Market Design for a Monopsonistic Labor Market*

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This paper asks whether centralized matching can make monopsonistic labor markets more efficient. We construct a job-matching model with fungible workers in which each firm must pay all its workers the same salary. This restriction generates monopsonistic inefficiencies: while an efficient stable allocation will always exist, inefficient allocations can be stable as well. Workers prefer one efficient stable allocation over any other stable allocation. Firms prefer inefficient stable allocations in which they pay lower salaries and thus extract greater profits. When production technologies are public information, a strategyproof mechanism can elicit how workers value employment, and thus implement an efficient stable allocation. However, no strategyproof mechanism can elicit firms' production technologies. Thus centralized matching can improve monopsonistic labor markets when the market designer observes production.

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1 Introduction

When a worker searches for a job, she rarely has a lot of options. Most labor markets are dominated by only a few firms (Azar, Marinescu, & Steinbaum, 2020). The worker will prefer one firm over others – perhaps because it is located near her home, or because it has an unusually active fantasy football pool. Firms thus face upward-sloping labor supply curves. If one firm wants to employ more workers, it will have to increase the salary that it pays. Its optimal salary will be less than its marginal product of labor, and it will employ fewer workers than would be efficient (Robinson, 1933; Boal & Ransom, 1997; Manning, 2011). Economists are increasingly blaming this 'monopsonistic distortion' for stagnant salaries and the distorted allocation of labor across the economy (Bivens, Mishel, & Schmitt, 2018; Berger, Herkenhoff, & Mongey, 2019).

In this paper we ask whether monopsonistic distortions can be addressed through a centralized matching mechanism. To do so, we unify the job matching and labor monopsony literatures. Following the job matching literature, we study stable allocations: matchings of workers to firms, along with a salary schedule, from which no worker-firm coalition can profitably deviate. Canonical job matching models lack monopsonistic distortions because they assume that each worker's salary can be set independently of her colleagues' salaries (Kelso & Crawford, 1982; Hatfield & Milgrom, 2005). We instead follow the labor monopsony literature by requiring that each firm pays all its workers the same salary. This requirement introduces inefficiencies that mimic the monopsonistic distortions found in existing labor monospony models. We study whether centralized matching can ameliorate these distortions.

We make two high-level contributions. First, we demonstrate when and how centralized matching can address monopsonistic distortions. Inefficient allocations can be stable. Thus a decentralized labor market need not function efficiently. However, at least one efficient allocation will always be stable. If a market designer could stipulate that allocation, no coalition of workers and firms could profitably destabilize it. Moreover, there is always an efficient stable allocation that workers prefer over any other stable allocation. Thus the market designer's desire for efficiency is compatible with her solidarity with the workers.

Our model contains two potential sources of private information: the idiosyncratic amenities (or disamenities) that workers receive from employment, and the production technologies with which firms produce their outputs. When firms' production technologies are private information, no strategyproof mechanism can implement an efficient allocation. That is because firms will want to shade their reports, reporting that they are less productive so that they pay a lower salary. Firms can find this profitable, even if it means that the mechanism matches them to inefficiently few workers. However, when firms' production technologies are public information, a strategyproof mechanism can elicit workers' idiosyncratic amenities and implement an efficient allocation. We thus conclude that a centralized matching mechanism can always address monopsonistic distortions when production technologies are observable.

Our second high-level contribution is to characterize how monopsonistic distortions arise from a restriction on the set of available transfers. The canonical Kelso and Crawford (1982) job-matching model predicts that allocations are always efficient. This prediction contrasts with labor monopsony models, which predict inefficient equilibria. It is not prima facie obvious why these literatures disagree. Our results imply that the disagreement cannot be explained by the different solution concepts that they adopt. By showing that our model produces inefficient stable allocations, we demonstrate that monopsonistic inefficiencies are inherent in market incentives with restricted transfers. In other words, monopsonistic inefficiencies are caused by the requirement that each firm pays all its workers the same salary and are not an artifact of a particular extensive form.

Two restrictions differentiate our model from the canonical Kelso-Crawford job-matching model. The first is that *a firm cannot pay different salaries to different workers*. The labor monopsony literature typically makes two justifications for that assumption. First, the firm may not know workers' idiosyncratic amenities and so would lack the information required to set worker-specific salaries (Card, Cardoso, Heining, & Kline, 2018). Second, workers shirk when they perceive that they are being paid less than their colleagues, especially when pay differentials are not based on easily-observed productivity differences (Breza, Kaur, & Shamdasani, 2017; Akerlof & Yellen, 1990).

A second difference between our model and the Kelso-Crawford model is that we assume *workers are fungible*: a firm's output depends only on the number of workers it employs, not on the workers' identities. (Kelso and Crawford impose this restriction briefly when exploring their gross substitutes condition.) Our results are thus most applicable to labor markets in which workers are interchangeable. This includes both high-education occupations like pharmacists (Goldin & Katz, 2016) and low-education occupations like textile manufacturing. Our results are less relevant to markets in which workers' productivities are heterogeneous. We show in Appendix C that when workers are not fungible, but salaries are still constant within each firm, a stable allocation may not exist.

1.1 A summary of our results

In Section 2 we present our model. Our model comprises workers and firms. Each worker can be employed by at most one firm, while each firm can hire any number of workers. A worker's utility depends quasi-linearly on her salary and on a firm-specific idiosyncratic amenity. The magnitudes, correlations, and signs of the amenities are unrestricted. In particular they might be positive (perhaps reflecting an exciting office environment) or negative (perhaps reflecting commuting costs). We model each firm's technology with a production function. Workers are fungible in production, in the sense that production functions depend on the number of workers that a firm employs but not on their identity.

Following the job-matching literature, we require that firms treat workers as gross substitutes. This means that, if a firm is willing to employ N workers at some salary, it must be willing to employ $N-1$ workers at that salary. We show that this is equivalent to firms' production functions having decreasing differences.

An allocation comprises a matching of workers to firms and a salary schedule associating each firm with a salary. Our baseline solution concept is stability. An allocation is stable if no firm and set of workers can deviate from the allocation and be no worse off, with some worker or firm strictly better off.

Worker and firm preferences are both quasi-linear in salaries. This facilitates a simple definition of match value: the sum of firm production and worker amenities. A matching is efficient if it has maximal value. A matching has hedonic efficiency if it maximizes value conditional on firm sizes.

Section 2 concludes with a simple example comprising one firm and two workers. The example demonstrates that multiple matchings can be stable, and that the value of these matchings can differ. In the stable allocations with an inefficient matching, salaries are lower and firm profits are higher. Our model thus exhibits monopsonistic distortions. This motivates the study of how centralized matching can ameliorate these distortions, our first high-level contribution. As we discuss in subsection 1.2, these distortions are not present in models in which a firm can pay different salaries to different workers. The example thus also relates to our second high-level contribution, which is to characterize how monopsonistic distortions arise from a restriction on transfers.

In Section 3, we characterize stable allocations. An allocation has No Envy if, given the prevailing salaries, each worker prefers her firm to any other firm. An allocation has No Firing if, given its salary, no firm would be better off being matched to one fewer worker. An allocation has No Poaching if no firm can increase its salary and make at least as much profit by attracting more workers. An allocation is stable if and only if it has No Envy, No Firing and No Poaching. This characterization provides a transparent interpretation of our solution concept. We regularly use it in the proofs of our later results.

Given a discrete production function, a firm's marginal product can be defined either as the increase in output from hiring an additional worker or as the decrease in output from firing a single worker. Given that production functions have decreasing differences, the former definition will be no larger than the latter. We say an allocation has Marginal Product Salaries if every firm's salary lies within those two bounds. Section 3 shows if an allocation has Marginal Product Salaries, then it will have No Firing and No Poaching. As a corollary, if an allocation has Marginal Product Salaries and No Envy, then it is stable.

In Section 4 we study the efficiency of stable allocations. We first introduce a piece of mathematical machinery. A replacement chain moves a sequence of workers from firm to firm such that each successive worker displaces the next. This means that a replacement chain changes each firm's size by at most one worker. It follows from our gross substitutes condition that, if some matching is inefficient, its value can be increased by a replacement chain. Moreover if the matching was in a stable allocation, this value-increasing replacement chain is acyclic: it begins and ends at different firms. These lemmas turn out to be quite powerful. We use them to show that every stable allocation will have hedonic efficiency, and that every stable allocation with Marginal Product Salaries will be efficient.

In the latter half of Section 4 we show that every efficient matching is in a stable allocation. We construct a Shapley and Shubik (1971) assignment game, which assigns workers to job *openings*. By defining the value of a worker-opening assignment appropriately, we can rely on Shapley and Shubik's results to construct a salary schedule which has No Envy and Marginal Product Salaries. This implies that the allocation is stable. This is an important component of our first high-level contribution, showing how centralized matching can ameliorate monopsonistic distortions: there is an efficient allocation that, if stipulated by a market designer, no coalition of workers and firms could profitably destabilize.

In Section 5 we discuss worker and firm welfare across stable allocations. We first show that some efficient stable allocation has greater salaries than any other stable allocation. We then show that all workers will prefer one stable allocation to another if and only if the former allocation has greater salaries than the latter. In combination, these results show that there exists an efficient stable allocation preferred by workers over any other stable allocation. We next show that if one stable allocation is preferred by all workers to another, all firms prefer the latter allocation to the former. Thus firms generally prefer inefficient stable allocations over the worker-optimal efficient stable allocation. This suggests that firms engineer monopsonistic distortions to increase their profits. In doing so they harm workers and shrink social surplus.

In Section 6 we ask whether an efficient stable allocation can be implemented through a strategyproof mechanism. When firms' production functions are private information, it cannot be. Firms can claim that they are less productive than they actually are. This results in them paying lower salaries, and thus can be profitable even if it means that they are matched to inefficiently few workers. However, when firms' production functions are public information, a strategyproof mechanism can elicit workers' idiosyncratic amenities and implement an efficient allocation. This demonstrates when centralized matching can ameliorate monopsonistic distortions: when firms' production functions are observed.

In Section 7 we study the sources of monopsonistic inefficiencies. We first consider a context in which amenities have 'common value': the amenity one worker receives from some firm equals the amenity that any other worker receives from the same firm. In this context, every stable allocation is efficient. We next consider the effect of duplicating firms. Two firms are duplicates if they have the same production function and provide the same amenities. When every firm has a duplicate, every stable allocation is efficient. These results clarify the sources of monopsonistic distortions. This is part of our second high-level contribution, characterizing labor market monopsony. It is also part of our first high-level contribution: by understanding when labor markets can be distorted, we show when centralized matching can potentially play an ameliorative role.

The final result in Section 7 identifies the precise coalition that could destabilize an inefficient stable allocation, were transfers unrestricted. In every inefficient stable allocation, some worker is willing to work at some new firm for a salary less than her marginal product. However, that new firm does not hire her, because doing so would require that the firm pay its existing workers more.

Appendix A proves the results in the main body of the text. Latter appendices contain auxiliary results. In Appendix B, we relate some definitions and results to those in other matching papers. In Appendix C, we prove that a stable allocation may not exist when workers are not fungible. In Appendix D, we motivate our model as the limiting case of a model in which workers shirk when they are paid less than their colleagues. In Appendix E, we study how the set of stable allocations changes when each worker is duplicated. In Appendix F, we show that many plausible statistics fail to indicate which of two stable allocations is the more efficient.

1.2 How our results relate to the existing literature

Our first high-level contribution is to assess how centralized matching can address monopsonistic distortions. To make that assessment, we connect the job-matching literature to the labor monopsony literature.

The intellectual antecedent of job-matching models is the Gale and Shapley (1962) college admissions model, in which students' preferences over colleges are combined with colleges' preferences over students to construct a stable matching: that is, a matching from which no set of students and colleges can profitably defect. Job-matching models extend the college admissions model by pairing each matching with transfers from one side of the market to the other (Shapley & Shubik, 1971; Crawford & Knoer, 1981; Kelso & Crawford, 1982; Fleiner, 2003; Hatfield & Milgrom, 2005).

The canonical job-matching model is that of Kelso and Crawford (1982). Kelso and Crawford show that, if firms treat workers as 'gross substitutes' and transfers are unrestricted, then a stable allocation will always exist. When workers' utilities are quasi-linear in salaries, every stable allocation of the Kelso and Crawford model is efficient. Kelso and Crawford assume that each worker's salary can be set independently of the salaries paid to that worker's colleagues. Thus a blocking coalition consisting of one worker and one firm will not affect the transfers paid to other workers whom that firm employs. Such coalitions block any inefficient allocation.

The labor monopsony literature descends from Robinson's (1933) study of imperfect competition. Modern monopsony models adopt functional form restrictions more frequently than the job matching literature. For example, some models postulate a representative worker with CES labor disutility (Berger et al., 2019). Others postulate a continuum of workers with Gumbel-distributed firm amenities (Card et al., 2018; Azar, Berry, & Marinescu, 2019; Lamadon, Mogstad, & Setzler, 2019; Kroft, Luo, Mogstad, & Setzler, 2020). Firms interact in

Bertrand or Cournot competition, and a firm either pays an identical salary to all its workers or discriminates solely on the basis of productivity. A recurring theme is that firms' strategic behavior distorts the labor market: unemployment is too high, productive firms are too small and unproductive firms are too large (Boal & Ransom, 1997; Manning, 2011; Berger et al., 2019; Lamadon et al., 2019).

These distortions are not found in Kelso and Crawford's model. Given the different modelling assumptions made by the job matching and monopsony literatures, it is not *prima facie* obvious why that is: Is it because of the functional forms they impose? Because of the solution concepts they employ? Or is it only because the job matching literature allows within-firm salary discrimination? By unifying the two literatures, this paper demonstrates how monopsonistic distortions arise from restrictions on transfers – the second of our two high-level contributions.

By studying market power in job matching, we follow Bulow and Levin (2006); Kojima (2007); and Azevedo (2014). Bulow and Levin study market power in centralized labor markets like those matching hospitals to doctors. They consider a stylized context in which each hospital sets an anonymous salary and is then matched to a single doctor. They assume that the efficient match is assortative: 'better' hospitals should be matched to 'better' doctors. Hospitals set salaries in mixed strategy equilibrium. Ex ante, salaries are lower than the competitive equilibrium. Ex post, the resultant match can be inefficient and unstable because better hospitals may happen to set lower salaries than worse hospitals. Bulow and Levin consider only one-to-one matching. Their model thus lacks the monopsonistic mechanism which stabilizes inefficient matchings in our model.

Kojima (2007) comments on the Bulow and Levin model. Kojima argues that Bulow and Levin's results need not extend to contexts in which each hospital is matched to many doctors. In particular, Kojima points out that strategic salary setting by firms can benefit inframarginal workers, as firms increase salaries to compete for marginal workers. Kojima limits his comparisons to the firm-optimal competitive equilibrium. Our results in Section 5 and Appendix B suggest that this perspective is limiting: no worker benefits from firms' strategic salary setting when it constitutes a departure from the worker-optimal competitive equilibrium.

Azevedo (2014) constructs a market with a finite set of firms and a continuum of workers. Firms choose quantities in Cournot competition. Azevedo first considers exogenous salaries. Exogenous salaries mean that Azevedo's model lacks the monopsonistic mechanism emphasised by our model. Despite this, Azevedo's model does produce inefficiencies. A firm might avoid hiring a relatively unproductive worker. The unproductive worker may then replace a worker matched to another firm. The ensuing chain of replacements can eventually result in the original firm being matched to a more productive worker. This can benefit the initial firm while hampering efficient employment. Our model lacks this mechanism because it assumes that workers are fungible in production.

Azevedo also considers endogenous salaries. When doing so he lets salaries vary between the workers employed by a given firm. As in Kelso and Crawford's model, this forecloses monopsonistic distortions.

2 A Model of Monopsonistic Distortions

A labor market $(\mathbf{F}, \mathbf{W})$ comprises a finite set of firms $\mathbf{F}$ and a finite set of workers $\mathbf{W}$. Each worker can be employed by at most one firm while each firm can hire any number of workers.

Each firm F is endowed with a production technology, which we represent with a non-decreasing function $y_F : \mathbb{N} \to \mathbb{R}^+$. Note that production depends only on the number of workers employed and not on their identity.

We normalize $y_F(0) = 0$. Each firm pays the same salary to all its workers: there is no salary discrimination within any firm's workforce. Firms face a competitive product market in which their good has price normalized to one. Thus if firm F employs N workers at salary s, its profit will be

$$\pi_F(N, s) \equiv y_F(N) - sN.$$

A worker $w \in \mathbf{W}$ employed at firm $F \in \mathbf{F} \cup \{\emptyset\}$ has quasi-linear preferences

$$u_w(F, s) \equiv \alpha_w(F) + s,$$

where $\alpha_w(F)$ is the idiosyncratic amenity that worker w receives from working at firm F. The amenity $\alpha_w(F)$ may be positive, negative, or zero. It encompasses any fixed benefit or cost the worker incurs from working at a given firm. Being employed at the empty set denotes unemployment, and we normalize $\alpha_w(\emptyset) = 0$.

2.1 Matchings and allocations

A **matching** is a function $\mu : \mathbf{F} \cup \mathbf{W} \to \mathscr{P}(\mathbf{F} \cup \mathbf{W})$ such that:

- For all workers $w \in \mathbf{W}$: $\left|\mu(w)\right| \leq 1$ and $\mu(w) \subseteq \mathbf{F}$.
- For all firms $F \in \mathbf{F}$: $\mu(F) \subseteq \mathbf{W}$.
- For all workers $w \in \mathbf{W}$ and all firms $F \in \mathbf{F}$: $w \in \mu(F)$ if and only if $\mu(w) = \{F\}$.

We use the matching to represent employment: a worker w is employed at firm F if and only if $\mu(w) = \{F\}$. Since workers are matched to at most one firm, we abuse notation and write $\mu(w) = F$ rather than $\mu(w) = \{F\}$.

An **allocation** (μ, s) comprises a matching μ and a salary function $s : \mathbf{F} \cup \emptyset \to \mathbb{R}^+$ associating each firm with a salary. We require all salaries to be non-negative, and we normalize $s(\emptyset) = 0$. To simplify our results we require that for any allocation (μ, s), for any firm F, if $\mu(F) = \emptyset$, then $s(F) = y_F(1)$. This is without loss of generality because the salary paid by an unmatched firm does not affect its profit.

An allocation (μ, s) is **individually rational** if

- for all workers $w \in \mathbf{W}$: $u_w\left(\mu(w), s(\mu(w))\right) \geq 0$, and
- for all firms $F \in \mathbf{F}$: $\pi_F\left(\left|\mu(F)\right|, s(F)\right) \geq 0$.

A coalition (F, C, s^*), with $F \in \mathbf{F}$, $C \subseteq \mathbf{W}$, and $s^* \in \mathbb{R}^+$, **blocks** allocation (μ, s) if

- $\pi_F(|C|, s^*) \geq \pi_F\left(\left|\mu(F)\right|, s(F)\right)$, and
- for all workers $w \in C$: $u_w(F, s^*) \geq u_w\left(\mu(w), s(\mu(w))\right)$,

where the inequality is strict for the firm or one of the workers. It is without loss of generality to consider only blocking coalitions comprised of a single firm: any coalition containing more than one firm could be be broken into multiple coalitions, each with only one firm.

Our solution concept is stability. An allocation (μ, s) is **stable** if it is individually rational and not blocked by any coalition.

We follow the matching literature in using a cooperative solution concept. Decentralized labor markets will produce stable allocations provided that firms and workers can freely sever existing relationships and form new relationships. This does not imply that all stable allocations are equally realistic as market equilibria. Equilibrium selection will depend upon the institutions of the market in question. For example, one implication of stability is that firms cannot decrease their salaries without the consent of their current workers; the plausibility of some stable allocations will thus depend on whether contracts or regulations enforce this feature. In contrast to our approach, the labor literature often adopts non-cooperative solution concepts like Bertrand competition. We will revisit this distinction in Appendix B.

2.2 Production functions

We required above that firms' production functions are non-decreasing and normalized such that for all firms F: $y_F(0) = 0$. Again, following the matching literature (Kelso & Crawford, 1982; Hatfield & Milgrom, 2005) we additionally impose the following gross substitutes restriction.

Firm F treats workers as **gross substitutes** if for any salary $s \in \mathbb{R}^+$ and maximal workforce N:

$$N \in \underset{M \leq N}{\arg\max}\, \pi_F(M, s) \implies N - 1 \in \underset{M \leq N-1}{\arg\max}\, \pi_F(M, s).$$

Assumption 1. *Every firm treats workers as gross substitutes.*

Theorem 6 in Kelso and Crawford shows that gross substitutes is equivalent to the production functions y_F having **decreasing differences**. Given that our notion of gross substitutes is slightly different from theirs, we provide our own proof. (Our proofs are in Appendix A.)

Lemma 1. *Firm F treats workers as gross substitutes if any only if y_F has decreasing differences.*

Typically, the gross substitutes condition is used to guarantee the existence of stable allocations (Kelso & Crawford, 1982; Hatfield & Milgrom, 2005). This condition is often compared to the concavity of utility or production functions. In our setting, this connection becomes even more clear: when workers are fungible and treated as gross substitutes, production functions exhibit diminishing returns to labor. In this way, the gross substitutes assumption is the discrete analogue of assuming concave production.

With a discrete production function, a firm's marginal product has two possible definitions. Given some matching μ, we let $\Delta^+_\mu(F)$ denote the increase in firm F's output from employing one worker *more* than the firm is employing at μ, and we let $\Delta^-_\mu(F)$ denote the decrease in firm F's output from employing one worker *fewer* than the firm is employing at μ:

$$\Delta^+_\mu(F) \equiv y_F\left(\left|\mu(F)\right| + 1\right) - y_F\left(\left|\mu(F)\right|\right);$$

$$\Delta^-_\mu(F) \equiv \begin{cases} y_F\left(\left|\mu(F)\right|\right) - y_F\left(\left|\mu(F)\right| - 1\right) & \text{if } \mu(F) \neq \emptyset; \\ \infty & \text{if } \mu(F) = \emptyset. \end{cases}$$

By Lemma 1, for any firm F and matching μ: $\Delta^+_\mu(F) \leq \Delta^-_\mu(F)$.

2.3 Definitions of efficiency

Given the quasi-linear setup, we can define the **value** of a matching μ as the sum of worker amenities and firm outputs:

$$\text{value}\left(\mu\right) \equiv \sum_{F \in \mathbf{F}} y_F\left(\left|\mu(F)\right|\right) + \sum_{w \in \mathbf{W}} \alpha_w\left(\mu(w)\right).$$

A matching μ^* is **efficient** if it has maximal value:

$$\mu^* \in \underset{\mu}{\arg\max} \left\{\text{value}\left(\mu\right)\right\}.$$

This notion of efficiency is sometimes referred to as utilitarian efficiency. We also define a more limited notion of efficiency. A matching μ^* has **hedonic efficiency** if it maximizes the sum of amenities, given firm sizes:

$$\mu^* \in \underset{\mu \text{ s.t.} \forall F: |\mu(F)| = |\mu^*(F)|}{\arg\max} \left\{ \sum_{w \in \mathbf{W}} \alpha_w \left(\mu(w)\right) \right\}.$$

By holding firm sizes fixed, hedonic efficiency speaks only to inefficiencies caused by a mismatch of workers to firms, rather than allocative inefficiencies in production. Note that hedonic efficiency is a strictly weaker requirement than efficiency: if μ^* is efficient, then μ^* has hedonic efficiency.

An allocation (μ, s) is efficient if its matching μ is efficient, and has hedonic efficiency if μ has hedonic efficiency.

2.4 An illustrative example

To elucidate our model, we present the following example of a monopsonistic labor market.

Example 1 (a simple monopsony). $\mathbf{F} = \{F\}$. $y_F(N) = 6N$. $\mathbf{W} = \{w_1, w_2\}$. $\alpha_{w_1}(F) = 0$. $\alpha_{w_2}(F) = -4$.

The stable allocations of Example 1 are presented in Figure 1. The stable allocations have one of these two matchings:

$$\mu^1 = \begin{pmatrix} w_1 & w_2 \\ F & \emptyset \end{pmatrix}; \quad \mu^2 = \begin{pmatrix} w_1 & w_2 \\ F & F \end{pmatrix}.$$

The matching μ^1 will be a stable allocation when composed with a salary $s^1(F) \in [0,2)$. The firm makes profit $\pi_F\left(1, s^1(F)\right) > 6-2 > 0$, worker w_1 has utility $u_{w_1}\left(F, s^1(F)\right) = s^1(F) \geq 0$, and worker w_2 has utility $u_{w_2}(\emptyset, 0) = 0$. Thus the allocation is individually rational.

We now verify that $\left(\mu^1, s^1\right)$ is not blocked by any coalition $\left(F, C, s'\right)$. If $C = \{w_1\}$, either both F and w are indifferent between the coalition and μ^1, or one is strictly better off while the other is strictly worse off. So a blocking coalition must include w_2. It is not individually rational for w_2 to work at any salary strictly less than 4, so $w_2 \in C$ requires $s' \geq 4$. The firm would be strictly worse off paying $s' > 2$ and only employing w_2, so we must have $C = \{w_1, w_2\}$. Thus the firm must be weakly better off employing 2 workers at s'. But $\pi_F\left(2, s'\right) \leq 2 \times 6 - 2 \times 4 = 4 < \pi_F\left(\left|\mu^1(F)\right|, s^1(F)\right)$. Therefore, there exists no blocking coalition, and $\left(\mu^1, s^1\right)$ is indeed stable.

The matching μ^2 will be a stable allocation when composed with a salary $s^2(F) \in [4,6]$. The firm makes profit $\pi_F\left(2, s^2(F)\right) = 2 \times 6 - 2 \times s^2(F) \geq 0$, worker w_1 has utility $u_{w_1}\left(F, s^2(F)\right) = s^2(F) \geq 4 \geq 0$, and worker w_2 has utility $u_{w_2}\left(F, s^2(F)\right) \geq s^2(F) - 4 \geq 0$. Thus the allocation is individually rational.

Again, we verify that there exists no blocking coalition $\left(F, C, s'\right)$. Given both workers are employed by F at μ^2, there can be no blocking coalition $\left(F, C, s'\right)$ in which $s' < s^2(F)$. If $C = \{w_1, w_2\}$ and $s' > s^2(F)$, the firm will

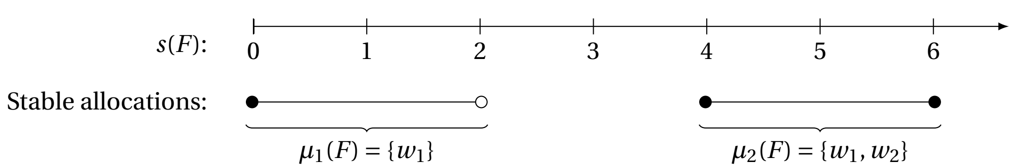


Figure 1: Stable allocations in Example 1

be strictly worse off. If $|C| = 1$ and $s' = s^2(F)$, then the firm will be weakly worse off, and either worker will be indifferent. Finally, if $|C| = 1$ and $s' > s^2(F)$, then the firm will be strictly worse off. Therefore, there exists no blocking coalition, and $\left(\mu^2, s^2\right)$ is indeed a stable allocation.

