ) + s^*(F^\circ)$$

with inequality (23), this implies that $s^*(F^\circ) < s(F^\circ)$. Let $\mathscr{I} = \{F : s(F) > s^*(F)\}$. We have shown that $F^\circ \in \mathscr{I}$. We will prove the contradiction that $\mathscr{I} = \emptyset$. The proof from this point is similar to that for Proposition 5 (which showed that there existed an efficient stable outcome with maximal salaries). The only difference is that we will here require No Reported Envy rather than No Envy.

Step 1: $\forall w \in \mathbf{W} : \mu^*(w) \in \mathscr{I} \implies \mu(w) \in \mathscr{I}$.

Proof of Step 1: We showed above that $F^\circ = \mu(\hat{w}) \in \mathscr{I}$, and thus it remains to show that the claim is true for all $w \neq \hat{w}$. Consider a worker $w \neq \hat{w}$ for whom $\mu^*(w) \in \mathscr{I}$ and a firm F such that $F \notin \mathscr{I}$. Since (μ^*, s^*) has No Reported Envy, it must be the case that $\alpha_w^\circ(\mu^*(w)) + s^*(\mu^*(w)) \geq \alpha_w^\circ(F) + s^*(F)$. $F \notin \mathscr{I}$ implies that $s^*(F) \geq s(F)$, while $\mu^*(w) \in \mathscr{I}$ implies that $s(\mu^*(w)) > s^*(\mu^*(w))$. Combining these inequalities implies that

$$\alpha_w^\circ(\mu^*(w)) + s(\mu^*(w)) > \alpha_w^\circ(F) + s(F).$$

Thus, $\mu(w) = F$ would imply (μ, s) lacks No Reported Envy – a contradiction.

Step 2: $\sum_{F \in \mathscr{I}} |\mu^*(F)| \leq \sum_{F \in \mathscr{I}} |\mu(F)|$.

Step 2 follows directly from Step 1.

Step 3: $\forall F \in \mathscr{I} : \left|\mu^*(F)\right| \geq \left|\mu(F)\right|$.

The proof of Step 3 is identical to the proof of Step 3 of Proposition 5.

Step 4: $\forall F \in \mathscr{I}, \left|\mu^*(F)\right| = \left|\mu(F)\right|$.

Step 4 follows from Steps 2 and 3.

Step 5: $\forall w \in \mathbf{W}$: $\mu^*(w) \in \mathscr{I} \iff \mu(w) \in \mathscr{I}$.

The proof of Step 5 is identical to the proof of Step 5 of Proposition 5.

Step 6: If for some $G \in \mathbf{F}$ there exists $F \in \mathscr{I}$ and $w \in \mu^*(G)$ such that $\alpha^\circ_w(F) + s^*(F) = \alpha^\circ_w(G) + s^*(G)$, then $G \in \mathscr{I}$.

Proof of Step 6: We showed above that $F^\circ = \mu(\hat{w}) \in \mathscr{I}$, and thus it remains to show that the claim is true for all $w \neq \hat{w}$. Given that $F \in \mathscr{I}$, $\alpha^\circ_w(F) + s(F) > \alpha^\circ_w(F) + s^*(F)$. By No Reported Envy for (μ, s): $\alpha^\circ_w(\mu(w)) + s(\mu(w)) \geq \alpha^\circ_w(F) + s(F)$ and by No Reported Envy for (μ^*, s^*): $\alpha^\circ_w(G) + s^*(G) \geq \alpha^\circ_w(\mu(w)) + s^*(\mu(w))$.

Combining these inequalities with the equality $\alpha^\circ_w(G) + s^*(G) = \alpha^\circ_w(F) + s^*(F)$ implies that

$$\alpha^\circ_w(\mu(w)) + s(\mu(w)) > \alpha^\circ_w(\mu(w)) + s^*(\mu(w)).$$

and thus $\mu(w) \in \mathscr{I}$. By Step 5 and the fact that $G = \mu^*(w)$: $G \in \mathscr{I}$.

Step 7: $\mathscr{I} = \emptyset$.

The proof of Step 7 is identical to the proof of Step 7 of Proposition 5.

Step 7 contradicts our earlier result that $F^\circ \in \mathscr{I}$, completing the proof. □

Proposition 7. *If every firm has common value amenities, then every stable outcome is efficient.*

Proof. Let every firm F have common value amenity α_F:

$$\forall w \in \mathbf{W}: \alpha_w(F) = \alpha(F),$$

and assume towards a contradiction that there exists a stable outcome (μ, s), where μ is inefficient.

First, note that it follows from the stability of (μ, s) and Proposition 2 that (μ, s) has No Envy. In turn, No Envy implies that:

$$\forall F, F' \in \mathbf{F} \text{ such that } \mu(F) \neq \emptyset: \alpha(F) + s(F) \geq \alpha(F') + s(F'). \tag{24}$$

By Lemma 3, there exists a replacement chain $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ from μ to an efficient matching such that $\text{value}(\mu + \chi) > \text{value}(\mu)$. By Lemma 4, χ is acyclic, and thus:

$$\begin{aligned}\text{value}(\mu + \chi) - \text{value}(\mu) &= \Delta^+_\mu(F_N) - \Delta^-_\mu(F_0) + \sum_{k=0}^{N-1}\left[\alpha(F_{k+1}) - \alpha(F_k)\right] \\ &= \Delta^+_\mu(F_N) - \Delta^-_\mu(F_0) + \alpha(F_N) - \alpha(F_0).\end{aligned}$$

Given that $\text{value}(\mu + \chi) > \text{value}(\mu)$, this implies that

$$\alpha(F_0) - \alpha(F_N) < \Delta^+_\mu(F_N) - \Delta^-_\mu(F_0). \tag{25}$$

By the definition of a replacement chain, $\mu(F_0) \neq \emptyset$. Thus, by expression (24), $s(F_N) - s(F_0) \leq \alpha(F_0) - \alpha(F_N)$. With inequality (25), this implies that

$$s(F_N) - s(F_0) < \Delta^+_\mu(F_N) - \Delta^-_\mu(F_0). \tag{26}$$

Since (μ, s) is stable, Proposition 2 tells us that it has No Firing. As such, $s(F_0) \leq \Delta^-_\mu(F_0)$. With inequality (26) this implies that $s(F_N) < \Delta^+_\mu(F_N)$. Thus, firm F_N would strictly benefit from hiring an additional worker at its current salary.

If $\mu(F_N) \neq \emptyset$, it follows from (24) that

$$\forall F \in \mathbf{F}: \alpha(F_N) + s(F_N) \geq \alpha(F) + s(F).$$

By the definition of an acyclic replacement chain, $w_0 \notin \mu(F_N)$. Thus, worker w_0 would be willing to work at firm F_N at salary $s(F_N)$. Thus, the coalition $(F_N, \mu(F_N) \cup \{w_0\}, s(F_N))$ would block (μ, s), contradicting (μ, s) being stable. Thus, it must be the case that $\mu(F_N) = \emptyset$.

However, if $\mu(F_N) = \emptyset$, then F_N must be making zero profit. It could thus offer to employ worker w_0 at salary $\Delta^+_\mu(F_N)$ and still make zero profit. By inequality (25), $\alpha(F_0) + \Delta^-_\mu(F_0) < \alpha(F_N) + \Delta^+_\mu(F_N)$. Recall that $s(F_0) \leq \Delta^-_\mu(F_0)$. Thus, worker w_0 would be strictly better off. The coalition $\left(F_N, \{w_0\}, \Delta^+_\mu(F_N)\right)$ blocks (μ, s), contradicting (μ, s) being stable.

Given that both $\mu(F_N) \neq \emptyset$ and $\mu(F_N) = \emptyset$ yield contradictions, there can be no stable outcome (μ, s), where μ is inefficient.

□

Proposition 8. *If every firm has a duplicate, then every stable outcome is efficient.*

Proof. Let (μ, s) be a stable outcome. Let F' be the duplicate of F. That (μ, s) has No Envy implies

$$\forall w \in \mu(F): \alpha_w(F) + s(F) \geq \alpha_w(F') + s(F') = \alpha_w(F) + s(F'),$$

where the equality follows from the assumption that F, F' are duplicates. Thus: $\mu(F) \neq \emptyset \implies s(F) \geq s(F')$.

If $\mu(F) = \emptyset$ then by construction $s(F) = y_F(1)$. As F, F' are duplicates: $y_F(1) = y_{F'}(1)$. By Lemma 1, $y_{F'}$ has decreasing differences, and so $y_{F'}(1) \geq \Delta^-_\mu(F')$. By Proposition 2, (μ, s) has No Firing, which requires $\Delta^-_\mu(F') \geq s(F')$. Combining these expressions we see that $\mu(F) = \emptyset$ implies $s(F) \geq s(F')$. Given the prior paragraph, this implies $s(F) \geq s(F')$ for all duplicates F, F'. Symmetrically, $s(F') \geq s(F)$. Therefore, $s(F) = s(F')$.

If $\mu(F') \neq \emptyset$ and $s(F) < \Delta^+_\mu(F)$, then F would be strictly better off being additionally matched to $w \in \mu(F')$ at salary $s(F)$, while w would be indifferent (because $s(F) = s(F')$). Thus, $(F, \mu(F) \cup \{w\}, s(F))$ would block (μ, s), contradicting the assumption that (μ, s) is a stable outcome. Thus, if $\mu(F') \neq \emptyset$, then $s(F) \geq \Delta^+_\mu(F)$. If $\mu(F') = \emptyset$, then $s(F) = s(F') = y_{F'}(1) = y_F(1) \geq \Delta^+_\mu(F)$, with the last inequality following from y_F having decreasing differences (by Lemma 1). Thus, if $\mu(F') = \emptyset$, then $s(F) \geq \Delta^+_\mu(F)$. In summary, for all F: $s(F) \geq \Delta^+_\mu(F)$.

By (μ, s) having No Firing, $s(F) \leq \Delta^-_\mu(F)$. We have shown that for any firm F with a duplicate, $s(F) \in \left[\Delta^+_\mu(F), \Delta^-_\mu(F)\right]$. When all firms have a duplicate, this implies that (μ, s) has Marginal Product Salaries. By Proposition 3, this implies that μ is efficient. □

Proposition 9. *Consider an inefficient stable outcome (μ, s). There exists a salary s', a firm F, and a worker w such that $s' < \Delta^+_\mu(F)$, and w strictly prefers to work for F at salary s' than for $\mu(w)$ at salary $s(\mu(w))$.*

Proof. Consider an inefficient stable outcome (μ, s). We will show that there exists a worker w and a firm $F \neq \mu(w)$ such that

$$\alpha_w(\mu(w)) + s(\mu(w)) < s' + \alpha_w(F).$$

and that $s' < \Delta^+_\mu(F)$. Combining inequalities, this is equivalent to

$$\alpha_w\left(\mu(w)\right) + s\left(\mu(w)\right) - \alpha_w(F) < \Delta^+_\mu(F). \tag{27}$$

In what follows, let $\left(\mu^*, s^*\right)$ be a worker-optimal efficient stable outcome, the existence of which is guaranteed by Theorem 2. By Lemmas 3 and 4 there exists an acyclic replacement chain $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ from μ to μ^* such that $\text{value}\left(\mu+\chi\right) > \text{value}\left(\mu\right)$. We will show that inequality (27) holds for the replacement chain χ's last firm F_N and its last worker w_{N-1}:

$$\alpha_{w_{N-1}}\left(\mu(w_{N-1})\right) + s\left(\mu(w_{N-1})\right) - \alpha_{w_{N-1}}(F_N) < \Delta^+_\mu(F_N). \tag{28}$$

Doing so will take four steps.

