

# A Finite-Geometric Theory Kernel from W33

Toward a Unified Algebra–Topology–Quantum Computation–Cryptography Framework

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## Abstract

This document consolidates the W33 tower into a single, self-contained theory kernel. Starting from the symplectic phase space  $V = \mathbb{F}_3^4$ , we construct the symplectic generalized quadrangle  $W(3, 3)$  and its point graph  $W33 = \text{SRG}(40, 12, 2, 4)$ . Over  $\mathbb{F}_2$ , the adjacency satisfies  $A^2 \equiv 0$ , producing a canonical code  $[40, 24, 6]$  and an intrinsic homology space  $H = \ker(A)/\text{im}(A) \cong \mathbb{F}_2^8$ . The nonsingular orbit in  $H$  yields a 120-element “root shell” with  $\text{SRG}(120, 56, 28, 24)$  adjacency, a 240 signed lift admitting global gauge fixing, and a quotient closure back to 40 points as  $Q = \overline{W33}$ . The quotient carries a canonical  $\mathbb{Z}_3$  holonomy, with flat faces classified exactly by the 90 non-isotropic projective lines. Over  $\mathbb{Z}_3$ , the clique complex of  $Q$  has  $H^3 \cong (\mathbb{Z}_3)^{89}$ , whose 88D core is identified (up to a canonical sign character) with the augmentation quotient on the 90 non-isotropic lines. Finally, the holonomy field  $F$  is sourced:  $J = dF$  is a 3-cochain supported on 3008 tetrahedra, and explicit sparse transfer operators map  $J$  to observed vacuum line responses.

## Remark

**What is meant by “theory of everything” here.** This manuscript presents a mathematically closed kernel in which geometry, algebra, topology, computation, and cryptography are realized as different functorial views of the same finite symplectic/projective object. Claims about physical constants require an additional scaling/continuum layer and are not asserted as part of the kernel.

## Contents

# Master Equation Summary

## Key Result

**Discrete gauge kernel (minimal equations).** Let  $Q = \overline{W33}$  be the quotient graph and  $\text{Cl}(Q)$  its clique complex.

**Field strength (holonomy).**  $F \in C^2(\text{Cl}(Q); \mathbb{Z}_3)$  is the computed triangle holonomy.

**Sources.**  $J := dF \in C^3(\text{Cl}(Q); \mathbb{Z}_3)$  is the sourced 3-cochain (supported on 3008 tetrahedra).

**Vacuum response (exact constitutive laws).** There exist explicit sparse operators

$$M, Z : \mathbb{Z}_3^{9450} \rightarrow \mathbb{Z}_3^{90}$$

such that the observed line fields satisfy

$$m_{\text{line}} = MJ, \quad z_{\text{line}} = ZJ$$

exactly.

**Vacuum harmonics.** The 90-line sector admits five canonical joint modes under the involution  $S$  and meet adjacency  $A_{\text{meet}}$ :

$$(+, 32)^1, (+, 2)^{24}, (+, -4)^{20}, (-, 8)^{15}, (-, -4)^{30}.$$

Bulk and boundary source classes inject into different harmonic mixtures (mode-response tables).

## 1 Master Equations and Couplings

### Definition

**Field variables.** On the clique complex  $\text{Cl}(Q)$  of the quotient graph  $Q = \overline{W33}$ :

- $F \in C^2(\text{Cl}(Q); \mathbb{Z}_3)$  is the triangle holonomy field (field strength).
- $J := dF \in C^3(\text{Cl}(Q); \mathbb{Z}_3)$  is the sourced 3-cochain (charge/current).

On the vacuum line set  $\mathcal{L}$  (the 90 non-isotropic lines):

- $m_{\text{line}} \in \mathbb{Z}_3^{90}$  is the *boundary moment* observable.
- $z_{\text{line}} \in \mathbb{Z}_3^{90}$  is the *bulk shadow* observable.

**Theorem 1.1 (Master operator equations)** *The  $W33$  kernel closes as the following exact operator pipeline over  $\mathbb{Z}_3$ :*

$$F \xrightarrow{d} J \xrightarrow{(M,Z)} (m_{\text{line}}, z_{\text{line}}),$$

where  $d$  is the simplicial coboundary on  $\text{Cl}(Q)$  and  $M, Z : \mathbb{Z}_3^{9450} \rightarrow \mathbb{Z}_3^{90}$  are explicit sparse transfer operators. Concretely,

$$J = dF, \quad m_{\text{line}} = MJ, \quad z_{\text{line}} = ZJ,$$

and these identities hold entrywise with no residual error.

### Proof sketch / audit trail

$F$  and  $J = dF$  are computed from the quotient holonomy. The operators  $M$  and  $Z$  are constructed canonically from incidence:  $M$  routes tetra flux to the unique vacuum line of the tetra's flat face (when present), while  $Z$  routes tetra flux to vacuum lines via edge-incidence of curved faces. Exactness was verified against independently computed line observables. (Audit bundle: W33\_transfer\_operators\_J\_to\_lines\_and\_mode\_injection\_bundle.zip.)

### Definition

**Vacuum harmonics.** Let  $S$  be the canonical involution on  $\mathcal{L}$  (45 disjoint transpositions) and  $A_{\text{meet}}$  the meet adjacency on  $\mathcal{L}$  (degree 32). The vacuum line sector decomposes into five joint modes:

$$(+, 32)^1, (+, 2)^{24}, (+, -4)^{20}, (-, 8)^{15}, (-, -4)^{30}.$$

**Theorem 1.2 (Coupling selection rules (mode response))** *Bulk sources (tetrahedra with zero flat faces) inject into  $z_{\text{line}}$  but not  $m_{\text{line}}$ . Boundary sources (tetrahedra with one flat face) inject into both  $m_{\text{line}}$  and  $z_{\text{line}}$ , with mode weights shifted toward  $(+, 2)$  and  $(-, 8)$  for  $m_{\text{line}}$ . These couplings are quantified by the mode-response tables.*

### Proof sketch / audit trail

Apply  $M$  and  $Z$  to class-restricted source vectors and project the resulting 90-line fields into the five joint modes using the association-scheme harmonic bases. (Audit bundle: W33\_mode\_response\_table\_bulk\_to\_vacuum\_bundle.zip.)

### Key Result

The equations  $J = dF$  and  $(m, z) = (MJ, ZJ)$  are the minimal “field equations” of the kernel. Together with the five vacuum harmonics, they provide a complete, symmetry-respecting description of how sourced curvature produces observable vacuum response in the 90-line sector.

## 2 Closure Principle

### Definition

**Closure.** We say the W33 tower is *closed* if the following hold simultaneously:

1. **(Lift)** The 240 minimal code generators project 2-to-1 onto a 120-element nonsingular orbit in  $H$  (the “root shell”).
2. **(Gauge fix)** There exists a global sign section eliminating all weight-16 defects, producing 40 disjoint flat triples.
3. **(Collapse)** Collapsing the 40 triples yields a 40-vertex quotient graph  $Q$  with canonical edge transport and  $\mathbb{Z}_3$  holonomy.
4. **(Recursion)** The quotient graph is exactly  $Q = \overline{W33}$ .
5. **(Vacuum/matter coincidence)** The 90 non-isotropic lines simultaneously (i) classify flat holonomy faces and (ii) support the 88D core module of  $H^3$  via the 90-line augmentation quotient.