Note that value $\left(\mu^2\right) = 8 >$ value $\left(\mu^1\right) = 6$. Thus this example suffices to prove the following two results.

Proposition 1. *There can exist multiple stable allocations, which contain different matchings with different values.*

Corollary 1. *Stable allocations can contain inefficient matchings.*

Proposition 1 demonstrates that our model behaves very differently from the Kelso and Crawford model, which always predicts efficient matchings. Note that the inefficient matching is stable directly because of the restriction on transfers. If the firm could pay different salaries to each worker, there would exist a blocking coalition to any allocation $\left(\mu^1, s^1\right)$, where $s^1(F) \in [0,2)$: the firm could continue to pay salary $s^1(F)$ to worker w_1 and offer w_2 her reservation salary, profitably employing both workers. Thus it is exactly the restriction on transfers that creates the monopsonistic distortion: the firm prefers employing inefficiently few workers over paying all its workers a higher salary.

Yet the restriction on transfers does not exclude the efficient matching from being stable. Although the firm prefers $\left(\mu^1, s^1\right)$ to $\left(\mu^2, s^2\right)$, a firm which found itself in allocation $\left(\mu^2, s^2\right)$ could not unilaterally decrease its salary to below w_2's reservation salary. An efficient allocation can thus be stable. Whether the efficient stable allocation actually occurs will depend on whether the firm has the power to unilaterally decrease its salary at its current workers' expense. Labor market efficiency thus depends directly on whether labor market institutions empower workers or firms.

Example 1 exhibits other key features of our model that we will show hold more generally. Though μ^1 is inefficient, μ^1 does have hedonic efficiency: given that only one worker is employed, it is more efficient for that worker to be worker w_1. When $s(F) = 6$, a higher salary than any other stable allocation, the allocation is efficient, is better for all workers than any other stable allocation, and yields lower firm profit than any other stable allocation.

3 A Characterization of Stable Allocations

In this section, we show that stability is equivalent to three simple conditions. We then define a condition on salary schedules that at we call 'Marginal Product Salaries', which we relate to stability. This provides an easily interpretable characterization of our solution concept. We will often use it in the proofs of later results.

An allocation (μ, s) has **No Envy** if $\forall w \in \mathbf{W}, F \in \mathbf{F} \cup \emptyset$:

$$u_w\left(\mu(w), s(\mu(w))\right) \geq u_w\left(F, s(F)\right).$$

The No Envy condition states that no worker would prefer to be matched to another firm, given the prevailing salaries.

An allocation (μ, s) has **No Firing** if $\forall F \in \mathbf{F}$:

$$s(F) \leq \Delta_\mu^-(F).$$

The No Firing condition states that no firm would be better off being matched to fewer workers while paying the same salary.

Consider an allocation in which firm F pays salary s_F and each other firm F' pays $s\left(F'\right)$, i.e. the corresponding element of the salary schedule s. The maximal labor-supply available for firm F is

$$L_F(s_F, s) \equiv \left|\left\{w : \alpha_w(F) + s_F \geq \max_{G \in \mathbf{F} \cup \{\emptyset\}} \{\alpha_w(G) + s(G)\}\right\}\right|.$$

Note that the maximal labor-supply functions allocate a worker to multiple firms when that worker is indifferent between them. An allocation (μ, s) has **No Poaching** if $\forall F \in \mathbf{F} : \nexists s_F > s(F)$, and $L \in \{1, ..., L_F(s_F, s)\}$ such that

$$\pi_F(L, s_F) \geq \pi_F\left(\left|\mu(F)\right|, s(F)\right).$$

The No Poaching condition states that no firm can increase its salary and, by attracting additional workers, make at least as much profit as it did previously.

Proposition 2. *An allocation is stable if and only if it has No Envy, No Firing, and No Poaching.*

If the No Envy condition fails, some 'envious' worker would prefer employment at some other firm, at that firm's existing salary. The envious worker could form a blocking coalition with that firm and all-but-one of that firm's existing workers, at the existing salary. The envious worker would be better off, while the firm and the existing workers would be no worse off. The fact that the firm is no worse off follows from our assumption that production depends only on the number of workers employed and not on their identity. As we will show, No Envy can be used to impose a great deal of structure on the set of stable allocations.

If the No Firing condition fails, some firm would be losing money on its marginal worker. The allocation would be blocked by a coalition comprising that firm and all-but-one of the firm's existing workers, at the existing salary. That coalition would leave the still employed workers no worse off and the firm would make strictly more profit.

If the No Poaching condition fails, some firm could form a blocking coalition with its existing workers and some new workers at a higher salary than it currently pays. The firm and the new workers would be no worse off, and the existing workers would be strictly better off.

The above summarizes why a stable allocation must have No Envy, No Firing, and No Poaching. To show that having these conditions is sufficient for an allocation to be stable, we first show that an allocation with No Envy and No Firing will also be individually rational for the workers and firms respectively. We then show that if any blocking coalition exists, then the original allocation must not satisfy the No Envy, No Firing, or No Poaching conditions.

Recall from Section 2 that a firm's marginal product can be defined either as the increase in output from being matched to one worker more ($\Delta_\mu^+(F)$) or as the reduction in output from being matched to one worker fewer ($\Delta_\mu^-(F)$); by decreasing differences $\Delta_\mu^+(F) \leq \Delta_\mu^-(F)$. We say that an allocation (μ, s) has **Marginal Product Salaries** if every firm's salary is sandwiched between these two bounds:

$$\forall F : s(F) \in \left[\Delta_\mu^+(F), \Delta_\mu^-(F)\right].$$

A recurring idea in this paper is that an allocation with Marginal Product Salaries will be 'pricing' labor efficiently, and will incentivize both workers and firms towards efficient outcomes. The following result is an example:

Lemma 2. *An allocation with Marginal Product Salaries will also have No Firing and No Poaching.*

If firm F's salary is at least $\Delta_\mu^+(F)$, then any higher salary will be strictly greater than $\Delta_\mu^+(F)$, and thus at such a salary the firm would be worse off hiring an additional worker. Thus $s(F) \geq \Delta_\mu^+(F)$ implies (μ, s) will have No Poaching. The No Firing condition is identical to the requirement that each firm F's salaries be less than $\Delta_\mu^-(F)$.

Proposition 2 tells us that an allocation with No Envy, No Firing, and No Poaching is stable, and Lemma 2 tells us that an allocation with Marginal Product Salaries also has No Firing and No Poaching. Combined, these results yields the following corollary:

Corollary 2. *If an allocation has No Envy and Marginal Product Salaries, then it is a stable allocation.*

The Marginal Product Salaries condition tells us that firms will not benefit from hiring or firing workers, at their current salaries. The No Envy condition tells us that workers do not want to change firms, at their current salaries. In combination, having these conditions is sufficient for an allocation to be stable.

Note that Example 1 shows that having Marginal Product Salaries is not necessary for an allocation to be stable. That is because firms need not lose their workers when they pay less than their marginal products.

4 The Efficiency of Stable Allocations

In this section, we consider the efficiency of stable allocations. We first introduce 'replacement chains', a piece of mathematical machinery that helps us formally compare allocations. A replacement chain represents moving a sequence of workers from their current firm to the following worker's firm. We then use this machinery to prove this section's substantive results. A stable allocation with Marginal Product Salaries will be efficient. While a stable allocation may be inefficient, it will nonetheless have hedonic efficiency. These results illustrate the nature of monopsonistic distortions.

Finally, this section shows that each efficient matchings will be in some stable allocation. This is an important step towards asking how centralized matching can address monopsony power: the fact that there will always be a stable efficient allocation means that, if a market designer could stipulate it, no coalition of firms and workers could profitably destabilize it.

4.1 Replacement chains and their economic implications

A **replacement chain** comprises a sequence of workers $(w_k)_{k=0}^{N-1} \subseteq \mathbf{W}$ and a sequence of firms $(F_k)_{k=0}^{N} \subseteq \mathbf{F} \cup \{\varnothing\}$ such that no worker is repeated:

$$\forall k \neq j : w_k \neq w_j,$$

and no adjacent firms are the same:

$$\forall k : F_k \neq F_{k+1}.$$

If μ is a matching and $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ is a replacement chain such that

$$\forall k \in 0, \ldots, N-1 : w_k \in \mu(F_k),$$

then $\mu + \chi$ is the matching constructed by moving each worker w_k from F_k to F_{k+1}:

$$(\mu + \chi)(w) = \begin{cases} F_{k+1} & \text{if } w = w_k; \\ \mu(w) & \text{if } w \notin (w_k)_{k=0}^{N-1}. \end{cases}$$

Similarly, if χ is such that

$$\forall k \in 0, \ldots, N-1 : w_k \in \mu(F_{k+1}),$$

then $\mu - \chi$ is the matching constructed by moving each worker w_k from F_{k+1} to F_k:

$$(\mu - \chi)(w) = \begin{cases} F_k & \text{if } w = w_k; \\ \mu(w) & \text{if } w \notin (w_k)_{k=0}^{N-1}. \end{cases}$$

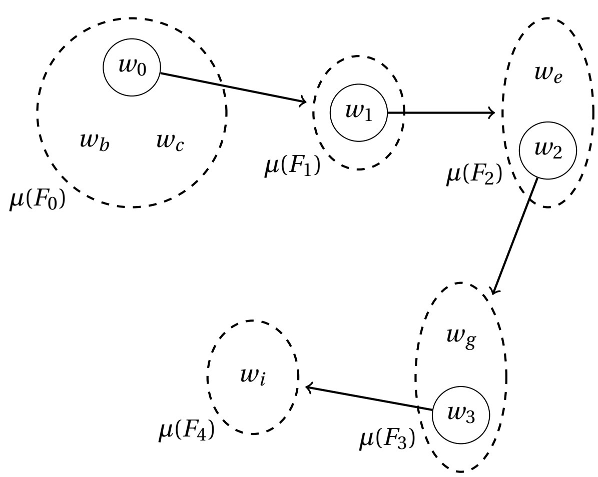
We say that χ is a replacement chain from μ to μ' if

$$\forall k \in 0, \ldots, N-1 : w_k \in \mu(F_k) \cap \mu'(F_{k+1}).$$

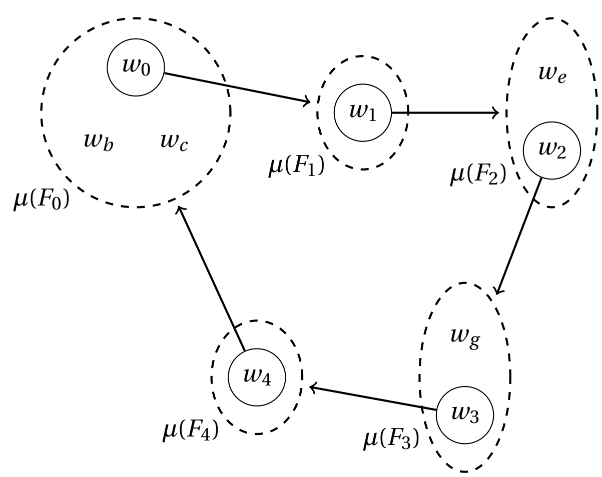
Note that if χ is a replacement chain from μ to μ' then both $\mu + \chi$ and $\mu' - \chi$ are well-defined.

If $\mu \neq \mu'$, there will exists some replacement chain from μ to μ'. For example, if there exists a worker w such that $\mu(w) \neq \mu'(w)$, then the trivial replacement chain $\left((w), \left(\mu(w), \mu'(w)\right)\right)$ is a replacement chain from μ to μ'. Note that if χ is a replacement chain from μ to μ', it need not be the case that $\mu + \chi = \mu'$. Rather, it is necessarily the case that there will exist a sequence of replacement chains $\chi_1, \chi_2, \ldots, \chi_k$ such that $\mu + \chi_1 + \chi_2 + \ldots + \chi_k = \mu'$.

Consider some replacement chain $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$. χ is **cyclic** if $F_0 = F_N$. χ is **acyclic** if $F_0 \neq F_N$. χ is a **maximal** chain from μ to μ' if it cannot be extended in either direction: $\mu'(F_0) \subset \mu(F_0)$ and $\mu(F_N) \subset \mu'(F_N)$. (These subsets are strict because if $\mu'(F_0) = \mu(F_0)$ or $\mu(F_N) = \mu'(F_N)$ then no worker would be moved by χ.)



(a) An acyclic replacement chain



(b) A cyclic replacement chain

Figure 2: Two replacement chains

The notion of a replacement chain is depicted in Figure 2. The replacement chain in Panel (a) moves worker w_0 from firm F_0 to firm F_1, moves worker w_1 from firm F_1 to firm F_2, moves worker w_2 from firm F_2 to firm F_3, and moves worker w_3 from firm F_3 to firm F_4. It is acyclic because it starts and ends at different firms.

The replacement chain in Panel (b) is identical, except that it additionally moves worker w_4 from firm F_4 to firm F_0. It is cyclic because it starts and ends at the same firm.

Our first replacement chain result claims that, from any inefficient matching, there exists a replacement chain that increases value. This is a consequence of our gross substitutes assumption (Assumption 1). For example consider two matchings μ, μ^* such that value$\left(\mu^*\right) >$ value$\left(\mu\right)$, and μ and μ^* only differ for two workers w_1, w_2, who are both matched to firm F_1 in μ and F_2 in μ^*:

$$\{w_1, w_2\} = \mu(F_1) \cap \mu^*(F_2);\ F_1 \neq F_2.$$

There is no replacement chain χ such that $\mu^* = \mu + \chi$. However, given Assumption 1, if moving *both* w_1 and w_2 from F_1 to F_2 increases value, then either moving w_1 from F_1 to F_2 increases value, or moving w_2 from F_1 to F_2 increases value (or both). It is this underlying connection to gross substitutes that makes replacement chains so useful for analyzing our model. Replacement chains change firm sizes by at most one, and thus the way they change the value of a given matching is very simple to describe.

Lemma 3. *Let μ and μ^* be matchings such that value$\left(\mu^*\right) >$ value$\left(\mu\right)$. There exists a replacement chain χ from μ to μ^* such that value$\left(\mu + \chi\right) >$ value$\left(\mu\right)$. Moreover, for each firm F: $\left|\left(\mu + \chi\right)(F)\right| \leq \max\left\{\left|\mu(F)\right|, \left|\mu^*(F)\right|\right\}$.*

The proof of Lemma 3 formalizes the above intuition. We present an algorithm which necessarily finds the required replacement chain. The fact that, for each firm F, $\left|\left(\mu + \chi\right)(F)\right| \leq \max\left\{\left|\mu(F)\right|, \left|\mu^*(F)\right|\right\}$ means that the value-improving replacement chain need not grow any firm to be bigger than its size in μ^*. We will use this fact in the proof of Theorem 1.

Our next result demonstrates which replacement chains can increase value from a stable allocation.

Lemma 4. *Let $\left(\mu, s\right)$ be a stable allocation. There exists no cyclic replacement chain χ such that value$\left(\mu + \chi\right) >$ value$\left(\mu\right)$.*

Given that firms care only about the number of workers that they employ, and not the identity of those workers, cyclic replacement chains do not change firms' outputs. Thus if a cyclic replacement chain increased an allocation's value, it would have to do so by improving workers' total amenities. Such a reshuffling would also increase workers' total utility, because the total salary paid would not change. Thus there must be some worker who would have higher utility after the reshuffle. This would imply $\left(\mu, s\right)$ does not have No Envy. This would imply that $\left(\mu, s\right)$ is not a stable allocation.

We can now show that *any* stable allocation with Marginal Product Salaries will be efficient. Consider first an allocation $\left(\mu, s\right)$ such that the value of μ can be increased by moving one worker w from F to F':

$$0 < \Delta^+_\mu(F') - \Delta^-_\mu(F) + \alpha_w(F') - \alpha_w(F).$$

If $\left(\mu, s\right)$ has Marginal Product Salaries, then $s(F') \geq \Delta^+_\mu(F')$ and $s(F) \leq \Delta^-_\mu(F)$, and thus:

$$0 < s(F') - s(F) + \alpha_w(F') - \alpha_w(F),$$

which shows that $\left(\mu, s\right)$ does not have No Envy, and so is not a stable allocation.

An inefficient matching cannot necessarily be improved by moving only one worker. However, Lemmas 3 and 4 guarantee the existence of an acyclic replacement chain with which the above argument can be extended to any inefficient allocation. This is formalized in the proof of the following proposition.

Proposition 3. *If (μ, s) is a stable allocation with Marginal Product Salaries, then μ is efficient.*

Proposition 3 tells us that, if we can stipulate a stable allocation in which workers are paid their marginal products, we can be assured that the allocation is efficient. Of course, knowing whether workers are paid their marginal products would require us to know firms' production functions, which may be the firms' private information. We will ask whether these production functions can be elicited in Section 6.

Lemma 4 told us that stable allocations never have a value-improving cyclic replacement chain. Proposition 4 captures the economic meaning of this result: the inefficiency of a stable allocation arises only through inefficient firm sizes rather than through a mismatch of workers to firms.

Proposition 4. *Every stable allocation has hedonic efficiency.*

Proposition 4 contrasts with recent criticisms of centralized labor markets like the National Resident Matching Program. Proposed reforms to the National Resident Matching Program focus on improving the match between workers and firms, given firm sizes (Crawford, 2008). In contrast, Proposition 4 suggests that, when workers are fungible, firm sizes are the *only* problem with an inefficient stable allocation.

4.2 Efficient matchings and stability

We know from Example 1 that while some stable allocations can be efficient, not all stable allocations will necessarily be so. In this subsection we show that at least one efficient allocation will be stable. In other words, different stable allocations can contain different matchings, some of which *can* be inefficient, and one of which *will always* be efficient.

Theorem 1. *Every efficient matching is in a stable allocation.*

To prove Theorem 1, we construct an auxiliary one-to-one 'job assignment game', in which job openings are matched to workers. At each firm, the number of job openings equals the number of workers to which the firm is matched in the efficient matching. The value of each match equals the worker amenity plus the production value of a marginal opening being filled.

The job assignment game is a Shapley and Shubik (1971) assignment game. Shapley and Shubik's solution concept is the core. We use Shapley and Shubik's results, along with our results about replacement chains, to show the job assignment game's core supports an assignment isomorphic to the efficient matching. Shapley and Shubik also tell us that the core contains a worker-optimal vector of payoffs. We decompose each worker's payoff into an idiosyncratic component, which equals her amenity, and a common component, which is equal across all workers assigned to openings at a given firm. We set salaries equal to the common components of the worker-optimal payoff vector, ensuring that each firm is paying all its workers the same salary.

That this payoff vector is in the core of the job assignment game implies that the efficient matching composed with this salary vector will have No Envy. That this payoff vector is core also implies that the openings have non-negative payoffs, which in turn implies that each firm F's salary is no greater than $\Delta^-_\mu(F)$. Finally, we use the fact that we picked the worker-optimal payoff vector to show that each firm F's salary is no less than $\Delta^+_\mu(F)$. Given that the allocation has No Envy and Marginal Product Salaries, Corollary 2 ensures that the allocation is stable.

That there is always an efficient stable allocation suggests that market incentives *can* support efficiency. However, inefficient allocations can also be stable, and so decentralized labor markets may not actually be efficient. Combined, these findings motivate the use of centralized matching mechanisms to guide the market towards efficiency. Before exploring such mechanisms, the multiplicity of stable allocations motivates further exploration of how worker and firm welfare varies across stable allocations. This will identify the incentives through which a centralized mechanism must operate.

5 Worker and Firm Welfare across Stable Allocations

The previous sections have demonstrated that stable allocations can have differing values. At least one will be efficient, while others may not be. In this section, we relate these results to worker and firm welfare. We show that there always exists an efficient stable allocation that all workers prefer to any other stable allocation. We also show that if all workers prefer one stable allocation to another, all firms must prefer the latter allocation to the former. It follows that the worker-optimal efficient stable allocation is worse for all firms than any other stable allocation.

5.1 The alignment of efficiency and worker welfare

We first show that there exists a stable allocation in which every firm pays the highest salary that it pays in any stable allocation. Moreover, this allocation is efficient.

Proposition 5. *There exists a stable allocation $\left(\mu^*, s^*\right)$ such that μ^* is efficient and for all stable allocations $\left(\mu, s\right)$: $s^* \geqq s$.*

Given that (by Lemma 1) firms' production functions have decreasing marginal products, one might worry that an allocation in which a firm is matched to inefficiently few workers would have higher salaries than an efficient allocation. However, such an allocation would make some workers, who are not matched to that firm, envious, destabilizing this allocation. The full proof of Proposition 5 exploits the fact that, in the proof of Theorem 1, we chose the worker-optimal Shapley-Shubik payoff vector. The proof shows that this payoff vector must correspond to maximal stable salaries.

Define the binary relation $\succeq_{\mathbf{W}}$ as representing workers' unanimous preferences across allocations:

$$\left(\mu, s\right) \succeq_{\mathbf{W}} \left(\mu', s'\right) \iff \forall\, w : u_w\left(\mu(w), s(\mu(w))\right) \geq u_w\left(\mu'(w), s'(\mu'(w))\right).$$

If one allocation has greater salaries than another, a worker matched to the same firm in both allocations will necessarily prefer the former allocation over the latter. Of course, workers who switch firms between the two allocations could prefer either allocation. However, if both allocations are stable, No Envy guarantees that each worker prefers the firm she is matched to over any other firm, given that firm's salaries. By combining inequalities, we can prove that if one *stable* allocation has greater salaries than another, all workers must prefer the former allocation over the latter.

Lemma 5. *For any two stable allocations $\left(\mu, s\right), \left(\mu', s'\right) : s \geqq s' \iff \left(\mu, s\right) \succeq_{\mathbf{W}} \left(\mu', s'\right)$.*

Combining Lemma 5 and Proposition 5 yields the following result:

Corollary 3. *There exists a stable allocation $\left(\mu^*, s^*\right)$ such that μ^* is efficient and for all stable allocations $\left(\mu, s\right)$: $\left(\mu^*, s^*\right) \succeq_{w} \left(\mu, s\right)$.*

Corollary 3 makes two claims. The first claim is that, across all stable allocations, there is one which all workers prefer over any other. In other words, workers' interests are globally aligned. The second claim is that this worker-optimal stable allocation is efficient. In other words, solidarity with workers is consistent with economic efficiency. This is because departures from efficiency arise from firms exploiting their monopsonistic labor market power; when they do not exploit that power, workers are better off.

5.2 Firm welfare and worker welfare

As with workers, we can define a binary relation representing firms preferences. We define $\succeq_{\mathbf{F}}$ as:

$$\left(\mu, s\right) \succeq_{\mathbf{F}} \left(\mu', s'\right) \iff \forall F : \pi_F \left(\left|\mu(F)\right|, s(F)\right) \geq \pi_F \left(\left|\mu'(F)\right|, s'(F)\right).$$

If one stable allocation is preferred by all workers over another, then no firm could strictly prefer the former allocation to the latter: if they did, they could block the latter allocation by forming a coalition with the workers to which they are matched in the former coalition. This is expressed in the following lemma.

Lemma 6. *For any two stable allocations $\left(\mu, s\right), \left(\mu', s'\right)$: $\left(\mu, s\right) \succeq_{\mathbf{W}} \left(\mu', s'\right) \implies \left(\mu', s'\right) \succeq_{\mathbf{F}} \left(\mu, s\right)$.*

Given Corollary 3, Lemma 6 implies the existence of a worker-optimal, firm-pessimal stable allocation, which has higher salaries than any other stable allocation and is efficient. We summarize these facts in the following theorem.

Theorem 2. *There exists a stable allocation $\left(\mu^*, s^*\right)$ such that, in comparison to any other stable allocation $\left(\mu, s\right)$:*

1. *it is more efficient: value$\left(\mu^*\right) \geq$ value$\left(\mu\right)$;*
2. *it has greater salaries: $s^* \geqq s$;*
3. *it is preferred by workers: $\left(\mu^*, s^*\right) \succeq_{\mathbf{W}} \left(\mu, s\right)$; and*
4. *it is less preferred by firms: $\left(\mu, s\right) \succeq_{\mathbf{F}} \left(\mu^*, s^*\right)$.*

In summary: across stable allocations, worker interests are aligned with efficiency whereas firm interests are not. Beyond its normative power, Theorem 2 has interesting implications for market design. It suggests three indicators of market efficiency that can be targeted (provided that the labor market remains stable): (1) high salaries, (2) worker welfare, and (3) firms making minimal profits. Theorem 2 also gives us some insight into the cause of labor market inefficiency: inefficiency arises because firms prefer it.

Theorem 2 also may explain aggregate unemployment. Generically, a worker employed at the worker-optimal stable allocation will strictly prefer that allocation over unemployment. (Given individual rationality, the only exception is the knife-edge case in which her maximal salary exactly offsets her disamenity of employment.) This means that, generically, a worker employed in any stable allocation will be employed in the worker-optimal efficient stable allocation. The worker-optimal efficient stable allocation will thus have the lowest unemployment level of any stable allocation. By exploiting their monopsony power, firms create aggregate unemployment.

Perhaps surprisingly, the converse of Lemma 6 does not hold: given two stable allocations, it is possible that one is better for all firms and a subset of the workers. That is because in many-to-one matching, a worker might not be able to form a blocking coalition with only a firm: she might need the support of her fellow workers as well. We require that each firm pays all its workers the same salary, and thus the salary sufficiently generous to earn the support of her fellow workers may cost her the support of the firm. The following example illustrates this phenomenon.

Example 2 (a worker prefers the firm-preferred allocation). $\mathbf{F} = \{F_1, F_2\}$. $y_{F_1}(N) = 5N$. $y_{F_2}(N) = 3N$. $\mathbf{W} = \{w_1, w_2, w_3\}$. *Amenities are given by this table:*

	F_1	F_2
w_1	5	0
w_2	−1	0
w_3	0	5

We consider two stable allocations: (μ^1, s^1) and (μ^2, s^2):

$$\mu^1 = \begin{pmatrix} w_1 & w_2 & w_3 \\ F_1 & F_1 & F_2 \end{pmatrix},\ s^1(F_1) = 5,\ s^1(F_2) = 1; \quad \mu^2 = \begin{pmatrix} w_1 & w_2 & w_3 \\ F_1 & F_2 & F_2 \end{pmatrix},\ s^2(F_1) = 0,\ s^2(F_2) = 2.$$

The corresponding profits are

$$\pi_{F_1}\left(\left|\mu^1(F_1)\right|, s^1(F_1)\right) = 2 \times (5-5) = 0, \quad \pi_{F_2}\left(\left|\mu^1(F_2)\right|, s^1(F_2)\right) = 1 \times (3-1) = 2;$$
$$\pi_{F_1}\left(\left|\mu^2(F_1)\right|, s^2(F_1)\right) = 1 \times (5-0) = 5, \quad \pi_{F_2}\left(\left|\mu^2(F_2)\right|, s^2(F_2)\right) = 2 \times (3-2) = 2.$$

Thus $\left(\mu^2, s^2\right) \succeq_{\mathbf{F}} \left(\mu^1, s^1\right)$. However, worker w_3 strictly prefers $\left(\mu^2, s^2\right)$ to $\left(\mu^1, s^1\right)$.