Step 1: If worker w_{N-1} strictly prefers $\left(\mu^*, s^*\right)$ to $\left(\mu, s\right)$, then inequality (28) holds.

Proof of Step 1: Worker w_{N-1}'s strict preference for $\left(\mu^*, s^*\right)$, in which they are matched to firm F_N, over $\left(\mu, s\right)$ implies that

$$\alpha_{w_{N-1}}(F_N) + s^*(F_N) > \alpha_{w_{N-1}}\left(\mu(w_{N-1})\right) + s\left(\mu(w_{N-1})\right).$$

Since $\left(\mu^*, s^*\right)$ is stable, Proposition 2 tells us that it has No Firing. Thus, $s^*(F_N) \le \Delta^-_{\mu^*}(F_N)$. Lemma 3 assured us that $\left|\left(\mu+\chi\right)(F_N)\right| \le \left|\mu^*(F_N)\right|$, while Lemma 1 assures us that y_{F_N} has decreasing differences; in combination these imply that $\Delta^-_{\mu^*}(F_N) \le \Delta^-_{\mu+\chi}(F_N)$. By the fact that χ is acyclic, $\Delta^-_{\mu+\chi}(F_N) = \Delta^+_\mu(F_N)$. Combining these expressions and rearranging yields inequality (28).

Step 2: If the replacement chain χ contains a worker w_k who strictly prefers $\left(\mu^*, s^*\right)$ to $\left(\mu, s\right)$, then the last worker w_{N-1} will strictly prefer $\left(\mu^*, s^*\right)$ over $\left(\mu, s\right)$.

Proof of Step 2: Let w_k strictly prefer $\left(\mu^*, s^*\right)$ (in which they are matched to firm F_{k+1}) over $\left(\mu, s\right)$ (in which they are matched to firm F_k):

$$\alpha_{w_k}(F_{k+1}) + s^*(F_{k+1}) > \alpha_{w_k}(F_k) + s(F_k).$$

Since $\left(\mu, s\right)$ is stable, Proposition 2 tells us that it has No Envy, and so

$$\alpha_{w_k}(F_{k+1}) + s(F_{k+1}) \le \alpha_{w_k}(F_k) + s(F_k).$$

Combining inequalities we have that $s^*(F_{k+1}) > s(F_{k+1})$.

Now consider worker w_{k+1}, who in $\left(\mu, s\right)$ is matched to F_{k+1} and who in $\left(\mu^*, s^*\right)$ is matched to F_{k+2}. Since $\left(\mu^*, s^*\right)$ is stable, Proposition 2 tells us that it has No Envy, and so

$$\alpha_{w_{k+1}}(F_{k+2}) + s^*(F_{k+2}) \ge \alpha_{w_{k+1}}(F_{k+1}) + s^*(F_{k+1}).$$

Given that $s^*(F_{k+1}) > s(F_{k+1})$, this implies that w_{k+1} strictly prefers $\left(\mu^*, s^*\right)$ to $\left(\mu, s\right)$:

$$\alpha_{w_{k+1}}(F_{k+2}) + s^*(F_{k+2}) > \alpha_{w_{k+1}}(F_{k+1}) + s(F_{k+1}).$$

By induction, each worker w_{k+j}, with $j \ge 0$, strictly prefers $\left(\mu^*, s^*\right)$ to $\left(\mu, s\right)$. That includes the last worker w_{N-1}.

Step 3: If the replacement chain χ contains no worker w_k who strictly prefers $\left(\mu^*, s^*\right)$ to $\left(\mu, s\right)$, then inequality (28) holds.

Proof of Step 3: Assume that the replacement chain χ contains no worker w_k who strictly prefers $\left(\mu^*, s^*\right)$ to

(μ, s). By Theorem 2, this implies that each worker in the replacement chain is indifferent between (μ^*, s^*) and (μ, s), since the former is worker-optimal. As such:

$$\sum_{k=0}^{N-1} \left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k) + s^*(F_{k+1}) - s(F_k)\right] = 0$$

Note that

$$\sum_{k=0}^{N-1} \left[s^*(F_{k+1}) - s(F_k)\right] = \sum_{k=1}^{N-1} \left[s^*(F_k) - s(F_k)\right] + s^*(F_N) - s(F_0),$$

and that, by Theorem 2, for each k: $s^*(F_k) \geq s(F_k)$. Thus,

$$\sum_{k=0}^{N-1} \left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k)\right] + s^*(F_N) - s(F_0) \leq 0.$$

Since (μ, s) is stable, Proposition 2 tells us that it has No Firing, and so $s(F_0) \leq \Delta_\mu^-(F_0)$. Thus:

$$\sum_{k=0}^{N-1} \left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k)\right] + s^*(F_N) - \Delta_\mu^-(F_0) \leq 0. \tag{29}$$

Given that value$(\mu + \chi) >$ value(μ), it must be the case that

$$\sum_{k=0}^{N-1} \left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k)\right] + \Delta_\mu^+(F_N) - \Delta_\mu^-(F_0) > 0. \tag{30}$$

Inequalities (29) and (30) imply that $\Delta_\mu^+(F_N) > s^*(F_N)$.

Worker w_{N-1} is indifferent between (μ^*, s^*), in which she is matched to firm F_N, and (μ, s):

$$\alpha_{w_{N-1}}\left(\mu(w_{N-1})\right) + s\left(\mu(w_{N-1})\right) = \alpha_{w_{N-1}}(F_N) + s^*(F_N).$$

With $\Delta_\mu^+(F_N) > s^*(F_N)$, this implies inequality (28).

Step 4: Inequality (28) holds.

Proof of Step 4: By Step 2, if any worker in the replacement chain χ strictly prefers (μ^*, s^*) to (μ, s), then w_{N-1} will strictly prefer (μ^*, s^*) to (μ, s). By Step 1, if w_{N-1} strictly prefers (μ^*, s^*) to (μ, s), then inequality (28) holds.

On the other hand, if no worker in the replacement chain χ strictly prefers (μ^*, s^*) to (μ, s), then Step 3 tells us that inequality (28) holds. Thus, regardless of whether a worker in the replacement chain χ strictly prefers (μ^*, s^*) to (μ, s), inequality (28) must hold.

□

B Relationships Between Our Model and Others

In this appendix, we contrast results and assumptions made by our model to results and assumptions made in the existing matching literature.

B.1 A worker can prefer a stable outcome preferred by all firms

Payoffs in matching models frequently form bounded lattices with payoffs on one side of the market dual to payoffs on the other side (Knuth, 1976; Shapley & Shubik, 1971; Hatfield & Milgrom, 2005; Blair, 1988). We will now show that this duality fails in our model: a worker and all firms can all benefit from the shift from one stable outcome to another.

Example B.1 (a worker prefers the firm-preferred outcome). $\mathbf{F} = \{F_1, F_2\}$. $y_{F_1}(N) = 5N$. $y_{F_2}(N) = 3N$. $\mathbf{W} = \{w_1, w_2, w_3\}$. *Amenities are given by this table:*

	F_1	F_2
w_1	5	0
w_2	-1	0
w_3	0	5

We consider two stable outcomes: (μ^1, s^1) and (μ^2, s^2):

$$\mu^1 = \begin{pmatrix} w_1 & w_2 & w_3 \\ F_1 & F_1 & F_2 \end{pmatrix},\ s^1(F_1) = 5,\ s^1(F_2) = 1; \quad \mu^2 = \begin{pmatrix} w_1 & w_2 & w_3 \\ F_1 & F_2 & F_2 \end{pmatrix},\ s^2(F_1) = 0,\ s^2(F_2) = 2.$$

The corresponding profits are

$$\pi_{F_1}\left(\left|\mu^1(F_1)\right|, s^1(F_1)\right) = 2 \times (5-5) = 0, \quad \pi_{F_2}\left(\left|\mu^1(F_2)\right|, s^1(F_2)\right) = 1 \times (3-1) = 2;$$
$$\pi_{F_1}\left(\left|\mu^2(F_1)\right|, s^2(F_1)\right) = 1 \times (5-0) = 5, \quad \pi_{F_2}\left(\left|\mu^2(F_2)\right|, s^2(F_2)\right) = 2 \times (3-2) = 2.$$

Thus, $\left(\mu^2, s^2\right) \succeq_{\mathbf{F}} \left(\mu^1, s^1\right)$. However, worker w_3 strictly prefers $\left(\mu^2, s^2\right)$ to $\left(\mu^1, s^1\right)$.

We now confirm that both outcomes are stable. Both have No Firing and No Envy. The only plausible threat to $\left(\mu^1, s^1\right)$ having No Poaching would be if F_2 poaches w_2. This would require that F_2 pay $s' \geq 5 - 1$ which is greater than its marginal product 3. The only plausible threat to $\left(\mu^2, s^2\right)$ having No Poaching would be if F_1 poaches w_2. This would require that F_1 pay $s' \geq 3$, making profit no greater than 4. This is less than its profit under $\left(\mu^2, s^2\right)$.

The intuition behind Example B.1 is that moving from one stable outcome to another can make some firms grow while making others shrink, in a manner that all firms benefit. The growing firm increases its salaries to attract marginal workers. This benefits inframarginal workers. The shrinking firms decrease their salaries, which harms those firms' workers.

A final point to emphasise about Example B.1 is its consistency with Proposition 8, Lemma 5 and Theorem 2. Neither $\left(\mu^1, s^1\right)$ nor $\left(\mu^2, s^2\right)$ is the worker-optimal stable outcome. In the worker-optimal stable outcome $\left(\mu^*, s^*\right)$: $s^*(F_1) = 5$ and $s^*(F_2) = 3$. Neither firm makes profits at $\left(\mu^*, s^*\right)$, and all workers are at least as well off as they are in $\left(\mu^1, s^1\right)$ and $\left(\mu^2, s^2\right)$. While Theorem 2 implies that workers' preferences are aligned *globally* – there is some outcome which is best for all of them – they need not be aligned *locally*.

B.2 There is no firm-optimal or worker-pessimal stable outcome

The previous subsection demonstrated that payoffs in our model lack the dual lattice structure commonly found in matching models. We will now demonstrate another reason why payoffs in out model lack a dual-lattice structure: there may not be a worker-pessimal or firm-optimal outcome.