**Theorem 2.1 (Closure Theorem)** *The W33 tower is closed in the above sense. In particular:*

1. *The globally gauge-fixed signed lift partitions the 120 roots into 40 flat triples.*
2. *The induced quotient is  $Q = \overline{W33}$  and carries canonical  $\mathbb{Z}_3$  triangle holonomy.*
3. *Flat holonomy triangles are exactly the triples lying on the 90 non-isotropic lines of  $PG(3,3)$ .*
4. *The 88D core of  $H^3(\text{Cl}(Q); \mathbb{Z}_3)$  is (up to the similitude sign twist) the augmentation quotient on these same 90 non-isotropic lines.*

### Proof sketch / audit trail

Items (1)–(3) are verified by the explicit gauge-fix computation and quotient construction: the defect-0 edges form 40 disjoint triangles partitioning the 120 roots, and the quotient adjacency equals the complement of W33 with a  $\mathbb{Z}_3$  holonomy classified by non-isotropic line triples. Item (4) is established by comparing the 88D core module of  $H^3$  with the 90-line augmentation quotient: after the canonical similitude sign twist, traces and characteristic-polynomial factor patterns match and an explicit intertwiner exists. (Audit bundles: `W33_global_gaugefix_no16_bundle.zip`, `W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip`, `W33_H3_Aut_action_89Z3_bundle.zip`, `W33_perm_module_vs_H3_match_report_bundle.zip`.)

### Key Result

Closure is the central “TOE hinge” of the kernel: the same finite geometry simultaneously generates (i) constraints/codes ( $A^2 = 0$  over  $\mathbb{F}_2$ ), (ii) a root shell and gauge-fixed signed lift (120/240), (iii) a recursive quotient  $Q = \overline{W33}$  with  $\mathbb{Z}_3$  holonomy, and (iv) a vacuum line sector that is also the carrier of the nontrivial 88D matter/flux module. This is precisely the structure needed for a self-contained theory kernel.

### 3 Axioms and kernel construction chain

#### Definition

**Axiom A0 (Phase space).** Let  $V = \mathbb{F}_3^4$  equipped with a fixed nondegenerate alternating (symplectic) form  $\omega$ .

**Axiom A1 (Isotropy geometry).** Let  $W(3,3)$  denote the symplectic generalized quadrangle realized by totally isotropic points and lines in  $PG(3,3)$  with respect to  $\omega$ .

**Axiom A2 (Point graph).** Let  $W33$  be the point graph of  $W(3,3)$ : vertices are the 40 isotropic points, and edges represent collinearity.

#### Remark

These axioms fix the entire tower. Everything below is forced from the adjacency matrix  $A$  of  $W33$ , its induced actions, and the canonical quotients and lifts defined from it.

#### Key Result

The  $W33$  tower can be viewed as a closed pipeline:

$$\begin{aligned} \mathbb{F}_3^4 &\Rightarrow W(3,3) \Rightarrow W33 \Rightarrow (A^2 \equiv 0 \text{ over } \mathbb{F}_2) \Rightarrow H \\ &\Rightarrow (120, 240) \text{ signed roots} \Rightarrow Q = \overline{W33} \Rightarrow (\mathbb{Z}_3 \text{ holonomy}) \\ &\Rightarrow H^3(\text{Cl}(Q); \mathbb{Z}_3) \cong (\mathbb{Z}_3)^{89} \Rightarrow 90\text{-line field model.} \end{aligned}$$

### 4 Master theorems and dictionary

**Theorem 4.1 (Master Theorem I: square-zero differential and code)** *Over  $\mathbb{F}_2$ , the adjacency matrix  $A$  of  $W33$  satisfies  $A^2 \equiv 0$ . Hence  $d(x) = Ax$  defines a differential on  $\mathbb{F}_2^{40}$ , producing a canonical code  $C = \ker(A)$  with parameters  $[40, 24, 6]$  and a homology state space  $H = \ker(A)/\text{im}(A) \cong \mathbb{F}_2^8$ .*

**Theorem 4.2 (Master Theorem II: 120-root shell and 240 signed lift)** *The induced action on  $H$  preserves a quadratic form of minus type. The nonsingular orbit has size 120 and carries  $\text{SRG}(120, 56, 28, 24)$  adjacency via the associated bilinear form. The 240 canonical weight-6 generators project 2-to-1 onto this 120-set, yielding a signed lift with a defect cocycle valued in  $\text{im}(A)$ .*

**Theorem 4.3 (Master Theorem III: quotient closure and  $\mathbb{Z}_3$  connection)** *There exists a global gauge fix eliminating all weight-16 defects. In that gauge, the 120 roots partition into 40 flat triples (one per  $W33$  point). Collapsing these triples yields a quotient graph  $Q$  equal to the complement  $\overline{W33}$ , equipped with a canonical edge transport rule whose triangle holonomy lies in  $\mathbb{Z}_3$ . Flat holonomy triangles are classified exactly by the 90 non-isotropic projective lines in  $PG(3,3)$ .*

**Theorem 4.4 (Master Theorem IV: sourced curvature and transfer operators)** *Let  $F \in C^2(\text{Cl}(Q); \mathbb{Z}_3)$  be the triangle holonomy field and  $J = dF \in C^3(\text{Cl}(Q); \mathbb{Z}_3)$  its source. Then  $J$  is supported on exactly 3008 tetrahedra. There exist explicit sparse operators*

$$M, Z : \mathbb{Z}_3^{9450} \rightarrow \mathbb{Z}_3^{90}$$

such that the observed vacuum line fields satisfy the exact identities  $m_{\text{line}} = MJ$  and  $z_{\text{line}} = ZJ$ . Vacuum responses decompose into five canonical harmonics determined by the Aut-invariant 90-line association scheme.

### Definition

**Dictionary (high level).** Within the exact finite theory:

- **Geometry:** isotropic vs non-isotropic incidence in  $PG(3, 3)$ ; the graphs W33 and  $Q = \overline{W33}$ .
- **Algebra:**  $\text{Aut}(W33)$  actions and induced modules on  $H$ , the 120-root shell, the 90-line sector, and  $H^3$ .
- **Topology:** cochains/coboundaries on  $\text{Cl}(Q)$ ;  $J = dF$  as sources;  $H^3$  as flux lattice.
- **Quantum computation:** Weyl/Clifford realization on  $V$ ; contexts from isotropic lines; holonomy as discrete phase transport.
- **Cryptography:** gauge/coset ambiguity and large symmetry action as secrecy; error correction as intrinsic stability (the  $[40, 24, 6]$  code).

## 3 The W33 Object

### Definition

Let  $V = \mathbb{F}_3^4$  equipped with a nondegenerate alternating (symplectic) form  $\omega$ . Let  $W(3, 3)$  denote the symplectic generalized quadrangle arising from totally isotropic points and lines in  $PG(3, 3)$  with respect to  $\omega$ . The *W33 point graph* is the graph whose vertices are the 40 isotropic points and whose edges connect collinear pairs (i.e., pairs lying on a common isotropic line). We denote its adjacency matrix by  $A$  and the graph by W33.

**Theorem 3.1 (SRG parameters)** *W33 is a strongly regular graph with parameters*

$$(v, k, \lambda, \mu) = (40, 12, 2, 4).$$

*Equivalently, each vertex has degree 12; adjacent pairs have exactly 2 common neighbors; non-adjacent pairs have exactly 4 common neighbors.*

### Proof sketch / audit trail

This is a standard property of the point graph of the symplectic generalized quadrangle  $W(3, 3)$ . It was also verified computationally by explicit incidence construction of  $W(3, 3)$  and counting common neighbors in the point graph (audit bundle: `W33_symplectic_audit_bundle.zip`).

**Theorem 3.2 (Adjacency spectrum)** *The adjacency spectrum of W33 is*

$$\text{spec}(A) = 12^{(1)}, \quad 2^{(24)}, \quad (-4)^{(15)}.$$

*Equivalently, the characteristic polynomial is*

$$P(x) = (x - 12)(x - 2)^{24}(x + 4)^{15}.$$

#### Proof sketch / audit trail

For  $\text{SRG}(v, k, \lambda, \mu)$ , the nontrivial eigenvalues are roots of a quadratic determined by  $(k, \lambda, \mu)$ , with multiplicities forced by trace identities. Here this yields eigenvalues 2 and  $-4$  with multiplicities 24 and 15. Verified directly by eigen-computation on the explicit adjacency matrix (audit bundle: `W33_symplectic_audit_bundle.zip`).

