A Finite-Geometric Theory Kernel from W33

Toward a Unified Algebra–Topology–Quantum Computation–Cryptography Framework

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Abstract

This document consolidates the W33 tower into a single, self-contained theory kernel. Starting from the symplectic phase space $V = \mathbb{F}_3^4$, we construct the symplectic generalized quadrangle $W(3,3)$ and its point graph $\mathrm{W33} = \mathrm{SRG}(40, 12, 2, 4)$. Over $\mathbb{F}_2$, the adjacency satisfies $A^2 \equiv 0$, producing a canonical code $[40, 24, 6]$ and an intrinsic homology space $H = \ker(A)/\mathrm{im}(A) \cong \mathbb{F}_2^8$. The nonsingular orbit in H yields a 120-element "root shell" with $\mathrm{SRG}(120, 56, 28, 24)$ adjacency, a 240 signed lift admitting global gauge fixing, and a quotient closure back to 40 points as $Q = \overline{\mathrm{W33}}$. The quotient carries a canonical $\mathbb{Z}_3$ holonomy, with flat faces classified exactly by the 90 non-isotropic projective lines. Over $\mathbb{Z}_3$, the clique complex of Q has $H^3 \cong (\mathbb{Z}_3)^{89}$, whose 88D core is identified (up to a canonical sign character) with the augmentation quotient on the 90 non-isotropic lines. Finally, the holonomy field F is sourced: $J = dF$ is a 3-cochain supported on 3008 tetrahedra, and explicit sparse transfer operators map J to observed vacuum line responses.

Remark

What is meant by "theory of everything" here. This manuscript presents a mathematically closed kernel in which geometry, algebra, topology, computation, and cryptography are realized as different functorial views of the same finite symplectic/projective object. Claims about physical constants require an additional scaling/continuum layer and are not asserted as part of the kernel.

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Master Equation Summary

Key Result

Discrete gauge kernel (minimal equations). Let $Q = \overline{\mathrm{W33}}$ be the quotient graph and $\mathrm{Cl}(Q)$ its clique complex.

Field strength (holonomy). $F \in C^2(\mathrm{Cl}(Q); \mathbb{Z}_3)$ is the computed triangle holonomy.

Sources. $J := dF \in C^3(\mathrm{Cl}(Q); \mathbb{Z}_3)$ is the sourced 3-cochain (supported on 3008 tetrahedra).

Vacuum response (exact constitutive laws). There exist explicit sparse operators

$$M, Z : \mathbb{Z}_3^{9450} \to \mathbb{Z}_3^{90}$$

such that the observed line fields satisfy

$$m_{\text{line}} = MJ, \qquad z_{\text{line}} = ZJ$$

exactly.

Vacuum harmonics. The 90-line sector admits five canonical joint modes under the involution S and meet adjacency A_{meet}:

$$(+, 32)^1,\ (+, 2)^{24},\ (+, -4)^{20},\ (-, 8)^{15},\ (-, -4)^{30}.$$

Bulk and boundary source classes inject into different harmonic mixtures (mode-response tables).

1 Master Equations and Couplings

Definition

Field variables. On the clique complex $\mathrm{Cl}(Q)$ of the quotient graph $Q = \overline{\mathrm{W33}}$:

- $F \in C^2(\mathrm{Cl}(Q); \mathbb{Z}_3)$ is the triangle holonomy field (field strength).
- $J := dF \in C^3(\mathrm{Cl}(Q); \mathbb{Z}_3)$ is the sourced 3-cochain (charge/current).

On the vacuum line set $\mathcal{L}$ (the 90 non-isotropic lines):

- $m_{\text{line}} \in \mathbb{Z}_3^{90}$ is the *boundary moment* observable.
- $z_{\text{line}} \in \mathbb{Z}_3^{90}$ is the *bulk shadow* observable.

Theorem 1.1 (Master operator equations) *The W33 kernel closes as the following exact operator pipeline over $\mathbb{Z}_3$:*

$$F \xrightarrow{d} J \xrightarrow{(M,Z)} (m_{\text{line}}, z_{\text{line}}),$$

where d is the simplicial coboundary on $\mathrm{Cl}(Q)$ and $M, Z : \mathbb{Z}_3^{9450} \to \mathbb{Z}_3^{90}$ are explicit sparse transfer operators. Concretely,

$$J = dF, \qquad m_{\text{line}} = MJ, \qquad z_{\text{line}} = ZJ,$$

and these identities hold entrywise with no residual error.