Example B.2. $\mathbf{F} = \{F_1, F_2\}$. $\mathbf{W} = \{w_1, w_2, w_3\}$. $y_{F_1}(N) = y_{F_2}(N) = 4N$. *Amenities are given by this table:*

	F_1	F_2
w_1	5	0
w_2	0	5
w_3	0	0

Upon inspection, it is clear that in every stable outcome, all workers will be employed, with $\mu(w_1) = F_1$, and $\mu(w_2) = F_2$.

Let's first consider stable outcomes (μ, s) in which $\mu(F_1) = \{w_1, w_3\}$ and $\mu(F_2) = \{w_2\}$. Having No Poaching requires that firm F_2 be unwilling to pay salary $s(F_1)$ to poach worker w_3:

$$\pi_{F_2}\big(2, s(F_1)\big) < \pi_{F_2}\big(1, s(F_2)\big).$$

Given the firms' production functions, this is equivalent to the requirement that $8-2s(F_1) < 4-s(F_2)$, which in turn is equivalent to the requirement that $4+s(F_2) < 2s(F_1)$. As such, given that $\mu(F_1) = \{w_1, w_3\}$ and $\mu(F_2) = \{w_2\}$, the minimal value of $s(F_2)$ consistent with having No Poaching is obtained when $s(F_2) = 0, s(F_1) > 2$. Such salaries would also imply the outcome (μ, s) has No Envy and No Firing provided $s(F_1) \leq 4$. These salaries yield firm F_2 profit $\pi_{F_2}(1,0) = 4$ and yield firm F_1 profit $\pi_{F_1}\big(2, s(F_1)\big) = 8-2s(F_2) < 4$.

Let's now consider stable outcomes (μ', s') with $\mu'(F_1) = \{w_1\}$ and $\mu'(F_2) = \{w_2, w_3\}$. Symmetrically, such outcomes will be stable when $s'(F_2) > 2$, and $s'(F_1) = 0$. This yields firm F_1 profit $\pi_{F_1}(1,0) = 4$ and yields firm F_2 profit $\pi_{F_2}\big(2, s'(F_2)\big) < 4$. This demonstrates that the stable outcome which is optimal for firm F_1 differs from the stable outcome which is optimal for firm F_2. Thus, while Theorem 3 told us that there is a worker-optimal stable outcome, we see here that there is no firm-optimal stable outcome.

Given that in every stable outcome worker w_1 is matched to firm F_1 and worker w_2 is matched to firm F_2, these workers preferences over stable outcomes depend only on $s(F_1)$ and $s(F_2)$ respectively. In this example, there are stable outcomes in which each of $s(F_1)$ and $s(F_2)$ are equal to 0, but no stable outcome in which they are both equal to 0. Thus, the example also demonstrates that there is no worker-pessimal stable outcome.

B.3 Existing substitutes conditions

Hatfield and Milgrom (2005) present a model which nests both the Gale and Shapley (1962) college admissions model and the Kelso and Crawford (1982) job matching model. They show that a substitutes condition guarantees the existence of a stable matching. In the same model, Hatfield and Kojima (2008) demonstrate a sense in which a weaker substitutes condition is necessary to guarantee the existence of a stable matching. In this subsection, we show that our gross substitutes condition (Assumption 1) implies neither the Hatfield and Milgrom (2005) substitutes condition nor the Hatfield and Kojima (2008) weak substitutes condition.

The Hatfield and Milgrom model studies contracting between a set of 'hospitals' (i.e., firms) and 'doctors' (i.e., workers). A **contract** $x \in X$ is 'bilateral', and is thus associated with a single doctor x_D and a single hospital x_H. Contracts may be also associated with additional characteristics, such as a salary. Given any hospital h and subset of contracts $X' \subseteq X$, the **chosen set** $C_h(X') \subseteq X'$ represents h's preferred subset of contracts. Hospital h's **rejected set** $R_h(X')$ is the complement of its chosen set: $R_h(X') \equiv X' \setminus C_h(X')$.

Contracts are **substitutes** for hospital h if for all subsets $X' \subseteq X'' \subseteq X$ we have $R_h(X') \subseteq R_h(X'')$. Contracts are **weak substitutes** for hospital h if for those subsets $X' \subseteq X'' \subseteq X$ such that, for all $x, y \in X''$, $x_D = y_D$ implies $x = y$, we have $R_h(X') \subseteq R_h(X'')$.

In other words, contracts are not substitutes if expanding the set of potential contracts means that some contract is no longer rejected. The weak substitutes condition is identical, but it only considers expanded sets of potential contracts containing each doctor at most once.

Our model can be represented in the Hatfield and Milgrom framework as follows. Let a contract x be a hospital-doctor-salary tuple $(x_H, x_D, x_s) \in X = \mathbf{F} \times \mathbf{W} \times \mathbb{R}^+$. A hospital $h \in \mathbf{F}$ selects the chosen set

$$C_h(X') = \underset{X'' \subseteq X'}{\arg\max} \left\{ y_h\left(\left|X''\right|\right) - \sum_{x \in X''} x_s \right\} \text{ subject to } \forall x \in X'' : x_H = h;$$
$$\forall x, x' \in X'' : x \neq x' \implies x_D \neq x'_D;$$
$$\forall x, x' \in X'' : x_s = x'_s.$$

The conditions $\forall x \in X'' : x_H = h$ (requiring that a hospital picks only contracts involving itself) and $\forall x, x' \in X'' : x \neq x' \implies x_D \neq x'_D$ (requiring that a hospital picks only one contract involving each doctor) are imposed by Hatfield and Milgrom. Our additional requirement $\forall x, x' \in X'' : x_s = x'_s$ requires that hospitals set homogeneous salaries.

Assumption 1 does not guarantee that these chosen sets will satisfy the Hatfield and Milgrom substitutes condition. For example, consider a hospital h with constant marginal product $y_h(N) = 4N$. (Given a constant marginal product, Lemma 1 tells us that Assumption 1 is satisfied.) Let there be three doctors d_1, d_2, d_3. Consider the set of contracts

$$X'' = \{(h, d_1, 1), (h, d_2, 2), (h, d_3, 2)\}.$$

X'' represents a context in which the hospital can pay salary 1 to hire doctor d_1 or can pay salary 2 to hire either doctor d_2 or d_3. Hiring only doctor d_1 at salary 1 yields the hospital a profit of $4 - 1 = 3$ whereas hiring both d_2 and d_3 at salary 2 yields the hospital a profit of $8 - 4 = 4$. Thus, $C_h(X'') = \{(h, d_2, 2), (h, d_3, 2)\}$ and $R_h(X'') = \{(h, d_1, 1)\}$. However, if the hospital can only hire either d_1 or d_2, as represented by

$$X' = \{(h, d_1, 1), (h, d_2, 2)\},$$

then it will prefer to do so at the minimal possible salary: $C_h(X') = \{(h, d_1, 1)\}$ and $R_h(X') = \{(h, d_2, 2)\}$. Thus, $X' \subseteq X''$ but $R_h(X') \nsubseteq R_h(X'')$, breaching both the substitutes condition and the weak substitutes condition.

That our gross substitutes condition does not imply the Hatfield and Milgrom substitutes condition means that our results do not follow trivially from existing work.

That our gross substitutes condition does not imply the Hatfield and Kojima weak substitutes condition may be surprising: Hatfield and Kojima show that, when some hospital's preferences fail the weak substitutes condition, one can construct an example such that a stable outcome does not exist. This might seem to contradict our Theorem 1, which implied that a stable outcome always exists. The supposed contradition is resolved by noting that Hatfield and Kojima's result requires the existence of a second hospital with strict preferences over different doctors. Such a hospital is ruled out by our model, which assumes that hospitals view doctors as interchangeable.

B.4 The core

An outcome is in the core if no coalition of workers and firms can deviate from it, producing more value than the sum of their payoffs in the original outcome. In many matching models, the core coincides with stability, leading to the two terms to be used somewhat interchangeably. For example, Kelso and Crawford refer to their solution concept as the core. Our definition of stability requires that, in a deviating coalition, every worker receives the same salary. This restriction on transfers means that the core does not coincide with stability. One implication is that stable outcomes can be inefficient (Corollary 1). A core outcome can never be inefficient: if it were, the coalition consisting of every worker and firm could deviate from it.

B.5 Pairwise stability

Pairwise stability requires that no worker-firm pair can unilaterally deviate such that both are better off (Roth & Sotomayor, 1990). In some many-to-one matching models, an outcome (or, in models without salaries, simply a matching) is stable if and only if it is pairwise stable. That equivalence between pairwise stability and stability does not hold in our model.

Proposition 2 told us that stable outcomes must satisfy No Poaching: no firm can unilaterally increase its salary, attract more workers, and make at least as much profit. Higher salaries must be paid to a firm's existing workers as well as to the workers that it poaches. A firm may be willing to increase its salary when doing so would attract many workers, but not when doing so would attract only a single worker. Thus, pairwise stability is a weaker requirement than stability.

B.6 Competitive equilibria

An outcome (μ, s) is a **competitive equilibrium** if

$$\forall F \in \mathbf{F} : \left|\mu(F)\right| \in \underset{L \in \mathbb{N}}{\arg\max} \left\{\pi_F\big(L, s(F)\big)\right\}, \tag{31}$$

$$\forall w \in \mathbf{W} : \mu(w) \in \underset{F \in \mathbf{F} \cup \{\emptyset\}}{\arg\max} \left\{\alpha_w(F) + s(F)\right\}. \tag{32}$$

In a competitive equilibrium, firms choose quantities taking salaries as fixed, while workers choose firms taking salaries as fixed.

Lemma B.1. *An outcome is a competitive equilibrium if and only if it has both Marginal Product Salaries and No Envy.*

Proof. Expression (32) is equivalent to requiring that the outcome has No Envy. It thus remains for us to show that expression (31) is equivalent to requiring that the outcome has Marginal Product Salaries.

We first show that an outcome satisfying equation (31) has Marginal Product Salaries. Let (μ, s) be an outcome. If for some firm F: $s(F) > \Delta^-_\mu(F) = y_F\left(\left|\mu(F)\right|\right) - y_F\left(\left|\mu(F)\right| - 1\right)$ then $y_F\left(\left|\mu(F)\right|\right) - y_F\left(\left|\mu(F)\right| - 1\right) - s(F) < 0$. Thus

$$\begin{aligned}
\pi_F\left(\left|\mu(F)\right|, s(F)\right) &= y_F\left(\left|\mu(F)\right|\right) - s(F)\left|\mu(F)\right| \\
&= y_F\left(\left|\mu(F)\right| - 1\right) - s(F)\left(\left|\mu(F)\right| - 1\right) + y_F\left(\left|\mu(F)\right|\right) - y_F\left(\left|\mu(F)\right| - 1\right) - s(F) \\
&< y_F\left(\left|\mu(F)\right| - 1\right) - s(F)\left(\left|\mu(F)\right| - 1\right) \\
&= \pi_F\left(\left|\mu(F)\right| - 1, s(F)\right),
\end{aligned}$$

and thus equation (31) fails to hold. A similar contradiction arises if $s(F) < \Delta^+_\mu(F) = y_F\left(\left|\mu(F)\right| + 1\right) - y_F\left(\left|\mu(F)\right|\right)$. By the contrapositive, an outcome satisfying equation (31) has Marginal Product Salaries.