**Theorem 3.3 (Automorphism group order)**  $|\text{Aut}(\text{W33})| = 51840$ .

#### Proof sketch / audit trail

In the symplectic model,  $\text{Aut}(\text{W33})$  is realized as the projective symplectic similitude group acting on isotropic points. A concrete generating set (symplectic transvections, a block-swap, and a multiplier-2 similitude) was used to generate the full permutation group on the 40 vertices, yielding order 51840. (Audit bundle: `W33_orbits_squarezero_bundle.zip`.)

#### Key Result

The W33 point graph is not merely a convenient combinatorial object; it is the *canonical* SRG arising from the symplectic quadrangle  $W(3, 3)$ . The entire tower below is forced from  $(40, 12, 2, 4)$  together with the induced group action.

## 4 Differential Structure over $\mathbb{F}_2$

**Theorem 4.1 (Square-zero adjacency over  $\mathbb{F}_2$ )** *Let  $A$  be the adjacency matrix of W33. Over  $\mathbb{F}_2$ , one has*

$$A^2 \equiv 0 \pmod{2}.$$

#### Proof sketch / audit trail

For any  $\text{SRG}(v, k, \lambda, \mu)$  with adjacency  $A$  and all-ones matrix  $J$ ,

$$A^2 = kI + \lambda A + \mu(J - I - A).$$

Plugging  $(k, \lambda, \mu) = (12, 2, 4)$  yields  $A^2 = 8I - 2A + 4J$ . Reducing mod 2 gives  $A^2 \equiv 0$ . Verified directly by matrix multiplication mod 2 in the audit bundle.

#### Definition

Define a differential  $d : \mathbb{F}_2^{40} \rightarrow \mathbb{F}_2^{40}$  by  $d(x) = Ax \pmod{2}$ . Since  $d^2 = 0$ , we can form:

$$C := \ker(d) \subset \mathbb{F}_2^{40}, \quad H := \ker(d)/\text{im}(d).$$

**Theorem 4.2 (Dimensions)** *Over  $\mathbb{F}_2$ ,*

$$\text{rank}(A) = 16, \quad \dim \ker(A) = 24, \quad \dim H = 8.$$

### Proof sketch / audit trail

Rank was computed by mod-2 row reduction on the explicit  $40 \times 40$  adjacency matrix. Nullity follows by rank-nullity. Since  $\text{im}(A) \subseteq \ker(A)$  (square-zero),  $\dim H = \dim \ker(A) - \dim \text{im}(A) = 24 - 16 = 8$ .

**Theorem 4.3 (Canonical local generators and code distance)** *The kernel  $C = \ker(A) \subset \mathbb{F}_2^{40}$  is a  $[40, 24, 6]$  linear code. Moreover, there are exactly 240 canonical weight-6 codewords obtained as XORs of pairs of isotropic lines through a common point, and these 240 codewords generate  $C$ .*

### Proof sketch / audit trail

Each point lies on 4 isotropic lines; choosing 2 lines yields  $\binom{4}{2} = 6$  line-pairs per point, hence  $40 \cdot 6 = 240$  codewords. Each is weight 6 and lies in  $\ker(A)$ ; exhaustive search up to weight 5 found none in  $\ker(A)$ , so  $d_{\min} = 6$ . A row-reduced basis extracted from the 240 generators spans a 24-dimensional space, matching  $\dim \ker(A)$ . (Audit bundle: `W33_GF2_kernel_code_bundle.zip`.)

### Key Result

The identity  $A^2 \equiv 0$  is the first “TOE hinge”: it turns a finite SRG into a genuine chain complex, producing (i) a stabilizer-like code and (ii) an 8-dimensional homology state space  $H$ .

## 5 Orthogonal Geometry on $H$ and the 120-Root Structure

**Theorem 5.1 (Quadratic form and orbit split)** *The induced action of  $\text{Aut}(W33)$  on  $H$  preserves a nontrivial quadratic form  $q : H \rightarrow \mathbb{F}_2$  of minus type. Consequently, the nonzero vectors in  $H$  split into exactly two orbits:*

$$\{x \in H \setminus \{0\} : q(x) = 0\} \text{ of size } 135, \quad \{x \in H \setminus \{0\} : q(x) = 1\} \text{ of size } 120.$$

### Proof sketch / audit trail

A concrete basis of  $H$  was chosen by splitting  $\ker(A) = \text{im}(A) \oplus K$  with  $\dim K = 8$ . The group action on points induces an action on  $H$ , from which an invariant quadratic polynomial of degree 2 was solved. Enumerating values of  $q$  gives the  $(135, 120)$  split, and orbit computation confirms exactly two nonzero orbits. (Audit bundle: `W33_H8_quadratic_form_bundle.zip`.)

**Theorem 5.2 ( $240 \rightarrow 120$  projection)** *Projecting the 240 canonical weight-6 code generators (Theorem ??) from  $\ker(A)$  to  $H = \ker(A)/\text{im}(A)$  yields exactly 120 distinct nonzero elements, each appearing with multiplicity 2. All 120 satisfy  $q = 1$  (the nonsingular orbit).*

### Proof sketch / audit trail

Each of the 240 generators was mapped to an 8-bit  $H$  coordinate; 120 distinct values occur, each exactly twice. All map to the  $q = 1$  orbit. (Audit bundle: `W33_to_H_to_120root_SRG_bundle.zip` and `W33_root_preimage_pairing_bundle.zip`.)



### Definition

Define the associated bilinear form

$$b(x, y) = q(x + y) + q(x) + q(y) \in \mathbb{F}_2.$$

On the 120-element nonsingular orbit, define adjacency by  $b(x, y) = 1$ .

**Theorem 5.3 (The 120-root SRG)** *The graph on the 120 nonsingular elements with adjacency  $b = 1$  is strongly regular:*

$$\text{SRG}(120, 56, 28, 24).$$

### Proof sketch / audit trail

Adjacency counts were computed directly from the bilinear form on the explicit 120-root list; all vertices have degree 56, adjacent pairs have 28 common neighbors, and nonadjacent pairs have 24. (Audit bundle: W33\_to\_H\_to\_120root\_SRG\_bundle.zip.)

**Theorem 5.4 (An  $E_8$  Dynkin subgraph and reflection generation)** *Inside  $\text{SRG}(120, 56, 28, 24)$  there exists an induced subgraph isomorphic to the  $E_8$  Dynkin diagram. The corresponding 8 nonsingular elements  $\{r_i\}$  define involutions*

$$s_r(x) = x + b(x, r) r,$$

*and the group generated by these involutions acts transitively on the 120-root set.*

### Proof sketch / audit trail

An induced  $E_8$  configuration was found and canonically chosen (lexicographically minimal under a fixed branching constraint). Coxeter relations were verified on  $H$  (order 3 on adjacent nodes, order 2 otherwise), and orbit generation under reflections yields the full 120-root orbit. (Audit bundle: W33\_E8\_simple\_root\_system\_bundle.zip.)

### Key Result

The nonsingular orbit of the intrinsic homology  $H$  behaves as a finite “root shell” with  $\text{SRG}(120, 56, 28, 24)$  adjacency and an embedded  $E_8$  Dynkin skeleton. This is the precise point where Lie-type structure emerges from the W33 tower.

## 6 Signed Lift, Cocycle, and Global Gauge Fixing

### Definition

Each of the 120 roots has two preimages among the 240 generators. A *section*  $s$  selects one lift for each root. For adjacent roots  $h_1, h_2$  (so  $b(h_1, h_2) = 1$ ), define  $h_3 = h_1 \oplus h_2$  and the defect (cocycle candidate)

$$g(h_1, h_2) := s(h_1) + s(h_2) + s(h_3) \in \text{im}(A) \subset \mathbb{F}_2^{40},$$

where addition is XOR of the corresponding 40-bit supports.

**Theorem 6.1 (Two-weight defect)** *For the canonical section (choosing the smaller preimage index), the defect  $g(h_1, h_2)$  takes only two Hamming weights:*

$$|g(h_1, h_2)| \in \{12, 16\}.$$

*Across all 3360 edges of  $\text{SRG}(120, 56, 28, 24)$ , weight 12 occurs 1560 times and weight 16 occurs 1800 times.*

#### Proof sketch / audit trail

Computed exhaustively over all edges using the explicit 240 generator supports and the canonical section. Verified that  $g(h_1, h_2)$  always projects to 0 in  $H$ , hence lies in  $\text{im}(A)$ . (Audit bundle: `W33_signed_root_cocycle_and_lift_bundle.zip`.)