We now confirm that both allocations are stable. Both have No Firing and No Envy. The only plausible threat to $\left(\mu^1, s^1\right)$ having No Poaching would be if F_2 poaches w_2. This would require that F_2 pay $s' \geq 5-1$ which is greater than its marginal product 3. The only plausible threat to $\left(\mu^2, s^2\right)$ having No Poaching would be if F_1 poaches w_2. This would require that F_1 pay $s' \geq 3$, making profit no greater than 4. This is less than its profit under $\left(\mu^2, s^2\right)$.

The intuition behind Example 2 is that moving from one stable allocation to another can make some firms grow while making others shrink, in a manner that all firms benefit. The growing firm increases its salaries to attract marginal workers. This benefits inframarginal workers. The shrinking firms decrease their salaries, which harms those firms' workers.

A final point to emphasise about Example 2 is its consistency with Proposition 5 and Lemma 5. Neither $\left(\mu^1, s^1\right)$ nor $\left(\mu^2, s^2\right)$ is the worker-optimal allocation. In the worker optimal allocation $\left(\mu^*, s^*\right)$: $s^*(F_1) = 5$ and $s^*(F_2) = 3$. Neither firm makes profits at $\left(\mu^*, s^*\right)$, and all workers are at least as well off as they are in $\left(\mu^1, s^1\right)$ and $\left(\mu^2, s^2\right)$.

Example 2 implies that payoffs in our model lack the dual lattice structure commonly found in matching models (Knuth, 1976; Shapley & Shubik, 1971; Hatfield & Milgrom, 2005; Blair, 1988). While Theorem 2 implies that workers' preferences are aligned *globally* – there is some allocation which is best for all of them – they need not be aligned *locally*. We show in Appendix B that neither worker nor firm payoffs form a lattice.

6 Designing a Centralized Labor Market

This paper asks how a centralized matching mechanism can make monopsonistic labor markets more efficient. Production functions and amenities might not be directly observed by the mechanism, in which case these must be elicited. In this section, we ask whether a strategyproof mechanism can implement an efficient stable allocation. We show that such a mechanism cannot elicit firms' production functions, but it can elicit workers' amenities.

We first consider eliciting firms' production functions:

Proposition 6. *When firms' production functions are private information, there may not exist a dominant strategy mechanism that implements an efficient stable allocation.*

Proposition 6 can be proved with the following example.

Example 3 (a slightly more general simple monopsony). $\mathbf{F} = \{F\}$. $y_F(N) = \beta N$; $\beta \in \{1, 6\}$. $\mathbf{W} = \{w_1, w_2\}$. $\alpha_{w_1}(F) = 0$. $\alpha_{w_2}(F) = -4$.

Example 3 generalizes Example 1, which was introduced in Section 2. When $\beta = 6$, Example 3 is identical to Example 1, and we showed earlier that the efficient matching $\mu^6(w_1) = \mu^6(w_2) = F$ will be supported by a salary $s^6(F) \in [4, 6]$. When $\beta = 1$, the efficient matching is $\mu^1(w_1) = F; \mu^1(w_2) = \emptyset$. This will be supported by a salary $s^1(F) \in [0, 1]$.

Consider the mechanism design problem of implementing (μ^6, s^6) when $\beta = 6$ and (μ^1, s^1) when $\beta = 1$; where the value of β is known only to the firm. By the revelation principle, we can consider only mechanisms in which the firm reports its type. If it reports $\beta = 1$, it will be matched to one worker, pay salary $s^1(F) \in [0, 1]$, and thus receive profit $\beta - s^1(F) \leq \beta - 1$. If it reports $\beta = 6$, it will be matched to two workers, pay salary $s^3(F) \in [4, 6]$, and thus receive profit $2\left(\beta - s^3(F)\right) \geq 2(\beta - 4)$. In particular, when the true value of the firm's productivity is $\beta = 6$, it would receive at least profit 5 from reporting $\beta = 1$ while it would receive at most profit 4 from reporting $\beta = 6$. It will thus not report truthfully. By the revelation principle, this constitutes a proof of Proposition 6.

In some contexts, firms' production functions will be known to the mechanism designer, while the amenities that workers receive from firms will not. Our next result shows that, in such cases, the mechanism designer can implement the worker-optimal allocation, which Theorem 2 tells us is efficient.

Theorem 3. *When firms' production functions are public information, there exists a strategyproof mechanism that implements an efficient stable allocation.*

The intuition for Theorem 3 is as follows. By Theorem 2, there exists a worker-optimal allocation with maximal salaries across all stable allocations. Thus for each firm F, either the No Firing constraint $s(F) \leq \Delta^-_\mu(F)$ is binding, or there exists a worker $w \notin \mu(F)$ such that the No Envy constraint $s(F) + \alpha_w(F) \leq s\left(\mu(w)\right) + \alpha_w\left(\mu(w)\right)$ is binding. A worker cannot affect the No Firing constraint by misreporting their amenities. A worker also cannot affect the No Envy constraints for the firm to which they are matched. It follows that no worker can increase salaries at the firm to which they are matched. By misreporting, a worker *can* move themselves to another firm, but by No Envy doing so cannot make them better off. Workers thus have no incentive to misreport their amenities.

This sketch suggests representing our mechanism as a VCG mechanism *a la* Green and Laffont (1977), where the firm-worker matching is the 'public good' chosen by the workers. As with our mechanism, each player in a VCG mechanism receives a transfer that depends only on the reports of the other players. This ensures that players lack an incentive to inflate their transfers by misreporting.

Proposition 6 and Theorem 3 echo earlier results in the many-to-one matching literature: there is no mechanism which is strategyproof for both sides of the market; and implementing the worker-optimal matching is strategyproof for workers but not firms (Roth & Sotomayor, 1990). Our context has the added twist that the worker-optimal allocation is efficient while other allocations need not be.

An efficient matching can be implemented through a dominant strategy mechanism provided that the mechanism designer observes firms' production functions. Whether the designer can observe firms' production functions will depend on context. Recall that we require that workers enter into firms' production functions fungibly. This reduces the informational complexity of a firm's production function. In contexts where workers are fungible, such as pharmacies, manufacturing assembly lines, and the construction trades, production functions could plausibly be inferred from engineering or accounting data.

7 The Sources of Monopsonistic Inefficiencies

We have shown in the previous sections how restrictions on transfers *can* permit inefficient stable allocations. In this section, we explore more directly *why* these inefficiencies arise. Each of the next two subsections explores a condition which guarantees that all stable allocations will be efficient. These results illuminate which market conditions mediate monopsonistic inefficiencies. The third subsection reveals exactly how our restriction on transfers – i.e., the requirement that a firm pay all its workers the same salary – allows inefficient stable allocations to exist.

7.1 Common value amenities

As Example 1 made clear, the different amenities that workers receive from the same firm can cause monopsonistic inefficiencies. Proposition 7 formalizes this intuition by showing that, when there is no within-firm heterogeneity in amenities, all stable allocations are efficient.

We say that firm F has **common value amenities** if every worker receives the same amenity from working at F:

$$\forall w, w' \in \mathbf{W}: \alpha_w(F) = \alpha_{w'}(F).$$

Proposition 7. *If every firm has common value amenities, then every stable allocation is efficient.*

When all firms have common value amenities, the No Envy condition equalizes workers' utilities. This pins down the relationship between firms' salaries and firms' amenities. Lemmas 3 and 4 tell us that, if a stable allocation is inefficient, its value could be improved by an acyclic replacement chain. Given the relationship between firms' salaries and firms' amenities, and the relationship between firms' salaries and their marginal products, that replacement chain can be used to construct a blocking coalition for the allocation. This implies that the inefficient allocation could not have been stable.

This result exposes one source of monopsonistic inefficiency. Firms want to lower their salaries to price out 'expensive' workers, even when employing those workers is efficient. This incentive does not arise when firms have common value amenities since no worker is relatively more expensive than any other: if a firm lost one worker when it reduced its salary, it would lose them all. This intuition accords with the labor monopsony literature: differences in amenities across workers generate upward sloping labor supply curves that in turn generate monopsonistic inefficiencies.

7.2 Duplicate firms

In this subsection, we explore the effect of duplicating firms. As with the previous subsection, this will help us better understand how firms derive their distortionary market power.

We say that two firms $F \neq F'$ are **duplicates** if:

$$\forall N \in \mathbb{N} : y_F(N) = y_{F'}(N),$$
$$\text{and } \forall w \in \mathbf{W} : \alpha_w(F) = \alpha_w(F').$$

Proposition 8. *If every firm has a duplicate, then every stable allocation is efficient.*

The proof of Proposition 8 first notes that, by the No Envy condition, if two firms are duplicates they must pay the same salary. This means that, if two firms are duplicates, either could poach any worker from the other by paying an infinitesimally higher salary. No Poaching requires that doing so would be unprofitable, which implies that each firm F's salary be at least equal to their increased output from hiring another worker (i.e. $\Delta_\mu^+(F)$). In addition, the No Firing condition requires that firm F's salary be no less than the decrease in output from firing a marginal worker (i.e. $\Delta_\mu^-(F)$). Thus each firm will have a salary $s(F) \in \left[\Delta_\mu^+(F), \Delta_\mu^-(F)\right]$, which means that the allocation has Marginal Product Salaries. By Proposition 3, the matching must be efficient.

As is suggested by the proof, duplicate firms will fruitlessly compete over their shared pool of workers. That competition drives up salaries, and so the duplicate firms will lack the monopsonistic incentive to exclude expensive workers. In the absence of that monopsonistic incentive, the labor market can allocate workers efficiently. When firms lack duplicates they can exploit their heterogeneity to decrease salaries and increase profits. In doing so, they may distort the market, rendering the match inefficient.

In Appendix E, we ask what happens when each *worker* has a duplicate. In contrast to the results above, duplicating workers has no effect on the set of stable allocations (provided that firms' production functions are 'stretched' appropriately). Theorem 2 showed that workers prefer an undistorted market, and thus it is firm power, rather than worker power, which distorts the labor market. This subsection provides additional support for this principle: distortions are eliminated when each firm has a duplicate. When each worker has a duplicate, distortions remain.

7.3 How restricted transfers maintain distortions

Our final result demonstrates precisely how our restriction on transfers allows inefficient allocations to remain stable.

Proposition 9. *Consider an inefficient stable allocation (μ, s). There exists a salary s', a firm F, and a worker w such that $s' < \Delta_\mu^+(F)$ and w strictly prefers to work for F at salary s' than for $\mu(w)$ at salary $s(\mu(w))$.*

To prove Proposition 9, we return to the replacement chain machinery which we introduced in Section 4. By Lemmas 3 and 4, for every inefficient matching, there must be an acyclic, value-increasing replacement chain from that matching to the efficient matching. Because the replacement chain is acyclic, it increases the size of the last firm. Because the replacement chain is value-increasing, the marginal product of moving the last worker to the last firm must be greater than the salary needed to induce the worker to move.

Proposition 9 tells us that, in every inefficient stable allocation, some worker would be willing to work at some new firm for a salary less than her marginal product. The firm refuses to hire her, because doing so would require that the firm increase the pay of its existing workers.

8 Conclusion

For many firms, employing more workers would require paying higher wages. In classic job matching models, these higher wages need only be paid to the firm's new workers; the firm can leave its existing workers' wages unchanged. In other words, these models assume that labor markets exhibit perfect price discrimination. This paper has argued that, without price discrimination, labor markets can suffer from monopsonistic distortions.

Why do we describe this inefficiency as 'monopsonistic'? As in traditional labor monopsony models, a marginal worker would be willing to work at some firm for a salary less than her marginal product. That firm, however, refuses to employ her, because doing so would require that it pay its existing workers more.

This model of monopsony has proven insightful. We showed that only firm sizes are distorted: conditional on firm sizes, the matching of workers to firms is efficient. We showed that monopsonistic distortions are beneficial to firms and are harmful to workers. Further, we showed that monopsonistic distortions stem from firms exploiting two-sided heterogeneity: when each firm has a duplicate, or when each firm's amenities are equally appreciated by all workers, every stable allocation is efficient.

We have used this characterization to assess a potential solution to monopsonistic distortions: a centralized matching mechanism. To be successful, such a mechanism would have to implement an allocation that is both efficient and stable. We showed that one efficient allocation is indeed always stable. To identify an efficient allocation, such a mechanism may also need to elicit the non-pecuniary amenities that workers receive from employment, or elicit the technologies with which firms produce output. We showed that a strategyproof mechanism can elicit the former but not the latter.

Ideally, a lighter-touch policy could implement an efficient allocation. For example, in a pure monopsony, a minimum wage can incentivize efficient employment. However, with multiple firms an efficient minimum wage must be firm-specific: a uniform minimum wage cannot generally allocate labor efficiently between a high amenity, low-productivity firm and a low-amenity, high-productivity firm.

While a market designer could impose firm-specific minimum wages, doing so is not trivial. We show that stable allocations with Marginal Product Salaries will be efficient. However, to know what the right marginal products are, the market designer must know what the efficient matching is. Given that these minimum wages require that the designer know an efficient matching, it seems simpler to just impose that matching directly.

Our baseline model requires that workers be fungible in production. When workers are not fungible in production, but salaries are still required to be constant within a firm, a stable allocation may not exist. On the other hand, our results would trivially extend to cover labor markets comprising multiple occupations provided that firms could set different salaries to different occupations, that each worker could only work in one occupation, and that firms' production was additively separable over occupations. Tractable models with heterogeneous worker productivity would be a valuable goal for future work. Similarly, it would be interesting to derive the minimal restrictions on worker heterogeneity, given restricted transfers, such that a stable allocation will always exist.

Monopsony is a persistent feature of many labor markets. This paper has characterized the problems monopsony can cause and begun to address how these problems can be ameliorated.

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A Proofs of main results

Lemma 1. *Firm F treats workers as gross substitutes if any only if y_F has decreasing differences.*

Proof. If a firm's production function has decreasing differences, then that firm treats workers as gross substitutes:

$$\begin{aligned} N \in \underset{M\leq N}{\arg\max}\, \pi_F(M,s) &\implies s \leq y_F(N) - y_F(N-1) \\ &\implies \forall M \leq N:\ s \leq y_F(M) - y_F(M-1) \text{ (by decreasing differences)} \\ &\implies N-1 \in \underset{M\leq N-1}{\arg\max}\, \pi_F(M,s). \end{aligned}$$

The proof of the converse is only slightly more involved. Assume towards a contradiction that some firm's production function does not have decreasing differences:

$$\exists F \in \mathbf{F}, N \in \mathbb{N} \text{ such that } y_F(N) - y_F(N-1) > y_F(N-1) - y_F(N-2). \tag{1}$$

Without loss of generality, let N be the smallest integer such that the inequality (1) holds for firm F.

Consider the salary $s_\epsilon \equiv y_F(N-1) - y_F(N-2) + \epsilon$, where $\epsilon \geq 0$. Given that $y_F(0) = 0$:

$$y_F(N) = \sum_{i=1}^{N} \left[y_F(i) - y_F(i-1) \right],$$

and thus firm F's profit from employing N workers at salary s_ϵ is

$$\pi_F(N, s_\epsilon) = \sum_{i=1}^{N} \left[y_F(i) - y_F(i-1) - s_\epsilon \right].$$

This implies that when $\epsilon = 0$, the marginal profit from hiring the ith worker is

$$\begin{aligned} &y_F(i) - y_F(i-1) - s_0 \\ =\ &y_F(i) - y_F(i-1) - \big(y_F(N-1) - y_F(N-2)\big). \end{aligned}$$

By the assumption that N is the smallest integer such that inequality (1) holds, the firm's marginal profit will be weakly positive for $i < N$ and strictly positive for $i = N$. Thus:

$$\forall M < N:\ \pi_F(N, s_0) > \pi_F(M, s_0).$$

Moreover, the continuity of the profit function with respect to the salary implies that the inequality will continue to hold for all ϵ sufficiently close to 0:

$$\exists \epsilon > 0 : \forall M < N:\ \pi_F(N, s_\epsilon) > \pi_F(M, s_\epsilon). \tag{2}$$

However, for any $\epsilon > 0$, the marginal profit from hiring the $(N-1)$th worker is negative. Thus:

$$\forall \epsilon > 0:\ \pi_F(N-1, s_\epsilon) < \pi_F(N-2, s_\epsilon). \tag{3}$$

In combination, expressions (2) and (3) contradict the gross substitutes assumption. □

Proposition 2. *An allocation is stable if and only if it has No Envy, No Firing, and No Poaching.*

Proof. We prove Proposition 2 in six steps.

Step 1: If an allocation is stable, then it has No Firing.

Proof of Step 1: If $\mu(F) = \emptyset$, then by definition $\Delta_{\mu}^{-}(F) = \infty$, and thus it is trivial that (μ, s) has No Firing. Assume $\mu(F) \neq \emptyset$. If the allocation lacks No Firing, then the firm is making a loss on its marginal worker. It would be better off being matched to one worker less at the same salary:

$$\forall w \in \mu(F) : \pi_F\left(\left|\mu(F) \setminus \{w\}\right|, s(F)\right) > \pi_F\left(\left|\mu(F)\right|, s(F)\right).$$

If $\left|\mu(F)\right| = 1$, the left hand size of the above inequality is 0, and thus the candidate allocation is not individually rational for the firm. If $\left|\mu(F)\right| > 1$, then for any worker $w \in \mu(F)$: $\left(F, \mu(F) \setminus \{w\}, s(F)\right)$ blocks (μ, s) because firm F would be strictly better off in the blocking coalition and every worker in $\mu(F) \setminus \{w\}$ would be indifferent. Thus an allocation without No Firing cannot be stable.

Step 2: If an allocation is stable, then it has No Envy.

Proof of Step 2: Assume towards a contradiction that (μ, s) is stable but does not have No Envy:

$$\exists w \in \mathbf{W}, F \in \mathbf{F} \cup \{\emptyset\} : u_w\left(\mu(w), s(\mu(w))\right) < u_w(F, s(F)).$$

If $F = \emptyset$, the right hand side of that inequality is 0, and thus (μ, s) is not individually rational for w. Thus if (μ, s) is stable but lacks No Envy, then $F \neq \emptyset$.

If $\mu(F) \neq \emptyset$, since workers are fungible, the firm could indifferently switch out any worker in its employ, keeping the same salary. Thus, for any $w' \in \mu(F)$, $\left(F, \mu(F) \cup \{w\} \setminus \{w'\}, s(F)\right)$ blocks (μ, s).

If $\mu(F) = \emptyset$, then $s(F) = y_F(1)$ by definition. Thus:

$$\alpha_w(F) + y_F(1) > s\left(\mu(w)\right) + \alpha_w\left(\mu(w)\right),$$

while

$$\pi_F\left(|\{w\}|, y_F(1)\right) = \pi_F\left(\left|\mu(F)\right|, s(F)\right) = 0.$$

Thus $\left(F, \{w\}, y_F(1)\right)$ blocks (μ, s).

Step 3: If an allocation is stable, then it has No Poaching.

Proof of Step 3: Assume towards a contradiction that (μ, s) is stable but lacks No Poaching: there exists F, $s' > s(F)$, and $L \in \mathbb{N}$ with $0 < L \leq L_F\left(s', s\right)$ such that

$$\pi_F\left(L, s'\right) = y_F(L) - Ls' \geq \pi_F\left(\left|\mu(F)\right|, s(F)\right) = y_F\left(\left|\mu(F)\right|\right) - \left|\mu(F)\right| s(F).$$

If $\mu(F) = \emptyset$, then $s' > s(F) = y_F(1)$, meaning firm F would make negative profit if matched to one worker. By decreasing differences, F makes a negative profit when matched to any number of workers. Thus $L = 0$. This contradicts $L > 0$.

If $\mu(F) \neq \emptyset$, every worker $w \in \mu(F)$ is strictly better off being employed at salary s' rather than salary $s(F)$. By assumption, there is a size-L set of workers C who weakly prefer being matched to F at salary s' over their current match. As $L > 0$ we can require that $\mu(F) \cap C \neq \emptyset$. Thus there is a worker in C who strictly prefers being matched to F at salary s' over their current match. By assumption, firm F weakly prefers being matched to C at salary s' over being matched to $\mu(F)$ at salary $s(F)$. Thus $\left(F, C, s'\right)$ blocks (μ, s).

Step 4: If an allocation has No Envy, then it is individually rational for workers.

Proof of Step 4: If an allocation has No Envy, then $\forall w$:

$$u_w\left(\mu(w), s(\mu(w))\right) \geq u_w(\emptyset, 0) = 0.$$

Step 5: If an allocation has No Firing, then it is individually rational for firms.

Proof of Step 5: That an allocation with No Firing is individually rational for unmatched firms is trivial: if $\mu(F) = \emptyset$, then $\pi_F\left(\left|\mu(F)\right|, s(F)\right) = \pi_F\left(0, y_F(1)\right) = 0$.

An allocation having No Firing requires that for all matched firms F: $s(F) \leq y_F\left(\left|\mu(F)\right|\right) - y_F\left(\left|\mu(F)\right| - 1\right)$. Thus:

$$\begin{aligned}\pi_F\left(\left|\mu(F)\right|, s(F)\right) &= \sum_{i=1}^{|\mu(F)|} \left[y_F(i) - y_F(i-1) - s(F)\right] \\ &\geq \sum_{i=1}^{|\mu(F)|} \left[y_F\left(\left|\mu(F)\right|\right) - y_F\left(\left|\mu(F)\right| - 1\right) - s(F)\right] \quad \text{(by decreasing differences)} \\ &\geq 0.\end{aligned}$$

Step 6: If an allocation has No Envy, No Firing and No Poaching, then there are no coalitions that block it.

Proof of Step 6: Assume towards a contradiction that $\left(F, C, s'\right)$ blocks $\left(\mu, s\right)$, where C is a nonempty subset of $\mathbf{W}$. If $s' < s(F)$, any worker previously matched to F would be strictly worse off in the coalition. Thus $C \cap \mu(F) = \emptyset$. Consider a worker $w \in C$. Since $\left(\mu, s\right)$ has No Envy,

$$\alpha_w\left(\mu(w)\right) + s\left(\mu(w)\right) \geq \alpha_w(F) + s(F) > \alpha_w(F) + s',$$

and thus the worker prefers the original allocation over being matched to F at salary s'.

Now consider the case $s' = s(F)$. Since $\left(\mu, s\right)$ has No Envy, $\forall w \in C$:

$$\alpha_w\left(\mu(w)\right) + s\left(\mu(w)\right) \geq \alpha_w(F) + s(F) = \alpha_w(F) + s',$$

and thus workers in C are at best indifferent between the original allocation and being matched to F at salary s'. Thus firm F must be strictly better off. Thus the firm would also be strictly better off being matched to C at salary slightly higher than s'. This contradicts that the allocation has No Poaching.

Finally consider the case $s' > s(F)$. As the firm is no worse off,

$$\pi_F\left(|C|, s'\right) \geq \pi_F\left(\left|\mu(F)\right|, s(F)\right).$$

As all workers in C are no worse off, $|C| \leq L_F\left(s', s\right)$. This contradicts that the allocation has No Poaching. □

Lemma 2. *An allocation with Marginal Product Salaries will also have No Firing and No Poaching.*

Proof. That an allocation with Marginal Product Salaries has No Firing follows from $s(F) \leq \Delta_\mu^-(F)$.

Fix a firm F and a salary $s' > s(F)$. Because $s(F) \geq \Delta_\mu^+(F)$:

$$s' > \Delta_\mu^+(F) \equiv y_F\left(\left|\mu(F)\right| + 1\right) - y_F\left(\left|\mu(F)\right|\right). \tag{4}$$

To show that the allocation has No Poaching, we must show that there exists no employment level $L \in \{1, \ldots, L_F(s_F, s)\}$ such that

$$\pi_F(L, s') \geq \pi_F\left(\left|\mu(F)\right|, s(F)\right).$$

First consider $L = \left|\mu(F)\right|$:

$$\begin{aligned}\pi_F\left(L, s'\right) &= \sum_{i=1}^{L} \left[y_F(i) - y_F(i-1) - s'\right] \\ &< \sum_{i=1}^{L} \left[y_F(i) - y_F(i-1) - s(F)\right] = \pi_F\left(\left|\mu(F)\right|, s(F)\right),\end{aligned}$$

where the inequality follows from $s' > s(F)$.

Next consider $L > \left|\mu(F)\right|$:

$$\pi_F\left(L, s'\right) = \sum_{i=1}^{L}\left[y_F(i) - y_F(i-1) - s'\right] = \sum_{i=1}^{|\mu(F)|}\left[y_F(i) - y_F(i-1) - s'\right] + \sum_{i=|\mu(F)|+1}^{L}\left[y_F(i) - y_F(i-1) - s'\right]$$
$$< \sum_{i=1}^{|\mu(F)|}\left[y_F(i) - y_F(i-1) - s(F)\right] + \sum_{i=|\mu(F)|+1}^{L}\left[y_F(i) - y_F(i-1) - s'\right], \qquad (5)$$

where the inequality follows from $s(F) < s'$. By decreasing differences:

$$\sum_{i=|\mu(F)|+1}^{L}\left[y_F(i) - y_F(i-1) - s'\right] \leq \sum_{i=|\mu(F)|+1}^{L}\left[y_F\left(\left|\mu(F)\right| + 1\right) - y_F\left(\left|\mu(F)\right|\right) - s'\right],$$

which, in combination with expression (4), implies that

$$\sum_{i=|\mu(F)|+1}^{L}\left[y_F(i) - y_F(i-1) - s'\right] < 0.$$

With inequality (5), this implies that

$$\pi_F\left(L, s'\right) < \sum_{i=1}^{|\mu(F)|}\left[y_F(i) - y_F(i-1) - s(F)\right] = \pi_F\left(\left|\mu(F)\right|, s(F)\right).$$

Finally, consider $L \in \left\{1, \ldots, \left|\mu(F)\right| - 1\right\}$.

$$\pi_F\left(\left|\mu(F)\right|, s(F)\right) = \sum_{i=1}^{|\mu(F)|}\left[y_F(i) - y_F(i-1) - s(F)\right] = \sum_{i=1}^{L}\left[y_F(i) - y_F(i-1) - s(F)\right] + \sum_{i=L+1}^{|\mu(F)|}\left[y_F(i) - y_F(i-1) - s(F)\right]$$
$$> \sum_{i=1}^{L}\left[y_F(i) - y_F(i-1) - s'\right] + \sum_{i=L+1}^{|\mu(F)|}\left[y_F(i) - y_F(i-1) - s(F)\right], \qquad (6)$$

where the inequality follows from $s(F) < s'$. By decreasing differences:

$$\sum_{i=L+1}^{|\mu(F)|}\left[y_F(i) - y_F(i-1) - s(F)\right] \geq \sum_{i=L+1}^{|\mu(F)|}\left[y_F\left(\left|\mu(F)\right|\right) - y_F\left(\left|\mu(F)\right| - 1\right) - s(F)\right].$$

Since (μ, s) has Marginal Product Salaries, $s(F) \leq y_F\left(\left|\mu(F)\right|\right) - y_F\left(\left|\mu(F)\right| - 1\right)$, and so

$$\sum_{i=L+1}^{|\mu(F)|}\left[y_F(i) - y_F(i-1) - s(F)\right] \geq 0.$$

With inequality (6), this implies that

$$\pi_F\left(\left|\mu(F)\right|, s(F)\right) > \sum_{i=1}^{L}\left[y_F(i) - y_F(i-1) - s'\right] = \pi_F\left(L, s'\right).$$

We conclude there can be no $s' > s(F)$ and $L \in \{1, \ldots, L_F(s_F, s)\}$ such that

$$\pi_F(L, s') \geq \pi_F\left(\left|\mu(F)\right|, s(F)\right).$$

□

Lemma 3. *Let μ and μ^* be matchings such that value$\left(\mu^*\right) >$ value$\left(\mu\right)$. There exists a replacement chain χ from μ to μ^* such that value$\left(\mu+\chi\right) >$ value$\left(\mu\right)$. Moreover, for each firm F: $\left|\left(\mu+\chi\right)(F)\right| \leq \max\left\{\left|\mu(F)\right|, \left|\mu^*(F)\right|\right\}$.*

Proof. Our proof is algorithmic. The state of an algorithm is a matching μ°. The algorithm is as follows:

1. $\mu^\circ \leftarrow \mu^*$. Then go to 2.
2. If there exists a cyclic replacement chain from μ to μ°, go to 3. Otherwise, go to 4.
3. Let χ be a cyclic replacement chain from μ to μ°. If value$\left(\mu+\chi\right) >$ value$\left(\mu\right)$, then χ is the required replacement chain and the algorithm can terminate. If not, set $\mu^\circ \leftarrow \mu^\circ - \chi$, and go to 2.
4. If there exists a replacement chain from μ to μ°, go to 5. Otherwise, terminate the algorithm.
5. Let χ be a maximal replacement chain from μ to μ°. (Such a chain exists at this point of the algorithm because there exists no cyclic replacement chain from μ to μ°.) If value$\left(\mu+\chi\right) >$ value$\left(\mu\right)$, then χ is the required replacement chain and the algorithm can terminate. If not, set $\mu^\circ \leftarrow \mu^\circ - \chi$, and go to 2.