We next show that if an outcome has Marginal Product Salaries, then it satisfies equation (31). Let equation (31) fail for (μ, s): there exists L' such that

$$\pi_F\left(L', s(F)\right) > \pi_F\left(\left|\mu(F)\right|, s(F)\right). \tag{33}$$

Consider first the case where $L' > \left|\mu(F)\right|$. Inequality (33) implies that $y_F(L') - y_F\left(\left|\mu(F)\right|\right) > s(F)\left(L' - \left|\mu(F)\right|\right)$. By Lemma 1, y_F has decreasing differences, which implies that $y_F(L') - y_F\left(\left|\mu(F)\right|\right) \le \Delta^+_\mu(F)\left(L' - \left|\mu(F)\right|\right)$. Together, these inequalities imply that

$$\Delta_\mu^+(F)\left(L' - \left|\mu(F)\right|\right) > s(F)\left(L' - \left|\mu(F)\right|\right),$$

which with $L' > \left|\mu(F)\right|$ implies that $\Delta_\mu^+(F) > s(F)$. The outcome thus lacks Marginal Product Salaries.

Next consider the case where $L' < \left|\mu(F)\right|$. Inequality (33) implies that $y_F\left(\left|\mu(F)\right|\right) - y_F(L') < s(F)\left(\left|\mu(F)\right| - L'\right)$. By Lemma 1, y_F has decreasing differences, which implies that $\Delta_\mu^-(F)\left(\left|\mu(F)\right| - L'\right) \le y_F\left(\left|\mu(F)\right|\right) - y_F(L')$. Together, these inequalities imply that

$$\Delta_\mu^-(F)\left(\left|\mu(F)\right| - L'\right) < s(F)\left(\left|\mu(F)\right| - L'\right),$$

which with $L' < \left|\mu(F)\right|$ implies that $\Delta_\mu^-(F) < s(F)$. This is again is inconsistent with the outcome having Marginal Product Salaries. Thus, an outcome failing equation (31) must lack Marginal Product Salaries. By the contrapositive, if an outcome has Marginal Product Salaries, then it must also satisfy equation (31). □

Given Lemma B.1, we can rely on earlier results to characterize the set of competitive equilibria: by Corollary 2 they are stable, by Proposition 3 they are efficient, and by Lemma 6 they exist. Thus, we have the following corollary:

Corollary B.1. *A competitive equilibrium exists. Moreover, if (μ, s) is a competitive equilibrium, then (μ, s) is a stable outcome, and μ is efficient.*

Competitive equilibria treat firms as naïve. Given an efficient outcome, a firm does not realize that hiring inefficiently few workers would let it pay lower salaries. Given an inefficient outcome, a firm will demand more workers than are willing to work at the prevailing salary. Thus, competitive equilibria are necessarily efficient.

In a stable outcome, firms cannot unilaterally reduce salaries: doing so would require the consent of their existing workers. In this sense, stability also forces firms to take the salary level as given (although a firm can always increase its salary). Theorem 1 told us that this mechanism prevents firms from destabilizing an efficient outcome. However, stability does let firms understand that they cannot employ arbitrarily many workers at the prevailing salary. Firms can thus resist the temptation to destabilize an inefficient outcome: while an efficient outcome is stable, inefficient outcomes may be as well. This suggests that inefficient stable outcomes are caused by firms failing to take salaries as given.

Although Corollary B.1 connects competitive equilibria and efficient stable outcomes, they are not equivalent. Every competitive equilibrium is an efficient stable outcome, but not every efficient stable outcome is a competitive equilibrium. For example, a monopsonist might be able to employ all available workers at a salary below their marginal product. This could be an efficient stable outcome but could never be a competitive equilibrium because, at that salary, the monopsonist would prefer to employ additional workers. Thus, stable outcomes can yield efficient quantities without getting prices 'right'.

Similarly, it's worth noting that there could be many competitive equilibria. Because they are efficient, they will generically have the same matching of workers to firms, but they can have differing salary schedules. This is a limitation of Kojima (2007)'s analysis. Kojima argues that strategic salary-setting by firms can be better for inframarginal workers than a competitive equilibrium, as firms increase salaries to compete for marginal workers. Kojima limits his comparisons to the firm-optimal competitive equilibrium. This perspective is limiting: no worker benefits from firms' strategic salary-setting when it differs from the worker-optimal stable outcome presented in Theorem 3. By Corollary B.1, the worker-optimal stable outcome is also a competitive equilibrium.

B.7 Bertrand equilibria

The Bertrand salary-setting game is a two-stage game. In the first period, firms simultaneously choose salaries. In the second period, each worker chooses a firm. Thus, each firm F's strategy is $s_F \in \mathbb{R}$ while each worker w's strategy is a function, which takes as input the vector of salaries and selects a firm:

$$\mathrm{Ch}_w : \mathbb{R}^{|\mathbf{F}|} \to \mathbf{F} \cup \{\emptyset\}.$$

Let $s \equiv (s_F)_{F \in \mathbf{F}}$ denote the vector of salaries chosen by firms. Let $\mathrm{Ch} \equiv (\mathrm{Ch}_w)_{w \in \mathbf{W}}$ denote the vector of choice functions chosen by workers. Let $L_F^{\circ}(s_F, s_{-F}, \mathrm{Ch})$ denote the number of workers for whom $\mathrm{Ch}_w = F$, given the vector of salaries with firm F's element equal to s_F and other elements equal to the corresponding element of $s_{-F} \equiv (s_{F'})_{F' \neq F}$. Note that $L_F^{\circ}(\cdot)$ differs from our definition of $L_F(\cdot)$ in Section 3: $L_F(\cdot)$ allocated a worker indifferent between two firms to both, whereas $L_F^{\circ}(\cdot)$ allocates such a worker to only one.

A **Bertrand equilibrium** (Ch^*, s^*) is a subgame perfect pure strategy Nash equilibrium of the Bertrand salary-setting game. It comprises a vector of choice functions Ch^* and a vector of salaries s^*. Firms set salaries optimally, given the other firms' salaries and the workers' choice functions:

$$\forall F : s_F^* \in \underset{s \in \mathbb{R}}{\arg\max} \left\{ \pi_F \left(L_F^{\circ} \left(s, s_{-F}^*, \mathrm{Ch}^* \right), s \right) \right\}. \tag{34}$$

Workers' choice functions are optimal given all possible salaries:

$$\forall w : \forall s : \mathrm{Ch}_w^*(s) \in \underset{F \in \mathbf{F} \cup \emptyset}{\arg\max} \left\{ \alpha_w(F) + s_F \right\}. \tag{35}$$

To connect this solution concept to our earlier analysis, we say that an outcome (μ, s) is a Bertrand equilibrium if there exists a Bertrand equilibrium (Ch, s) such that $\forall w : \mathrm{Ch}_w(s) = \mu(w)$.

We will focus on the case where all firms have constant returns to scale. The below lemma shows that this yields a simple characterization of Bertrand equilibria. We will use that characterization to show that Bertrand equilibria are stable.

Lemma B.2. *Let all firms have constant returns to scale $y_F(N) = \Delta_F N$. Let (μ, s) be a Bertrand equilibrium. For every firm F: either $\mu(F) = \emptyset$, $s(F) = 0$, or there exists a firm $F' \neq F$ and a worker $w \in \mu(F)$ such that $s(F) = \alpha_w(F') - \alpha_w(F) + \Delta_{F'}$.*

Proof. Consider a firm F with $\mu(F) \neq \emptyset$ and $s(F) > 0$. (Recall salaries are non-negative by definition.) For such a firm to be worse off unilaterally decreasing its salary, such a decrease must cause it to lose a worker. Thus, there must be some worker w and some firm $F' \neq F$ such that $\mathrm{Ch}_w(s(F), s_{-F}) = F$ but, for all $r < s(F)$: $\mathrm{Ch}_w(r, s_{-F}) = F'$. By expression (35) $\alpha_F + s(F) = \alpha_{F'} + s(F')$. The firm F' would benefit by slightly increasing its salary and poaching worker w unless its salary equals its marginal product $\Delta_{F'}$. Thus, $s(F') = \Delta_{F'}$. Combining these two expressions implies that $s(F) = \alpha_w(F') - \alpha_w(F) + \Delta_{F'}$. □

Lemma B.2 highlights the effects of competition in the Bertrand game. No firm will pay a salary such that its workers strictly prefer that firm over other firms: if it did so, it could profitably decrease its salary. Thus, at each firm there will be some worker who is indifferent between working at that firm and working at another firm. That other firm could poach the worker by paying an infinitesimally higher salary; for that to be unprofitable, its salary must already equal its marginal product.

We will use Lemma B.2 to argue that Bertrand equilibria can be more efficient than some stable outcomes. Before doing so, we use it to prove the following proposition, which argues that Bertrand equilibria are themselves generically stable outcomes.

Proposition B.1. *Let each firm F have constant returns to scale $y_F(N) = \Delta_F N$. For almost all technologies Δ_F and amenities $\alpha_w(F)$, all Bertrand equilibria are stable outcomes.*

Proof. Consider a Bertrand equilibrium (μ, s). We will show that (μ, s) has No Envy and No Firing. We will then use Lemma B.2 to show that (μ, s) will only lack No Poaching in a knife-edge case. By Proposition 2, this implies Proposition B.1.

Step 1: (μ, s) has No Envy.

Step 1 follows immediately from expression (35).

Step 2: (μ, s) has No Firing.

Proof of Step 2: Given constant returns to scale, $s(F) > \Delta_\mu^-(F)$ implies that firm F makes negative profits. F would be better off choosing salary $s(F) = 0$ and making non-negative profits. Thus, in every Bertrand equilibrium, $s(F) \leq \Delta_\mu^-(F)$.

Step 3: There is no firm $F \in \mathbf{F}$, salary $s_F > s(F)$ and employment level L such that $\left|\mu(F)\right| < L \leq L_F(s_F, s)$, and that $\pi_F\left(L, s'\right) > \pi_F\left(\left|\mu(F)\right|, s(F)\right)$.

Proof of Step 3: Assume towards a contradiction that there is a firm $F \in \mathbf{F}$, salary $s_F > s(F)$ and employment level L such that $\left|\mu(F)\right| < L \leq L_F(s_F, s)$, and that $\pi_F\left(L, s'\right) > \pi_F\left(\left|\mu(F)\right|, s(F)\right)$. By the assumption that production functions have constant returns to scale:

$$(\Delta_F - s_F)L > \left(\Delta_F - s(F)\right)\left|\mu(F)\right|.$$

Given that $\left|\mu(F)\right| < L \leq L_F(s_F, s)$, this implies that

$$(\Delta_F - s_F)L_F(s_F, s) > \left(\Delta_F - s(F)\right)\left|\mu(F)\right|.$$

Given that this inequality is strict, there exists an $s'_F > s_F$ such that

$$(\Delta_F - s'_F)L_F(s_F, s) > \left(\Delta_F - s(F)\right)\left|\mu(F)\right|.$$

If a worker is indifferent between working at F and some other firm F' at salary s_F, she will strictly prefer working at F at salary s'_F. By expression (35), $L^\circ_F(s'_F, s) \geq L_F(s_F, s)$ and so

$$(\Delta_F - s'_F)L^\circ_F(s'_F, s) > \left(\Delta_F - s(F)\right)\left|\mu(F)\right|,$$

implying that (μ, s) is not a Bertrand equilibrium.