Proof sketch / audit trail

F and $J = dF$ are computed from the quotient holonomy. The operators M and Z are constructed canonically from incidence: M routes tetra flux to the unique vacuum line of the tetra's flat face (when present), while Z routes tetra flux to vacuum lines via edge-incidence of curved faces. Exactness was verified against independently computed line observables. (Audit bundle: `W33_transfer_operators_J_to_lines_and_mode_injection_bundle.zip`.)

Definition

Vacuum harmonics. Let S be the canonical involution on $\mathcal{L}$ (45 disjoint transpositions) and A_{meet} the meet adjacency on $\mathcal{L}$ (degree 32). The vacuum line sector decomposes into five joint modes:

$$(+,32)^1,\ (+,2)^{24},\ (+,-4)^{20},\ (-,8)^{15},\ (-,-4)^{30}.$$

Theorem 1.2 (Coupling selection rules (mode response)) *Bulk sources (tetrahedra with zero flat faces) inject into z_{line} but not m_{line}. Boundary sources (tetrahedra with one flat face) inject into both m_{line} and z_{line}, with mode weights shifted toward $(+,2)$ and $(-,8)$ for m_{line}. These couplings are quantified by the mode-response tables.*

Proof sketch / audit trail

Apply M and Z to class-restricted source vectors and project the resulting 90-line fields into the five joint modes using the association-scheme harmonic bases. (Audit bundle: `W33_mode_response_table_bulk_to_vacuum_bundle.zip`.)

Key Result

The equations $J = dF$ and $(m, z) = (MJ, ZJ)$ are the minimal "field equations" of the kernel. Together with the five vacuum harmonics, they provide a complete, symmetry-respecting description of how sourced curvature produces observable vacuum response in the 90-line sector.

2 Closure Principle

Definition

Closure. We say the W33 tower is *closed* if the following hold simultaneously:

1. (**Lift**) The 240 minimal code generators project 2-to-1 onto a 120-element nonsingular orbit in H (the "root shell").
2. (**Gauge fix**) There exists a global sign section eliminating all weight-16 defects, producing 40 disjoint flat triples.
3. (**Collapse**) Collapsing the 40 triples yields a 40-vertex quotient graph Q with canonical edge transport and $\mathbb{Z}_3$ holonomy.
4. (**Recursion**) The quotient graph is exactly $Q = \overline{\mathrm{W33}}$.
5. (**Vacuum/matter coincidence**) The 90 non-isotropic lines simultaneously (i) classify flat holonomy faces and (ii) support the 88D core module of H^3 via the 90-line augmentation quotient.

Theorem 2.1 (Closure Theorem) *The W33 tower is closed in the above sense. In particular:*

1. *The globally gauge-fixed signed lift partitions the 120 roots into 40 flat triples.*
2. *The induced quotient is $Q = \overline{\mathrm{W33}}$ and carries canonical $\mathbb{Z}_3$ triangle holonomy.*
3. *Flat holonomy triangles are exactly the triples lying on the 90 non-isotropic lines of $PG(3,3)$.*
4. *The 88D core of $H^3(\mathrm{Cl}(Q); \mathbb{Z}_3)$ is (up to the similitude sign twist) the augmentation quotient on these same 90 non-isotropic lines.*

Proof sketch / audit trail

Items (1)–(3) are verified by the explicit gauge-fix computation and quotient construction: the defect-0 edges form 40 disjoint triangles partitioning the 120 roots, and the quotient adjacency equals the complement of W33 with a $\mathbb{Z}_3$ holonomy classified by non-isotropic line triples. Item (4) is established by comparing the 88D core module of H^3 with the 90-line augmentation quotient: after the canonical similitude sign twist, traces and characteristic-polynomial factor patterns match and an explicit intertwiner exists. (Audit bundles: `W33_global_gaugefix_no16_bundle.zip`, `W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip`, `W33_H3_Aut_action_89Z3_bundle.zip`, `W33_perm_module_vs_H3_match_report_bundle.zip`.)

Key Result

Closure is the central "TOE hinge" of the kernel: the same finite geometry simultaneously generates (i) constraints/codes ($A^2 = 0$ over $\mathbb{F}_2$), (ii) a root shell and gauge-fixed signed lift (120/240), (iii) a recursive quotient $Q = \overline{\mathrm{W33}}$ with $\mathbb{Z}_3$ holonomy, and (iv) a vacuum line sector that is also the carrier of the nontrivial 88D matter/flux module. This is precisely the structure needed for a self-contained theory kernel.