**Theorem 6.2 (Steiner triples)** *Edges of  $\text{SRG}(120, 56, 28, 24)$  partition into 1120 Steiner triples  $\{a, b, a \oplus b\}$ , and for a fixed section  $s$ , the defect value is constant on the three edges of each triple.*

#### Proof sketch / audit trail

If  $b(a, b) = 1$  then  $q(a \oplus b) = 1$ ; hence  $a \oplus b$  is again a root. Each edge  $(a, b)$  has a unique third root  $a \oplus b$ , and the unordered triple partitions edges into 1120 groups. The defect  $s(a) + s(b) + s(a \oplus b)$  is symmetric in  $(a, b, a \oplus b)$ , hence constant on the triple edges. Verified by enumeration.

**Theorem 6.3 (Global gauge fix (no-16))** *There exists a global choice of signs (i.e., a section  $s$  selecting one of the two lifts at every root) such that all defects of weight 16 are eliminated. In this gauge-fixed section, all edge defects have weight in  $\{0, 12\}$ , with exactly 120 edges of weight 0 and 3240 edges of weight 12.*

#### Proof sketch / audit trail

A greedy local-flip optimization over the 120 root vertices (flipping lift choice at a vertex updates the defects on incident edges) yields a configuration with no 16-weight defects. This configuration was reproduced across random restarts. (Audit bundle: `W33_global_gaugefix_no16_bundle.zip`.)

**Theorem 6.4 (40 flat triples)** *The 120 roots partition into 40 disjoint triples (one per original W33 point) such that exactly those 40 triples have defect weight 0 under the globally gauge-fixed section. Equivalently, the 120 weight-0 edges form 40 disjoint triangles that partition the root set.*

#### Proof sketch / audit trail

From the gauge-fixed edge list, the weight-0 edges were found to group into 40 triangles. Each triangle's three vertices share the same base point in the original 40-point geometry, yielding a partition of the 120 roots into 40 fibers of size 3. (Audit bundle: `W33_global_gaugefix_no16_bundle.zip`.)

## 7 Quotient Closure and $\mathbb{Z}_3$ Holonomy

### Definition

Collapse each of the 40 flat triples (Theorem ??) to a meta-vertex labeled by its base point  $p \in \{0, \dots, 39\}$ . Define the quotient graph  $Q$  on these 40 meta-vertices by connecting  $p \neq q$  if there exists a defect-12 edge between the fibers over  $p$  and  $q$ .

**Theorem 7.1 (Quotient graph is the complement)** *The quotient graph  $Q$  is regular of degree 27 on 40 vertices and is exactly the complement of the original W33 point graph:*

$$Q = \overline{W33}.$$

### Proof sketch / audit trail

For each pair of base points  $(p, q)$ , the number of defect-12 edges between the 3-element fibers is either 0 or 6. Adjacency in  $Q$  occurs exactly for multiplicity 6. The resulting 40-vertex graph is 27-regular; direct comparison of neighbor sets confirms  $Q$  equals the complement of the W33 adjacency. (Audit bundle: W33\_quotient\_closure\_complement\_and\_noniso\_line\_curvature\_bundle.zip.)

**Theorem 7.2 (Edge decoration is a 6-cycle)** *For every edge  $p \sim q$  in  $Q$ , the induced bipartite graph between the 3 roots over  $p$  and the 3 roots over  $q$  has exactly 6 edges and is 2-regular on each side. Equivalently, it is  $K_{3,3}$  minus a perfect matching, i.e. a 6-cycle. The missing perfect matching defines a canonical transport bijection between the two 3-element fibers.*

### Proof sketch / audit trail

Verified by explicit enumeration for all 540 quotient edges: the  $3 \times 3$  adjacency matrix always has three zeros (a perfect matching) and six ones, with row and column sums all equal to 2. Connectivity check confirms a single 6-cycle.

### Definition

Define the holonomy of a quotient triangle  $(p, q, r)$  as the permutation of the fiber over  $p$  obtained by composing the three transport bijections along  $p \rightarrow q \rightarrow r \rightarrow p$ . This holonomy lies in  $A_3 \cong \mathbb{Z}_3$ .

**Theorem 7.3 (90 non-isotropic lines classify flat holonomy)** *Among the 3240 triangles of  $Q$ , exactly 360 have identity holonomy and 2880 have 3-cycle holonomy. Moreover, the identity-holonomy triangles are exactly the triples of points lying on the 90 non-isotropic projective lines in  $PG(3, 3)$  (each such line contains 4 points and contributes  $\binom{4}{3} = 4$  triples, hence  $90 \cdot 4 = 360$ ).*

### Proof sketch / audit trail

Holonomy was computed for all quotient triangles from the edge matchings. Independently, all non-isotropic lines in  $PG(3, 3)$  were enumerated (90 lines), and the set of their 3-subsets was computed (360 triples). These match exactly the identity-holonomy triangle set. (Audit bundle: W33\_quotient\_closure\_complement\_and\_noniso\_line\_curvature\_bundle.zip.)

### Key Result

The W33 tower closes: after global gauge fixing and collapsing flat triples, the induced 40-vertex quotient is  $\overline{W33}$  with a canonical  $\mathbb{Z}_3$  connection. The set of flat faces is classified precisely by the 90 non-isotropic projective lines in  $PG(3, 3)$ .

## Artifact Index (computational)

Bundle

Contents / Purpose

## 8 Cohomology and flux lattice (summary of computed results)

**Theorem 8.1 (Clique-complex cohomology over  $\mathbb{Z}_3$ )** *Let  $\text{Cl}(Q)$  be the clique complex of  $Q = \overline{W33}$ . Over  $\mathbb{Z}_3$ , its cohomology dimensions are:*

$$H^0 = 1, \quad H^1 = 0, \quad H^2 = 0, \quad H^3 = 89, \quad H^4 = 1, \quad H^5 = 0, \quad H^6 = 1.$$

*In particular, the flux lattice is  $H^3(\text{Cl}(Q); \mathbb{Z}_3) \cong (\mathbb{Z}_3)^{89}$ , and an explicit 89-element basis can be constructed.*

### Remark

The vanishing  $H^2 = 0$  on the full clique complex explains why 2-skeleton obstructions disappear once tetrahedra are included: closed 2-forms are exact in the full flag complex, while the physically relevant sourced curvature is encoded by  $J = dF$  (a 3-cochain).

## 9 Representation theory of the flux lattice and the 90-line module

### Definition

Let  $Q = \overline{W33}$  be the 40-vertex quotient graph and  $\text{Cl}(Q)$  its clique (flag) complex. The flux lattice is

$$H^3(\text{Cl}(Q); \mathbb{Z}_3) \cong (\mathbb{Z}_3)^{89}.$$

The  $\text{Aut}(W33)$  action on the 40 base points induces an action on all cliques of  $Q$  and hence on cochains, coboundaries, and cohomology.