When the algorithm does not terminate on lines 3 or 5, the state μ° becomes more similar to μ. When $\mu^\circ = \mu$, there is no replacement chain from μ to μ°, and so the algorithm will terminate at line 4. Because the matching is discrete, this means that the algorithm must terminate eventually.

Lemma 3 will hold provided that the algorithm never terminates at line 4, and that for each replacement chain χ proposed in lines 3 and 5: χ is a replacement chain from μ to μ^*, and for each firm F: $\left|\left(\mu+\chi\right)(F)\right| \leq \max\left\{\left|\mu(F)\right|, \left|\mu^*(F)\right|\right\}$. We prove these results in turn.

Step 1: The algorithm never terminates at line 4.

Proof of Step 1: We first show that value$\left(\mu^\circ\right)$ is weakly increasing as the algorithm proceeds. The state matching μ° is altered in lines 3 and 5. In line 3, the replacement chain $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ is cyclic, and thus it does not change the number of workers matched to any firm. Thus:

$$\text{value}\left(\mu^\circ\right) - \text{value}\left(\mu^\circ - \chi\right) = \sum_{k=0}^{N-1}\left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k)\right] = \text{value}\left(\mu+\chi\right) - \text{value}\left(\mu\right),$$

which is non-positive if the algorithm does not terminate. Thus value$\left(\mu^\circ\right) \leq$ value$\left(\mu^\circ - \chi\right)$.

In line 5, the replacement chain $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ is acyclic. It thus removes a worker from F_0 and adds a worker to F_N. As such:

$$\text{value}\left(\mu^\circ\right) - \text{value}\left(\mu^\circ - \chi\right) = \Delta^-_{\mu^\circ}(F_N) - \Delta^+_{\mu^\circ}(F_0) + \sum_{k=0}^{N-1}\left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k)\right];$$

$$\text{value}\left(\mu+\chi\right) - \text{value}\left(\mu\right) = \Delta^+_{\mu}(F_N) - \Delta^-_{\mu}(F_0) + \sum_{k=0}^{N-1}\left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k)\right].$$

Note that because χ is maximal, $\mu(F_N) \subset \mu^\circ(F_N)$ and $\mu^\circ(F_0) \subset \mu(F_0)$. (Recall that these subsets are strict.) Thus:

$$\begin{aligned}\Delta^-_{\mu^\circ}(F_N) &\equiv y_{F_N}\left(\left|\mu^\circ(F_N)\right|\right) - y_{F_N}\left(\left|\mu^\circ(F_N)\right| - 1\right)\\ &\leq y_{F_N}\left(\left|\mu(F_N)\right| + 1\right) - y_{F_N}\left(\left|\mu(F_N)\right|\right) = \Delta^+_{\mu}(F_N),\end{aligned}$$

where the second line follows from decreasing differences and $\left|\mu(F_N)\right| < \left|\mu^\circ(F_N)\right|$. Similarly, $\Delta^+_{\mu^\circ}(F_0) \geq \Delta^-_{\mu}(F_0)$. As such

$$\text{value}\left(\mu^\circ\right) - \text{value}\left(\mu^\circ - \chi\right) \leq \text{value}\left(\mu+\chi\right) - \text{value}\left(\mu\right),$$

which is non-positive if the algorithm does not terminate. Thus value$\left(\mu^\circ\right) \leq$ value$\left(\mu^\circ - \chi\right)$. This completes the proof that value$\left(\mu^\circ\right)$ is weakly increasing as the algorithm proceeds.

Initially, value$\left(\mu^\circ\right) =$ value$\left(\mu^*\right) >$ value$\left(\mu\right)$. Thus at every stage of the algorithm: value$\left(\mu^\circ\right) >$ value$\left(\mu\right)$. But if at line 4 there are no replacement chains from μ to μ°, then $\mu^\circ = \mu$ and thus value$\left(\mu^\circ\right) =$ value$\left(\mu\right)$. Thus there will always be at least one replacement chain from μ to μ°.

Step 2: Each replacement chain proposed in lines 3 and 5 is a replacement chain from μ to μ^*.

Proof of Step 2: Let $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ be a candidate replacement chain proposed in lines 3 or 5. We must show that $\forall k: w_k \in \mu(F_k) \cap \mu^*(F_{k+1})$. Given that χ is a replacement chain from μ to μ°: $w_k \in \mu(F_k) \cap \mu^\circ(F_{k+1})$. Given that $F_k \neq F_{k+1}$, that $F_k = \mu(w_k)$, and that $F_{k+1} = \mu^\circ(w_k)$, this implies that $\mu^\circ(w_k) \neq \mu(w_k)$. But when the state μ° is updated in lines 3 and 5, workers are only ever moved from their match in μ^* to their match in μ. Thus given that $\mu^\circ(w_k) \neq \mu(w_k)$ it must be the case that $\mu^\circ(w_k) = \mu^*(w_k)$. Given that $F_{k+1} = \mu^\circ(w_k)$, this completes the proof that, for each k,

$$w_k \in \mu(F_k) \cap \mu^*(F_{k+1}).$$

Step 3: As the algorithm runs, the state matching μ° is such that $\forall F: \left|\mu^\circ(F)\right| \leq \max\left\{\left|\mu(F)\right|, \left|\mu^*(F)\right|\right\}$.

Proof of Step 3: Assume towards a contradiction that, at some point of the algorithm, this is not the case. Given that μ° is initially set equal to μ^*, this requires that there be a point in the algorithm such that

$$\forall F: \left|\mu^\circ(F)\right| \leq \max\left\{\left|\mu(F)\right|, \left|\mu^*(F)\right|\right\}; \quad \exists F_0: \left|\left(\mu^\circ - \chi\right)(F_0)\right| > \max\left\{\left|\mu(F_0)\right|, \left|\mu^*(F_0)\right|\right\}, \tag{7}$$

where $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ is a replacement chain proposed in lines 3 or 5. Replacement chains proposed in line 3 are cyclic and thus do not change the number of workers employed at any firm. Thus it must be the case that χ is proposed in line 5.

Replacement chains proposed in line 5 are maximal from μ to μ°, and so $\mu^\circ(F_0) \subset \mu(F_0)$. Thus $\left|\mu^\circ(F_0)\right| < \left|\mu(F_0)\right|$. Subtracting the replacement chain χ moves at most one worker to firm F_0, and so $\left|\left(\mu^\circ - \chi\right)(F_0)\right| \leq \left|\mu(F_0)\right| \leq \max\left\{\left|\mu(F_0)\right|, \left|\mu^*(F_0)\right|\right\}$, which contradicts expression (7).

Step 4: Each replacement chain χ proposed in lines 3 and 5 is such that for each firm F: $\left|\left(\mu + \chi\right)(F)\right| \leq \max\left\{\left|\mu(F)\right|, \left|\mu^*(F)\right|\right\}$.

Proof of Step 4: This proof is similar to that of Step 3. Assume towards a contradiction that some replacement chain $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ is such that

$$\left|\left(\mu + \chi\right)(F_N)\right| > \max\left\{\left|\mu(F_N)\right|, \left|\mu^*(F_N)\right|\right\}. \tag{8}$$

Replacement chains proposed in line 3 are cyclic and thus do not change the number of workers employed at any firm. Thus it must be the case that χ is proposed in line 5.

Replacement chains proposed in line 5 are maximal from μ to μ°, and so $\mu(F_N) \subset \mu^\circ(F_N)$. Thus $\left|\mu(F_N)\right| < \left|\mu^\circ(F_N)\right|$. The replacement chain χ moves at most one worker to firm F_N, and so $\left|\left(\mu + \chi\right)(F_N)\right| \leq \left|\mu^\circ(F_N)\right|$. By Step 3, $\left|\mu^\circ(F_N)\right| \leq \max\left\{\left|\mu(F_N)\right|, \left|\mu^*(F_N)\right|\right\}$. Combining inequalities, $\left|\left(\mu + \chi\right)(F_N)\right| \leq \max\left\{\left|\mu(F_N)\right|, \left|\mu^*(F_N)\right|\right\}$. This contradicts expression (8).

□

Lemma 4. *Let $\left(\mu, s\right)$ be a stable allocation. There exists no cyclic replacement chain χ such that value$\left(\mu + \chi\right) >$ value$\left(\mu\right)$.*

Proof. Assume towards a contradiction that there exists a cyclic replacement chain χ such that value$(\mu+\chi) >$ value(μ). Given that χ is cyclic, it does not change the number of workers employed by any firm. Thus the only difference between value$(\mu+\chi)$ and value(μ) is workers' amenities. It thus follows from value$(\mu+\chi) >$ value(μ) that

$$\begin{aligned}
&\sum_{k=0}^{N-1}\left[\alpha_{w_k}(F_{k+1})-\alpha_{w_k}(F_k)\right]>0 \\
\implies &\sum_{k=0}^{N-1}\left[\alpha_{w_k}(F_{k+1})-\alpha_{w_k}(F_k)+s(F_{k+1})-s(F_k)\right]>0 && \text{because } F_N=F_0 \\
\implies &\exists k \text{ such that } \alpha_{w_k}(F_{k+1})-\alpha_{w_k}(F_k)+s(F_{k+1})-s(F_k)>0 \\
\implies &\exists k \text{ such that } \alpha_{w_k}(F_{k+1})+s(F_{k+1})>\alpha_{w_k}\left(\mu(w_k)\right)+s\left(\mu(w_k)\right) && \text{because } \mu(w_k)=F_k.
\end{aligned}$$

This implies (μ,s) lacks No Envy and thus cannot be stable. □

Proposition 3. *If (μ,s) is a stable allocation with Marginal Product Salaries, then μ is efficient.*

Proof. By Lemma 3, if μ is not efficient, there exists a replacement chain $\chi=\left((w_k)_{k=0}^{N-1},(F_k)_{k=0}^{N}\right)$ such that value$(\mu+\chi) >$ value(μ). By Lemma 4 and the fact that (μ,s) is stable, χ is acyclic. It follows that

$$\begin{aligned}
0 &< \text{value}(\mu+\chi)-\text{value}(\mu) \\
&= \Delta_\mu^+(F_N)-\Delta_\mu^-(F_0)+\sum_{k=0}^{N-1}\left[\alpha_{w_k}(F_{k+1})-\alpha_{w_k}(F_k)\right] \\
&\le \Delta_\mu^+(F_N)-\Delta_\mu^-(F_0)+\sum_{k=0}^{N-1}\left[s(F_k)-s(F_{k+1})\right] && \text{(by No Envy: } \alpha_{w_k}(F_{k+1})-\alpha_{w_k}(F_k)\le s(F_k)-s(F_{k+1})) \\
&= \Delta_\mu^+(F_N)-\Delta_\mu^-(F_0)+s(F_0)-s(F_N) \\
&\le 0 && \text{(by Marginal Product Salaries: } s(F_N)\ge\Delta_\mu^+(F_N) \\
& && \text{and } s(F_0)\le\Delta_\mu^-(F_0)).
\end{aligned}$$

$0<0$ is the desired contradiction. □

Proposition 4. *Every stable allocation has hedonic efficiency.*

Proof. Two replacement chains $\chi_A=\left((w_k)_{k=0}^{N_A-1},(F_k)_{k=0}^{N_A}\right)$ and $\chi_B=\left((w_l)_{l=0}^{N_B-1},(F_l)_{l=0}^{N_B}\right)$ are **worker-disjoint** if they share no worker: $\{w_k\}_{k=0}^{N_A-1}\cap\{w_l\}_{l=0}^{N_B-1}=\emptyset$.

Step 1: If χ_A and χ_B are worker-disjoint cyclic replacement chains from μ to μ', then

$$\text{value}(\mu+\chi_A+\chi_B)=\text{value}(\mu+\chi_A)+\text{value}(\mu+\chi_B)-\text{value}(\mu).$$

Proof of Step 1: Note that $(\mu+\chi_A+\chi_B)$ and $(\mu+\chi_B+\chi_A)$ are both well-defined since each replacement chain moves a worker from μ to μ' not yet moved by the other chain.

Let $\chi_A=\left((w_k)_{k=0}^{N_A-1},(F_k)_{k=0}^{N_A}\right)$ and let $\chi_B=\left((w_l)_{l=0}^{N_B-1},(F_l))_{l=0}^{N_B}\right)$. Since χ_A and χ_B are both cyclic replacement chains from μ to μ':

$$\text{value}\left(\mu+\chi_A\right)-\text{value}\left(\mu\right)=\sum_{w\in(w_k)_{k=0}^{N_A-1}}\left[\alpha_w\left(\mu'(w)\right)-\alpha_w\left(\mu(w)\right)\right];$$

$$\text{value}\left(\mu+\chi_B\right)-\text{value}\left(\mu\right)=\sum_{w\in(w_l)_{l=0}^{N_B-1}}\left[\alpha_w\left(\mu'(w)\right)-\alpha_w\left(\mu(w)\right)\right];$$

$$\text{value}\left(\mu+\chi_A+\chi_B\right)-\text{value}\left(\mu\right)=\sum_{w\in(w_k)_{k=0}^{N_A-1}\cup(w_l)_{l=0}^{N_B-1}}\left[\alpha_w\left(\mu'(w)\right)-\alpha_w\left(\mu(w)\right)\right].$$

Since χ_A and χ_B are worker-disjoint, $(w_k)_{k=0}^{N_A-1}\cap(w_l)_{l=0}^{N_B-1}=\emptyset$, which implies that:

$$\sum_{w\in(w_k)_{k=0}^{N_A-1}\cup(w_l)_{l=0}^{N_B-1}}\left[\alpha_w\left(\mu'(w)\right)-\alpha_w\left(\mu(w)\right)\right]=\sum_{w\in(w_k)_{k=0}^{N_A-1}}\left[\alpha_w\left(\mu'(w)\right)-\alpha_w\left(\mu(w)\right)\right]+\sum_{w\in(w_l)_{l=0}^{N_B-1}}\left[\alpha_w\left(\mu'(w)\right)-\alpha_w\left(\mu(w)\right)\right],$$

Combining the above expressions implies that

$$\text{value}\left(\mu+\chi_A+\chi_B\right)-\text{value}\left(\mu\right)=\text{value}\left(\mu+\chi_A\right)+\text{value}\left(\mu+\chi_B\right)-2\cdot\text{value}\left(\mu\right).$$

Thus $\text{value}\left(\mu+\chi_A+\chi_B\right)=\text{value}\left(\mu+\chi_A\right)+\text{value}\left(\mu+\chi_B\right)-\text{value}\left(\mu\right)$.

Step 2: Every stable allocation has hedonic efficiency.

Proof of Step 2: Assume towards a contradiction that $\left(\mu^\circ,s^\circ\right)$ is a stable allocation and μ° lacks hedonic efficiency. Let

$$\mu^*\in\underset{\mu\text{ s.t.}\forall F:\left|\mu(F)\right|=\left|\mu^\circ(F)\right|}{\arg\max}\left\{\sum_{w\in\mathbf{W}}\alpha_w\left(\mu(w)\right)\right\}$$

be a matching with the same firm sizes as μ° but which does have hedonic efficiency.

Select an arbitrary worker w_0 for whom $\mu^\circ(w_0)\neq\mu^*(w_0)$; let $F_0=\mu^\circ(w_0)$ and let $F_1=\mu^*(w_0)$. For every firm F: $\left|\mu^\circ(F)\right|=\left|\mu^*(F)\right|$. Thus, there must be some worker $w_1\in\mu^\circ(F_1)$ such that $\mu^\circ(w_1)\neq\mu^*(w_1)$. We can iteratively continue to identify new worker-firm pairs w_j,F_j such that $w_j\in\mu^\circ(F_j)\cap\mu^*(F_{j+1})$. Because the number of firms is finite we must eventually find a firm F_N such that $F_N=F_i$ with $i<N$. We have constructed the cyclic replacement chain $\chi_1=\left(\left(w_j\right)_{j=i}^{N-1},\left(F_j\right)_{j=i}^{N}\right)$. Now repeat the above process to find a sequence of cyclic worker-disjoint replacement chains $\{\chi_m\}_{m=1}^{M}$ from μ° to μ^* such that $\left(\mu^\circ+\chi_1+\chi_2+...+\chi_M\right)=\mu^*$. Thus

$$\text{value}\left(\mu^*\right)=\text{value}\left(\mu^\circ+\chi_1+\chi_2+...+\chi_M\right).\tag{9}$$

Iterating Step 1 implies that

$$\text{value}\left(\mu^\circ+\chi_1+\chi_2+...+\chi_M\right)=\text{value}\left(\mu^\circ+\chi_1\right)+\text{value}\left(\mu^\circ+\chi_2\right)+...+\text{value}\left(\mu^\circ+\chi_M\right)-(M-1)\cdot\text{value}\left(\mu^\circ\right).\tag{10}$$

Each χ_m is cyclic and thus, by Lemma 4 , for all $m\in\{1,...,M\}$: $\text{value}\left(\mu^\circ+\chi_m\right)\leq\text{value}\left(\mu^\circ\right)$. With equations (9) and (10), this implies that

$$\text{value}\left(\mu^*\right)\leq M\cdot\text{value}\left(\mu^\circ\right)-(M-1)\cdot\text{value}\left(\mu^\circ\right)=\text{value}\left(\mu^\circ\right).$$

This contradicts the assumption that $\text{value}\left(\mu^\circ\right)<\text{value}\left(\mu^*\right)$. □

Theorem 1. *Every efficient matching is in a stable allocation.*

Proof. We construct the salary schedule required by Corollary 2 by considering an auxiliary one-to-one matching game in which each worker is matched to a job *opening*. This is a Shapley and Shubik (1971) assignment game, and thus the efficient matching of workers to openings is in the core of this auxiliary game. The proof is completed by showing that, salaries constructed from the worker-optimal core payoffs are sufficiently high for Marginal Product Salaries to hold. In what follows, we assume that all workers are matched to some firm; this assumption is without loss of generality because unmatched workers can be thought of as matched to a firm $\emptyset$ with $\forall N : y_{\emptyset}(N) = 0$ and $\forall w : \alpha_w(\emptyset) = 0$.

The job assignment game. Fix an efficient matching μ. Given that matching, the job assignment game is defined as follows.

Players. For each firm F, construct $\max\left\{\left|\mu(F)\right|, 1\right\}$ openings. The total number of openings is thus $|\mathbf{W}| + \left|\{F \in \mathbf{F} : \mu(F) = \emptyset\}\right|$. For each opening o, let $F(o)$ denote the associated firm. Let $\mathbf{O}$ denote the set of openings. The players of the job assignment game are $\mathbf{O} \cup \mathbf{W}$.

Valuations. The value of opening o being assigned to worker w is defined as

$$a_{ow} = \begin{cases} \Delta_{\mu}^{-}(F(o)) + \alpha_w(F(o)) & \text{if } \mu(F(o)) \neq \emptyset; \\ \Delta_{\mu}^{+}(F(o)) + \alpha_w(F(o)) & \text{if } \mu(F(o)) = \emptyset. \end{cases}$$

Solution concepts. An **assignment** x comprises $|\mathbf{O} \times \mathbf{W}|$ real numbers x_{ow} such that

- $\forall o \in \mathbf{O}, w \in \mathbf{W}$: $x_{ow} \in [0,1]$,
- $\forall o \in \mathbf{O}$: $\sum_{w \in \mathbf{W}} x_{ow} \leq 1$,
- $\forall w \in \mathbf{W}$: $\sum_{o \in \mathbf{O}} x_{ow} \leq 1$.

An assignment x^* is **optimal** if $x^* \in \arg\max_x \sum_{o,w \in \mathbf{O} \times \mathbf{W}} x_{ow} a_{ow}$. A pair of vectors $u \in \mathbb{R}^{|\mathbf{O}|}$, $v \in \mathbb{R}^{|\mathbf{W}|}$ are **feasible payoffs** if there exists an assignment x such that $\sum_{o \in \mathbf{O}} u_o + \sum_{w \in \mathbf{W}} v_w = \sum_{o,w \in \mathbf{O} \times \mathbf{W}} x_{ow} a_{ow}$. We then say that (u, v) is **compatible** with x. A feasible payoff u, v is **core** if $u \geqq 0, v \geqq 0$ and $\forall o \in \mathbf{O}, w \in \mathbf{W}$: $u_o + v_w \geq a_{ow}$. (Shapley and Shubik show that this definition also rules out larger defecting coalitions.)

Shapley and Shubik show that an optimal assignment will be integer-valued; for simplicity, we consider only integer-valued assignments.

Isomorphisms. The assignment x is **isomorphic** to a matching m if $\forall w, o : x_{ow} = 1 \iff m(w) = F(o)$. Given the job assignment game for μ, we are interested in the subset of feasible payoffs P_μ which are compatible with some assignment isomorphic to μ:

$$\begin{aligned} P_\mu \equiv \{(u, v) :& (u, v) \text{ are core payoffs and there exists a feasible assignment} \\ & x \text{ such that } (u, v) \text{ is compatible with } x\text{, and } x \text{ is isomorphic to } \mu\}. \end{aligned}$$

We prove Theorem 1 in nine steps.

Step 1: Any assignment isomorphic to μ is optimal.

Proof of Step 1: Let x be isomorphic to μ and assume towards a contradiction that x is not optimal. That means there exists another feasible assignment x' such that

$$\sum_{o,w \in \mathbf{O} \times \mathbf{W}} x'_{ow} a_{ow} > \sum_{o,w \in \mathbf{O} \times \mathbf{W}} x_{ow} a_{ow}. \tag{11}$$

Consider the *linearized labor market* comprising the same firms and workers as the original labor market but with firms' production functions $y_F(\cdot)$ replaced with linear production functions $y_F^\circ(\cdot)$, defined by

$$y_F^\circ(N) \equiv \begin{cases} N\Delta_\mu^-(F) & \text{if } \mu(F) \neq \emptyset; \\ N\Delta_\mu^+(F) & \text{if } \mu(F) = \emptyset. \end{cases}$$

Amenities in the linearized labor market are identical to those in the original labor market. For a matching m, let value$^\circ(m)$ be the value of m defined using the linearized production functions y_F° rather than the original production functions y_F. (We will continue to use value(m) for the value of m defined using the original production functions y_F.) Let μ' be the matching to which the assignment x' is isomorphic.

The linear production functions mean that the function value$^\circ(m)$ is additively separable over worker-firm pairs, and thus value$^\circ\left(\mu\right) = \sum_{o,w\in\mathbf{O}\times\mathbf{W}} x_{ow}a_{ow}$, and value$^\circ\left(\mu'\right) = \sum_{o,w\in\mathbf{O}\times\mathbf{W}} x'_{ow}a_{ow}$. By inequality (11), this means that value$^\circ\left(\mu'\right) >$ value$^\circ\left(\mu\right)$. By Lemma 3, there must be a replacement chain χ from μ to μ' such that

$$\text{value}^\circ\left(\mu+\chi\right) > \text{value}^\circ\left(\mu\right) \tag{12}$$

and that for each firm F:

$$\left|\left(\mu+\chi\right)(F)\right| \leq \max\left\{\left|\mu(F)\right|, \left|\mu'(F)\right|\right\}. \tag{13}$$

The assignment x' is feasible in the job assignment game for μ. Thus for every firm F, if $\mu(F) \neq \emptyset$ then $\left|\mu'(F)\right| \leq \left|\mu(F)\right|$. Given that χ is a replacement chain from μ to μ', it follows from inequality (13) that for every firm F such that $\mu(F) \neq \emptyset$: $\left|\left(\mu+\chi\right)(F)\right| \in \left\{\left|\mu(F)\right|, \left|\mu(F)\right|-1\right\}$. Thus:

$$\begin{aligned}\forall F \in \mathbf{F} \text{ such that } \mu(F) \neq \emptyset:\ y_F^\circ\left(\left(\mu+\chi\right)(F)\right) - y_F^\circ\left(\mu(F)\right) &= -\mathbb{1}\left\{\left(\mu+\chi\right)(F) < \mu(F)\right\}\Delta_\mu^-(F) \\ &= y_F\left(\left(\mu+\chi\right)(F)\right) - y_F\left(\mu(F)\right).\end{aligned}$$

A replacement chain can only increase the number of workers matched to a firm by 1, and so if $\mu(F) = \emptyset$ then $\left|\left(\mu+\chi\right)(F)\right| \in \left\{\left|\mu(F)\right|, \left|\mu(F)\right|+1\right\}$. Thus:

$$\begin{aligned}\forall F \in \mathbf{F} \text{ such that } \mu(F) = \emptyset:\ y_F^\circ\left(\left(\mu+\chi\right)(F)\right) - y_F^\circ\left(\mu(F)\right) &= \mathbb{1}\left\{\left(\mu+\chi\right)(F) > \mu(F)\right\}\Delta_\mu^+(F) \\ &= y_F\left(\left(\mu+\chi\right)(F)\right) - y_F\left(\mu(F)\right).\end{aligned}$$

Combining results: for every firm F, $y_F^\circ\left(\left(\mu+\chi\right)(F)\right) - y_F^\circ\left(\mu(F)\right) = y_F\left(\left(\mu+\chi\right)(F)\right) - y_F\left(\mu(F)\right)$. Amenities in the linearized labor market are identical to those in the original labor market. Thus

$$\text{value}^\circ\left(\mu+\chi\right) - \text{value}^\circ\left(\mu\right) = \text{value}\left(\mu+\chi\right) - \text{value}\left(\mu\right).$$

With inequality (12), this implies that value$\left(\mu+\chi\right) >$ value$\left(\mu\right)$. This contradicts the efficiency of μ.