3 Functorial Field Theory View

Definition

Clique category. Let $Q = \overline{\mathrm{W33}}$ and $\mathrm{Cl}(Q)$ its clique (flag) complex. Define a small category $\mathcal{C}(Q)$ as follows:

- Objects are cliques $\sigma \subseteq V(Q)$ (equivalently simplices of $\mathrm{Cl}(Q)$), including vertices, edges, triangles, tetrahedra, etc.
- Morphisms are inclusions $\tau \hookrightarrow \sigma$ (face maps).

Thus $\mathcal{C}(Q)$ encodes the full incidence/facial structure of the quotient geometry.

Definition

Cochain functors. Fix a coefficient ring R (typically $R = \mathbb{Z}_3$). For each $k \geq 0$, define a functor

$$\mathsf{C}^k_R : \mathcal{C}(Q)^{\mathrm{op}} \to \mathsf{Mod}_R$$

by assigning to each k-simplex σ the free rank-one R-module generated by σ, and to each face inclusion the corresponding restriction map. The usual coboundary $d : \mathsf{C}^k_R \to \mathsf{C}^{k+1}_R$ is a natural transformation determined by alternating sums of face restrictions (with orientation conventions).

Definition

Vacuum line functor. Let $\mathcal{L}$ be the 90 non-isotropic lines in $PG(3,3)$, which are also the 90 flat K_4 cells in Q. Define the vacuum sector as the permutation module

$$\mathsf{V} := \mathbb{Z}_3^{\mathcal{L}},$$

together with its canonical 88D augmentation quotient $\mathsf{V}_{88} = \mathrm{Aug}(\mathcal{L})/\langle \mathbf{1} \rangle$ (up to the similitude sign twist).

Theorem 3.1 (Kernel as a functorial gauge system) *The W33 tower admits a functorial formulation in which geometry, topology, computation, and quantum structure are different functorial shadows of the same underlying incidence data:*

1. *(**Geometry→ Topology**) The clique category $\mathcal{C}(Q)$ determines cochain functors $\mathsf{C}^k_{\mathbb{Z}_3}$ and a natural coboundary d. The holonomy field F is an element of $\mathsf{C}^2_{\mathbb{Z}_3}$ and the source field is $J = dF \in \mathsf{C}^3_{\mathbb{Z}_3}$.*
2. *(**Topology→ Vacuum response**) The transfer operators M and Z are natural, Aut(W33)-equivariant linear maps from the tetra-source module to the vacuum module:*

$$M, Z : \mathbb{Z}_3^{\{tetrahedra\}} \to \mathbb{Z}_3^{\mathcal{L}},$$

giving exact observables $(m_{\mathrm{line}}, z_{\mathrm{line}}) = (MJ, ZJ)$.

3. *(**Computation**) Over $\mathbb{F}_2$, the W33 adjacency defines a square-zero differential on $\mathbb{F}_2^{40}$, yielding the intrinsic code $\ker(A)$ and homology $H = \ker(A)/\mathrm{im}(A)$; these are functorial with respect to the Aut(W33) action.*

4. (***Quantum***) *The phase space axiom $V = \mathbb{F}_3^4$ defines the 2-qutrit Weyl functor (Weyl labels and commutator phase) and a projectivized Clifford action by $PGSp(4, 3)$ on $PG(3, 3)$; isotropic lines correspond to maximal commuting contexts.*

Moreover, the representation-theoretic identification $H^3(\mathrm{Cl}(Q); \mathbb{Z}_3)_{88} \cong \mathsf{V}_{88}$ (up to twist) provides an explicit equivalence between the flux-lattice core and the vacuum line module.

Proof sketch / audit trail

Each item is backed by explicit constructions: (1) and (2) follow from the computed holonomy F, sources $J = dF$, and the sparse transfer operators M, Z built from incidence (Section 11 and associated bundles). (3) follows from the SRG identity implying A^2
$equiv 0$ over
$mathbbF_2$ and the explicit kernel-code computation (Section 4). (4) follows from the standard Weyl/Clifford construction on V and the identification of W33 points/lines with projective points/isotropic lines in $PG(3, 3)$ (Section 10). The module equivalence is established by comparing the Aut(W33) actions and constructing an explicit intertwiner (Section 9).

Key Result

This functorial view is the cleanest "TOE statement" available at the kernel level: a single finite incidence object induces, via natural functors, (i) a sourced gauge field (F, J), (ii) exact response laws (M, Z) into the vacuum line sector, (iii) an intrinsic error-correcting code over
$mathbbF_2$, and (iv) a 2-qutrit Weyl/Clifford quantum structure over
$mathbbF_3$. These are not separate theories but compatible projections of the same kernel.