Step 4: (μ, s) will almost always have No Poaching.

Proof of Step 4: Given Step 3, if (μ, s) lacks No Poaching, it will only do so by equality:

$$\exists F_1 \in \mathbf{F}, s_{F_1} > s(F_1), L \in \mathbb{N} \text{ with } \left|\mu(F)\right| < L \leq L_{F_1}(s_{F_1}, s) \text{ such that } \pi_{F_1}(L, s_{F_1}) = \pi_{F_1}\left(\left|\mu(F_1)\right|, s(F_1)\right). \tag{36}$$

Note that if there was some $s'_{F_1} \in (s(F), s_{F_1})$ such that $L_{F_1}(s'_{F_1}, s) = L_{F_1}(s_{F_1}, s)$, Step 3 would fail. Thus, at least one worker $w_1 \notin \mu(F_1)$ must be indifferent between working for some firm $F_2 \in \mathbf{F} \cup \{\varnothing\} \setminus \{F_1\}$ at salary $s(F_2)$ and for firm F_1 at salary s_{F_1}:

$$s_{F_1} = s(F_2) + \alpha_{w_1}(F_2) - \alpha_{w_1}(F_1). \tag{37}$$

If $\mu(F_1) = \emptyset$, then $s(F_1) = y_F(1_1) = \Delta_{F_1}$ and so firm F_1 could never raise its salary, hire workers and make non-negative profit. Thus, $\mu(F_1) \neq \emptyset$. Given Lemma B.2, this tells us that:

$$s(F_1) = 0 \text{ or } \exists F_3 \in \mathbf{F} \cup \{\emptyset\} \setminus \{F_1\}, w_2 \in \mu(F_1) \text{ such that } s(F_1) = \alpha_{w_2}(F_3) - \alpha_{w_2}(F_1) + \Delta_{F_3} \tag{38}$$

$$\text{and } s(F_2) = 0 \text{ or } \exists F_4 \in \mathbf{F} \cup \{\emptyset\} \setminus \{F_2\}, w_3 \in \mu(F_2) \text{ such that } s(F_2) = \alpha_{w_3}(F_4) - \alpha_{w_3}(F_2) + \Delta_{F_4}. \tag{39}$$

Combining equations (36)-(39), the following must hold:

$$\begin{aligned}&\left(L - \left|\mu(F_1)\right|\right)\Delta_{F_1} - L\left(\alpha_{w_1}(F_2) - \alpha_{w_1}(F_1)\right)\\ &\qquad = \begin{cases} 0 & \text{if } s(F_1) = s(F_2) = 0;\\ -\left|\mu(F_1)\right|\left(\alpha_{w_2}(F_3) - \alpha_{w_2}(F_1) + \Delta_{F_3}\right) & \text{if } s(F_1) \neq 0, s(F_2) = 0;\\ L\left(\alpha_{w_3}(F_4) - \alpha_{w_3}(F_2) + \Delta_{F_4}\right) & \text{if } s(F_1) = 0, s(F_2) \neq 0;\\ L\left(\alpha_{w_3}(F_4) - \alpha_{w_3}(F_2) + \Delta_{F_4}\right) - \left|\mu(F_1)\right|\left(\alpha_{w_2}(F_3) - \alpha_{w_2}(F_1) + \Delta_{F_3}\right) & \text{if } s(F_1) \neq 0, s(F_2) \neq 0.\end{cases}\end{aligned}$$

That condition is, admittedly, quite opaque. For our purposes, its critical property is that is expressed solely in terms of technologies, amenities and the integer-valued L and $\left|\mu(F_1)\right|$. Thus, the amenities and technologies for which such an expression can hold have measure 0. This implies that for almost all amenities and technologies, if the Bertrand equilibrium lacks No Poaching, then it lacks it by strict inequality. We showed above that if the an outcome lacks No Poaching by strict inequality, then that outcome is not a Bertrand equilibrium. □

The critical distinction between a Bertrand equilibrium and other stable outcomes is that, in a Bertrand equilibrium, firms can unilaterally decrease their salaries. Interestingly, this does not imply that Bertrand equilibria are less efficient or better for firms than other stable outcomes.

This point can be demonstrated by revisiting Example B.2. Let (μ, s) be a Bertrand equilibrium. No firm will pay strictly more than the other: if $s(F_1) > s(F_2)$, for example, firm F_1 would retain its current workers by paying any salary $s' \in \left(s(F_2), s(F_1)\right)$. Such a salary would increase firm F_1's profit. Thus, $s(F_1) = s(F_2)$, and so worker w_3 will be indifferent between the two firms. If $s(F_1) < 4$ and $\mu(w_3) = F_2$, firm F_1 could pay an infinitesimally higher salary $s' \in \left(s(F_1), 4\right)$, employ w_3 and make profit $2 \times (4 - s') > 4 - s(F_1)$. Similarly, if $s(F_2) < 4$ and $\mu(w_3) = F_1$ then firm F_2 could profit by paying an infinitesimally higher salary. Thus, in every Bertrand equilibrium, $s(F_1) = s(F_2) = 4$, and both firms make zero profit.

Consider this other outcome:

$$\mu' = \begin{pmatrix} w_1 & w_2 & w_3 \\ F_1 & F_2 & F_2 \end{pmatrix}, \; s'(F_1) = 0, \; s'(F_2) = 3.$$

At outcome (μ', s'), both firms make positive profit: $\pi_{F_1} = 4$; $\pi_{F_2} = 2$. Yet (μ', s') is a stable outcome: that this outcome has No Envy and No Firing is self-evident, while it has No Poaching because firm F_1 would have to pay salary 3 to poach w_2, which would yield F_1 a profit of 2.

In Bertrand competition, firms are constantly tempted to decrease their salaries. This can destabilise profitable outcomes, eventually making all firms worse off. In a stable outcome, firms cannot decrease their salaries while retaining their existing workers. Example B.2 shows that firms can benefit from an inability to decrease their salaries.

However, Bertrand equilibria are not necessarily efficient or worker-optimal. In Example 1, the unique Bertrand equilibrium has the firm paying salary zero. The efficiency of Bertrand equilibria depends on whether firms can be induced to compete for marginal workers.

C No Stable Outcome with Non-Interchangeable Workers

Throughout the main text we assumed that workers are interchangeable in production: a firm's output depended only on the number of workers it employed. In this appendix, we show that a stable outcome may not exist when that assumption is dropped. In fact, a stable outcome may not exist even in the simple case where firms have homogeneous technologies which are additive in workers' productivities, and where there are no worker-firm amenities. We present an example in which a stable outcome does not exist. To simplify our exposition — and because the case may be of independent interest — we first characterize stable outcomes in this simple case.

This model again comprises a set of firms $\mathbf{F}$ and a set of workers $\mathbf{W}$. There are fewer firms than workers. Each worker $w \in \mathbf{W}$ is endowed with a productivity $\rho_w > 0$. Firms are symmetric and have output equal to the sum of the productivities of the workers to which they are matched. Thus, if firm F employs workers $C \subseteq W$ at salary s its profit will be

$$\pi_F(C,s) = \sum_{w \in C} \left[\rho_w - s\right].$$

Workers care only about their salary: $u_w(F,s) = s$. We consider the generic case where each worker's productivity is different to that of every other: $w \neq w' \implies \rho_w \neq \rho_{w'}$.

Matchings and outcomes are defined as in the main text, and the definition of a stable outcome is unchanged.

Proposition C.1. *Any stable outcome can be characterized by a labelling of the M firms $1,2,\ldots,M$ and a set of intervals $\left\{[0,s_1),[s_1,s_2),\ldots,[s_M,s_{M+1})\right\}$, where $s_j < s_{j+1}$ and $s_{M+1} = \infty$. The firm labelled j will pay salary s_j and will hire all workers with productivity in $[s_j, s_{j+1})$. Workers with productivity in $[0,s_1)$ will be unemployed. Moreover, all firms must make the same profit and firms must be making profit no less than the sum of unemployed worker productivities.*

Proof: Let (μ, s) denote a stable outcome. We prove Proposition C.1 in seven steps.

Step 1: $\forall F : \forall w \in \mu(F) : \rho_w \geq s(F)$.

Proof of Step 1: Otherwise, $\left(F, \mu(F) \setminus \{w\}, s(F)\right)$ would block (μ, s).

Step 2: If $s(F) < s(F')$ and $\mu(w) = F$, $s(F') > \rho_w$.

Proof of Step 2: Otherwise, $\left(F', \mu(F') \cup \{w\}, s(F')\right)$ would block (μ, s).

Step 3: $\forall F : \mu(F) \neq \emptyset$.

Proof of Step 3: By Step 1, $\forall w : \rho_w \geq s\left(\mu(w)\right)$. Moreover, each ρ_w differs and there are fewer firms than workers. Thus, there is some worker w for whom $\rho_w > s\left(\mu(w)\right)$. If $\exists F : \mu(F) = \emptyset$, $\left(F, \{w\}, \rho_w\right)$ would block (μ, s).

Step 4: $\forall F \neq F' : s(F) \neq s(F')$.

Proof of Step 4: Assume $s(F) = s(F')$. By Step 3, both firms are matched to at least one worker. There is at most one worker with $\rho_w = s(F)$. Thus, by Step 1, at least one of F or F' must be matched to a worker w' with $\rho_{w'} > s(F) = s(F')$. If $\mu(w') = F$ then $\left(F', \mu(F') \cup \{w'\}, s(F')\right)$ would block (μ, s). If $\mu(w') = F'$ then $\left(F, \mu(F) \cup \{w'\}, s(F)\right)$ would block (μ, s).

Step 5: If $\mu(w) = \emptyset : \forall F : \rho_w < s(F)$.

Proof of Step 5: Otherwise, $\left(F, \mu(F) \cup \{w\}, s(F)\right)$ would block (μ, s).

Step 6: All firms must make the same profit.

Proof of Step 6: If firm F made more profit than firm F', then $\left(F', \mu(F), s(F)\right)$ would block (μ, s).

Step 7: Each firm's profit must be no less than the sum of unemployed worker productivities.

Proof of Step 7: Let C denote the set of unemployed workers. If $\sum_{w\in C}\rho_w > \pi_F\left(\mu(F), s(F)\right)$, then $(F, C, 0)$ would block $\left(\mu, s\right)$.