Step 2. $P_\mu \neq \emptyset$. In fact, P_μ contains **worker-optimal** payoffs: vectors $(u^*, v^*) \in P_\mu$ such that $\forall (u,v) \in P_\mu$: $v^* \geqq v$. Finally, for any assignment x compatible with some $(u,v) \in P_\mu$:

$$\forall o \in \mathbf{O}, w \in \mathbf{W}: x_{ow} = 1 \implies v_w + u_o = a_{ow}. \tag{14}$$

Proof of Step 2: Let x be an assignment isomorphic to μ. By Step 1, x is optimal. Shapley and Shubik's Theorem 2 claims that the "core of an assignment game is precisely the set of solutions of the LP dual of the corresponding assignment problem." This means that there exist core payoffs compatible with any optimal assignment

x. The existence of the worker-optimal payoffs is guaranteed by Shapley and Shubik's Theorem 3. Finally, the claim that $x_{ow} = 1 \implies v_w + u_o = a_{ow}$ is Roth and Sotomayor (1990)'s Lemma 8.5.

In what follows, let (u^*, v^*) denote the worker-optimal elements of P_μ.

Step 3: For all $(u, v) \in P_\mu$: $F(o) = F(o') \implies u_o = u_{o'}$.

Proof of Step 3: Fix payoffs $(u, v) \in P_\mu$. Assume towards a contradiction that there exists openings o, o' such that $o \neq o'$ with $F(o) = F(o')$ and $u_o > u_{o'}$.

Given that $(u, v) \in P_\mu$, there must be a feasible assignment x such that (u, v) is compatible with x, and x is isomorphic to μ. Given that firm $F(o)$ has at least two openings, $\mu(F(o)) \neq \emptyset$. Thus if x is isomorphic to μ, then both o and o' must be matched in x. Let them be matched to w, w' respectively. By expression (14) : $u_o + v_w = a_{ow}$. Moreover, given that $F(o) = F(o')$ it must be the case that $a_{ow} = \Delta^-_\mu\big(F(o)\big) + \alpha_w\big(F(o)\big) = \Delta^-_\mu\left(F(o')\right) + \alpha_w\left(F(o')\right) = a_{o'w}$. Thus we have

$$v_w + u_{o'} = v_w + u_o - (u_o - u_{o'}) = a_{ow} - (u_o - u_{o'}) = a_{o'w} - (u_o - u_{o'}) < a_{o'w},$$

contradicting the assumption that (u, v) are core payoffs. This completes the proof of Step 3.

Given steps 2 and 3, we can define salaries as follows. Let $o(F)$ be an arbitrary opening corresponding to F. Recall that (u^*, v^*) are the worker-optimal payoffs. Salaries are given by

$$s(F) = \begin{cases} \Delta^-_\mu(F) - u^*_{o(F)} & \text{if } \mu(F) \neq \emptyset; \\ \Delta^+_\mu(F) & \text{if } \mu(F) = \emptyset. \end{cases} \tag{15}$$

Step 4: For all $F:\ s(F) \leq \Delta^-_\mu(F)$.

Proof of Step 4: If $\mu(F) = \emptyset$, then $\Delta^-_\mu(F) = \infty$ by definition and thus $s(F) \leq \Delta^-_\mu(F)$. If $\mu(F) \neq \emptyset$, equation (15) requires that $s\big(F(o)\big) = \Delta^-_\mu\big(F(o)\big) - u^*_o$. Thus $u^*_o \geq 0$ implies that $\Delta^-_\mu\big(F(o)\big) - s\big(F(o)\big) \geq 0$.

Step 5: For all $w: v^*_w = s\left(\mu(w)\right) + \alpha_w\left(\mu(w)\right) \geq 0$.

Proof of Step 5: Let x be isomorphic to μ. For each worker w, there exists an opening o such that $x_{ow} = 1$. By expression (14), if $x_{ow} = 1$, then $v^*_w = a_{ow} - u^*_o$. Also, if $x_{ow} = 1$, then $\mu(F(o)) \neq \emptyset$, and so $s(F(o)) = \Delta^-_\mu(F(o)) - u^*_o$ and $a_{ow} = \Delta^-_\mu(F(o)) + \alpha_w(F(o))$. Finally, recall that (u^*, v^*) being core requires that $v^* \geqq 0$. Combining these expressions yields

$$v^*_w = s\big(F(o)\big) + \alpha_w\big(F(o)\big) \geq 0.$$

If $x_{ow} = 1$ then $F(o) = \mu(w)$. Thus $v^*_w = s\left(\mu(w)\right) + \alpha_w\left(\mu(w)\right) \geq 0$.

Step 6: For all $w \in \mathbf{W}, F \in \mathbf{F}:\ s\left(\mu(w)\right) + \alpha_w\left(\mu(w)\right) \geq s(F) + \alpha_w(F)$.

Proof of Step 6: Consider any $w \in \mathbf{W}, F \in \mathbf{F}$. Consider first the case where $\mu(F) \neq \emptyset$. Let $F = F(o)$ for some opening o. Because (u^*, v^*) is core,

$$u^*_o + v^*_w \geq a_{ow} = \Delta^-_\mu(F) + \alpha_w(F).$$

Note that $\mu(F) \neq \emptyset \implies u^*_o = \Delta^-_\mu(F) - s(F)$. From Step 5 we know that $v^*_w = s\left(\mu(w)\right) + \alpha_w\left(\mu(w)\right)$. Thus:

$$\begin{aligned} u^*_o + v^*_w &= \Delta^-_\mu(F) - s(F) + s\left(\mu(w)\right) + \alpha_w\left(\mu(w)\right) \geq \Delta^-_\mu(F) + \alpha_w(F) \\ &\implies s\left(\mu(w)\right) + \alpha_w\left(\mu(w)\right) \geq s(F) + \alpha_w(F). \end{aligned}$$

Now consider an unmatched firm F, i.e., $\mu(F) = \emptyset$. Continue to let $F = F(o)$ for some opening o. Now $a_{ow} = \Delta^+_\mu(F) + \alpha_w(F)$, $s(F) = \Delta^+_\mu(F)$, and $u^*_o = 0$. It remains the case that $v^*_w = s\left(\mu(w)\right) + \alpha_w\left(\mu(w)\right)$. Thus $u^*_o + v^*_w \geq a_{ow}$ implies

$$s\left(\mu(w)\right) + \alpha_w\left(\mu(w)\right) \geq \Delta^+_\mu(F) + \alpha_w(F) = s(F) + \alpha_w(F).$$

This completes the proof of Step 6.

An observant reader will note that we have now proven that our salary schedule satisfies all of the requirements of Corollary 2 except $s(F) \geq \Delta^+_\mu(F)$, since Step 5 and Step 6 prove that (μ, s) has No Envy, and Step 4 proves that (μ, s) satisfies the upper bound in the definition of Marginal Product Salaries. She may also note that we have not yet exploited the fact that our salaries are defined with respect to the worker-optimal payoffs. The final steps of the proof will connect these remaining pieces.

Construct a directed graph $(\mathbf{F}, E)$ with firms as nodes and an edge $\langle F, F'\rangle$ existing if

$$\exists w \in \mu(F') \text{ such that } s(F) + \alpha_w(F) = s(F') + \alpha_w(F').$$

In words: an edge $\langle F, F'\rangle$ exists if there exists a worker matched to F' under μ who is indifferent between working at F' and F. Let $\mathscr{D}(F)$ be the union of F and the descendants of F:

$$\mathscr{D}(F) \equiv \{F\} \cup \left\{F' : \exists \text{ a directed path from } F \text{ to } F' \text{ in the graph } (\mathbf{F}, E)\right\}.$$

Step 7: For all $F \in \mathbf{F}$ such that $\mu(F) \neq \emptyset$: there exists a firm $F' \in \mathscr{D}(F)$ such that $s(F') = \Delta^-_\mu(F')$.

Proof of Step 7. Fix a firm F. Consider increasing the salary of every firm in $\mathscr{D}(F)$ by $\epsilon > 0$. Let (u^ϵ, v^ϵ) denote the resultant payoffs:

$$v^\epsilon_w = \begin{cases} v^*_w + \epsilon & \text{if } \mu(w) \in \mathscr{D}(F), \\ v^*_w & \text{otherwise;} \end{cases} \qquad u^\epsilon_o = \begin{cases} u^*_o - \epsilon & \text{if } F(o) \in \mathscr{D}(F), \\ u^*_o & \text{otherwise.} \end{cases}$$

The payoffs (u^*, v^*) are the worker-optimal element of P_μ. Thus $(u^\epsilon, v^\epsilon) \notin P_\mu$. The payoffs (u^*, v^*) are feasible and $\sum_{o\in\mathbf{O}} u^\epsilon_o + \sum_{w\in\mathbf{W}} v^\epsilon_w = \sum_{o\in\mathbf{O}} u^*_o + \sum_{w\in\mathbf{W}} v^*_w$. Thus (u^ϵ, v^ϵ) is feasible. By the definition of P_μ: (u^ϵ, v^ϵ) are not core.

We will now show that, if ϵ sufficiently small, then $\forall o \in \mathbf{O}, w \in \mathbf{W}$: $u^\epsilon_o + v^\epsilon_w \geq a_{ow}$. Assume towards a contradiction that there exists $o \in \mathbf{O}, w \in \mathbf{W}$ such that $u^\epsilon_o + v^\epsilon_w < a_{ow}$. Given that $v^*_w \geq v^\epsilon_w$ and $u^*_o + v^*_w \geq a_{ow}$, this implies that $u^\epsilon_o < u^*_o$, which in turn implies that $F(o) \in \mathscr{D}(F)$. If $\mu(w) \in \mathscr{D}(F)$, then

$$u^\epsilon_o + v^\epsilon_w = u^*_o - \epsilon + v^*_w + \epsilon = u^*_o + v^*_w \geq a_{ow},$$

so it must be the case that $\mu(w) \notin \mathscr{D}(F)$. By the definition of $\mathscr{D}(F)$ and the fact that $F(o) \in \mathscr{D}(F)$, this implies that $s\big(F(o)\big) + \alpha_w\big(F(o)\big) \neq s\big(\mu(w)\big) + \alpha_w\big(\mu(w)\big)$. By Step 6, this implies that

$$s\big(\mu(w)\big) + \alpha_w\big(\mu(w)\big) > s\big(F(o)\big) + \alpha_w\big(F(o)\big). \tag{16}$$

By Step 5, the left hand side of that inequality can be expressed as $s\big(\mu(w)\big) + \alpha_w\big(\mu(w)\big) = v^*_w$. By the definition of $\mathscr{D}(F)$, either $F(o) = F$ (in which case $\mu\big(F(o)\big) \neq \emptyset$ by the assumption of Step 7) or there exists a directed path from F to F' in the graph $(\mathbf{F}, E)$. By the definition of the graph $(\mathbf{F}, E)$, that would also imply that $\mu\big(F(o)\big) \neq \emptyset$. Thus $\mu\big(F(o)\big) \neq \emptyset$. As such, the right hand side of inequality (16) can be expressed as

$$\begin{aligned} s(F(o)) + \alpha_w(F(o)) &= \Delta^-_\mu(F(o)) - u^*_o + \alpha_w(F(o)) && \text{(by equation (15))} \\ &= a_{ow} - u^*_o && \text{(by the definition of } a_{ow}\text{).} \end{aligned}$$

Thus inequality (16) requires that $u^*_o + v^*_w > a_{ow}$. Since $\mu(w) \notin \mathscr{D}(F)$, it must be true that $u^*_o = u^\epsilon_o$. Moreover, since $v^\epsilon_w = v^*_w - \epsilon$, for ϵ sufficiently small, it must be the case that $u^\epsilon_o + v^\epsilon_w > a_{ow}$. This contradicts the hypothesis that $u^\epsilon_o + v^\epsilon_w < a_{ow}$, completing the proof that, if ϵ is sufficiently small, then

$$\forall o \in \mathbf{O}, w \in \mathbf{W}: \; u^\epsilon_o + v^\epsilon_w \geq a_{ow}. \tag{17}$$

Given that (u^ϵ, v^ϵ) is not core, (17) implies that either $u^\epsilon \ngeqq 0$ or $v^\epsilon \ngeqq 0$. The payoffs (u^*, v^*) are core and thus $v^* \geqq 0$. As such, $v^\epsilon > v^* \geqq 0$. So, $u^\epsilon \ngeqq 0$. It also follows from (u^*, v^*) being core that $u^* \geqq 0$. In summary, there exists $o \in \mathbf{O}$ such that $u^*_o \geq 0 > u^\epsilon_o = u^*_o - \epsilon$. Thus for all $\epsilon > 0$: $u^*_o \in [0, \epsilon)$. This implies that $u^*_o = 0$.

The firm $F(o)$ is in $\mathscr{D}(F)$. As argued above, this implies that $\mu\big(F(o)\big) \neq \emptyset$. Given the definition of the salary schedule s in equation (15) and the fact that $u^*_o = 0$, this implies that $s\big(F(o)\big) = \Delta^-_\mu\big(F(o)\big)$.

Step 8: For all $F \in \mathbf{F}$: $s(F) \geq \Delta^+_\mu(F)$.

Proof of Step 8. Assume towards a contradiction that there exists a firm F_N such that $s(F_N) < \Delta^+_\mu(F_N)$. By Step 7, in the graph $(\mathbf{F}, E)$, F_N must have a descendant F_0 such that $s(F_0) = \Delta^-_\mu(F_0)$. Let the path from F_N to F_0 be $F_N, F_{N-1}, ..., F_2, F_1, F_0$. Let w_j be the worker ensuring that the edge $\langle F_{j+1}, F_j\rangle$ exists:

$$w_j \in \mu(F_j) \text{ such that } s(F_{j+1}) + \alpha_{w_j}(F_{j+1}) = s(F_j) + \alpha_{w_j}(F_j).$$

Consider the replacement chain $\chi = \Big(\big(w_j\big)_{j=0}^{N-1}, \big(F_j\big)_{j=0}^{N}\Big)$:

$$\begin{aligned}
\text{value}\big(\mu + \chi\big) - \text{value}\big(\mu\big) =& \Delta^+_\mu(F_N) - \Delta^-_\mu(F_0) + \sum_{j=0}^{N-1} \big[\alpha_{w_j}(F_{j+1}) - \alpha_{w_j}(F_j)\big] \\
=& \Delta^+_\mu(F_N) - \Delta^-_\mu(F_0) + \sum_{j=0}^{N-1} \big[s(F_j) - s(F_{j+1})\big] \quad \text{because } s(F_{j+1}) + \alpha_{w_j}(F_{j+1}) = s(F_j) + \alpha_{w_j}(F_j) \\
=& \Delta^+_\mu(F_N) - \Delta^-_\mu(F_0) + [s(F_0) - s(F_N)] \\
=& \Delta^+_\mu(F_N) - s(F_N) \quad \text{because } \Delta^-_\mu(F_0) = s(F_0).
\end{aligned}$$

By assumption $\Delta^+_\mu(F_N) > s(F_N)$ and so value$\big(\mu + \chi\big) >$ value$\big(\mu\big)$. But μ is efficient. Thus we have contradicted our assumption that there exists a firm F_0 such that $s(F_0) < \Delta^+_\mu(F_0)$. This proves Step 8, which was the only requirement remaining from the hypothesis of Corollary 2. □

Proposition 5. *There exists a stable allocation $\big(\mu^*, s^*\big)$ such that μ^* is efficient and for all stable allocations $\big(\mu, s\big)$: $s^* \geqq s$.*

Proof. This proof relies on intermediate results from the proof of Theorem 1, and may thus be opaque to readers who have not read that proof. Let $\big(\mu^*, s^*\big)$ be the allocation constructed in the proof of Theorem 1 (i.e., the matching isomorphic to the worker-optimal elements of P_μ with salaries defined as in equation (15)). Consider any other stable allocation $\big(\mu, s\big)$. Let $\mathscr{I} = \big\{F : s(F) > s^*(F)\big\}$. We will show that $\mathscr{I} = \emptyset$.

Step 1: $\forall w \in \mathbf{W} : \mu^*(w) \in \mathscr{I} \implies \mu(w) \in \mathscr{I}$.

Proof of Step 1: Consider a worker w for whom $\mu^*(w) \in \mathscr{I}$ and a firm F such that $F \notin \mathscr{I}$. Since $\big(\mu^*, s^*\big)$ has No Envy, it must be the case that $\alpha_w\big(\mu^*(w)\big) + s^*\big(\mu^*(w)\big) \geq \alpha_w(F) + s^*(F)$. $F \notin \mathscr{I}$ implies that $s^*(F) \geq s(F)$, while $\mu^*(w) \in \mathscr{I}$ implies that $s\big(\mu^*(w)\big) > s^*\big(\mu^*(w)\big)$. Combining these inequalities implies that

$$\alpha_w\big(\mu^*(w)\big) + s\big(\mu^*(w)\big) > \alpha_w(F) + s(F).$$

Thus if $\mu(w) = F$, then $\big(\mu, s\big)$ would lack No Envy, constituting a contradiction.

Step 2: $\sum_{F \in \mathscr{I}} \big|\mu^*(F)\big| \leq \sum_{F \in \mathscr{I}} \big|\mu(F)\big|$.

Step 2 follows directly from Step 1.

Step 3: $\forall F \in \mathscr{I} : \big|\mu^*(F)\big| \geq \big|\mu(F)\big|$.

Proof of Step 3: Assume towards a contradiction that there exists a firm $F \in \mathscr{I}$ such that $\big|\mu(F)\big| > \big|\mu^*(F)\big|$. By Step

8 in the proof of Theorem 1, $s^*(F) \geq \Delta^+_{\mu^*}(F)$. By decreasing differences and $\left|\mu(F)\right| > \left|\mu^*(F)\right|$: $\Delta^+_{\mu^*}(F) \geq \Delta^-_{\mu}(F)$. Thus $s^*(F) \geq \Delta^-_{\mu}(F)$. Given that $F \in \mathscr{I} : s(F) > s^*(F)$. In summary:

$$s(F) > \Delta^-_{\mu}(F),$$

which contradicts the assumption that (μ, s) has No Firing.

Step 4: $\forall F \in \mathscr{I}, \left|\mu^*(F)\right| = \left|\mu(F)\right|$.

Step 4 is the conjunction of Steps 2 and 3.

Step 5: $\mu^*(w) \in \mathscr{I} \iff \mu(w) \in \mathscr{I}$.

Proof of Step 5: By Step 4: $\left|\left\{w : \mu(w) \in \mathscr{I}\right\}\right| = \left|\left\{w : \mu^*(w) \in \mathscr{I}\right\}\right|$. By Step 1: $\left\{w : \mu^*(w) \in \mathscr{I}\right\} \subseteq \left\{w : \mu(w) \in \mathscr{I}\right\}$. Thus $\left\{w : \mu^*(w) \in \mathscr{I}\right\} = \left\{w : \mu(w) \in \mathscr{I}\right\}$.

As in the proof of Theorem 1, construct a directed graph $(\mathbf{F}, E)$ with firms as nodes and an edge $\langle F, F' \rangle$ existing if

$$\exists w \in \mu^*(F') \text{ such that } s^*(F) + \alpha_w(F) = s^*(F') + \alpha_w(F').$$

Let $\mathscr{D}(F)$ be the union of F and the descendants of F:

$$\mathscr{D}(F) \equiv \{F\} \cup \left\{F' : \exists \text{ a directed path from } F \text{ to } F' \text{ in the graph } (\mathbf{F}, E)\right\}.$$

Step 6: $\forall F \in \mathscr{I} : \mathscr{D}(F) \subseteq \mathscr{I}$.

Proof of Step 6: Let $F \in \mathscr{I}$. Let the edge $\langle F, F' \rangle$ be in E. There thus exists a worker $w \in \mu^*(F')$ such that

$$s^*(F) + \alpha_w(F) = s^*(F') + \alpha_w(F').$$

Since (μ^*, s^*) has No Envy, it must be the case that for any firm F'': $s^*(F') + \alpha_w(F') \geq s^*(F'') + \alpha_w(F'')$. Given that $F \in \mathscr{I}$, $s(F) > s^*(F)$. Thus for all $F'' \notin \mathscr{I}$:

$$s(F) + \alpha_w(F) > s(F'') + \alpha_w(F''),$$

whereas the fact that (μ, s) has No Envy implies that

$$s\left(\mu(w)\right) + \alpha_w\left(\mu(w)\right) \geq s(F) + \alpha_w(F).$$

Thus $\mu(w) \in \mathscr{I}$. By Step 5, $F' \in \mathscr{I}$. $\mathscr{D}(F) \subseteq \mathscr{I}$ follows from iteratively applying this argument to the descendants of F, their descendants' descendants, and so on.

Step 7: If $\mathscr{I} \neq \emptyset$, then there exists a firm $F \in \mathscr{I}$ with $s(F) > \Delta^-_{\mu^*}(F)$.

Proof of Step 7: By Step 7 of the proof of Theorem 1, every set $\mathscr{D}(F')$ contains a firm F such that $s^*(F) = \Delta^-_{\mu^*}(F)$. With Step 6, if $\mathscr{I} \neq \emptyset$ then there exists a firm $F \in \mathscr{I}$ such that $s^*(F) = \Delta^-_{\mu^*}(F)$. Moreover, $s(F) > s^*(F)$ because $F \in \mathscr{I}$. Combining these inequalities implies that $s(F) > \Delta^-_{\mu^*}(F)$.

Step 8: $\mathscr{I} = \emptyset$.

Proof of Step 8: By Step 7, $\mathscr{I} \neq \emptyset$ implies the existence of a firm $F \in \mathscr{I}$ such that $s(F) > \Delta^-_{\mu^*}(F)$. By Step 4, $\Delta^-_{\mu^*}(F) = \Delta^-_{\mu}(F)$. Therefore, $s(F) > \Delta^-_{\mu}(F)$, which would contradict (μ, s) having No Firing, which in turn would contradict the assumption that (μ, s) is a stable allocation. □

Lemma 5. *For any two stable allocations $(\mu, s), (\mu', s') : s \geqq s' \iff (\mu, s) \succeq_{\mathbf{W}} (\mu', s')$.*

Proof. We first show $s \geqq s' \implies (\mu, s) \succeq_{\mathbf{W}} (\mu', s')$. Since (μ, s) has No Envy, it must be the case that for every worker w:

$$\alpha_w(\mu(w)) + s(\mu(w)) \geq \alpha_w(\mu'(w)) + s(\mu'(w)),$$

while $s \geqq s'$ implies that $\alpha_w(\mu'(w)) + s(\mu'(w)) \geq \alpha_w(\mu'(w)) + s'(\mu'(w))$.

We now show $(\mu, s) \succeq_{\mathbf{W}} (\mu', s') \implies s \geqq s'$. For every worker:

$$\alpha_w(\mu(w)) + s(\mu(w)) \geq \alpha_w(\mu'(w)) + s'(\mu'(w)) \geq \alpha_w(\mu(w)) + s'(\mu(w)),$$

where the first inequality follows from $(\mu, s) \succeq_{\mathbf{W}} (\mu', s')$, while the second follows from the fact that (μ', s') has No Envy. This implies $s(F) \geq s'(F)$ for every firm F such that $\mu(F) \neq \emptyset$.

For firms F such that $\mu(F) = \emptyset$, $s(F) = y_F(1)$ by definition. For such firms, either $\mu'(F) = \emptyset$ in which case by definition $s'(F) = y_F(1) = s(F)$, or $\mu'(F) \neq \emptyset$ in which case decreasing differences and the fact that (μ', s') has No Firing imply that $s'(F) \leq y_F(1) = s(F)$. □

Lemma 6. *For any two stable allocations $(\mu, s), (\mu', s')$: $(\mu, s) \succeq_{\mathbf{W}} (\mu', s') \implies (\mu', s') \succeq_{\mathbf{F}} (\mu, s)$.*

Proof. Let $(\mu, s) \succeq_{\mathbf{W}} (\mu', s')$ be two stable allocations. Assume towards a contradiction there exists a firm F such that $\pi_F(|\mu(F)|, s(F)) > \pi_F(|\mu'(F)|, s'(F))$. Since (μ', s') is individually rational for firms, $\pi_F(|\mu'(F)|, s'(F)) \geq 0$. Thus $\mu(F) \neq \emptyset$. By assumption, all workers in $\mu(F)$ weakly prefer (μ, s) to (μ', s'). Thus $(F, \mu(F), s(F))$ blocks (μ', s'), contradicting the assumption that (μ', s') is a stable allocation. □

Theorem 3. *When firms' production functions are public information, there exists a strategyproof mechanism that implements an efficient stable allocation.*

Proof. Theorem 2 told us that a worker-optimal efficient allocation always exists. Consider the mechanism which asks each worker for her amenities and then implements the corresponding worker-optimal efficient allocation. Note that regardless of the veracity of workers' reported amenities, the mechanism will use the true production functions since we assumed they are public information. We will show that, under such a mechanism, it is a weakly-dominant strategy for each worker to report her true amenities.

Let $\alpha_w \equiv (\alpha_w(F))_{F \in \mathbf{F}}$ concatenate each worker w's amenities. Let α_w° represent worker w's reported amenities. Fix a particular worker $\hat{w} \in \mathbf{W}$. Assume towards a contradiction that there exists a report $\alpha_{\hat{w}}^\circ \neq \alpha_{\hat{w}}$ such that $\hat{w}$ strictly benefits from reporting $\alpha_{\hat{w}}^\circ$, given the other workers' reports. Let (μ°, s°) denote the worker-optimal allocation given that all workers w (including $w = \hat{w}$) report α_w°. Let (μ^*, s^*) denote the worker-optimal allocation given that all workers $w \neq \hat{w}$ report α_w°, while worker $\hat{w}$ truthfully reports $\alpha_{\hat{w}}$. Let $F^* \equiv \mu^*(\hat{w})$ and let $F^\circ \equiv \mu^\circ(\hat{w})$. Our assumption that $\hat{w}$ benefits from misreporting requires:

$$\alpha_{\hat{w}}(F^*) + s^*(F^*) < \alpha_{\hat{w}}(F^\circ) + s^\circ(F^\circ). \tag{18}$$

Both (μ°, s°) and (μ^*, s^*) have No Envy for their respective reports, though not necessarily for the true amenities. For clarity, we will say an allocation has No Envy$^\circ$ if that allocation would have No Envy if reported amenities were true.

Since (μ^*, s^*) has No Envy$^\circ$:

$$\alpha_{\hat{w}}(F^*) + s^*(F^*) \geq \alpha_{\hat{w}}(F^\circ) + s^*(F^\circ)$$

with inequality (18), this implies that $s^*(F^\circ) < s^\circ(F^\circ)$. Let $\mathscr{I} = \{F : s^\circ(F) > s^*(F)\}$. We have shown that $F^\circ \in \mathscr{I}$. We will prove the contradiction that $\mathscr{I} = \emptyset$. The proof from this point is similar to that for Proposition 5 (which showed that there existed an efficient stable allocation with maximal salaries). The only difference is that we will here require No Envy° rather than No Envy.

Step 1: $\forall w \in \mathbf{W} : \mu^*(w) \in \mathscr{I} \implies \mu^\circ(w) \in \mathscr{I}$.