4 Continuum and Scaling Layer (Optional Program)

Remark

Status. Everything in Sections 1–12 is a finite, exact kernel. This section is explicitly labeled optional: it proposes principled scaling routes that could connect the finite kernel to effective continuum physics, without asserting any numerical "constant matching" as part of the kernel.

4.1 Three natural scaling parameters

Definition

Scaling routes. The W33 kernel suggests three canonical families:

1. (**Field size**) Replace $\mathbb{F}_3$ by $\mathbb{F}_q$ and study $V = \mathbb{F}_q^4$ with symplectic form, yielding $W(3, q)$ and its point graph.
2. (**Rank**) Replace $V = \mathbb{F}_q^4$ by $V = \mathbb{F}_q^{2n}$, studying $W(2n-1, q)$ and the resulting tower as n grows.
3. (**Covers / coarse graining**) Use regular covers of the quotient connection (e.g., minimal regular covers of transport/holonomy data) as lattice refinements, and study renormalization via pushforward/pullback of cochains.

4.2 Field-size family $W(3,q)$: exact SRG parameters and mod-2 square-zero for odd q

Theorem 4.1 (Symplectic $W(3,q)$ point-graph parameters) *Let q be a prime power and let G_q be the point graph of the symplectic generalized quadrangle $W(3,q)$ (points are projective points of $PG(3,q)$; edges are collinearity on totally isotropic lines). Then G_q is strongly regular with parameters:*

$$v = q^3 + q^2 + q + 1, \qquad k = q(q+1), \qquad \lambda = q - 1, \qquad \mu = q + 1.$$

Its adjacency spectrum is

$$k^{(1)}, \qquad r^{(q^2(q+1))}, \qquad s^{(q(q^2+1))},$$

where $r = q - 1$ and $s = -(q+1)$.

Proof sketch / audit trail

These are standard parameters for the symplectic polar space $W(3,q)$. For completeness, we verified them computationally for $q \in \{2,3,5,7\}$ by explicit enumeration of projective points and totally isotropic lines, building the point graph, and counting common neighbors; the rounded eigenvalue multiplicities match the stated spectrum.

Theorem 4.2 (Mod-2 square-zero for odd q) *Let A_q be the adjacency matrix of G_q and reduce it mod 2. If q is odd, then*

$$A_q^2 \equiv 0 \pmod 2.$$

Equivalently, the symplectic $W(3,q)$ point graph defines a canonical square-zero differential over $\mathbb{F}_2$ for every odd q.

Proof sketch / audit trail

For any $\mathrm{SRG}(v,k,\lambda,\mu)$,

$$A^2 = kI + \lambda A + \mu(J - I - A).$$

Reducing mod 2 yields $A^2 \equiv (k-\mu)I + (\lambda-\mu)A + \mu J \pmod 2$. For G_q, we have $k - \mu = (q-1)(q+1) = q^2 - 1$ (even for odd q), $\lambda - \mu = -2$ (even), and $\mu = q + 1$ (even for odd q). Hence $A^2 \equiv 0 \pmod 2$ for odd q. The case $q = 2$ fails as expected.

4.3 $q = 5$: root shell orbit and order-3 projectivization (first pass)

Remark

This subsection reports the first nontrivial lift-layer test for $q = 5$. It is not yet a full analog of the $q = 3$ signed 240→120 lift, but it reveals a closely related phenomenon: the local line-pair generators form a single large orbit in H_5 and admit a canonical order-3 "projectivization" induced by the endomorphism ring of the H_5 module.

Theorem 4.3 (A 2340-element root-shell orbit in H_5) *For $q = 5$, the local line-pair generators (XOR of two isotropic lines through a point) map injectively into the 24D homology module H_5 over $\mathbb{F}_2$, producing a set of 2340 distinct nonzero vectors. Under a symplectic subgroup action (generated by transvections), this 2340-set is a single orbit.*

Proof sketch / audit trail

We explicitly construct the $W(3,5)$ point graph (156 vertices, 156 isotropic lines), compute H_5 via $\ker(A_5)/\mathrm{im}(A_5)$ (dimension 24), map all 2340 line-pair generators into H_5 coordinates, and verify invariance/orbit transitivity under 20 symplectic generators. (Bundle: `W33_q5_lift_layer_first_pass_bundle.zip`.)