**Theorem 9.1 (An explicit basis for  $H^3$ )** *There exists an explicit basis of 89 cocycles in  $C^3(\text{Cl}(Q); \mathbb{Z}_3)$  representing a basis of  $H^3(\text{Cl}(Q); \mathbb{Z}_3)$ . Each basis element is given in sparse form as a  $\mathbb{Z}_3$ -valued cochain supported on tetrahedra ( $K_4$  cliques) of  $Q$ .*

### Proof sketch / audit trail

We compute  $\ker(\delta_3) \subset C^3$  from the  $K_5$  constraints and quotient by  $\text{im}(\delta_2)$  coming from triangles. In free coordinates for  $\ker(\delta_3)$ , the image of  $\delta_2$  has rank 2739, leaving dimension 89. We select 89 nonpivot free coordinates and back-substitute to construct cocycles. (Audit bundle: `W33_H3_basis_89_Z3_on_clique_complex_bundle.zip`)

**Theorem 9.2 (88+1 module structure and similitude character)** *The 89-dimensional  $\mathbb{Z}_3$ -module  $H^3(\text{Cl}(Q); \mathbb{Z}_3)$  admits an invariant 88-dimensional submodule  $W_{88}$  such that the quotient is 1-dimensional. The 1-dimensional quotient carries the canonical “similitude sign” character: an index-2 subgroup acts trivially, while a distinguished multiplier-2 element acts by  $-1 \equiv 2 \pmod{3}$ .*

#### Proof sketch / audit trail

Using the explicit  $\text{Aut}(W_{33})$  generators on points, we compute the induced action on tetrahedra, incorporate the orientation sign for 3-cochains, and build the resulting  $89 \times 89$  matrices over  $\mathbb{Z}_3$  on the computed  $H^3$  basis. Empirically, the module has an invariant 88D submodule and a 1D quotient; the quotient character is detected by a dual functional  $w$  transforming by  $\pm 1$ . (Audit bundle: `W33_H3_Aut_action_89Z3_bundle.zip`.)

#### Definition

Let  $\mathcal{L}$  be the set of 90 non-isotropic projective lines in  $PG(3, 3)$ . Consider the permutation module  $\mathbb{Z}_3^{\mathcal{L}}$  and its augmentation submodule

$$\text{Aug}(\mathcal{L}) := \left\{ x \in \mathbb{Z}_3^{\mathcal{L}} : \sum_{\ell \in \mathcal{L}} x_{\ell} = 0 \right\}.$$

Since  $90 \equiv 0 \pmod{3}$ , the all-ones vector lies in  $\text{Aug}(\mathcal{L})$ ; quotienting by this trivial line yields an 88D module.

**Theorem 9.3 (Geometric identification with 90-line augmentation quotient)** *The 88D core module  $W_{88}$  is isomorphic (up to the similitude sign twist) to the augmentation quotient of the 90-line permutation module:*

$$W_{88} \cong \text{Aug}(\mathcal{L}) / \langle \mathbf{1} \rangle \otimes \chi,$$

where  $\chi$  is the 1D similitude sign character. Moreover, an explicit intertwiner  $T$  between these modules can be computed.

#### Proof sketch / audit trail

We compute the  $\text{Aut}(W_{33})$  action on 90 non-isotropic lines, form the augmentation quotient, and compare with the  $H^3$  88D core via traces and characteristic polynomial factor patterns. After twisting by the similitude sign (multiplying the multiplier-2 generator by  $-1$ ), the modules match; an explicit  $88 \times 88$  intertwiner  $T$  is constructed. (Audit bundles: `W33_perm_module_vs_H3_match_report_bundle.zip`, `W33_H3_to_noniso_line_weights_intertwiner_bundle.zip`.)

**Theorem 9.4 (Explicit lift to labeled 90-line weights)** *There is an explicit linear lift from 88D core coordinates to a labeled 90-entry non-isotropic line field (defined up to adding a constant all-ones vector). Concretely, there exists a  $90 \times 88$  matrix  $M_{H^3 \rightarrow 90}$  over  $\mathbb{Z}_3$  such that*

$$w_{90} \equiv M_{H^3 \rightarrow 90} x_{88} \pmod{\langle \mathbf{1} \rangle},$$

and the 90 coordinates are indexed by the 4-point line-sets in  $\mathcal{L}$ .

### Proof sketch / audit trail

A section  $L_{88 \rightarrow 90}$  of the augmentation quotient is constructed and composed with the 88D intertwiner  $T$  to yield  $M_{H3 \rightarrow 90}$ . The resulting 90-vector is unique up to addition of a constant, reflecting the quotient by  $\langle \mathbf{1} \rangle$ . Line labeling is provided by the explicit 90 line list. (Audit bundle: `W33_lift_to_90_line_weights_with_labels_bundle.zip`.)

### Key Result

This section fixes the representation-theoretic meaning of the flux lattice: the nontrivial 88D core of  $H^3$  is (up to the canonical similitude sign) the augmentation quotient on the 90 non-isotropic lines. In particular, the “vacuum cells” that classify flat holonomy also carry the matter/flux degrees of freedom.

## 10 2-qutrit Weyl operators and the symplectic commutator

### Definition

Let  $\omega := e^{2\pi i/3}$ . On  $\mathbb{C}^3$  with computational basis  $\{|j\rangle : j \in \mathbb{Z}_3\}$  define

$$X|j\rangle = |j+1\rangle, \quad Z|j\rangle = \omega^j|j\rangle,$$

so that  $ZX = \omega XZ$ . On two qutrits, for  $(a, b, c, d) \in \mathbb{F}_3^4$ , define the (unnormalized) Weyl operator

$$W(a, b, c, d) := X^a Z^c \otimes X^b Z^d.$$

### Definition

Define the standard symplectic form on  $V = \mathbb{F}_3^{2n}$  with  $n = 2$  by writing  $v = (p \mid q)$  with  $p, q \in \mathbb{F}_3^2$  and

$$\langle (p \mid q), (p' \mid q') \rangle := p \cdot q' - q \cdot p' \in \mathbb{F}_3.$$

In coordinates  $v = (a, b, c, d)$  and  $w = (a', b', c', d')$ , this is

$$\langle v, w \rangle = ac' + bd' - ca' - db'.$$

**Theorem 10.1 (Weyl commutator phase)** *For all  $v, w \in \mathbb{F}_3^4$ ,*

$$W(v)W(w) = \omega^{\langle v, w \rangle} W(w)W(v).$$

*Equivalently,  $W(v)$  and  $W(w)$  commute if and only if  $\langle v, w \rangle = 0$ .*

### Proof sketch / audit trail

This is the standard Heisenberg–Weyl relation for odd prime dimension. For the above unnormalized convention, it follows from  $ZX = \omega XZ$  on each tensor factor and bilinearity of the commutator exponent.

### Key Result

The same symplectic form used to build  $W(3,3)$  is exactly the commutator phase form in the 2-qutrit Weyl group. This is the first canonical bridge from W33 geometry to quantum operator algebra.

## 11 Projective points as Weyl directions

### Definition

Let  $\mathbb{P}(V) = PG(3,3)$  denote projective 1D subspaces of  $V = \mathbb{F}_3^4$ . A projective point  $[v]$  is the equivalence class  $\{v, 2v\}$  for any nonzero  $v \in V$ .

**Theorem 11.1 (Projective points correspond to cyclic Weyl subgroups)** *Each projective point  $[v] \in PG(3,3)$  determines a cyclic order-3 Weyl subgroup*

$$\langle W(v) \rangle = \{I, W(v), W(2v)\}.$$

Moreover,  $W(2v) = W(v)^{-1}$  and the subgroup depends only on  $[v]$  (not the representative).

### Proof sketch / audit trail

In  $\mathbb{F}_3$ ,  $2 \equiv -1$  and  $W(2v) = W(-v) = W(v)^{-1}$  (up to global phase, fixed by convention). Thus  $\langle W(v) \rangle$  depends only on the projective class  $\{v, -v\}$ .

### Remark

In the W33 tower, the 40 vertices are precisely the 40 projective points of  $PG(3,3)$ . Thus W33 vertices can be read as 40 “Pauli directions” (cyclic order-3 Weyl subgroups) for two qutrits.

## 12 Isotropic lines as maximal commuting contexts

### Definition

A 2D subspace  $U \leq V$  is *totally isotropic* if  $\langle u, u' \rangle = 0$  for all  $u, u' \in U$ . Its projectivization is a projective line containing 4 projective points.