Proof of Step 1: We showed above that $F^\circ = \mu^\circ(\hat{w}) \in \mathscr{I}$, and thus it remains to show that the claim is true for all $w \neq \hat{w}$. Consider a worker $w \neq \hat{w}$ for whom $\mu^*(w) \in \mathscr{I}$ and a firm F such that $F \notin \mathscr{I}$. Since $\left(\mu^*, s^*\right)$ has No Envy°, it must be the case that $\alpha^\circ_w\left(\mu^*(w)\right) + s^*\left(\mu^*(w)\right) \geq \alpha^\circ_w(F) + s^*(F)$. $F \notin \mathscr{I}$ implies that $s^*(F) \geq s^\circ(F)$, while $\mu^*(w) \in \mathscr{I}$ implies that $s^\circ\left(\mu^*(w)\right) > s^*\left(\mu^*(w)\right)$. Combining these inequalities implies that

$$\alpha^\circ_w\left(\mu^*(w)\right) + s^\circ\left(\mu^*(w)\right) > \alpha^\circ_w(F) + s^\circ(F).$$

Thus $\mu^\circ(w) = F$ would imply $\left(\mu^\circ, s^\circ\right)$ lacks No Envy° – a contradiction.

Step 2: $\sum_{F \in \mathscr{I}} \left|\mu^*(F)\right| \leq \sum_{F \in \mathscr{I}} \left|\mu^\circ(F)\right|$.

Step 2 follows directly from Step 1.

Step 3: $\forall F \in \mathscr{I} : \left|\mu^*(F)\right| \geq \left|\mu^\circ(F)\right|$.

The proof of Step 3 is identical to the proof of Step 3 of Proposition 5.

Step 4: $\forall F \in \mathscr{I}, \left|\mu^*(F)\right| = \left|\mu^\circ(F)\right|$.

Step 4 is the conjunction of Steps 2 and 3.

Step 5: $\mu^*(w) \in \mathscr{I} \iff \mu^\circ(w) \in \mathscr{I}$.

The proof of Step 5 is identical to the proof of Step 5 of Proposition 5.

As in the proof of Theorem 1, construct a directed graph $(\mathbf{F}, E)$ with firms as nodes and an edge $\langle F, F'\rangle$ existing if

$$\exists w \in \mu^*(F') \text{ such that } \begin{cases} s^*(F) + \alpha_w(F) = s^*(F') + \alpha_w(F') & \text{if } w = \hat{w}, \text{ or} \\ s^*(F) + \alpha^\circ_w(F) = s^*(F') + \alpha^\circ_w(F') & \text{if } w \neq \hat{w}. \end{cases}$$

Let $\mathscr{D}(F)$ be the union of F and the descendants of F:

$$\mathscr{D}(F) \equiv \{F\} \cup \left\{F' : \exists \text{ a directed path from } F \text{ to } F' \text{ in the graph } (\mathbf{F}, E)\right\}.$$

Step 6: $\forall F \in \mathscr{I} : \mathscr{D}(F) \subseteq \mathscr{I}$.

Proof of Step 6: Let $F \in \mathscr{I}$. Let the edge $\langle F, F'\rangle$ be in E. There thus exists a worker $w \in \mu^*(F')$ such that

$$\begin{aligned} &s^*(F) + \alpha_w(F) = s^*(F') + \alpha_w(F') && \text{if } w = \hat{w}, \text{ or} \\ &s^*(F) + \alpha^\circ_w(F) = s^*(F') + \alpha^\circ_w(F') && \text{if } w \neq \hat{w}. \end{aligned}$$

We already know that $F^\circ = \mu^\circ(\hat{w}) \in \mathscr{I}$, so we turn to the case in which $w \neq \hat{w}$. Since $\left(\mu^*, s^*\right)$ has No Envy°, it must be true that for any firm F'': $s^*(F') + \alpha^\circ_w(F') \geq s^*(F'') + \alpha^\circ_w(F'')$. Given that $F \in \mathscr{I}$, $s^\circ(F) > s^*(F)$. Thus for all $F'' \notin \mathscr{I}$:

$$s^\circ(F) + \alpha^\circ_w(F) > s^\circ(F'') + \alpha^\circ_w(F''),$$

whereas $\left(\mu^\circ, s^\circ\right)$ having No Envy° requires that

$$s^\circ\left(\mu^\circ(w)\right) + \alpha^\circ_w\left(\mu^\circ(w)\right) \geq s^\circ(F) + \alpha^\circ_w(F).$$

Thus $\mu^\circ(w) \in \mathscr{I}$. By Step 5, $F' \in \mathscr{I}$. $\mathscr{D}(F) \subseteq \mathscr{I}$ follows from iteratively applying this argument to the descendants of F, their descendants' descendants, and so on.

Step 7: If $\mathscr{I} \neq \emptyset$, then there exists a firm $F \in \mathscr{I}$ with $s^\circ(F) > \Delta^-_{\mu^*}(F)$.

The proof of Step 7 is identical to the proof of Step 7 of Proposition 5.

Step 8: $\mathscr{I} = \emptyset$.

The proof of Step 8 is identical to the proof of Step 8 of Proposition 5.

Step 8 contradicts our earlier result that $F^\circ \in \mathscr{I}$, completing the proof. □

Proposition 7. *If every firm has common value amenities, then every stable allocation is efficient.*

Proof. Let every firm F have common value amenity α_F:

$$\forall w \in \mathbf{W}: \alpha_w(F) = \alpha(F),$$

and assume towards a contradiction that there exists a stable allocation (μ, s), where μ is inefficient.

First, note that it follows from (μ, s) having No Envy that:

$$\forall F, F' \in \mathbf{F} \text{ such that } \mu(F) \neq \emptyset: \alpha(F) + s(F) \geq \alpha(F') + s(F'). \tag{19}$$

By Lemma 3, there exists a replacement chain $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ from μ to an efficient matching such that value$(\mu + \chi) >$ value(μ). By Lemma 4, χ is acyclic, and thus:

$$\begin{aligned}\text{value}(\mu+\chi) - \text{value}(\mu) &= \Delta^+_\mu(F_N) - \Delta^-_\mu(F_0) + \sum_{k=0}^{N-1}\left[\alpha(F_{k+1}) - \alpha(F_k)\right] \\ &= \Delta^+_\mu(F_N) - \Delta^-_\mu(F_0) + \alpha(F_N) - \alpha(F_0).\end{aligned}$$

Given that value$(\mu + \chi) >$ value(μ), this implies that

$$\alpha(F_0) - \alpha(F_N) < \Delta^+_\mu(F_N) - \Delta^-_\mu(F_0). \tag{20}$$

By the definition of a replacement chain, $\mu(F_0) \neq \emptyset$. Thus by expression (19), $s(F_N) - s(F_0) \leq \alpha(F_0) - \alpha(F_N)$. With inequality (20), this implies that

$$s(F_N) - s(F_0) < \Delta^+_\mu(F_N) - \Delta^-_\mu(F_0). \tag{21}$$

Since (μ, s) has No Firing, $s(F_0) \leq \Delta^-_\mu(F_0)$. With inequality (21) this implies that $s(F_N) < \Delta^+_\mu(F_N)$. Thus firm F_N would strictly benefit from hiring an additional worker at its current salary.

If $\mu(F_N) \neq \emptyset$, it follows from (19) that

$$\forall F \in \mathbf{F}: \alpha(F_N) + s(F_N) \geq \alpha(F) + s(F).$$

By the definition of an acyclic replacement chain, $w_0 \notin \mu(F_N)$. Thus worker w_0 would be willing to work at firm F_N at salary $s(F_N)$. Thus the coalition $\left(F_N, \mu(F_N) \cup \{w_0\}, s(F_N)\right)$ would block (μ, s), contradicting (μ, s) being stable. Thus it must be the case that $\mu(F_N) = \emptyset$.

However, if $\mu(F_N) = \emptyset$, then F_N must be making zero profit. It could thus offer to employ worker w_0 at salary $\Delta^+_\mu(F_N)$ and still make zero profit. By inequality (20), $\alpha(F_0) + \Delta^-_\mu(F_0) < \alpha(F_N) + \Delta^+_\mu(F_N)$. Recall that $s(F_0) \leq \Delta^-_\mu(F_0)$. Thus worker w_0 would be strictly better off. The coalition $\left(F_N, \{w_0\}, \Delta^+_\mu(F_N)\right)$ blocks (μ, s), contradicting (μ, s) being stable.

Given that both $\mu(F_N) \neq \emptyset$ and $\mu(F_N) = \emptyset$ yield contradictions, there can be no stable allocation (μ, s), where μ is inefficient.

□

Proposition 8. *If every firm has a duplicate, then every stable allocation is efficient.*

Proof. Let (μ, s) be a stable allocation. Let F' be the duplicate of F. That (μ, s) has No Envy implies

$$\forall w \in \mu(F) : \alpha_w(F) + s(F) \geq \alpha_w(F') + s(F') = \alpha_w(F) + s(F'),$$

where the equality follows from the assumption that F, F' are duplicates. Thus: $\mu(F) \neq \emptyset \implies s(F) \geq s(F')$.

If $\mu(F) = \emptyset$ then by construction $s(F) = y_F(1)$. As F, F' are duplicates: $y_F(1) = y_{F'}(1)$. By decreasing differences, $y_{F'}(1) \geq \Delta^-_\mu(F')$, while (μ, s) having No Firing requires $\Delta^-_\mu(F') \geq s(F')$. Combining these expressions we see that $\mu(F) = \emptyset$ implies $s(F) \geq s(F')$. Given the prior paragraph, this implies $s(F) \geq s(F')$ for all duplicates F, F'. Symmetrically, $s(F') \geq s(F)$. Therefore $s(F) = s(F')$.

If $\mu(F') \neq \emptyset$ and $s(F) < \Delta^+_\mu(F)$, then F would be strictly better off being additionally matched to $w \in \mu(F')$ at salary $s(F)$, while w would be indifferent (because $s(F) = s(F')$). Thus $(F, \mu(F) \cup \{w\}, s(F))$ would block (μ, s), contradicting the assumption that (μ, s) is a stable allocation. Thus, if $\mu(F') \neq \emptyset$, then $s(F) \geq \Delta^+_\mu(F)$. If $\mu(F') = \emptyset$, then $s(F) = s(F') = y_{F'}(1) = y_F(1) \geq \Delta^+_\mu(F)$, with the last inequality following from decreasing differences. Thus, if $\mu(F') = \emptyset$, then $s(F) \geq \Delta^+_\mu(F)$. In summary, for all F: $s(F) \geq \Delta^+_\mu(F)$.

By (μ, s) having No Firing, $s(F) \leq \Delta^-_\mu(F)$. We have shown that for any firm F with a duplicate, $s(F) \in \left[\Delta^+_\mu(F), \Delta^-_\mu(F)\right]$. When all firms have a duplicate, this implies that (μ, s) has Marginal Product Salaries. By Proposition 3, this implies that μ is efficient. □

Proposition 9. *Consider an inefficient stable allocation (μ, s). There exists a salary s', a firm F, and a worker w such that $s' < \Delta^+_\mu(F)$ and w strictly prefers to work for F at salary s' than for $\mu(w)$ at salary $s(\mu(w))$.*

Proof. Consider an inefficient stable allocation (μ, s). We will show that there exists a worker w and a firm $F \neq \mu(w)$ such that

$$\alpha_w(\mu(w)) + s(\mu(w)) < s' + \alpha_w(F).$$

and that $s' < \Delta^+_\mu(F)$. Combining inequalities, this is equivalent to

$$\alpha_w(\mu(w)) + s(\mu(w)) - \alpha_w(F) < \Delta^+_\mu(F). \tag{22}$$

In what follows, let (μ^*, s^*) be a worker-optimal efficient stable allocation, the existence of which is guaranteed by Theorem 2. By Lemmas 3 and 4 there exists an acyclic replacement chain $\chi = \left((w_k)_{k=0}^{N-1}, (F_k)_{k=0}^{N}\right)$ from μ to μ^* such that value$(\mu + \chi) >$ value(μ). We will show that inequality (22) holds for the replacement chain χ's last firm F_N and its last worker w_{N-1}:

$$\alpha_{w_{N-1}}(\mu(w_{N-1})) + s(\mu(w_{N-1})) - \alpha_{w_{N-1}}(F_N) < \Delta^+_\mu(F_N). \tag{23}$$

Doing so will take four steps.

Step 1: If worker w_{N-1} strictly prefers (μ^*, s^*) to (μ, s), then inequality (23) holds.

Proof of Step 1: Worker w_{N-1}'s strict preference for (μ^*, s^*), in which they are matched to firm F_N, over (μ, s) implies that

$$\alpha_{w_{N-1}}(F_N) + s^*(F_N) > \alpha_{w_{N-1}}(\mu(w_{N-1})) + s(\mu(w_{N-1})).$$

Since (μ^*, s^*) has No Firing, $s^*(F_N) \leq \Delta^-_{\mu^*}(F_N)$. Lemma 3 assured us that $\left|(\mu + \chi)(F_N)\right| \leq \left|\mu^*(F_N)\right|$, and thus by decreasing differences, $\Delta^-_{\mu^*}(F_N) \leq \Delta^-_{\mu+\chi}(F_N)$. By the fact that χ is acyclic, $\Delta^-_{\mu+\chi}(F_N) = \Delta^+_\mu(F_N)$. Combining these expressions and rearranging yields inequality (23).

Step 2: If the replacement chain χ contains a worker w_k who strictly prefers (μ^*, s^*) to (μ, s), then the last worker w_{N-1} will strictly prefer (μ^*, s^*) over (μ, s).

Proof of Step 2: Let w_k strictly prefer (μ^*, s^*) (in which they are matched to firm F_{k+1}) over (μ, s) (in which they are matched to firm F_k):

$$\alpha_{w_k}(F_{k+1}) + s^*(F_{k+1}) > \alpha_{w_k}(F_k) + s(F_k).$$

Since (μ, s) has No Envy,

$$\alpha_{w_k}(F_{k+1}) + s(F_{k+1}) \leq \alpha_{w_k}(F_k) + s(F_k).$$

Combining inequalities we have that $s^*(F_{k+1}) > s(F_{k+1})$.

Now consider worker w_{k+1}, who in (μ, s) is matched to F_{k+1} and who in (μ^*, s^*) is matched to F_{k+2}. Since (μ^*, s^*) has No Envy,

$$\alpha_{w_{k+1}}(F_{k+2}) + s^*(F_{k+2}) \geq \alpha_{w_{k+1}}(F_{k+1}) + s^*(F_{k+1}).$$

Given that $s^*(F_{k+1}) > s(F_{k+1})$, this implies that w_{k+1} strictly prefers (μ^*, s^*) to (μ, s):

$$\alpha_{w_{k+1}}(F_{k+2}) + s^*(F_{k+2}) > \alpha_{w_{k+1}}(F_{k+1}) + s(F_{k+1}).$$

By induction, each worker w_{k+j}, with $j \geq 0$, strictly prefers (μ^*, s^*) to (μ, s). That includes the last worker w_{N-1}.

Step 3: If the replacement chain χ contains no worker w_k who strictly prefers (μ^*, s^*) to (μ, s), then inequality (23) holds.

Proof of Step 3: Assume that the replacement chain χ contains no worker w_k who strictly prefers (μ^*, s^*) to (μ, s). By Theorem 2, this implies that each worker in the replacement chain is indifferent between (μ^*, s^*) and (μ, s), since the former is worker-optimal. As such:

$$\sum_{k=0}^{N-1} \left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k) + s^*(F_{k+1}) - s(F_k)\right] = 0$$

Note that

$$\sum_{k=0}^{N-1} \left[s^*(F_{k+1}) - s(F_k)\right] = \sum_{k=1}^{N-1} \left[s^*(F_k) - s(F_k)\right] + s^*(F_N) - s(F_0),$$

and that, by Theorem 2, for each k: $s^*(F_k) \geq s(F_k)$. Thus

$$\sum_{k=0}^{N-1} \left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k)\right] + s^*(F_N) - s(F_0) \leq 0.$$

Since (μ, s) has No Firing, $s(F_0) \leq \Delta^-_\mu(F_0)$. Thus:

$$\sum_{k=0}^{N-1} \left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k)\right] + s^*(F_N) - \Delta^-_\mu(F_0) \leq 0. \tag{24}$$

Given that $\text{value}(\mu + \chi) > \text{value}(\mu)$, it must be the case that

$$\sum_{k=0}^{N-1} \left[\alpha_{w_k}(F_{k+1}) - \alpha_{w_k}(F_k)\right] + \Delta^+_\mu(F_N) - \Delta^-_\mu(F_0) > 0. \tag{25}$$

Inequalities (24) and (25) imply that $\Delta^+_\mu(F_N) > s^*(F_N)$. Worker w_{N-1} is indifferent between (μ^*, s^*), in which she is matched to firm F_N, and (μ, s):

$$\alpha_{w_{N-1}}\left(\mu(w_{N-1})\right) + s\left(\mu(w_{N-1})\right) = \alpha_{w_{N-1}}(F_N) + s^*(F_N).$$

With $\Delta^+_\mu(F_N) > s^*(F_N)$, this implies inequality (23).

Step 4: Inequality (23) holds.

Proof of Step 4: By Step 2, if any worker in the replacement chain χ strictly prefers (μ^*, s^*) to (μ, s), then w_{N-1} will strictly prefer (μ^*, s^*) to (μ, s). By Step 1, if w_{N-1} strictly prefers (μ^*, s^*) to (μ, s), then inequality (23) holds.

On the other hand, if no worker in the replacement chain χ strictly prefers (μ^*, s^*) to (μ, s), then Step 3 tells us that inequality (23) holds. Thus regardless of whether a worker in the replacement chain χ strictly prefers (μ^*, s^*) to (μ, s), inequality (23) must hold.

□

B Relationships Between Our Model and Others

In this appendix, we contrast results and assumptions made by our model to results and assumptions made in the existing matching literature.

B.1 There is no firm-optimal or worker-pessimal stable allocation

Payoffs in matching models frequently form bounded lattices with payoffs on one side of the market dual to payoffs on the other side (Knuth, 1976; Shapley & Shubik, 1971; Hatfield & Milgrom, 2005; Blair, 1988). We showed in Example 2 that this duality fails in our model: a worker and all firms can all benefit from the shift from one stable allocation to another. We will now show that the dual-lattice structure additionally fails for another reason: there may not be a worker-pessimal or firm-optimal allocation.

Example B.1. $\mathbf{F} = \{F_1, F_2\}$. $\mathbf{W} = \{w_1, w_2, w_3\}$. $y_{F_1}(N) = y_{F_2}(N) = 4N$. *Amenities are given by this table:*

	F_1	F_2
w_1	5	0
w_2	0	5
w_3	0	0

Upon inspection, it is clear that in every stable allocation, all workers will be employed, with $\mu(w_1) = F_1$, and $\mu(w_2) = F_2$.

Let's first consider stable allocations (μ, s) in which $\mu(F_1) = \{w_1, w_3\}$ and $\mu(F_2) = \{w_2\}$. Having No Poaching requires that firm F_2 be unwilling to pay salary $s(F_1)$ to poach worker w_3:

$$\pi_{F_2}\big(2, s(F_1)\big) < \pi_{F_2}\big(1, s(F_2)\big).$$

Given the firms' production functions, this is equivalent to the requirement that $8 - 2s(F_1) < 4 - s(F_2)$, which in turn is equivalent to the requirement that $4 + s(F_2) < 2s(F_1)$. As such, given that $\mu(F_1) = \{w_1, w_3\}$ and $\mu(F_2) = \{w_2\}$, the minimal value of $s(F_2)$ consistent with having No Poaching is obtained when $s(F_2) = 0, s(F_1) > 2$. Such salaries would also imply the allocation (μ, s) has No Envy and No Firing provided $s(F_1) \leq 4$. These salaries yield firm F_2 profit $\pi_{F_2}(1, 0) = 4$ and yield firm F_1 profit $\pi_{F_1}\big(2, s(F_1)\big) = 8 - 2s(F_2) < 4$.

Let's now consider stable allocations (μ', s') with $\mu'(F_1) = \{w_1\}$ and $\mu'(F_2) = \{w_2, w_3\}$. Symmetrically, such allocations will be stable when $s'(F_2) > 2$, and $s'(F_1) = 0$. This yields firm F_1 profit $\pi_{F_1}(1, 0) = 4$ and yields firm F_2 profit $\pi_{F_2}\big(2, s'(F_2)\big) < 4$. This demonstrates that the stable allocation which is optimal for firm F_1 differs from the stable allocation which is optimal for firm F_2. Thus while Theorem 3 told us that there is a worker-optimal stable allocation, we see here that there is no firm-optimal stable allocation.

Given that in every stable allocation worker w_1 is matched to firm F_1 and worker w_2 is matched to firm F_2, these workers preferences over stable allocations depend only on $s(F_1)$ and $s(F_2)$ respectively. In this example, there are stable allocations in which each of $s(F_1)$ and $s(F_2)$ are equal to 0, but no stable allocation in which they are both equal to 0. Thus the example also demonstrates that there is no worker-pessimal stable allocation.

B.2 Existing substitutes conditions

Hatfield and Milgrom (2005) present a model which nests both the Gale and Shapley (1962) college admissions model and the Kelso and Crawford (1982) job matching model. They show that a substitutes condition guarantees the existence of a stable matching. In the same model, Hatfield and Kojima (2008) demonstrate a sense in which a weaker substitutes condition is necessary to guarantee the existence of a stable matching. In this subsection, we show that our gross substitutes condition (Assumption 1) implies neither the Hatfield and Milgrom (2005) substitutes condition nor the Hatfield and Kojima (2008) weak substitutes condition.

The Hatfield and Milgrom model studies contracting between a set of 'hospitals' (i.e., firms) and 'doctors' (i.e., workers). A **contract** $x \in X$ is 'bilateral', and is thus associated with a single doctor x_D and a single hospital x_H. Contracts may be also associated with additional characteristics, such as a salary. Given any hospital h and subset of contracts $X' \subseteq X$, the **chosen set** $C_h(X') \subseteq X'$ represents h's preferred subset of contracts. Hospital h's **rejected set** $R_h(X')$ is the complement of its chosen set: $R_h(X') \equiv X' \setminus C_h(X')$.

Contracts are **substitutes** for hospital h if for all subsets $X' \subseteq X'' \subseteq X$ we have $R_h(X') \subseteq R_h(X'')$. Contracts are **weak substitutes** for hospital h if for all subsets $X' \subseteq X'' \subseteq X$ such that $x_D = y_D$ implies $x = y$ we have $R_h(X') \subseteq R_h(X'')$.

In other words, contracts are not substitutes if expanding the set of potential contracts means that some contract is no longer rejected. The weak substitutes condition is identical, but it only considers expanded sets of potential contracts containing each doctor at most once.

Our model can be represented in the Hatfield and Milgrom framework as follows. Let a contract x be a hospital-doctor-salary tuple $(x_H, x_D, x_s) \in X = \mathbf{F} \times \mathbf{W} \times \mathbb{R}^+$. A hospital $h \in \mathbf{F}$ selects the chosen set

$$
\begin{aligned}
C_h(X') = \underset{X'' \subseteq X'}{\arg\max} \left\{ y_h\left(|X''|\right) - \sum_{x \in X''} x_s \right\} \text{ subject to } & \forall x \in X'' : x_H = h; \\
& \forall x, x' \in X'' : x \neq x' \implies x_D \neq x'_D; \\
& \forall x, x' \in X'' : x_s = x'_s.
\end{aligned}
$$

The conditions $\forall x \in X'' : x_H = h$ (requiring that a hospital picks only contracts involving itself) and $\forall x, x' \in X'' : x \neq x' \implies x_D \neq x'_D$ (requiring that a hospital picks only one contract involving each doctor) are imposed by Hatfield and Milgrom. Our additional requirement $\forall x, x' \in X'' : x_s = x'_s$ requires that hospitals set homogeneous salaries.

Assumption 1 does not guarantee that these chosen sets will satisfy the Hatfield and Milgrom substitutes condition. For example, consider a hospital h with constant marginal product $y_h(N) = 4N$. (Given a constant marginal product, Lemma 1 tells us that Assumption 1 is satisfied.) Let there be three doctors d_1, d_2, d_3.

Consider the set of contracts

$$X'' = \{(h, d_1, 1), (h, d_2, 2), (h, d_3, 2)\}.$$

X'' represents a context in which the hospital can pay salary 1 to hire doctor d_1 or can pay salary 2 to hire either doctor d_2 or d_3. Hiring only doctor d_1 at salary 1 yields the hospital a profit of $4 - 1 = 3$ whereas hiring both d_2 and d_3 at salary 2 yields the hospital a profit of $8 - 4 = 4$. Thus $C_h(X'') = \{(h, d_2, 2), (h, d_3, 2)\}$ and $R_h(X'') = \{(h, d_1, 1)\}$. However, if the hospital can only hire either d_1 or d_2, as represented by

$$X' = \{(h, d_1, 1), (h, d_2, 2)\},$$

then it will prefer to do so at the minimal possible salary: $C_h(X') = \{(h, d_1, 1)\}$ and $R_h(X') = \{(h, d_2, 2)\}$. Thus $X' \subseteq X''$ but $R_h(X') \nsubseteq R_h(X'')$, breaching both the substitutes condition and the weak substitutes condition.

That our gross substitutes condition does not imply the Hatfield and Milgrom substitutes condition means that our results do not follow trivially from existing work.

That our gross substitutes condition does not imply the Hatfield and Kojima weak substitutes condition may be surprising: Hatfield and Kojima show that, when some hospital's preferences fail the weak substitutes condition, one can construct an example such that a stable allocation does not exist. This might seem to contradict our Theorem 1, which implied that a stable allocation always exists. The supposed contradition is resolved by noting that Hatfield and Kojima's result requires the existence of a second hospital with strict preferences over different doctors. Such a hospital is ruled out by our model, which assumes that hospitals view doctors as fungible.

B.3 The core

An allocation is in the core if no coalition of workers and firms can deviate from it, producing more value than the sum of their payoffs in the original allocation. In many matching models, the core coincides with stability, leading to the two terms to be used somewhat interchangeably. For example, Kelso and Crawford refer to their solution concept as the core. Our definition of stability requires that, in a deviating coalition, every worker receives the same salary. This restriction on transfers means that the core does not coincide with stability. One implication is that stable allocations can be inefficient (Corollary 1). A core allocation can never be inefficient: if it were, the coalition consisting of every worker and firm could deviate from it.

B.4 Pairwise stability

Pairwise stability requires that no worker-firm pair can unilaterally deviate such that both are better off (Roth & Sotomayor, 1990). In some many-to-one matching models, an allocation (or, in models without salaries, simply a matching) is stable if and only if it is pairwise stable. That equivalence between pairwise stability and stability does not hold in our model.

Proposition 2 told us that stable allocations must satisfy No Poaching: no firm can unilaterally increase its salary, attract more workers, and make at least as much profit. Higher salaries must be paid to a firm's existing workers as well as to the workers that it poaches. A firm may be willing to increase its salary when doing so would attract many workers, but not when doing so would attract only a single worker. Thus pairwise stability is a weaker requirement than the stability.