Step 4 implies that firm salaries form disjoint intervals $[s_1, s_2), [s_2, s_3), \ldots$. Steps 1, 2 and 5 imply that the firm paying salary s_j will hire all workers with productivity in $[s_j, s_{j+1})$. Steps 6 and 7 correspond to Proposition C.1's final sentence. □

Given Proposition C.1, it is relatively simple to produce an example without a stable outcome. The following is such an example.

Example C.1 (with non-interchangeable workers, a stable outcome need not exist). $\mathbf{F} = \{F_1, F_2, F_3\}$.$\mathbf{W} = \{1, 2, 5, 6\}$. *Each worker is labelled by their productivity:* $\forall\, w \in \mathbf{W} : \rho_w = w$.

We will now show that Example C.1 lacks a stable outcome. We will consider each candidate matching in turn, exploiting Proposition C.1 to limit the number of matchings we must consider. As firms are homogeneous it is without loss of generality to assume that firm F_1 employs the least productive employed worker(s) and that firm F_3 employs the most productive.

Candidate matching 1: $\begin{pmatrix} 1 & 2 & 5 & 6 \\ \emptyset & F_1 & F_2 & F_3 \end{pmatrix}$.

$\mu(1) = \emptyset$, and thus, by Proposition C.1, each firm must make profit no less than 1. However, it is impossible for firm F_3 to make profit no less than 1, as Proposition C.1 implies that $s(F_3) > 5$.

Candidate matching 2: $\begin{pmatrix} 1 & 2 & 5 & 6 \\ F_1 & F_1 & F_2 & F_3 \end{pmatrix}$.

By Proposition C.1, $s(F_1) \le 1$, and thus $\pi_{F_1} \ge 1$. However, it is again impossible for firm F_3 to make profit no less than 1, as Proposition C.1 implies that $s(F_3) > 5$.

Candidate matching 3: $\begin{pmatrix} 1 & 2 & 5 & 6 \\ F_1 & F_2 & F_2 & F_3 \end{pmatrix}$.

By Proposition C.1, $s(F_2) \le 2$, and thus $\pi_{F_2} \ge 3$. However, it is impossible for firm F_3 to make profit no less than 3, as Proposition C.1 implies that $s(F_3) > 5$.

Candidate matching 4: $\begin{pmatrix} 1 & 2 & 5 & 6 \\ F_1 & F_2 & F_3 & F_3 \end{pmatrix}$.

By Proposition C.1, $s(F_3) \le 5$, and thus $\pi_{F_3} \ge 1$. However, it is impossible for firm F_2 to make profit no less than 1, as Proposition C.1 implies that $s(F_2) > 1$. This argument completes the proof that Example C.1 lacks a stable outcome.

If the productivities in this example are perturbed, a stable outcome exists. This observation suggests that a stable outcome may almost-always exist. It also suggests that there might always exist a weak stable outcome, which is defined to consider only a breaking coalition of workers and firms who are *all* strictly better off compared to the candidate outcome. We leave investigation of these conjectures for future work.

D Representing our Model as the Limit of A Shirking Cost Model

We motivated our assumption that each firm pays all its workers the same salary by referencing the literature showing that within-firm salary inequality causes workers to shirk (Breza et al., 2017). In this appendix, we study these shirking effects explicitly. We consider a model in which a firm can pay some of its workers more than others (as in Kelso and Crawford's model) but doing so causes workers to shirk. We show that this model aligns with our baseline model when the shirking cost is arbitrarily large.

The model in this appendix again comprises a set of firms **F** and a set of workers **W**. As in our baseline model, each worker has quasi-linear preferences

$$u_w(F,s) \equiv \alpha_w(F) + s.$$

A firm can now pay different salaries to different workers, but doing so incurs a shirking cost. Its payoff is thus

$$\pi_F\Big(C, \big(s(w)\big)_{w\in C}; \phi\Big) \equiv y_F(|C|) - \sum_{w\in C} s(w) - \phi \operatorname{var}\Big(\big(s(w)\big)_{w\in C}\Big),$$

where C is the set of workers employed by the firm, $\big(s(w)\big)_{w\in C}$ concatenates the employed workers' salaries, y_F is the firm's production function (which as before depends only on the number of workers employed, implying that workers are interchangeable in production) and $\phi \operatorname{var}\Big(\big(s(w)\big)_{w\in C}\Big)$ is the shirking cost. For simplicity we assume that the shirking cost is proportional to the variance of employed workers' salaries, with constant of proportionality ϕ. We will refer to ϕ as the **shirking parameter**. The shirking parameter ϕ lies in the extended non-negative reals: $\phi \in [0,\infty]$. When $\phi = \infty$, we take the firm's payoff to be

$$\pi_F\Big(C, \big(s(w)\big)_{w\in C}; \infty\Big) = \begin{cases} y_F(|C|) - \sum_{w\in C} s(w) & \text{if } \operatorname{var}\Big(\big(s(w)\big)_{w\in C}\Big) = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

With an infinite shirking parameter $\phi = \infty$, no firm will tolerate $\operatorname{var}\Big(\big(s(w)\big)_{w\in C}\Big) \neq 0$, and thus the model coincides with our baseline model.

An outcome (μ, s) comprises a matching μ (defined as in our baseline model) and a salary schedule $s : \mathbf{W} \to \mathbb{R}^+$ which now associates each worker with a salary, rather than each firm. An outcome (μ, s) is **individually rational** given shirking parameter ϕ if

- for all workers $w \in \mathbf{W}$: $u_w\big(\mu(w), s(w)\big) \geq 0$, and
- for all firms $F \in \mathbf{F}$: $\pi_F\Big(\mu(F), \big(s(w)\big)_{w\in\mu(F)}; \phi\Big) \geq 0$.

An outcome (μ, s) is **stable** given shirking parameter ϕ if it is individually rational and not blocked by any coalition (F, C, s^*), with $F \in \mathbf{F}$, $C \subseteq \mathbf{W}$ and $s^* : C \to \mathbb{R}^+$, where

- $\pi_F\Big(C, \big(s^*(w)\big)_{w\in C}; \phi\Big) \geq \pi_F\Big(\big|\mu(F)\big|, \big(s(w)\big)_{w\in C}; \phi\Big)$, and
- for all workers $w \in C$, $u_w\big(F, s^*(w)\big) \geq u_w\big(\mu(w), s(w)\big)$,

and the inequality is strict for the firm or one of the workers.

Let M be a correspondence from shirking costs to outcomes defined as

$$M(\phi) = \big\{(\mu, s) : (\mu, s) \text{ is a stable outcome given shirking parameter } \phi\big\}.$$

We will now show that an infinite shirking cost approximates a very large shirking cost.

Proposition D.1. $\liminf_{\phi\to\infty} M(\phi) = M(\infty)$.

Proof: We will first show that $\liminf_{\phi\to\infty} M(\phi) \subseteq M(\infty)$ and then show that $M(\infty) \subseteq \liminf_{\phi\to\infty} M(\phi)$.

Step 1: $\liminf_{\phi\to\infty} M(\phi) \subseteq M(\infty)$.

Proof of Step 1: Let $(\mu, s) \notin M(\infty)$. Thus, either (μ, s) is not individually rational for some worker, is not individually rational for some firm, or is blocked by a coalition. We consider these three cases in turn.

Worker individual rationality does not depend on the shirking parameter. Thus, if (μ, s) is not individually rational for some worker given shirking parameter ∞, it also will not be individually rational for that worker given any shirking parameter. Thus, $(\mu, s) \notin \liminf_{\phi \to \infty} M(\phi)$.

Given shirking parameter ∞, individual rationality for firm F can fail either because $\text{var}\left(\left(s(w)\right)_{w \in \mu(F)}\right) \neq 0$ or because $y_F\left(\left|\mu(F)\right|\right) - \sum_{w \in \mu(F)} s(w) < 0$. If $\text{var}\left(\left(s(w)\right)_{w \in \mu(F)}\right) \neq 0$ then

$$\lim_{\phi \to \infty} \pi_F\left(\mu(F), (s(w))_{w \in \mu(F)}; \phi\right) = \lim_{\phi \to \infty}\left[y_F\left(\left|\mu(F)\right|\right) - \sum_{w \in \mu(F)} s(w) - \phi \text{var}\left(\left(s(w)\right)_{w \in \mu(F)}\right)\right] = -\infty.$$

Thus, (μ, s) would lack individual rationality for F given any sufficiently large ϕ. If $y_F\left(\left|\mu(F)\right|\right) - \sum_{w \in \mu(F)} s(w) < 0$ then for all $\phi > 0$:

$$\pi_F\left(\mu(F), \left(s(w)\right)_{w \in \mu(F)}; \phi\right) \leq y_F\left(\left|\mu(F)\right|\right) - \sum_{w \in \mu(F)} s(w) < 0$$

and thus (μ, s) lacks individual rationality for F given any positive shirking parameter. Either way, $(\mu, s) \notin \liminf_{\phi \to \infty} M(\phi)$.

Finally, consider the case where (μ, s) is individually rational but is blocked by a coalition (F, C, s^*) given shirking parameter ∞. Individual rationality for F requires $\text{var}\left(\left(s(w)\right)_{w \in \mu(F)}\right) = 0$. Similarly, $\pi_F\left(C, \left(s^*(w)\right)_{w \in C}; \infty\right) \geq \pi_F\left(\mu(F), \left(s(w)\right)_{w \in \mu(F)}; \infty\right)$ requires that $\text{var}\left(\left(s^*(w)\right)_{w \in C}\right) = 0$. Thus, $\pi_F\left(C, \left(s^*(w)\right)_{w \in C}; \phi\right) - \pi_F\left(\mu(F), \left(s(w)\right)_{w \in C}; \phi\right)$ does not depend on the shirking parameter ϕ. Worker welfare never depends on the shirking parameter. Thus, (F, C, s^*) blocks (μ, s) for any shirking parameter. Thus, $(\mu, s) \notin \liminf_{\phi \to \infty} M(\phi)$. This step concludes the proof that

$$(\mu, s) \notin M(\infty) \implies (\mu, s) \notin \liminf_{\phi \to \infty} M(\phi).$$

By the contrapositive, $\liminf_{\phi \to \infty} M(\phi) \subseteq M(\infty)$.

Step 2: $M(\infty) \subseteq \liminf_{\phi \to \infty} M(\phi)$.

Proof of Step 2: Assume towards a contradiction that there exists $(\mu, s) \in M(\infty) \setminus \liminf_{\phi \to \infty} M(\phi)$.

For every firm F: $\pi_F\left(\mu(F), \left(s(w)\right)_{w \in \mu(F)}; \phi\right)$ is non-decreasing in ϕ. Thus, if (μ, s) is not individually rational for firms given some shirking parameter ϕ, it will also not be individually rational for firms given an infinite shirking parameter. By assumption, $(\mu, s) \in M(\infty)$. Thus, by the contrapositive, (μ, s) is individually rational for firms for any shirking parameter.

Worker welfare does not depend on the shirking parameter. Thus if (μ, s) is not individually rational for workers for some shirking parameter ϕ, it will not be indivudally rational for workers given an infinite shirking parameter. By assumption, $(\mu, s) \in M(\infty)$. Thus, (μ, s) is individually rational for workers for any shirking parameter.