Theorem 4.4 (Order-3 centralizer and 780-cycle projectivization) *The induced 24D H_5 module has a 2D endomorphism ring under the tested symplectic subgroup, generated by the identity and an element X of order 3. The action of X permutes the 2340-element orbit without fixed points, partitioning it into 780 disjoint 3-cycles. Using the invariant 2D alternating-form space (F_0, F_1), pairs may be labeled by $(b_0, b_1) \in \mathbb{F}_2^2$, whose nonzero classes form a $\mathrm{GF}(4)^*$-like set of size 3. Declaring adjacency by nonzero label yields a regular quotient graph on 780 vertices of degree 504 whose pairwise common-neighbor counts split into two values on edges and two values on nonedges (a higher-rank association-scheme signature).*

Proof sketch / audit trail

We compute the centralizer of the subgroup action on H_5 by solving $XM = MX$ over $\mathbb{F}_2$ for the generator set, obtaining a 2D solution space and an order-3 element. Applying this element to the 2340 orbit yields 780 disjoint 3-cycles. Using two independent invariant alternating forms F_0, F_1, we label pairs by (b_0, b_1) and define adjacency by nonzero label; the resulting quotient graph is regular of degree 504 with a two-type adjacency/two-type nonadjacency common-neighbor signature. (Bundle: `W33_q5_root_shell_orbit_and_GF4_projectivization_bundle.zip`.)

Key Result

The $q = 5$ lift-layer reveals a strong analog of the $q = 3$ signed-lift mechanism: instead of a 2-to-1 sign lift, the natural commutant structure induces an order-3 projectivization on the root-shell orbit. This suggests the correct higher-q generalization is governed by endomorphism-ring structure (field extensions) rather than a fixed $\pm$ sign.

4.4 $q = 5$: 780-cycle association scheme and harmonics

Theorem 4.5 (Five-orbital scheme on the 780 projectivized root shell) *Under the induced symplectic subgroup action, the 780-cycle quotient carries a symmetric 5-orbital association scheme (commutant dimension 5). The corresponding symmetric relations have row degrees*

$$1,\ 4,\ 125,\ 150,\ 500,$$

where the degree-4 relation decomposes into 156 disjoint K_5 cliques (a canonical $156 \times K_5$ fibration of the 780 vertices).

Theorem 4.6 (Canonical q=5 harmonics via joint diagonalization) *Let A_4 denote the adjacency matrix of the degree-4 relation (the K_5 fiber graph) and let A_{500} denote the adjacency matrix of the degree-500 relation. Then A_4 has eigenvalues $4^{(156)}$ and $(-1)^{(624)}$. Restricting A_{500} to these eigenspaces yields a full five-mode decomposition of $\mathbb{R}^{780}$ into joint eigenspaces of (A_4, A_{500}) with dimensions:*

$$(4, 500)^1, \quad (4, 20)^{65}, \quad (4, -20)^{90}, \quad (-1, 25)^{104}, \quad (-1, -5)^{520}.$$

These are the $q = 5$ analogs of the "vacuum harmonics" in the $q = 3$ kernel.

Proof sketch / audit trail

We compute the orbitals (ordered-pair orbits) of the induced action on 780 vertices and obtain five symmetric relations. The degree-4 relation is verified to split into 156 components of size 5, each a complete K_5. Joint diagonalization is obtained by first diagonalizing A_4 (block-diagonal K5 spectrum) and then diagonalizing A_{500} restricted to each A_4 eigenspace. (Bundle: `W33_q5_780_association_scheme_harmonics_bundle.zip`.)

Key Result

The $q = 5$ projectivized root shell inherits the same structural signature that made the $q = 3$ vacuum sector decisive: a small commutant (dimension 5) and a canonical finite harmonic decomposition. This strongly supports the hypothesis that the W33 kernel is the $q = 3$ member of a universal symplectic ladder whose higher-q members replace the $\pm$ signed lift by extension-field projectivizations (here, order 3 / GF(4)*).

4.5 $q = 7$: root shell orbit and idempotent split (first pass)

Theorem 4.7 (A 11200-element root-shell orbit in H_7) *For $q = 7$, the local line-pair generators (XOR of two isotropic lines through a point) map injectively into the 48D homology module H_7 over $\mathbb{F}_2$, producing a set of 11200 distinct nonzero vectors. Under a symplectic subgroup action (generated by transvections), this 11200-set is a single orbit.*

Proof sketch / audit trail

We explicitly construct the $W(3, 7)$ point graph (400 vertices, 400 isotropic lines), compute H_7 via $\ker(A_7)/\mathrm{im}(A_7)$ (dimension 48), map all 11200 line-pair generators into H_7 coordinates, and verify invariance/orbit transitivity under 22 symplectic generators. (Bundle: `W33_q7_lift_layer_first_pass_bundle.zip`.)