B.5 Competitive equilibria

An allocation (μ, s) is a **competitive equilibrium** if

$$\forall F \in \mathbf{F} : |\mu(F)| \in \operatorname*{arg\,max}_{L \in \mathbb{N}} \{\pi_F(L, s(F))\}, \tag{26}$$

$$\forall w \in \mathbf{W} : \mu(w) \in \operatorname*{arg\,max}_{F \in \mathbf{F} \cup \{\emptyset\}} \{\alpha_w(F) + s(F)\}. \tag{27}$$

In a competitive equilibrium, firms choose quantities taking salaries as fixed, while workers choose firms taking salaries as fixed.

Lemma B.1. *An allocation is a competitive equilibrium if and only if it has both Marginal Product Salaries and No Envy.*

Proof. Expression (27) is equivalent to requiring that the allocation has No Envy. It thus remains for us to show that expression (26) is equivalent to requiring that the allocation has Marginal Product Salaries.

We first show that an allocation satisfying equation (26) has Marginal Product Salaries. Let (μ, s) be an allocation. If for some firm F: $s(F) > \Delta^-_\mu(F) = y_F\left(|\mu(F)|\right) - y_F\left(|\mu(F)| - 1\right)$ then $y_F\left(|\mu(F)|\right) - y_F\left(|\mu(F)| - 1\right) - s(F) < 0$. Thus

$$\begin{aligned}
\pi_F\left(|\mu(F)|, s(F)\right) &= y_F\left(|\mu(F)|\right) - s(F)\,|\mu(F)| \\
&= y_F\left(|\mu(F)| - 1\right) - s(F)\left(|\mu(F)| - 1\right) + y_F\left(|\mu(F)|\right) - y_F\left(|\mu(F)| - 1\right) - s(F) \\
&< y_F\left(|\mu(F)| - 1\right) - s(F)\left(|\mu(F)| - 1\right) \\
&= \pi_F\left(|\mu(F)| - 1, s(F)\right),
\end{aligned}$$

and thus equation (26) fails to hold. A similar contradiction arises if $s(F) < \Delta^+_\mu(F) = y_F\left(|\mu(F)| + 1\right) - y_F\left(|\mu(F)|\right)$. By the contrapositive, an allocation satisfying equation (26) has Marginal Product Salaries.

We next show that if an allocation has Marginal Product Salaries, then it satisfies equation (26). Let equation (26) fail for (μ, s): there exists L' such that

$$\pi_F\left(L', s(F)\right) > \pi_F\left(|\mu(F)|, s(F)\right).$$

If $L' > |\mu(F)|$, then decreasing differences implies that $y_F(L') - y_F\left(|\mu(F)|\right) < \Delta^+_\mu(F)\left(L' - |\mu(F)|\right)$. Thus,

$$\begin{aligned}
\pi_F\left(L', s(F)\right) > \pi_F\left(|\mu(F)|, s(F)\right) &\implies y_F(L') - y_F\left(|\mu(F)|\right) > s(F)\left(L' - |\mu(F)|\right) \\
&\implies \Delta^+_\mu(F)\left(L' - |\mu(F)|\right) > s(F)\left(L' - |\mu(F)|\right) \\
&\implies \Delta^+_\mu(F) > s(F),
\end{aligned}$$

and so the allocation lacks Marginal Product Salaries. Similarly if $L' < |\mu(F)|$, then decreasing differences implies that $\Delta^-_\mu(F)\left(|\mu(F)| - L'\right) < y_F\left(|\mu(F)|\right) - y_F(L')$, and thus

$$\begin{aligned}
\pi_F\left(L', s(F)\right) > \pi_F\left(|\mu(F)|, s(F)\right) &\implies y_F\left(|\mu(F)|\right) - y_F(L') < s(F)\left(|\mu(F)| - L'\right) \\
&\implies \Delta^-_\mu(F)\left(|\mu(F)| - L'\right) < s(F)\left(|\mu(F)| - L'\right) \\
&\implies \Delta^-_\mu(F) < s(F),
\end{aligned}$$

which again is inconsistent with the allocation having Marginal Product Salaries. Thus an allocation failing equation (26) must lack Marginal Product Salaries. By the contrapositive, if an allocation has Marginal Product Salaries, then it must also satisfy equation (26). □

Given Lemma B.1, we can rely on earlier results to characterize the set of competitive equilibria: by Corollary 2 they are stable, by Proposition 3 they are efficient, and by the proof of Theorem 1 they exist. Thus we have the following corollary:

Corollary B.1. *A competitive equilibrium exists. Moreover, if (μ, s) is a competitive equilibrium, then (μ, s) is a stable allocation, and μ is efficient.*

Competitive equilibria treat firms as naïve. Given an efficient allocation, a firm does not realize that hiring inefficiently few workers would let it pay lower salaries. Given an inefficient allocation, a firm will demand more workers than are willing to work at the prevailing salary. Thus competitive equilibria are necessarily efficient.

In a stable allocation, firms cannot unilaterally reduce salaries: doing so would require the consent of their existing workers. In this sense, stability also forces firms to take the salary level as given (although a firm can always increase its salary). Theorem 1 told us that this mechanism prevents firms from destabilizing an efficient allocation. However, stability does let firms understand that they cannot employ arbitrarily many workers at the prevailing salary. Firms can thus resist the temptation to destabilize an inefficient allocation: while an efficient allocation is stable, inefficient allocations may be as well. This suggests that inefficient stable allocations are caused by firms failing to take salaries as given. We formalize firms choosing salaries below.

Although Corollary B.1 connects competitive equilibria and efficient stable allocations, they are not equivalent. Every competitive equilibrium is an efficient stable allocation, but not every efficient stable allocation is a competitive equilibrium. For example, a monopsonist might be able to employ all available workers at a salary below their marginal products. This could be an efficient stable allocation but could never be a competitive equilibrium because, at that salary, the monopsonist would prefer to employ additional workers. Thus stable allocations can yield efficient quantities without getting prices 'right'.

Similarly, it's worth noting that there could be many competitive equilibria. Because they are efficient, they will generically have the same matching of workers to firms, but they can have differing salary schedules. This is a limitation of Kojima (2007)'s analysis. Kojima argues that strategic salary-setting by firms can be better for inframarginal workers than a competitive equilibrium, as firms increase salaries to compete for marginal workers. Kojima limits his comparisons to the firm-optimal competitive equilibrium. This perspective is limiting: no worker benefits from firms' strategic salary-setting when it differs from the worker-optimal stable allocation presented in Theorem 3. By Corollary B.1, the worker-optimal stable allocation is also a competitive equilibrium.

B.6 Bertrand equilibria

The Bertrand salary-setting game is a two-stage game. In the first period, firms simultaneously choose salaries. In the second period, each worker chooses a firm. Thus each firm F's strategy is $s_F \in \mathbb{R}$ while each worker w's strategy is a function, which takes as input the vector of salaries and selects a firm:

$$\mathrm{Ch}_w : \mathbb{R}^{|\mathbf{F}|} \to \mathbf{F} \cup \{\emptyset\}.$$

Let $s \equiv (s_F)_{F \in \mathbf{F}}$ denote the vector of salaries chosen by firms. Let $\mathrm{Ch} \equiv (\mathrm{Ch}_w)_{w \in \mathbf{W}}$ denote the vector of choice functions chosen by workers. Let $L_F^\circ(s_F, s_{-F}, \mathrm{Ch})$ denote the number of workers for whom $\mathrm{Ch}_w = F$, given the vector of salaries with firm F's element equal to s_F and other elements equal to the corresponding element of $s_{-F} \equiv (s_{F'})_{F' \neq F}$. Note that $L^{\circ}_F(\cdot)$ differs from our definition of $L_F(\cdot)$ in Section 3: $L_F(\cdot)$ allocated a worker indifferent between two firms to both, whereas $L^{\circ}_F(\cdot)$ allocates such a worker to only one.

A **Bertrand equilibrium** (Ch^*, s^*) is a subgame perfect pure strategy Nash equilibrium of the Bertrand salary-setting game. It comprises a vector of choice functions Ch^* and a vector of salaries s^*. Firms set salaries optimally, given the other firms' salaries and the workers' choice functions:

$$\forall F : s^*_F \in \operatorname*{arg\,max}_{s \in \mathbb{R}} \left\{ \pi_F \left(L^{\circ}_F \left(s, s^*_{-F}, \mathrm{Ch}^* \right), s \right) \right\}. \tag{28}$$

Workers' choice functions are optimal given all possible salaries:

$$\forall w : \forall s : \mathrm{Ch}^*_w(s) \in \operatorname*{arg\,max}_{F \in \mathbf{F} \cup \emptyset} \left\{ \alpha_w(F) + s_F \right\}. \tag{29}$$

To connect this solution concept to our earlier analysis, we say that an allocation (μ, s) is a Bertrand equilibrium if there exists a Bertrand equilibrium (Ch, s) such that $\forall w : \mathrm{Ch}_w(s) = \mu(w)$.

We will focus on the case where all firms have constant returns to scale. The below lemma shows that this yields a simple characterization of Bertrand equilibria. We will use that characterization to show that Bertrand equilibria are stable.

Lemma B.2. *Let all firms have constant returns to scale $y_F(N) = \Delta_F N$. Let (μ, s) be a Bertrand equilibrium. For every firm F: either $\mu(F) = \emptyset$, $s(F) = 0$, or there exists a firm $F' \neq F$ and a worker $w \in \mu(F)$ such that $s(F) = \alpha_w(F') - \alpha_w(F) + \Delta_{F'}$.*

Proof. Consider a firm F with $\mu(F) \neq \emptyset$ and $s(F) > 0$. (Recall salaries are non-negative by definition.) For such a firm to be worse off unilaterally decreasing its salary, such a decrease must cause it to lose a worker. Thus there must be some worker w and some firm $F' \neq F$ such that $\mathrm{Ch}_w(s(F), s_{-F}) = F$ but, for all $r < s(F)$: $\mathrm{Ch}_w(r, s_{-F}) = F'$. By equation (29) $\alpha_F + s(F) = \alpha_{F'} + s(F')$. The firm F' would benefit by slightly increasing its salary and poaching worker w unless its salary equals its marginal product $\Delta_{F'}$. Thus $s(F') = \Delta_{F'}$. Combining these two expressions implies that $s(F) = \alpha_w(F') - \alpha_w(F) + \Delta_{F'}$. □

Lemma B.2 highlights the effects of competition in the Bertrand game. No firm will pay a salary such that its workers strictly prefer that firm over other firms: if it did so, it could profitably decrease its salary. Thus at each firm there will be some worker who is indifferent between working at that firm and working at another firm. That other firm could poach the worker by paying an infinitesimally higher salary; for that to be unprofitable, its salary must already equal its marginal product.

We will use Lemma B.2 to argue that Bertrand equilibria can be more efficient than some stable allocations. Before doing so, we use it to prove the following proposition, which argues that Bertrand equilibria are themselves generically stable allocations.

Proposition B.1. *Let each firm F have constant returns to scale $y_F(N) = \Delta_F N$. For almost all technologies Δ_F and amenities $\alpha_w(F)$, all Bertrand equilibria are stable allocations.*

Proof. Consider a Bertrand equilibrium (μ, s). We will show that (μ, s) has No Envy and No Firing. We will then use Lemma B.2 to show that (μ, s) will only lack No Poaching in a knife-edge case.

Step 1: (μ, s) has No Envy.

Step 1 follows immediately from equation (29).

Step 2: (μ, s) has No Firing.

Proof of Step 2: Given constant returns to scale, $s(F) > \Delta^-_\mu(F)$ implies that firm F makes negative profits. F would be better off choosing salary $s(F) = 0$ and making non-negative profits. Thus in every Bertrand equilibrium, $s(F) \le \Delta^-_\mu(F)$.

Step 3: (μ, s) will almost always have No Poaching.

Proof of Step 3: Consider first the case where (μ, s) lacks No Poaching by strict inequality:

$$\exists F \in \mathbf{F}, s' > s(F), L \in \mathbb{N} \text{ with } 0 < L \le L_F(s', s) \text{ such that } \pi_F\left(L, s'\right) > \pi_F\left(\left|\mu(F)\right|, s(F)\right).$$

By constant returns to scale:

$$(\Delta_F - s')L_F(s', s) \ge (\Delta_F - s')L > \left(\Delta_F - s(F)\right)\left|\mu(F)\right|.$$

Given that this inequality is strict, there exists an $s'' > s'$ such that

$$(\Delta_F - s'')L_F(s', s) > \left(\Delta_F - s(F)\right)\left|\mu(F)\right|.$$

If a worker is indifferent between working at F and some other firm F' at salary s', she will strictly prefer working at F at salary s''. By equation (29), $L^\circ_F(s'', s) \ge L_F(s', s)$ and so

$$(\Delta_F - s'')L^\circ_F(s'', s) > \left(\Delta_F - s(F)\right)\left|\mu(F)\right|,$$

implying that (μ, s) is not a Bertrand equilibrium.

Now consider the case where (μ, s) only lacks the No Poaching by equality:

$$\exists F_1 \in \mathbf{F}, s' > s(F_1), L \in \mathbb{N} \text{ with } 0 < L \le L_{F_1}(s', s) \text{ such that } \pi_{F_1}(L, s') = \pi_{F_1}\left(\left|\mu(F_1)\right|, s(F_1)\right), \tag{30}$$

while that equality does not hold for any lower salary. That means that at least one worker $w_1 \notin \mu(F_1)$ must be indifferent between working for some firm $F_2 \in \mathbf{F} \cup \{\emptyset\} \setminus \{F_1\}$ at salary $s(F_2)$ and for firm F_1 at salary s':

$$s' = s(F_2) + \alpha_{w_1}(F_2) - \alpha_{w_1}(F_1). \tag{31}$$

If $\mu(F) = \emptyset$, then $s(F) = y_F(1) = \Delta_F$ and so firm F could never raise its salary and make positive profit. Thus $\mu(F) \ne \emptyset$. Given Lemma B.2, this tells us that:

$$s(F_1) = 0 \text{ or } \exists F_3 \in \mathbf{F} \cup \{\emptyset\} \setminus \{F_1\}, w_2 \in \mu(F_1) \text{ such that } s(F_1) = \alpha_{w_2}(F_3) - \alpha_{w_2}(F_1) + \Delta_{F_3} \tag{32}$$

$$\text{and } s(F_2) = 0 \text{ or } \exists F_4 \in \mathbf{F} \cup \{\emptyset\} \setminus \{F_2\}, w_3 \in \mu(F_2) \text{ such that } s(F_2) = \alpha_{w_3}(F_4) - \alpha_{w_3}(F_2) + \Delta_{F_4}. \tag{33}$$

Combining equations (30)-(33), the following must hold:

$$\begin{aligned}
&\left(L - \left|\mu(F_1)\right|\right)\Delta_{F_1} - L\left(\alpha_{w_1}(F_2) - \alpha_{w_1}(F_1)\right) \\
&\qquad = \begin{cases} 0 & \text{if } s(F_1) = s(F_2) = 0; \\ -\left|\mu(F_1)\right|\left(\alpha_{w_2}(F_3) - \alpha_{w_2}(F_1) + \Delta_{F_3}\right) & \text{if } s(F_1) \ne 0, s(F_2) = 0; \\ L\left(\alpha_{w_3}(F_4) - \alpha_{w_3}(F_2) + \Delta_{F_4}\right) & \text{if } s(F_1) = 0, s(F_2) \ne 0; \\ L\left(\alpha_{w_3}(F_4) - \alpha_{w_3}(F_2) + \Delta_{F_4}\right) - \left|\mu(F_1)\right|\left(\alpha_{w_2}(F_3) - \alpha_{w_2}(F_1) + \Delta_{F_3}\right) & \text{if } s(F_1) \ne 0, s(F_2) \ne 0. \end{cases}
\end{aligned}$$

That condition is, admittedly, quite opaque. For our purposes, its critical property is that is expressed solely in terms of technologies, amenities and the integer-valued L and $\left|\mu(F_1)\right|$. Thus the amenities and technologies for which such an expression can hold have measure 0. This implies that for almost all amenities and technologies, if the Bertrand equilibrium lacks No Poaching, then it lacks it by strict inequality. We showed above that if the an allocation lacks No Poaching by strict inequality, then that allocation is not a Bertrand equilibrium. $\square$

The critical distinction between a Bertrand equilibrium and other stable allocations is that, in a Bertrand equilibrium, firms can unilaterally decrease their salaries. Interestingly, this does not imply that Bertrand equilibria are less efficient or better for firms than other stable allocations.

This point can be demonstrated by revisiting Example B.1. Let (μ, s) be a Bertrand equilibrium. No firm will pay strictly more than the other: if $s(F_1) > s(F_2)$, for example, firm F_1 would retain its current workers by paying any salary $s' \in \left(s(F_2), s(F_1)\right)$. Such a salary would increase firm F_1's profit. Thus $s(F_1) = s(F_2)$, and so worker w_3 will be indifferent between the two firms. If $s(F_1) < 4$ and $\mu(w_3) = F_2$, firm F_1 could pay an infinitesimally higher salary $s' \in \left(s(F_1), 4\right)$, employ w_3 and make profit $2 \times (4 - s') > 4 - s(F_1)$. Similarly, if $s(F_2) < 4$ and $\mu(w_3) = F_1$ then firm F_2 could profit by paying an infinitesimally higher salary. Thus in every Bertrand equilibrium, $s(F_1) = s(F_2) = 4$, and both firms make zero profit.

Consider this other allocation:

$$\mu' = \begin{pmatrix} w_1 & w_2 & w_3 \\ F_1 & F_2 & F_2 \end{pmatrix},\ s'(F_1) = 0,\ s'(F_2) = 3.$$

At allocation (μ', s'), both firms make positive profit: $\pi_{F_1} = 4$; $\pi_{F_2} = 2$. Yet (μ', s') is a stable allocation: that this allocation has No Envy and No Firing is self-evident, while it has No Poaching because firm F_1 would have to pay salary 3 to poach w_2, which would yield F_1 a profit of 2.

In Bertrand competition, firms are constantly tempted to decrease their salaries. This can destabilise profitable allocations, eventually making all firms worse off. In a stable allocation, firms cannot decrease their salaries while retaining their existing workers. Example B.1 shows that firms can benefit from an inability to decrease their salaries.

However, Bertrand equilibria are not necessarily efficient or worker-optimal. In Example 1, the unique Bertrand equilibrium has the firm paying salary zero. The efficiency of Bertrand equilibria depends on whether firms can be induced to compete for marginal workers.

C No Stable Allocation with Non-Fungible Workers

Throughout the main text we assumed that workers are fungible: a firm's output depended only on the number of workers it employed. In this appendix, we show that a stable allocation may not exist when that assumption is dropped. In fact, a stable allocation may not exist even in the simple case where firms have homogeneous technologies which are additive in workers' productivities, and where there are no worker-firm amenities. We present an example in which a stable allocation does not exist. To simplify our exposition — and because the case may be of independent interest — we first characterize stable allocations in this simple case.

This model again comprises a set of firms $\mathbf{F}$ and a set of workers $\mathbf{W}$. There are fewer firms than workers. Each worker $w \in \mathbf{W}$ is endowed with a productivity $\rho_w > 0$. Firms are symmetric and have output equal to the sum of the productivities of the workers to which they are matched. Thus if firm F employs workers $C \subseteq W$ at salary s its profit will be

$$\pi_F(C,s) = \sum_{w \in C} \left[\rho_w - s\right].$$

Workers care only about their salary: $u_w(F,s) = s$. We consider the generic case where each worker's productivity is different to that of every other: $w \neq w' \implies \rho_w \neq \rho_{w'}$.

Matchings and allocations are defined as in the main text, and the definition of a stable allocation is unchanged.

Proposition C.1. *Any stable allocation can be characterized by a labelling of the M firms $1, 2, \ldots, M$ and a set of intervals $\left\{[0,s_1), [s_1,s_2), \ldots, [s_M, s_{M+1})\right\}$, where $s_j < s_{j+1}$ and $s_{M+1} = \infty$. The firm labelled j will pay salary s_j and will hire all workers with productivity in $[s_j, s_{j+1})$. Workers with productivity in $[0, s_1)$ will be unemployed. Moreover, all firms must make the same profit and firms must be making profit no less than the sum of unemployed worker productivities.*

Proof: Let (μ, s) denote a stable allocation. We prove Proposition C.1 in seven steps.

Step 1: $\forall F : \forall w \in \mu(F) : \rho_w \geq s(F)$.

Proof of Step 1: Otherwise, $\left(F, \mu(F) \setminus \{w\}, s(F)\right)$ would block (μ, s).

Step 2: If $s(F) < s(F')$ and $\mu(w) = F$, $s(F') > \rho_w$.

Proof of Step 2: Otherwise, $\left(F', \mu(F') \cup \{w\}, s(F')\right)$ would block (μ, s).

Step 3: $\forall F : \mu(F) \neq \emptyset$.

Proof of Step 3: By Step 1, $\forall w : \rho_w \geq s\left(\mu(w)\right)$. Moreover each ρ_w differs and there are fewer firms than workers. Thus there is some worker w for whom $\rho_w > s\left(\mu(w)\right)$. If $\exists F : \mu(F) = \emptyset$, $\left(F, \{w\}, \rho_w\right)$ would block (μ, s).

Step 4: $\forall F \neq F' : s(F) \neq s(F')$.

Proof of Step 4: Assume $s(F) = s(F')$. By Step 3, both firms are matched to at least one worker. There is at most one worker with $\rho_w = s(F)$. Thus by Step 1, at least one of F or F' must be matched to a worker w' with $\rho_{w'} > s(F) = s(F')$. If $\mu(w') = F$ then $\left(F', \mu(F') \cup \{w'\}, s(F')\right)$ would block (μ, s). If $\mu(w') = F'$ then $\left(F, \mu(F) \cup \{w'\}, s(F)\right)$ would block (μ, s).

Step 5: If $\mu(w) = \emptyset : \forall F : \rho_w < s(F)$.

Proof of Step 5: Otherwise, $\left(F, \mu(F) \cup \{w\}, s(F)\right)$ would block (μ, s).

Step 6: All firms must make the same profit.

Proof of Step 6: If firm F made more profit than firm F', then $\left(F', \mu(F), s(F)\right)$ would block (μ, s).

Step 7: Each firm's profit must be no less than the sum of unemployed worker productivities.

Proof of Step 7: Let C denote the set of unemployed workers. If $\sum_{w \in C} \rho_w > \pi_F\left(\mu(F), s(F)\right)$, then $(F, C, 0)$ would block (μ, s).

Step 4 implies that firm salaries form disjoint intervals $[s_1, s_2), [s_2, s_3), \ldots$. Steps 1, 2 and 5 imply that the firm paying salary s_j will hire all workers with productivity in $[s_j, s_{j+1})$. Steps 6 and 7 correspond to Proposition C.1's final sentence. □

Given Proposition C.1, it is relatively simple to produce an example without a stable allocation. The following is such an example.

Example C.1 (with nonfungible workers, a stable allocation need not exist). $\mathbf{F} = \{F_1, F_2, F_3\}$.$\mathbf{W} = \{1, 2, 5, 6\}$. *Each worker is labelled by their productivity: $\forall w \in \mathbf{W} : \rho_w = w$.*

We will now show that Example C.1 lacks a stable allocation. We will consider each candidate matching in turn, exploiting Proposition C.1 to limit the number of matchings we must consider. As firms are homogeneous it is without loss of generality to assume that firm F_1 employs the least productive employed worker(s) and that firm F_3 employs the most productive.

Candidate matching 1: $\begin{pmatrix} 1 & 2 & 5 & 6 \\ \emptyset & F_1 & F_2 & F_3 \end{pmatrix}$.

$\mu(1) = \emptyset$, and thus, by Proposition C.1, each firm must make profit no less than 1. This is impossible for firm F_3, as Proposition C.1 implies that $s(F_3) > 5$.

Candidate matching 2: $\begin{pmatrix} 1 & 2 & 5 & 6 \\ F_1 & F_1 & F_2 & F_3 \end{pmatrix}$.

By Proposition C.1, $s(F_1) \leq 1$, and thus $\pi_{F_1} \geq 1$. This is again impossible for firm F_3, as Proposition C.1 implies that $s(F_3) > 5$.

Candidate matching 3: $\begin{pmatrix} 1 & 2 & 5 & 6 \\ F_1 & F_2 & F_2 & F_3 \end{pmatrix}$.

By Proposition C.1, $s(F_2) \leq 2$, and thus $\pi_{F_2} \geq 3$. This is again impossible for firm F_3, as Proposition C.1 implies that $s(F_3) > 5$.

Candidate matching 4: $\begin{pmatrix} 1 & 2 & 5 & 6 \\ F_1 & F_2 & F_3 & F_3 \end{pmatrix}$.

By Proposition C.1, $s(F_3) \leq 5$, and thus $\pi_{F_3} \geq 1$. This is impossible for firm F_2, as Proposition C.1 implies that $s(F_2) > 1$. This completes the proof that Example C.1 lacks a stable allocation.

If the productivities in this example are perturbed, a stable allocation exists. This suggests that a stable allocation may almost-always exist. It also suggests that there might always exist a weak stable allocation, which is defined to consider only a breaking coalition of workers and firms who are *all* strictly better off compared to the candidate allocation. We leave investigation of these conjectures for future work.

D Representing our Model as the Limit of A Shirking Cost Model

We motivated our assumption that each firm pays all its workers the same salary by referencing the literature showing that within-firm salary inequality causes workers to shirk (Breza et al., 2017). In this appendix we study these shirking effects explicitly. We consider a model in which a firm can pay some of its workers more than others (as in Kelso and Crawford's model) but doing so causes workers to shirk. We show that this model aligns with our baseline model when the shirking cost is arbitrarily large.

The model in this appendix again comprises a set of firms $\mathbf{F}$ and a set of workers $\mathbf{W}$. As in our baseline model, each worker has quasi-linear preferences

$$u_w(F, s) \equiv \alpha_w(F) + s.$$

A firm can now pay different salaries to different workers, but doing so incurs a shirking cost. Its payoff is thus

$$\pi_F\Big(C, \big(s(w)\big)_{w \in C}; \phi\Big) \equiv y_F(|C|) - \sum_{w \in C} s(w) - \phi \operatorname{var}\Big(\big(s(w)\big)_{w \in C}\Big),$$

where C is the set of workers employed by the firm, $\big(s(w)\big)_{w \in C}$ concatenates the employed workers' salaries, y_F is the firm's production function (which as before depends only on the number of workers employed, implying that workers are fungible) and $\phi \operatorname{var}\Big(\big(s(w)\big)_{w \in C}\Big)$ is the shirking cost. For simplicity we assume that the shirking cost is proportional to the variance of employed workers' salaries, with constant of proportionality ϕ. We will refer to ϕ as the **shirking parameter**. The shirking parameter ϕ lies in the extended non-negative reals: $\phi \in [0, \infty]$. When $\phi = \infty$, we take the firm's payoff to be

$$\pi_F\left(C, \left(s(w)\right)_{w\in C}; \infty\right) = \begin{cases} y_F(|C|) - \sum_{w\in C} s(w) & \text{if } \operatorname{var}\left(\left(s(w)\right)_{w\in C}\right) = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

With an infinite shirking parameter $\phi = \infty$, no firm will tolerate $\operatorname{var}\left(\left(s(w)\right)_{w\in C}\right) \neq 0$, and thus the model coincides with our baseline model.