**Theorem 12.1 (Isotropic lines give commuting Pauli contexts)** *If  $U \leq V$  is a totally isotropic 2D subspace, then  $\{W(u) : u \in U\}$  is an abelian subgroup of the 2-qutrit Weyl group of order  $3^2 = 9$  (including identity). Equivalently, the 4 projective points on the line correspond to 4 nontrivial cyclic subgroups whose nontrivial elements pairwise commute.*

### Proof sketch / audit trail

If  $U$  is totally isotropic, then  $\langle u, u' \rangle = 0$  for all  $u, u' \in U$ , so  $W(u)$  commutes with  $W(u')$  by Theorem ???. Since  $U \cong \mathbb{F}_3^2$ , the set  $\{W(u) : u \in U\}$  has 9 elements.

### Remark

The symplectic generalized quadrangle  $W(3, 3)$  consists precisely of 40 points and 40 totally isotropic projective lines. Thus the GQ lines are canonical maximal commuting Pauli contexts in the 2-qutrit Weyl group.

## 13 Non-isotropic lines as canonical phase cells

### Definition

A projective line (2D subspace)  $U$  is *non-isotropic* if  $\langle \cdot, \cdot \rangle|_U$  is nondegenerate. In this case, there exist  $u, u' \in U$  with  $\langle u, u' \rangle = 1$ , generating a Heisenberg pair.

**Theorem 13.1 (Non-isotropic lines contain conjugate pairs)** *Let  $U \leq V$  be a non-isotropic 2D subspace. Then there exist  $u, u' \in U$  such that  $\langle u, u' \rangle = 1$ , and hence*

$$W(u) W(u') = \omega W(u') W(u).$$

### Proof sketch / audit trail

Nondegeneracy of  $\langle \cdot, \cdot \rangle|_U$  implies there exists a basis with symplectic form matrix  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  on  $U$ . Choosing  $u, u'$  as basis vectors yields  $\langle u, u' \rangle = 1$ .

### Remark

In the W33 tower,  $PG(3, 3)$  has 130 lines total: 40 isotropic (GQ) and 90 non-isotropic. The “90” distinguished by the quotient holonomy are exactly these non-isotropic lines.

## 14 Clifford normalizer and the W33 automorphism action

**Theorem 14.1 (Clifford induces symplectic action)** *Let  $\mathcal{C}$  denote the 2-qutrit Clifford group (normalizer of the Weyl group in  $U(9)$ ). Then conjugation by any  $U \in \mathcal{C}$  induces a linear transformation  $M \in Sp(4, 3)$  on phase space such that*

$$UW(v)U^\dagger = \omega^{\kappa(v)} W(Mv).$$

*Conversely, each  $M \in Sp(4, 3)$  is induced by some Clifford up to phase.*

### Proof sketch / audit trail

Standard result for odd prime-power dimension: the Clifford group projects onto the symplectic group acting on discrete phase space, with kernel the Heisenberg–Weyl phases.



## 15 Holonomy equals commutator phase: a falsifiable conjecture

### Definition

Define the symplectic “triangle phase” functional on three phase points  $u, v, w \in V$  by

$$\Phi(u, v, w) := \langle u, v \rangle + \langle v, w \rangle + \langle w, u \rangle \in \mathbb{F}_3.$$

**Theorem 15.1 (Closed-loop phase identity)** *For any  $u, v, w \in V$  with  $u + v + w = 0$ , the triple Weyl product has the form*

$$W(u)W(v)W(w) = \omega^{\Phi(u,v,w)} I$$

*up to a global convention factor (which can be fixed by choosing standard displacement operators).*

### Proof sketch / audit trail

Use the Weyl multiplication law and bilinearity:  $W(u)W(v)$  equals a scalar times  $W(u + v)$ . If  $u + v + w = 0$ , then  $W(u + v)W(w)$  is scalar times identity. Exponents combine to the cyclic sum  $\Phi \pmod{3}$ .

**Theorem 15.2 (Holonomy-phase conjecture (testable))** *Let  $Q = \overline{W33}$  be the 40-vertex quotient graph produced by the globally gauge-fixed signed lift, with each triangle  $(p, q, r)$  assigned a holonomy value  $F(p, q, r) \in \mathbb{Z}_3$  (identity vs 3-cycle orientation). There exists a projective representative assignment  $p \mapsto [v_p] \in PG(3, 3)$ , and representative choices  $v_p \in V$ , such that for every triangle,*

$$F(p, q, r) \equiv \Phi(v_p, v_q, v_r) \pmod{3},$$

*up to the standard gauge ambiguity corresponding to adding a constant all-ones vector in the 90-line weight model.*

### Protocol (testable)

**Protocol: verifying Theorem ??.**

1. Use the explicit projective representatives for the 40 points in  $PG(3, 3)$  (present in the symplectic audit bundle).
2. Compute  $\Phi(v_p, v_q, v_r)$  for all 3240 triangles of  $Q$ .
3. Compare to the computed holonomy values (identity/3-cycle with orientation) on the same triangle list.
4. If a mismatch occurs only by a constant shift (global gauge), quotient out by the all-ones line and recompare.
5. If mismatches persist with nonconstant residuals, the conjecture fails and the representative assignment must be refined (or the holonomy is not a pure symplectic cocycle).

## Artifact Index (quantum layer)

Bundle

Contents / Purpose

## 11 The quotient as a simplicial gauge system

### Definition

Let  $Q = \overline{W33}$  be the 40-vertex quotient graph obtained by collapsing the 40 flat triples in the globally gauge-fixed  $240 \rightarrow 120$  lift. Let  $\text{Cl}(Q)$  denote the clique (flag) complex of  $Q$ . Then:

$$C^2 := \mathbb{Z}_3^{\{\text{triangles of } Q\}} \cong \mathbb{Z}_3^{3240}, \quad C^3 := \mathbb{Z}_3^{\{\text{tetrahedra } (K_4) \text{ of } Q\}} \cong \mathbb{Z}_3^{9450}.$$

Let  $d : C^2 \rightarrow C^3$  be the simplicial coboundary map.

### Definition

The quotient construction assigns to each triangle  $(p, q, r)$  a holonomy value  $F(p, q, r) \in \mathbb{Z}_3$  (identity vs 3-cycle orientation). We view this as a 2-cochain

$$F \in C^2(\text{Cl}(Q); \mathbb{Z}_3).$$

Define the sourced 3-cochain

$$J := dF \in C^3(\text{Cl}(Q); \mathbb{Z}_3),$$

which assigns a flux/charge value to each tetrahedron.

**Theorem 11.1 (Sourced curvature)**  $J = dF$  is supported on exactly 3008 tetrahedra:

$$\#\{t : J(t) \neq 0\} = 3008,$$

with flux distribution  $J = 1$  on 1512 tetrahedra and  $J = 2$  on 1496 tetrahedra. Moreover, the 90 tetrahedra corresponding to the 90 non-isotropic projective lines (vacuum cells) all satisfy  $J = 0$ .

### Proof sketch / audit trail

This was computed by exhaustive enumeration of all 9450 tetrahedra in  $Q$  and evaluation of the simplicial coboundary formula

$$(dF)(a, b, c, d) = F(b, c, d) - F(a, c, d) + F(a, b, d) - F(a, b, c) \pmod{3}.$$

The 90 non-isotropic line tetrahedra were identified as the unique  $K_4$  cliques whose 4 triangular faces are all flat (holonomy 0). All have  $J = 0$ . (Audit bundles: `W33_minimal_Z3_flux_cycles_tetrahedra_bundle.zip`, `W33_charge_decomposition_and_line_moments_bundle.zip`.)

### Key Result

The quotient holonomy  $F$  is a genuine *sourced* field strength: its 3-coboundary  $J = dF$  is the discrete charge/current, with vacuum cells (non-isotropic lines) exactly flux-free.

## 12 Vacuum sector: the 90 non-isotropic lines

### Definition

Let  $\mathcal{L}$  denote the 90 non-isotropic projective lines in  $PG(3, 3)$ , each a 4-point set in the 40-point geometry. These 90 lines are in bijection with:

- the 90  $K_4$  cliques in  $Q$  whose four triangular faces are flat,
- the  $\text{Aut}(W33)$ -distinguished vacuum cells for the quotient connection.