We have shown that there exists $(\mu, s) \in M(\infty) \setminus \liminf_{\phi \to \infty} M(\phi)$ such that (μ, s) is individually rational for both workers and firms given any shirking parameter. Given that $(\mu, s) \notin \liminf_{\phi \to \infty} M(\phi)$: for any shirking parameter ϕ, there exists $\phi' > \phi$ such that (μ, s) is not a stable outcome given ϕ'. We can thus construct the increasing, unbounded sequence $(\phi_n)_{n=1}^{\infty}$ such that, for each n, (μ, s) is not a stable outcome given shirking parameter ϕ_n. By individual rationality, there exists a corresponding sequence of blocking coalitions $\left(F_n, C_n, s_n^*\right)_{n=1}^{\infty}$ such that each coalition $\left(F_n, C_n, s_n^*\right)$ blocks (μ, s) given shirking parameter ϕ_n. There are a finite number of firms F_n and of potential subsets of workers C_n. Thus, there must exist a pair comprising a firm and a set of workers which recurs infinitely often in the sequence $\left(F_n, C_n, s_n^*\right)_{n=1}^{\infty}$. Denote the infinitely-recurring firm as $\bar{F}$ and the infinitely-recurring subset of workers as $\bar{C}$. Let $\left(\bar{F}, \bar{C}, s_{n(m)}^*\right)_{m=1}^{\infty}$ be the subsequence of $\left(F_n, C_n, s_n^*\right)_{n=1}^{\infty}$ such that for each m: $F_{n(m)} = \bar{F}$ and $C_{n(m)} = \bar{C}$.

Individual rationality for the workers and firms imply that each salary schedule $s_{n(m)}^*$ is bounded. Thus, by the Bolzano–Weierstrass Theorem, $\left(s_{n(m)}^*\right)_{m=1}^{\infty}$ contains a convergent subsequence. Let $\left(s_{n(l)}^*\right)_{l=1}^{\infty}$ be that convergent subsequence, and let $s_{\infty}^* \equiv \lim_{l\to\infty} s_{n(l)}^*$ be its limit. We will show that $\left(\bar{F}, \bar{C}, s_{\infty}^*\right)$ blocks (μ, s) given shirking parameter $\phi = \infty$.

Payoffs are continuous in salaries and the shirking parameter, and thus

$$\begin{aligned}
&\lim_{l\to\infty} \pi_{\bar{F}}\left(\bar{C}, \left(s_{n(l)}^*(w)\right)_{w\in\bar{C}}; \phi_{n(l)}\right) = \pi_{\bar{F}}\left(\bar{C}, \left(s_{\infty}^*(w)\right)_{w\in\bar{C}}; \infty\right);\\
&\lim_{l\to\infty} \pi_{\bar{F}}\left(\mu\left(\bar{F}\right), \left(s(w)\right)_{w\in\mu(\bar{F})}; \phi_{n(l)}\right) = \pi_{\bar{F}}\left(\mu\left(\bar{F}\right), \left(s(w)\right)_{w\in\mu(\bar{F})}; \infty\right); \text{ and}\\
&\forall w \in \bar{C}: \lim_{l\to\infty} u_w\left(\bar{F}, s_{n(l)}^*(w)\right) = u_w\left(\bar{F}, s_{\infty}^*(w)\right).
\end{aligned}$$

That each $\left(\bar{F}, \bar{C}, s_{n(l)}^*\right)_{l=1}^{\infty}$ blocks (μ, s) implies that

$$\pi_{\bar{F}}\left(\bar{C}, \left(s_{\infty}^*(w)\right)_{w\in\bar{C}}; \infty\right) \geq \pi_{\bar{F}}\left(\mu\left(\bar{F}\right), \left(s(w)\right)_{w\in\mu(\bar{F})}; \infty\right); \text{ and} \tag{40}$$

$$\forall w \in \bar{C}: u_w\left(\bar{F}, s_{\infty}^*(w)\right) \geq u_w\left(\mu(w), s(w)\right). \tag{41}$$

It thus remains only to show either that either inequality (40) is strict, or inequality (41) is strict for some worker. We will show that, if (41) is strict for *no* worker (i.e., holds with equality for all workers), it must be strict for the firm. If (41) is holds with equality for all workers, then

$$\forall w \in \bar{C}: s_{\infty}^*(w) = \alpha_w\left(\mu(w)\right) + s(w) - \alpha_w\left(\bar{F}\right). \tag{42}$$

Given that (μ, s) is individually rational for the firm $\bar{F}$ given an infinite shirking parameter:

$$\pi_{\bar{F}}\left(\mu\left(\bar{F}\right), \left(s(w)\right)_{w\in\mu(\bar{F})}; \infty\right) \geq 0. \tag{43}$$

By inequality (40), this inequality implies that $\pi_{\bar{F}}\left(\bar{C}, \left(s_{\infty}^*(w)\right)_{w\in\bar{C}}; \infty\right) \geq 0$, which in turn implies that $\text{var}\left(\left(s_{\infty}^*(w)\right)_{w\in\bar{C}}\right) = 0$. Thus, with (42), inequality (40) holds with strict inequality if and only if

$$y_{\bar{F}}\left(\left|\bar{C}\right|\right) - \sum_{w\in\bar{C}}\left[\alpha_w\left(\mu(w)\right) + s(w) - \alpha_w\left(\bar{F}\right)\right] > y_{\bar{F}}\left(\left|\mu\left(\bar{F}\right)\right|\right) - \sum_{w\in\mu(\bar{F})} s(w). \tag{44}$$

Inequality (43) also implies that $\text{var}\left(\left(s(w)\right)_{w\in\mu(\bar{F})}\right) = 0$. Taking an arbitrary $\left(\bar{F}, \bar{C}, s_n^*\right)$ that blocks (μ, s) given shirking parameter ϕ_n,

$$\begin{aligned}
&y_{\bar{F}}\left(\left|\bar{C}\right|\right) - \sum_{w\in\bar{C}} s_n^*(w) - \phi_n \text{var}\left(\left(s_n^*(w)\right)_{w\in\bar{C}}\right) \geq y_{\bar{F}}\left(\left|\mu\left(\bar{F}\right)\right|\right) - \sum_{w\in\mu(\bar{F})} s(w); \text{ and}\\
&\forall w \in \bar{C}: \alpha_w\left(\bar{F}\right) + s_n^*(w) \geq \alpha_w\left(\mu(w)\right) + s(w).
\end{aligned}$$

These inequalities together imply inequality (44):

$$
\begin{aligned}
&\forall w \in \bar{C}: s_n^*(w) \geq \alpha_w\left(\mu(w)\right) + s(w) - \alpha_w\left(\bar{F}\right)\\
\therefore\quad &\sum_{w\in\bar{C}} s_n^*(w) \geq \sum_{w\in\bar{C}}\left[\alpha_w\left(\mu(w)\right) + s(w) - \alpha_w\left(\bar{F}\right)\right]\\
\therefore\quad &y_{\bar{F}}\left(|\bar{C}|\right) - \sum_{w\in\bar{C}}\left[\alpha_w\left(\mu(w)\right) + s(w) - \alpha_w\left(\bar{F}\right)\right] - \phi_n \mathrm{var}\Big(\left(s_n^*(w)\right)_{w\in\bar{C}}\Big)\\
&\qquad\qquad \geq y_{\bar{F}}\left(|\bar{C}|\right) - \sum_{w\in\bar{C}} s_n^*(w) - \phi_n \mathrm{var}\Big(\left(s_n^*(w)\right)_{w\in\bar{C}}\Big) \geq y_{\bar{F}}\left(|\mu\left(\bar{F}\right)|\right) - \sum_{w\in\mu(\bar{F})} s(w)\\
\therefore\quad &y_{\bar{F}}\left(\left|\bar{C}\right|\right) - \sum_{w\in\bar{C}}\left[\alpha_w\left(\mu(w)\right) + s(w) - \alpha_w\left(\bar{F}\right)\right] \geq y_{\bar{F}}\left(\left|\mu\left(\bar{F}\right)\right|\right) - \sum_{w\in\mu(\bar{F})} s(w).
\end{aligned}
$$

To recap: the above shows that $(\bar{C}, \bar{F}, s_\infty^*)$ blocks $\left(\mu, s\right)$ given shirking parameter $\phi = \infty$. We showed inequalities (40) and (41) hold with weak inequality. It remained to be shown that one held with strict inequality. We argued that if (41) does *not* hold with strict inequality, then inequality (40) *will* hold with strict inequality if and only if inequality (44) holds. We finally showed that inequality (44) does in fact hold. Thus, $\left(\bar{C}, \bar{F}, s_\infty^*\right)$ blocks $\left(\mu, s\right)$ given shirking parameter $\phi = \infty$. This observation contradicts our assumption that there exists $\left(\mu, s\right) \in M(\infty) \setminus \liminf_{\phi\to\infty} M(\phi)$. Thus, $M(\infty) \subseteq \liminf_{\phi\to\infty} M(\phi)$, completing the proof of Step 2 and of Proposition D.1. □

E Duplicating Workers

In subsection 8.2, we showed that, when each firm has a duplicate, every stable outcome is efficient. In this appendix, we show that the effect of duplicating workers is very different: in some sense, nothing happens when workers are duplicated. Of course, given decreasing returns to scale, increasing the number of workers may decrease firms' marginal products and thus decrease salaries. We focus on the effect of workers' 'market power' by duplicating workers while 'stretching' firms' production function appropriately. This duplication and stretching has no effect on the set of stable salary schedules.

Before formalizing that result we need to formalize the relationship between two labor markets. Consider two labor markets $(\mathbf{F}, \mathbf{W})$ and $(\mathbf{G}, \mathbf{X})$. Let $\mathscr{P}(\mathbf{X})$ denote the power set of $\mathbf{X}$. We say that $\psi: \mathbf{W}\cup\mathbf{F} \to \mathscr{P}(\mathbf{X}\cup\mathbf{G})$ is a **transformation** from $(\mathbf{F}, \mathbf{W})$ to $(\mathbf{G}, \mathbf{X})$ if $\left\{\psi(w) : w \in \mathbf{W}\right\}$ partitions $\mathbf{X}$ while $\left\{\psi(F) : F \in \mathbf{F}\right\}$ partitions $\mathbf{G}$.