An allocation (μ, s) comprises a matching μ (defined as in our baseline model) and a salary schedule $s : \mathbf{W} \to \mathbb{R}^+$ which now associates each worker with a salary, rather than each firm. An allocation (μ, s) is **individually rational** given shirking parameter ϕ if

- for all workers $w \in \mathbf{W}$: $u_w\left(\mu(w), s(w)\right) \geq 0$, and
- for all firms $F \in \mathbf{F}$: $\pi_F\left(\mu(F), \left(s(w)\right)_{w\in\mu(F)}; \phi\right) \geq 0$.

An allocation (μ, s) is **stable** given shirking parameter ϕ if it is individually rational and not blocked by any coalition (F, C, s^*), with $F \in \mathbf{F}$, $C \subseteq \mathbf{W}$ and $s^* : C \to \mathbb{R}^+$, where

- $\pi_F\left(C, \left(s^*(w)\right)_{w\in C}; \phi\right) \geq \pi_F\left(\left|\mu(F)\right|, \left(s(w)\right)_{w\in C}; \phi\right)$, and
- for all workers $w \in C$, $u_w\left(F, s^*(w)\right) \geq u_w\left(\mu(w), s(w)\right)$,

and the inequality is strict for the firm or one of the workers.

Let M be a correspondence from shirking costs to allocations defined as

$$M(\phi) = \left\{(\mu, s) : (\mu, s) \text{ is a stable allocation given shirking parameter } \phi\right\}.$$

We will now show that an infinite shirking cost approximates a very large shirking cost.

Proposition D.1. $\liminf_{\phi\to\infty} M(\phi) = M(\infty)$.

Proof: We will first show that $\liminf_{\phi\to\infty} M(\phi) \subseteq M(\infty)$ and then show that $M(\infty) \subseteq \liminf_{\phi\to\infty} M(\phi)$.

Step 1: $\liminf_{\phi\to\infty} M(\phi) \subseteq M(\infty)$.

Proof of Step 1: Let $(\mu, s) \notin M(\infty)$. Thus either (μ, s) is not individually rational for some worker, is not individually rational for some firm, or is blocked by a coalition. We consider these three cases in turn.

Worker individual rationality does not depend on the shirking parameter. Thus if (μ, s) is not individually rational for some worker given shirking parameter ∞, it also will not be individually rational for that worker given any shirking parameter. Thus $(\mu, s) \notin \liminf_{\phi\to\infty} M(\phi)$.

Given shirking parameter ∞, individual rationality for firm F can fail either because $\operatorname{var}\left(\left(s(w)\right)_{w\in\mu(F)}\right) \neq 0$ or because $y_F\left(\left|\mu(F)\right|\right) - \sum_{w\in\mu(F)} s(w) < 0$. If $\operatorname{var}\left(\left(s(w)\right)_{w\in\mu(F)}\right) \neq 0$ then

$$\lim_{\phi\to\infty} \pi_F\left(\mu(F), (s(w))_{w\in\mu(F)}; \phi\right) = \lim_{\phi\to\infty}\left[y_F\left(\left|\mu(F)\right|\right) - \sum_{w\in\mu(F)} s(w) - \phi \operatorname{var}\left(\left(s(w)\right)_{w\in\mu(F)}\right)\right] = -\infty.$$

Thus (μ, s) would lack individual rationality for F given any sufficiently large ϕ. If $y_F\left(\left|\mu(F)\right|\right) - \sum_{w\in\mu(F)} s(w) < 0$ then for all $\phi > 0$:

$$\pi_F\left(\mu(F), \left(s(w)\right)_{w\in\mu(F)}; \phi\right) \leq y_F\left(\left|\mu(F)\right|\right) - \sum_{w\in\mu(F)} s(w) < 0$$

and thus $\left(\mu,s\right)$ lacks individual rationality for F given any positive shirking parameter. Either way, $\left(\mu,s\right)\notin \liminf_{\phi\to\infty} M(\phi)$.

Finally, consider the case where $\left(\mu,s\right)$ is individually rational but is blocked by a coalition (F,C,s^*) given shirking parameter ∞. Individual rationality for F requires $\mathrm{var}\left(\left(s(w)\right)_{w\in\mu(F)}\right)=0$. Similarly, $\pi_F\left(C,\left(s^*(w)\right)_{w\in C};\infty\right)\geq \pi_F\left(\mu(F),\left(s(w)\right)_{w\in\mu(F)};\infty\right)$ requires that $\mathrm{var}\left(\left(s^*(w)\right)_{w\in C}\right)=0$. Thus $\pi_F\left(C,\left(s^*(w)\right)_{w\in C};\phi\right)-\pi_F\left(\mu(F),\left(s(w)\right)_{w\in C};\phi\right)$ does not depend on the shirking parameter ϕ. Worker welfare never depends on the shirking parameter. Thus (F,C,s^*) blocks $\left(\mu,s\right)$ for any shirking parameter. Thus $\left(\mu,s\right)\notin \liminf_{\phi\to\infty} M(\phi)$. This concludes the proof that

$$\left(\mu,s\right)\notin M(\infty) \implies \left(\mu,s\right)\notin \liminf_{\phi\to\infty} M(\phi).$$

By the contrapositive, $\liminf_{\phi\to\infty} M(\phi)\subseteq M(\infty)$.

Step 2: $M(\infty)\subseteq \liminf_{\phi\to\infty} M(\phi)$.

Proof of Step 2: Assume towards a contradiction that there exists $\left(\mu,s\right)\in M(\infty)\setminus \liminf_{\phi\to\infty} M(\phi)$.

For every firm F: $\pi_F\left(\mu(F),\left(s(w)\right)_{w\in\mu(F)};\phi\right)$ is non-decreasing in ϕ. Thus if $\left(\mu,s\right)$ is not individually rational for firms given some shirking parameter ϕ, it will also not be individually rational for firms given an infinite shirking parameter. By assumption, $\left(\mu,s\right)\in M(\infty)$. Thus by the contrapositive, $\left(\mu,s\right)$ is individually rational for firms for any shirking parameter.

Worker welfare does not depend on the shirking parameter. Thus if $\left(\mu,s\right)$ is not individually rational for workers for some shirking parameter ϕ, it will not be indivudally rational for workers given an infinite shirking parameter. By assumption, $\left(\mu,s\right)\in M(\infty)$. Thus $\left(\mu,s\right)$ is individually rational for workers for any shirking parameter.

We have shown that there exists $\left(\mu,s\right)\in M(\infty)\setminus \liminf_{\phi\to\infty} M(\phi)$ such that $\left(\mu,s\right)$ is individually rational for both workers and firms given any shirking parameter. Given that $\left(\mu,s\right)\notin \liminf_{\phi\to\infty} M(\phi)$, for any shirking parameter ϕ, there exists $\phi'>\phi$ such that $\left(\mu,s\right)$ is not a stable allocation given ϕ'. We can thus construct the increasing, unbounded sequence $(\phi_n)_{n=1}^{\infty}$ such that, for each n, $\left(\mu,s\right)$ is not a stable allocation given shirking parameter ϕ_n. By individual rationality, there exists a corresponding sequence of blocking coalitions $\left(F_n,C_n,s_n^*\right)_{n=1}^{\infty}$ such that each coalition $\left(F_n,C_n,s_n^*\right)$ blocks $\left(\mu,s\right)$ given shirking parameter ϕ_n. There are a finite number of firms F_n and of potential subsets of workers C_n. Thus there must exist a pair comprising a firm and a set of workers which recurs infinitely often in the sequence $\left(F_n,C_n,s_n^*\right)_{n=1}^{\infty}$. Denote the infinitely-recurring firm as $\bar{F}$ and the infinitely-recurring subset of workers as $\bar{C}$. Let $\left(\bar{F},\bar{C},s_{n(m)}^*\right)_{m=1}^{\infty}$ be the subsequence of $\left(F_n,C_n,s_n^*\right)_{n=1}^{\infty}$ such that for each m: $F_{n(m)}=\bar{F}$ and $C_{n(m)}=\bar{C}$.

Individual rationality for the workers and firms imply that each salary schedule $s_{n(m)}^*$ is bounded. Thus by the Bolzano–Weierstrass theorem, $\left(s_{n(m)}^*\right)_{m=1}^{\infty}$ contains a convergent subsequence. Let $\left(s_{n(l)}^*\right)_{l=1}^{\infty}$ be that convergent subsequence, and let $s_\infty^*\equiv \lim_{l\to\infty} s_{n(l)}^*$ be its limit. We will show that $\left(\bar{F},\bar{C},s_\infty^*\right)$ blocks $\left(\mu,s\right)$ given shirking parameter $\phi=\infty$.

Payoffs are continuous in salaries and the shirking parameter, and thus

$$\lim_{l\to\infty}\pi_{\bar{F}}\left(\bar{C},\left(s_{n(l)}^*(w)\right)_{w\in\bar{C}};\phi_{n(l)}\right)=\pi_{\bar{F}}\left(\bar{C},\left(s_\infty^*(w)\right)_{w\in\bar{C}};\infty\right);$$
$$\lim_{l\to\infty}\pi_{\bar{F}}\left(\mu\left(\bar{F}\right),\left(s(w)\right)_{w\in\mu(\bar{F})};\phi_{n(l)}\right)=\pi_{\bar{F}}\left(\mu\left(\bar{F}\right),\left(s(w)\right)_{w\in\mu(\bar{F})};\infty\right);\text{ and}$$
$$\forall w\in\bar{C}:\lim_{l\to\infty}u_w\left(\bar{F},s_{n(l)}^*(w)\right)=u_w\left(\bar{F},s_\infty^*(w)\right).$$

Given that each $\left(\bar{F}, \bar{C}, s^*_{n(l)}\right)_{l=1}^{\infty}$ blocks (μ, s), this implies that

$$\pi_{\bar{F}}\left(\bar{C}, \left(s^*_\infty(w)\right)_{w\in\bar{C}}; \infty\right) \geq \pi_{\bar{F}}\left(\mu\left(\bar{F}\right), \left(s(w)\right)_{w\in\mu(\bar{F})}; \infty\right); \text{ and} \tag{34}$$

$$\forall w \in \bar{C} : u_w\left(\bar{F}, s^*_\infty(w)\right) \geq u_w\left(\mu(w), s(w)\right). \tag{35}$$

It thus remains only to show either that either inequality (34) is strict, or inequality (35) is strict for some worker. We will show that, if (35) is strict for *no* worker (i.e., holds with equality for all workers), it must be strict for the firm. If (35) is holds with equality for all workers:

$$\forall w \in \bar{C} : s^*_\infty(w) = \alpha_w\left(\mu(w)\right) + s(w) - \alpha_w\left(\bar{F}\right). \tag{36}$$

Given that (μ, s) is individually rational for the firm $\bar{F}$ given an infinite shirking parameter:

$$\pi_{\bar{F}}\left(\mu\left(\bar{F}\right), \left(s(w)\right)_{w\in\mu(\bar{F})}; \infty\right) \geq 0. \tag{37}$$

By inequality (34), this implies that $\pi_{\bar{F}}\left(\bar{C}, \left(s^*_\infty(w)\right)_{w\in\bar{C}}; \infty\right) \geq 0$, which in turn implies that $\mathrm{var}\left(\left(s^*_\infty(w)\right)_{w\in\bar{C}}\right) = 0$. With (36), this means that inequality (34) holds with strict inequality if and only if

$$y_{\bar{F}}\left(|\bar{C}|\right) - \sum_{w\in\bar{C}}\left[\alpha_w\left(\mu(w)\right) + s(w) - \alpha_w\left(\bar{F}\right)\right] > y_{\bar{F}}\left(\left|\mu\left(\bar{F}\right)\right|\right) - \sum_{w\in\mu(\bar{F})} s(w). \tag{38}$$

Inequality (37) also implies that $\mathrm{var}\left(\left(s(w)\right)_{w\in\mu(\bar{F})}\right) = 0$. Taking an arbitrary $\left(\bar{F}, \bar{C}, s^*_n\right)$ that blocks (μ, s), this implies that

$$y_{\bar{F}}\left(|\bar{C}|\right) - \sum_{w\in\bar{C}} s^*_n(w) - \phi_n \mathrm{var}\left(\left(s^*_n(w)\right)_{w\in\bar{C}}\right) \geq y_{\bar{F}}\left(\left|\mu\left(\bar{F}\right)\right|\right) - \sum_{w\in\mu(\bar{F})} s(w); \text{ and}$$
$$\forall w \in \bar{C} : \alpha_w\left(\bar{F}\right) + s^*_n(w) \geq \alpha_w\left(\mu(w)\right) + s(w).$$

These together imply inequality (38):

$$\begin{aligned}
& \forall w \in \bar{C} : s^*_n(w) \geq \alpha_w\left(\mu(w)\right) + s(w) - \alpha_w\left(\bar{F}\right) \\
\therefore\ & \sum_{w\in\bar{C}} s^*_n(w) \geq \sum_{w\in\bar{C}}\left[\alpha_w\left(\mu(w)\right) + s(w) - \alpha_w\left(\bar{F}\right)\right] \\
\therefore\ & y_{\bar{F}}\left(|\bar{C}|\right) - \sum_{w\in\bar{C}}\left[\alpha_w\left(\mu(w)\right) + s(w) - \alpha_w\left(\bar{F}\right)\right] - \phi_n \mathrm{var}\left(\left(s^*_n(w)\right)_{w\in\bar{C}}\right) \\
& \qquad \geq y_{\bar{F}}\left(|\bar{C}|\right) - \sum_{w\in\bar{C}} s^*_n(w) - \phi_n \mathrm{var}\left(\left(s^*_n(w)\right)_{w\in\bar{C}}\right) \geq y_{\bar{F}}\left(|\mu\left(\bar{F}\right)|\right) - \sum_{w\in\mu(\bar{F}} s(w) \\
\therefore\ & y_{\bar{F}}\left(|\bar{C}|\right) - \sum_{w\in\bar{C}}\left[\alpha_w\left(\mu(w)\right) + s(w) - \alpha_w\left(\bar{F}\right)\right] \geq y_{\bar{F}}\left(\left|\mu\left(\bar{F}\right)\right|\right) - \sum_{w\in\mu(\bar{F}} s(w).
\end{aligned}$$

To recap: the above shows that $(\bar{C}, \bar{F}, s^*_\infty)$ blocks (μ, s) given shirking parameter $\phi = \infty$. We showed inequalities (34) and (35) hold with weak inequality. It remained to be shown that one held with strict inequality. We argued that if (35) does *not* hold with strict inequality, then inequality (34) *will* hold with strict inequality if and only if inequality (38) holds. We finally showed that inequality (38) does in fact hold. Thus $\left(\bar{C}, \bar{F}, s^*_\infty\right)$ blocks (μ, s) given shirking parameter $\phi = \infty$. This contradicts our assumption that there exists $(\mu, s) \in M(\infty) \setminus \liminf_{\phi\to\infty} M(\phi)$. Thus $M(\infty) \subseteq \liminf_{\phi\to\infty} M(\phi)$, completing the proof of Step 2 and of Proposition D.1. □

E Duplicating Workers

In subsection 7.2 we showed that, when each firm has a duplicate, every stable allocation is efficient. In this appendix, we show that the effect of duplicating workers is very different: in some sense, nothing happens when workers are duplicated. Of course, given decreasing returns to scale, increasing the number of workers may decrease firms' marginal products and thus decrease salaries. We focus on the effect of workers' 'market power' by duplicating workers while 'stretching' firms' production function appropriately. This has no effect on the set of stable salary schedules.

Before formalizing that result we need to formalize the relationship between two labor markets. Consider two labor markets $(\mathbf{F},\mathbf{W})$ and $(\mathbf{G},\mathbf{X})$. Let $\mathscr{P}(\mathbf{X})$ denote the power set of $\mathbf{X}$. We say that $\psi:\mathbf{W}\cup\mathbf{F}\to\mathscr{P}(\mathbf{X}\cup\mathbf{G})$ is a **transformation** from $(\mathbf{F},\mathbf{W})$ to $(\mathbf{G},\mathbf{X})$ if $\{\psi(w):w\in\mathbf{W}\}$ partitions $\mathbf{X}$ while $\{\psi(F):F\in\mathbf{F}\}$ partitions $\mathbf{G}$.

We are interested in comparing labor markets in which all workers are duplicated and all firms are stretched. A transformation ψ from $(\mathbf{F},\mathbf{W})$ to $(\mathbf{G},\mathbf{X})$ *duplicates workers and stretches firms* if

$$\forall F\in\mathbf{F}:\ \left|\psi(F)\right|=1;$$
$$\forall w\in\mathbf{W}:\ \left|\psi(w)\right|=2;$$
$$\forall w\in\mathbf{W},x\in\psi(w),F\in\mathbf{F}:\alpha_w(F)=\alpha_x\left(\psi(F)\right);$$
$$\forall F\in\mathbf{F},N\in\mathbb{N}:y_{\psi(F)}(N)=\sum_{i=1}^{N}\left[y_F\left(\lceil i\div 2\rceil\right)-y_F\left(\lceil i\div 2\rceil-1\right)\right],\text{ where }\lceil\cdot\rceil\text{ is the ceiling function.}$$

The first condition requires that there be one firm in $\mathbf{G}$ for every firm in $\mathbf{F}$. The second condition requires that there be two workers in $\mathbf{X}$ for every worker in $\mathbf{W}$. The third condition requires that firms in $\mathbf{G}$ provide workers the same amenities as the corresponding firms in $\mathbf{F}$. The fourth condition requires that firms in $\mathbf{G}$ have production functions similar to those in $\mathbf{F}$ but stretched so that each marginal product can be produced by each of two workers.

Proposition E.1. *Let ψ be a transformation from $(\mathbf{F},\mathbf{W})$ to $(\mathbf{G},\mathbf{X})$ that duplicates workers and stretches firms. Let (μ,s) be an allocation in the labor market $(\mathbf{F},\mathbf{W})$, and let (μ',s') be an allocation in the labor market $(\mathbf{G},\mathbf{X})$ such that*

$$\forall F\in\mathbf{F}:s(F)=s'\left(\psi(F)\right);\ \textit{and}\ \forall w\in\mathbf{W},x\in\psi(w):\mu'(x)=\psi\left(\mu(w)\right).$$

(μ,s) is a stable allocation if and only if (μ',s') is a stable allocation.

Proof. We will show that the No Envy, No Firing and No Poaching conditions are equivalent across the two allocations.

Step 1: (μ,s) has No Envy if and only if (μ',s') has No Envy.

Proof of Step 1: Consider workers $w\in\mathbf{W},x\in\psi(w)$ and firms $F,F'\in\mathbf{F},G=\psi(F),G'=\psi(F')$. Given that $s(F)=s'(G)$, $s(F')=s'(G')$, $\alpha_w(F)=\alpha_x(G)$ and $\alpha_w(F')=\alpha_x(G')$:

$$\alpha_w(F)+s(F)\geq\alpha_w(F')+s(F')\iff\alpha_x(G)+s'(G)\geq\alpha_x(G')+s'(G').$$

Thus the No Envy conditions in the two allocations are equivalent.

Step 2: (μ, s) has No Firing if and only if (μ', s') has No Firing.
Proof of Step 2: Letting $G = \psi(F)$:

$$\begin{aligned}
\Delta^{-}_{\mu'}(G) &= y_F\left(\lceil|\mu'(G)| \div 2\rceil\right) - y_F\left(\lceil|\mu'(G)| \div 2\rceil - 1\right)\\
&= y_F\left(|\mu'(F)|\right) - y_F\left(|\mu'(F)| - 1\right)\\
&= \Delta^{-}_{\mu}(F),
\end{aligned}$$

where the second equality follows from two workers being matched to G for every one matched to F, and thus $\lceil|\mu'(G)| \div 2\rceil = \lceil 2 \times |\mu'(F)| \div 2\rceil = |\mu'(F)|$. Given that $s(F) = s'(G)$, it follows that

$$s(F) \leq \Delta^{-}_{\mu}(F) \iff s'(G) \leq \Delta^{-}_{\mu'}(G).$$

Step 3: (μ, s) has No Poaching if and only if (μ', s') has No Poaching.
Proof of Step 3: For some firm $F \in \mathbf{F}$, let $G = \psi(F)$. Note that for any $L \in \mathbb{N}$:

$$\begin{aligned}
y_G(2L) &= \sum_{i=1}^{2L}\left[y_F\left(\lceil i \div 2\rceil\right) - y_F\left(\lceil i \div 2\rceil - 1\right)\right]\\
&= \sum_{j=1}^{L} 2\left[y_F\left(\lceil j\rceil\right) - y_F\left(\lceil j\rceil - 1\right)\right] = 2y_F(L).
\end{aligned}$$

It follows that for any salary r:

$$\pi_G(2L, r) = y_G(2L) - r \times 2L = 2\pi_F(L, r).$$

In particular, given that $|\mu'(G)| = 2|\mu(F)|$ and $s(F) = s'(G)$: $\pi_G\left(|\mu'(G)|, s'(G)\right) = 2\pi_F\left(|\mu(F)|, s(F)\right)$.

Let (μ, s) lack No Poaching because of firm F: there exists $r > s(F)$ and $L \leq L_F(r, s)$ with $L > 0$ such that $\pi_F(L, r) \geq \pi_F\left(|\mu(F)|, s(F)\right)$. By the above, $\pi_G(2L, r) = 2\pi_F(L, r)$ and $\pi_G\left(|\mu'(G)|, s'(G)\right) = 2\pi_F\left(|\mu'(F)|, s(F)\right)$. Thus:

$$\pi_G(2L, r) \geq \pi_G\left(|\mu'(G)|, s'(G)\right).$$

Moreover, $L_G(r, s') = 2L_F(r, s)$. Thus $2L \leq L_G(r, s')$. Thus (μ', s') lacks No Poaching. By the contrapositive, if (μ', s') has No Poaching, then (μ, s) has No Poaching. The proof of the converse is symmetric. □

If firms have constant returns to scale – as they do in all of our examples – 'stretching' firms does not change them. This motivates a simpler version of Proposition E.1: Assume that each firm has constant returns to scale. If each worker is duplicated while each firm is unchanged, the set of stable allocations will be unchanged, except that the two duplicate workers take the place of the one original worker. This means that every example in this paper can be extended to involve arbitrarily many workers, with the nature of the example unchanged.

F Comparing the Efficiency of Two Stable Allocations

In Section 7 we discussed conditions under which all stable allocations would be efficient. In this appendix, we take a different tack: given two stable allocations, we ask whether it can be known which has greater value. If production and amenities are both observed, then match value can be calculated directly. However, as discussed in Section 6, such observations might be difficult to obtain.

We are thus interested in whether simpler statistics can indicate whether one allocation has greater value than another. For example, Proposition 3 told us that any stable allocation in which firms pay Marginal Product Salaries must be efficient. Unfortunately, this appendix will present an example suggesting that many plausible heuristics can fail. Rather, comparisons of match value seem to generally require observing (or restricting) amenities.

Example F.1 (insufficient statistics for efficiency). $\mathbf{F} = \{F_1, F_2\}.\mathbf{W} = \{w_1, w_2, w_3\}$. $y_{F_1}(N) = y_{F_2}(N) = 4N$. *Amenities are given by this table:*

	F_1	F_2
w_1	10	0
w_2	0	10
w_3	β	0

Example F.1 comprises three workers. Worker w_1 has a strong preference towards working for firm F_1 while worker w_2 has a strong preference towards working for firm F_2. We represent worker w_3's preferences with the parameter β: when $\beta > 0$ it is more efficient for w_3 to be matched to F_1; when $\beta < 0$ it is more efficient for w_3 to be matched to F_2.

For β close to zero, there exist both stable allocations in which w_3 is matched to F_1 and stable allocations in which w_3 is matched to F_2. As both firms have constant marginal products, that an allocation has No Firing just requires that $s(F_1)$ and $s(F_2)$ both be at most 4. When w_3 is matched to F_1 and β is sufficiently close to zero, that the allocation has No Envy is implied by $s(F_1) > s(F_2)$. If w_3 is matched to F_1, then F_2 would have to pay $s(F_1) + \beta$ to poach w_3. Thus such an allocation has No Poaching provided that

$$\left(4 - \left[s(F_1) + \beta\right]\right) \times 2 < \left(4 - s(F_2)\right) \times 1.$$

For β sufficiently close to zero, this is implied by $s(F_1) > 2 + \frac{s(F_2)}{2}$. We have shown that for β sufficiently close to zero, an allocation which matches w_3 to F_1 is a stable allocation provided that

$$s(F_1) > \max\left\{2 + \frac{s(F_2)}{2}, s(F_2)\right\};\ s(F_1) \leq 4;\ s(F_2) \leq 4. \tag{39}$$

That system of inequalities has many solutions. For example, one is $s(F_2) = 1$, $s(F_1) = 3$. Symmetrically, for $\beta \approxeq 0$, worker w_3 will be matched to F_2 if

$$s(F_2) > \max\left\{2 + \frac{s(F_1)}{2}, s(F_1)\right\};\ s(F_1) \leq 4;\ s(F_2) \leq 4. \tag{40}$$

Consider a shift from a stable allocation in which w_3 is matched to F_1 to a stable allocation in which w_3 is matched to F_2. In both allocations there will be one firm matched to 2 workers and one firm matched to 1 worker, and total output will be $3 \times 6 = 18$. Thus no measure of firm concentration (like a Herfindahl–Hirschman Index) or of total output could tell us whether the shift increased the value of the match.

In fact, one can perturb the example by adding additional workers or by making one firm more productive than the other, such that the more efficient match has greater firm concentration or less total output.

Our discussion thus far has ignored salaries. Can salaries tell us whether one allocation is more efficient than another? Unfortunately not. Proposition 5 told us that there was an efficient allocation with maximal salaries. But away from the maximum, higher salaries do not always imply greater efficiency. Returning to Example F.1, let $\beta > 0$ so that it is more efficient for w_3 to be matched to F_1. Contrast these two allocations: (μ^1, s^1) and (μ^2, s^2):

$$\mu_1 = \begin{pmatrix} w_1 & w_2 & w_3 \\ F_1 & F_2 & F_1 \end{pmatrix},\ s^1(F_1) = 2.5,\ s^1(F_2) = 0; \quad \mu_2 = \begin{pmatrix} w_1 & w_2 & w_3 \\ F_1 & F_2 & F_2 \end{pmatrix},\ s^2(F_1) = 3,\ s^2(F_2) = 4.$$

The allocations (μ^1, s^1) and (μ^2, s^2) satisfy inequalities (39) and (40) respectively, and thus both are stable. While $s^2 \geqq s^1$, the value of μ_1 is greater.

Similarly, comparing worker and firm welfare between (μ^1, s^1) and (μ^2, s^2) demonstrates that worker and firm welfare do not reveal which is more efficient. Theorem 2 told us that *some* efficient stable allocation is better for workers than any other stable allocation, but this does not imply that *every* efficient stable allocation is better for workers than every inefficient stable allocation.

Our discussion of Example F.1 suggests that the relative efficiency of two allocations cannot generally be known without knowing amenities. This suggests that there would be value in inferring amenities using the mechanism suggested in Section 6 to diagnose inefficiencies caused by market power, as well as to solve them.