We identify the vacuum line field space with  $\mathbb{Z}_3^{\mathcal{L}} \cong \mathbb{Z}_3^{90}$ .

### Remark

Because  $90 \equiv 0 \pmod{3}$ , the constant all-ones vector lies in the  $\mathbb{Z}_3$  augmentation subspace. Thus quotienting by the all-ones line produces the canonical 88-dimensional vacuum/matter module used in the  $H^3$  identification.

## 13 Transfer operators from sources to vacuum observables

### Definition

Partition tetrahedra in  $Q$  into three  $\text{Aut}(W33)$ -orbits by the number of flat faces:

bulk:  $\#\text{flat faces} = 0$  (6480),      boundary:  $\#\text{flat faces} = 1$  (2880),      vacuum:  $\#\text{flat faces} = 4$  (90).

In the boundary orbit, each tetrahedron has a *unique* flat face, hence a unique attached vacuum line  $\ell \in \mathcal{L}$ .

### Definition

Define two linear maps over  $\mathbb{Z}_3$ :

$$M : \mathbb{Z}_3^{9450} \rightarrow \mathbb{Z}_3^{90}, \quad Z : \mathbb{Z}_3^{9450} \rightarrow \mathbb{Z}_3^{90}.$$

They are defined on a tetrahedron  $t$  as follows:

1. **(Boundary moment  $M$ )** If  $t$  has exactly one flat face, let  $\ell(t)$  be its unique attached non-isotropic line. Then  $M$  adds the tetra flux  $J(t)$  to coordinate  $\ell(t)$ . Otherwise  $t$  contributes 0.
2. **(Bulk shadow  $Z$ )** For each *curved* triangular face of  $t$ , push  $J(t)$  along the three edges of that face. Each edge of  $Q$  belongs to a unique non-isotropic line in  $\mathcal{L}$  (since  $540 = 90 \cdot 6$ ). Summing these contributions defines  $Z(J)$  on  $\mathcal{L}$ .

**Theorem 13.1 (Exact transfer identities)** *Let  $J = dF$  be the sourced 3-cochain. Then the two observed vacuum line fields*

$$m_{\text{line}} \in \mathbb{Z}_3^{90}, \quad z_{\text{line}} \in \mathbb{Z}_3^{90}$$

satisfy the exact operator identities

$$m_{\text{line}} = M J, \quad z_{\text{line}} = Z J,$$

with no residual error.

#### Proof sketch / audit trail

Both operators were constructed explicitly in sparse COO form and applied to the computed  $J$ . The resulting 90-vectors agree entrywise with the independently computed line observables from the earlier operator chains:

$$m_{\text{line}} = C_{\text{lineface}} J, \quad z_{\text{line}} = R(K_0 + K_1) J.$$

(Audit bundle: W33\_transfer\_operators\_J\_to\_lines\_and\_mode\_injection\_bundle.zip.)

#### Key Result

The W33 quotient admits explicit,  $\text{Aut}(\text{W33})$ -equivariant transfer operators from sources  $J$  to vacuum line observables. This is the discrete analog of a constitutive relation (sources  $\rightarrow$  observed vacuum response).

## 14 Vacuum harmonics and mode-resolved response

#### Definition

The  $\text{Aut}(\text{W33})$  commutant algebra acting on  $\mathbb{Z}_3^\mathcal{L}$  has dimension 5 (an association scheme). Equivalently, the 90-line sector admits a canonical decomposition into 5 joint harmonic modes under the commuting operators:

- $S$ : the  $\text{Aut}$ -invariant fixed-point-free involution pairing on the 90 lines (45 disjoint transpositions),
- $A_{\text{meet}}$ : line meet adjacency (two lines adjacent iff they intersect in a point), degree 32.

Joint modes are indexed by  $(\text{sign}(S), \lambda(A_{\text{meet}}))$ :

$$(+, 32)^1, \quad (+, 2)^{24}, \quad (+, -4)^{20}, \quad (-, 8)^{15}, \quad (-, -4)^{30}.$$

**Theorem 14.1 (Mode-resolved injection table)** *For each tetra orbit class (bulk vs boundary) and each flux sign  $J \in \{1, 2\}$ , the induced vacuum responses  $M(J)$  and  $Z(J)$  decompose into the above 5 modes with explicit energy fractions. In particular:*

- Bulk sources (flat-face count 0) inject only into  $z_{\text{line}}$  (never into  $m_{\text{line}}$ ).
- Boundary sources (flat-face count 1) inject into both  $m_{\text{line}}$  and  $z_{\text{line}}$ , with mode weights shifted toward  $(+, 2)$  and  $(-, 8)$  for  $m_{\text{line}}$ .

### Proof sketch / audit trail

This was computed by restricting  $J$  to each class+flux, applying the exact transfer operators  $M$  and  $Z$ , mapping  $\mathbb{Z}_3$  entries to real values  $\{-1, 0, 1\}$  (with  $2 \mapsto -1$ ), removing the mean, and projecting onto the orthonormal joint-mode bases. The resulting mode-energy fractions are tabulated. (Audit bundle: `W33_mode_response_table_bulk_to_vacuum_bundle.zip`.)

### Key Result

The vacuum sector is not a single “channel”: bulk and boundary sources excite different vacuum harmonics. This explains why no Aut-equivariant line-only operator can strongly predict  $m$  from  $z$  (they are distinct projections of the same bulk source field).

## Artifact Index (field-equation layer)

Bundle	Contents / Purpose
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## 15 Vacuum association scheme and canonical harmonics

**Theorem 15.1 (90-line association scheme and involution)** *The  $\text{Aut}(W33)$  action on the 90 non-isotropic lines induces an association scheme with commutant dimension 5 (five orbitals on ordered pairs). One orbital is the diagonal; another is a fixed-point-free involution  $\sigma$  pairing the 90 lines into 45 disjoint transpositions, with each paired lineset disjoint (skew).*

**Theorem 15.2 (Five canonical harmonics)** *Let  $S$  be the permutation matrix of  $\sigma$  and let  $A_{\text{meet}}$  be the adjacency of the line-meet graph (degree 32). Then  $S$  and  $A_{\text{meet}}$  commute and admit a joint decomposition into five modes:*

$$(+, 32)^1, \quad (+, 2)^{24}, \quad (+, -4)^{20}, \quad (-, 8)^{15}, \quad (-, -4)^{30}.$$

*These modes provide the canonical “vacuum harmonics” for line fields.*

### Remark

This harmonic analysis explains why distinct vacuum observables (e.g., boundary moment  $m$  vs bulk shadow  $z$ ) are not related by a single Aut-equivariant line-only operator: they occupy different mixtures of the canonical modes. The correct dynamics closes only when bulk source variables  $J = dF$  and the transfer operators  $M, Z$  are included.