We are interested in comparing labor markets in which all workers are duplicated and all firms are stretched. A transformation ψ from $(\mathbf{F}, \mathbf{W})$ to $(\mathbf{G}, \mathbf{X})$ *duplicates workers and stretches firms* if

$$
\begin{aligned}
&\forall F \in \mathbf{F}: \left|\psi(F)\right| = 1;\\
&\forall w \in \mathbf{W}: \left|\psi(w)\right| = 2;\\
&\forall w \in \mathbf{W}, x \in \psi(w), F \in \mathbf{F}: \alpha_w(F) = \alpha_x\left(\psi(F)\right);\\
&\forall F \in \mathbf{F}, N \in \mathbb{N}: y_{\psi(F)}(N) = \sum_{i=1}^{N}\left[y_F\left(\lceil i \div 2\rceil\right) - y_F\left(\lceil i \div 2\rceil - 1\right)\right], \text{ where } \lceil\cdot\rceil \text{ is the ceiling function.}
\end{aligned}
$$

The first condition requires that there be one firm in $\mathbf{G}$ for every firm in $\mathbf{F}$. The second condition requires that there be two workers in $\mathbf{X}$ for every worker in $\mathbf{W}$. The third condition requires that firms in $\mathbf{G}$ provide workers the same amenities as the corresponding firms in $\mathbf{F}$. The fourth condition requires that firms in $\mathbf{G}$

have production functions similar to those in $\mathbf{F}$ but stretched so that each marginal product can be produced by each of two workers.

Proposition E.1. *Let ψ be a transformation from $(\mathbf{F},\mathbf{W})$ to $(\mathbf{G},\mathbf{X})$ that duplicates workers and stretches firms. Let (μ,s) be an outcome in the labor market $(\mathbf{F},\mathbf{W})$, and let (μ',s') be an outcome in the labor market $(\mathbf{G},\mathbf{X})$ such that*

$$\forall F \in \mathbf{F} : s(F) = s'\left(\psi(F)\right);\ \textit{and}\ \forall w \in \mathbf{W}, x \in \psi(w) : \mu'(x) = \psi\left(\mu(w)\right).$$

(μ,s) is a stable outcome if and only if (μ',s') is a stable outcome.

Proof. We will show that the No Envy, No Firing, and No Poaching conditions are equivalent across the two outcomes. By Proposition 2, this equivalence implies Proposition E.1.

Step 1: (μ,s) has No Envy if and only if (μ',s') has No Envy.

Proof of Step 1: Consider workers $w \in \mathbf{W}, x \in \psi(w)$ and firms $F, F' \in \mathbf{F}, G = \psi(F), G' = \psi(F')$. Given that $s(F) = s'(G)$, $s(F') = s'(G')$, $\alpha_w(F) = \alpha_x(G)$ and $\alpha_w(F') = \alpha_x(G')$:

$$\alpha_w(F) + s(F) \geq \alpha_w(F') + s(F') \iff \alpha_x(G) + s'(G) \geq \alpha_x(G') + s'(G').$$

Thus, the No Envy conditions for the two outcomes are equivalent.

Step 2: (μ,s) has No Firing if and only if (μ',s') has No Firing.

Proof of Step 2: Letting $G = \psi(F)$:

$$\begin{aligned}\Delta^-_{\mu'}(G) &= y_F\left(\left\lceil \left|\mu'(G)\right| \div 2\right\rceil\right) - y_F\left(\left\lceil \left|\mu'(G)\right| \div 2\right\rceil - 1\right)\\ &= y_F\left(\left|\mu'(F)\right|\right) - y_F\left(\left|\mu'(F)\right| - 1\right)\\ &= \Delta^-_{\mu}(F),\end{aligned}$$

where the second equality follows from two workers being matched to G for every one matched to F, and thus $\left\lceil \left|\mu'(G)\right| \div 2\right\rceil = \left\lceil 2 \times \left|\mu'(F)\right| \div 2\right\rceil = \left|\mu'(F)\right|$. Given that $s(F) = s'(G)$, it follows that

$$s(F) \leq \Delta^-_{\mu}(F) \iff s'(G) \leq \Delta^-_{\mu'}(G).$$

Step 3: (μ,s) has No Poaching if and only if (μ',s') has No Poaching.

Proof of Step 3: For some firm $F \in \mathbf{F}$, let $G = \psi(F)$. Note that for any $L \in \mathbb{N}$:

$$\begin{aligned}y_G(2L) &= \sum_{i=1}^{2L} \left[y_F\left(\lceil i \div 2\rceil\right) - y_F\left(\lceil i \div 2\rceil - 1\right)\right]\\ &= \sum_{j=1}^{L} 2\left[y_F\left(\lceil j\rceil\right) - y_F\left(\lceil j\rceil - 1\right)\right] = 2y_F(L).\end{aligned}$$

It follows that for any salary r:

$$\pi_G(2L,r) = y_G(2L) - r \times 2L = 2\pi_F(L,r).$$

In particular, given that $\left|\mu'(G)\right| = 2\left|\mu(F)\right|$ and $s(F) = s'(G)$: $\pi_G\left(\left|\mu'(G)\right|, s'(G)\right) = 2\pi_F\left(\left|\mu(F)\right|, s(F)\right)$.

Let (μ,s) lack No Poaching because of firm F: there exists $r > s(F)$ and $L \leq L_F(r,s)$ with $L > \left|\mu(F)\right|$ such that $\pi_F(L,r) \geq \pi_F\left(\left|\mu(F)\right|, s(F)\right)$. By the above, $\pi_G(2L,r) = 2\pi_F(L,r)$ and $\pi_G\left(\left|\mu'(G)\right|, s'(G)\right) = 2\pi_F\left(\left|\mu'(F)\right|, s(F)\right)$. Thus:

$$\pi_G(2L,r) \geq \pi_G\left(\left|\mu'(G)\right|, s'(G)\right).$$

Moreover, $L_G(r,s') = 2L_F(r,s)$. Thus, $2L \leq L_G(r,s')$. Thus, (μ',s') lacks No Poaching. By the contrapositive, if (μ',s') has No Poaching, then (μ,s) has No Poaching. The proof of the converse is symmetric. □

If firms have constant returns to scale – as they do in all of our examples – 'stretching' firms does not change them. This observation motivates a simpler version of Proposition E.1: Assume that each firm has constant returns to scale. If each worker is duplicated while each firm is unchanged, the set of stable outcomes will be unchanged, except that the two duplicate workers take the place of the one original worker. Thus, every example in this paper can be extended to involve arbitrarily many workers, with the nature of the example unchanged.

F Comparing the Efficiency of Two Stable Outcomes

In Section 8, we discussed conditions under which all stable outcomes would be efficient. In this appendix, we take a different tack: given two stable outcomes, we ask whether it can be known which has greater value. If production and amenities are both observed, then match value can be calculated directly. However, as discussed in Section 7, such observations might be difficult to obtain.

We are thus interested in whether simpler statistics can indicate whether one outcome has greater value than another. For example, Proposition 3 told us that any stable outcome in which firms pay Marginal Product Salaries must be efficient. Unfortunately, this appendix will present an example suggesting that many plausible heuristics can fail. Rather, comparisons of match value seem to generally require observing (or restricting) amenities.

Example F.1 (insufficient statistics for efficiency). $\mathbf{F} = \{F_1, F_2\}.\mathbf{W} = \{w_1, w_2, w_3\}$. $y_{F_1}(N) = y_{F_2}(N) = 4N$. *Amenities are given by this table:*

	F_1	F_2
w_1	10	0
w_2	0	10
w_3	β	0

Example F.1 comprises three workers. Worker w_1 has a strong preference towards working for firm F_1 while worker w_2 has a strong preference towards working for firm F_2. We represent worker w_3's preferences with the parameter β: when $\beta > 0$ it is more efficient for w_3 to be matched to F_1; when $\beta < 0$ it is more efficient for w_3 to be matched to F_2.

For β close to zero, there exist both stable outcomes in which w_3 is matched to F_1 and stable outcomes in which w_3 is matched to F_2. As both firms have constant marginal products, that an outcome has No Firing just requires that $s(F_1)$ and $s(F_2)$ both be at most 4. When w_3 is matched to F_1 and β is sufficiently close to zero, that the outcome has No Envy is implied by $s(F_1) > s(F_2)$. If w_3 is matched to F_1, then F_2 would have to pay $s(F_1) + \beta$ to poach w_3. Thus, such an outcome has No Poaching provided that

$$\left(4 - \left[s(F_1) + \beta\right]\right) \times 2 < \left(4 - s(F_2)\right) \times 1.$$

For β sufficiently close to zero, this condition is implied by $s(F_1) > 2 + \frac{s(F_2)}{2}$. We have shown that for β sufficiently close to zero, an outcome which matches w_3 to F_1 is a stable outcome provided that

$$s(F_1) > \max\left\{2 + \frac{s(F_2)}{2}, s(F_2)\right\};\ s(F_1) \leq 4;\ s(F_2) \leq 4. \tag{45}$$

That system of inequalities has many solutions. For example, one is $s(F_2) = 1$, $s(F_1) = 3$. Symmetrically, for $\beta \approxeq 0$, worker w_3 will be matched to F_2 if

$$s(F_2) > \max\left\{2 + \frac{s(F_1)}{2}, s(F_1)\right\};\ s(F_1) \le 4;\ s(F_2) \le 4. \tag{46}$$

Consider a shift from a stable outcome in which w_3 is matched to F_1 to a stable outcome in which w_3 is matched to F_2. In both outcomes there will be one firm matched to 2 workers and one firm matched to 1 worker, and total output will be $3 \times 6 = 18$. Thus, no measure of firm concentration (like a Herfindahl–Hirschman Index) or of total output could tell us whether the shift increased the value of the match.

In fact, one can perturb the example by adding additional workers or by making one firm more productive than the other, such that the more efficient match has greater firm concentration or less total output.

Our discussion thus far has ignored salaries. Can salaries tell us whether one outcome is more efficient than another? Unfortunately not. Proposition 5 told us that there was an efficient outcome with maximal salaries. But away from the maximum, higher salaries do not always imply greater efficiency. Returning to Example F.1, let $\beta > 0$ so that it is more efficient for w_3 to be matched to F_1. Contrast these two outcomes: $\left(\mu^1, s^1\right)$ and $\left(\mu^2, s^2\right)$:

$$\mu_1 = \begin{pmatrix} w_1 & w_2 & w_3 \\ F_1 & F_2 & F_1 \end{pmatrix},\ s^1(F_1) = 2.5,\ s^1(F_2) = 0; \quad \mu_2 = \begin{pmatrix} w_1 & w_2 & w_3 \\ F_1 & F_2 & F_2 \end{pmatrix},\ s^2(F_1) = 3,\ s^2(F_2) = 4.$$

The outcomes $\left(\mu^1, s^1\right)$ and $\left(\mu^2, s^2\right)$ satisfy inequalities (45) and (46) respectively, and thus both are stable. While $s^2 \geqq s^1$, the value of μ_1 is greater.

Similarly, comparing worker and firm welfare between $\left(\mu^1, s^1\right)$ and $\left(\mu^2, s^2\right)$ demonstrates that worker and firm welfare do not reveal which is more efficient. Theorem 2 told us that *some* efficient stable outcome is better for workers than any other stable outcome, but this result does not imply that *every* efficient stable outcome is better for workers than every inefficient stable outcome.

Our discussion of Example F.1 suggests that the relative efficiency of two outcomes cannot generally be known without knowing amenities. This observation suggests that there would be value in inferring amenities using the mechanism suggested in Section 7 to diagnose inefficiencies caused by market power, as well as to solve them.