## A Global Artifact Index

Bundle	Contents / Purpose
W33_symplectic_audit_bundle.zip	Explicit construction of $W(3,3)$ and W33; point/line incidence; $PG(3,3)$ points; isotropic vs nonisotropic line lists; verification of SRG parameters and spectrum.
W33_orbits_squarezero_bundle.zip	Aut(W33) generators (permutations and GF(3) matrices); orbit computations; square-zero and symmetry checkpoints.
W33_GF2_kernel_code_bundle.zip	The [40, 24, 6] kernel code $\ker(A)$ over $\mathbb{F}_2$ ; 240 weight-6 generators; code basis and supporting tables.
W33_H8_quadratic_form_bundle.zip	Basis of $H = \ker(A)/\text{im}(A)$ ; invariant quadratic form $q$ ; orbit split (135 singular / 120 nonsingular).
W33_to_H_to_120root_SRG_bundle.zip	The 120 nonsingular orbit list; SRG(120,56,28,24) edges/adjacency; mappings from code generators to $H$ .
W33_E8_simple_root_system_bundle.zip	Canonical induced $E_8$ Dynkin configuration inside the 120-root SRG; Coxeter checks; reflection orbit generation.
W33_signed_root_cocycle_and_lift_bundle.zip	Signed lift/cocycle computations on 120-root edges and Steiner triples; defect weights; gauge studies.
W33_global_gaugefix_no16_bundle.zip	Global sign/gauge fix removing all weight-16 defects; resulting 0,12 defect spectrum; 40 flat triples.
W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip	Quotient $Q = \overline{W33}$ ; edge matchings; triangle holonomy values; proof that flat holonomy triangles are exactly nonisotropic line triples.
W33_Z3_curvature_cohomology_on_quotient_bundle.zip	Triangle curvature cochain over $\mathbb{Z}_3$ ; non-exactness on the 2-skeleton; supporting tables.
W33_minimal_Z3_flux_cycles_tetrahedra_bundle.zip	Minimal-support flux cycles (tetrahedron boundaries) and flux statistics for $J = dF$ .
W33_flux_lattice_clique_complex_Z3_cohomology_bundle.zip	Clique-complex cohomology ranks and dimensions over $\mathbb{Z}_3$ ; $H^3$ dimension 89; higher cohomology signature.
W33_H3_basis_89_Z3_on_clique_complex_bundle.zip	Explicit 89-element basis for $H^3$ as sparse tetra-cochains; pivot/free coordinate metadata.
W33_H3_Aut_action_89Z3_bundle.zip	Aut(W33) action matrices on $H^3$ ; 88+1 decomposition; quotient functional and block form.
W33_perm_module_vs_H3_match_report_bundle.zip	Evidence and generators showing the 88D core matches the 90-line augmentation quotient up to the similitude sign twist.
W33_H3_to_noniso_line_weights_intertwiner_bundle.zip	Explicit intertwiner between $H^3$ 88D core and the twisted 90-line augmentation quotient.
W33_lift_to_90_line_weights_with_labels_bundle.zip	Explicit lift to labeled 90 nonisotropic line weights (mod all-ones gauge); line_id to 4-point set.
W33_holonomy_phase_test_bundle.zip	Holonomy vs symplectic triangle phase test; shows background closed 2-form vs sourced curvature.
W33_current_operator_C_lineface_bundle.zip	Operator $C_{\text{lineface}}$ and line-moment statistics (source attachments to vacuum cells).
W33_bulk_operator_KOK1_curved_triangle_current_bundle.zip	Bulk current operators on curved triangles ( $K_0, K_1$ ); outputs $y$ on the 2880 curved triangle orbit.
W33_curved_triangle_to_noniso_line_operator_R_bundle.zip	23Operator $R$ mapping curved-triangle current to 90-line aggregates via edge-incidence.
W33_charge_decomposition_and_line_moments_bundle.zip	Charge decomposition $J = dF$ ; point incidences; preliminary line moments and constraints.

## B Global Dictionary Table

Object	Interpretation	Algebra	Geometry/- Topology	Quantum computation	Crypto / security
$V = \mathbb{F}_3^4$	Finite phase space; 2-qutrit discrete symplectic phase space.	Vector space over $\mathbb{F}_3$ with symplectic form.	Underlying coordinate domain for projective geometry and Weyl operators.	Pauli/Weyl labels; Clifford acts by $Sp(4, 3)$ .	Key space for symplectic commutator phase.
$W(3, 3)$ / isotropic lines	Maximal commuting contexts.	Incidence geometry of totally isotropic points/lines.	Produces W33 as point graph.	Stabilizer contexts for two qutrits.	Basis for context-based protocols.
W33 SRG(40,12,2,4)	= Base combinatorial geometry.	Adjacency matrix $A$ with SRG identities.	Over $\mathbb{F}_2$ , yields differential $A^2 = 0$ .	Constraint graph / stabilizer structure.	Public structure; secrecy comes from gauge/coset choices.
$A^2 \equiv 0$ over $\mathbb{F}_2$	Chain-complex calculus.	Defines $d(x) = Ax$ with $d^2 = 0$ .	Produces code $\ker(A)$ and homology $H$ .	Error correction / stabilizer relations.	Syndromes / tamper detection.
$H = \ker(A)/\text{im}(A)$ (8D)	Intrinsic state space.	Carries invariant quadratic form; orbit split.	Nonsingular orbit gives 120-root shell.	Finite “root” degrees; phase classes.	Key reduction space for encoding.
120/240 roots	Finite root shell and signed lift.	SRG(120) adjacency via bilinear form; 2-to-1 lift.	Global gauge fixing yields flat triples.	Discrete gauge degrees; lift choices.	Keyed section choices = secrecy.
$Q = \overline{W33}$	Quotient spacetime / interaction graph.	40 meta-vertices after collapse; edge matchings.	Supports $\mathbb{Z}_3$ holonomy.	Transport/holon- omy = topological gate.	Holonomy checks = authentication.
Holonomy $F \in C^2(\text{Cl}(Q); \mathbb{Z}_3)$	Field strength / curvature.	Triangle cochain valued in $\mathbb{Z}_3$ .	Flat set classified by 90 nonisotropic lines.	Discrete phase curvature.	Consistency checks / signatures.
Sources $J = dF \in C^3$	Charge/current.	Supported on 3008 tetrahedra.	Generates vacuum responses via $M, Z$ .	Excitations / particles.	Error/fault injection model.
90 nonisotropic lines	Vacuum cells and matter carrier space.	Association scheme (5-mode harmonic analysis).	Line-weight field model (mod all-ones).	Contextual phase cells.	Share space for schemes; 88D core module.
Transfer operators $M, Z$	Constitutive laws.	Exact maps $J \mapsto (m, z)$ .	Mode-resolved response tables.	Measuremen- t/readout operators.	Encryption/read- out operators.

## C Reproducibility Checklist

### Remark

Short SHA-256 prefixes (first 16 hex characters) for primary bundles in the current workspace.



File	SHA-256 prefix
W33_symplectic_audit_bundle.zip	c8f7547649abdab1
W33_orbits_squarezero_bundle.zip	84835a9889e4380b
W33_GF2_kernel_code_bundle.zip	952858afb5d65007
W33_H8_quadratic_form_bundle.zip	de3a9a9b0afb6a37
W33_to_H_to_120root_SRG_bundle.zip	3257de84a4b9c466
W33_E8_simple_root_system_bundle.zip	d200bec6ff81f00a
W33_signed_root_cocycle_and_lift_bundle.zip	d33146ea2d96104f
W33_global_gaugefix_no16_bundle.zip	8de8d1182056ac00
W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip	8a6cda139ed0a0e6
W33_Z3_curvature_cohomology_on_quotient_bundle.zip	1a7804dd46ccb1b5
W33_minimal_Z3_flux_cycles_tetrahedra_bundle.zip	8d69efdc34b5a0e6
W33_flux_lattice_clique_complex_Z3_cohomology_bundle.zip	17f5bb8490fc2d36
W33_H3_basis_89_Z3_on_clique_complex_bundle.zip	2fa53b14fcd57da9
W33_H3_Aut_action_89Z3_bundle.zip	032be0e14f33c5cc
W33_perm_module_vs_H3_match_report_bundle.zip	535aa4d6b03264d9
W33_H3_to_noniso_line_weights_intertwiner_bundle.zip	da15db795acf478b
W33_lift_to_90_line_weights_with_labels_bundle.zip	81b9f049398d5f93
W33_holonomy_phase_test_bundle.zip	5991ca050359bc4b
W33_current_operator_C_lineface_bundle.zip	02e3566e1869ce07
W33_bulk_operator_KOK1_curved_triangle_current_bundle.zip	5953f1541d2793f1
W33_curved_triangle_to_noniso_line_operator_R_bundle.zip	633e86c28d6433cf
W33_charge_decomposition_and_line_moments_bundle.zip	d9c00f5e46ca2658
W33_nonisotropic_line_association_scheme_bundle.zip	ec4b4b8e10918586
W33_vacuum_line_scheme_mode_decomposition_bundle.zip	d8545a6b843ab310
W33_transfer_operators_J_to_lines_and_mode_injection_bundle.zip	647e18c9a6ac8f7c
W33_best_field_equation_operator_on_lines_bundle.zip	3494bf1e74c08f1b