Theorem 4.8 (Idempotent commutant and 24+24 splitting) *Under the tested symplectic subgroup, the endomorphism (centralizer) algebra of the 48D H_7 module has dimension 2 over $\mathbb{F}_2$, generated by the identity and a nontrivial idempotent projector P of rank 24. Hence H_7 splits into two invariant 24D submodules:*

$$H_7 \cong \mathrm{Im}(P) \;\oplus\; \mathrm{Im}(I - P).$$

Projecting the 11200-element root shell into either half yields a single orbit of size 2800, and $2800 = 7 \cdot 400$ *suggests a q-fibered structure over the 400-point base.*

Proof sketch / audit trail

We compute the centralizer by solving $XM = MX$ over $\mathbb{F}_2$ for the induced 48D action matrices, finding a 2D solution space. The nontrivial element satisfies $P^2 = P$ and has rank 24. We then project root-shell vectors via P and $I - P$, convert to 24D coordinates, and compute orbit decompositions. (Bundle: `W33_q7_root_shell_and_idempotent_split_bundle.zip`.)

Key Result

The $q = 7$ lift-layer exhibits a different "cheeky" generalization mechanism than $q = 5$: instead of an order-3 projectivization, the commutant produces an idempotent 24+24 split of the 48D homology module, with each half carrying a $2800 = 7 \cdot 400$ root-shell orbit. This strongly suggests that the higher-q lift structure is governed by commutant type (field extension vs idempotent splitting) rather than a universal $\pm$ sign.

4.6 $q = 7$: 2800-cycle association scheme and $400 \times K_7$ fibers (first pass)

Theorem 4.9 (Five-orbital scheme on the 2800 projected root shell) *The projected $q = 7$ root-shell orbit of size 2800 (in either 24D half-module of H_7) carries a 5-orbital association scheme (commutant dimension 5). Equivalently, the point stabilizer has five orbits on the 2800 points with sizes*

$$1,\ 6,\ 343,\ 392,\ 2058.$$

Theorem 4.10 (Degree-6 relation yields a canonical $400 \times K_7$ fibration) *The degree-6 relation in the above scheme is a disjoint union of 400 complete K_7 cliques, partitioning the 2800 vertices as*

$$2800 = 400 \times 7.$$

Thus the $q = 7$ projected root shell admits a canonical fiber structure with fiber size q over a 400-object base.

Proof sketch / audit trail

We compute a transitive permutation action on the 2800 projected orbit induced from the symplectic subgroup action on H_7, and compute stabilizer orbits via Schreier generators derived from a BFS transversal. The orbit-size list gives the five orbital degrees. The degree-6 relation is realized explicitly as the image of the stabilizer 6-orbit under the transversal, and its connected components are verified to be 400 disjoint K_7 cliques. (Bundle: `W33_q7_2800_association_scheme_first_pass_bundle.zip`.)

Key Result

The $q = 7$ half-module recovers the same "small commutant" signature as $q = 3$ and $q = 5$ (dimension 5), but with a fiber relation matching the field size: K_7 fibers over a 400-object base. This strongly supports a universal ladder where higher-q kernels produce a $v \times K_q$ fibration at the projectivized root-shell level.

4.7 $q = 7$: 2800-cycle harmonics (five primitive modes)

Theorem 4.11 (Five primitive harmonics on the 2800 projected root shell) *The 5-orbital association scheme on the 2800 projected $q = 7$ root shell admits five primitive harmonic modes with multiplicities:*

$$1,\ 224,\ 2100,\ 175,\ 300,$$

summing to 2800. Writing the nontrivial relation valencies as $(6, 343, 392, 2058)$*, the corresponding eigenvalues of the relation adjacencies on these five modes are:*

mode mult.	A_6	A_{343}	A_{392}	A_{2058}
1	6	343	392	2058
224	6	-7	42	-42
2100	-1	7	0	-7
175	6	7	-56	42
300	-1	-49	0	49

In particular, the fiber relation A_6 *has eigenvalues* 6 *and* -1 *with multiplicities* 400 *and* 2400*, matching the* $400 \times K_7$ *fibration.*

Proof sketch / audit trail

We compute intersection numbers p_{ij}^k using the stabilizer-orbit method: the relation class of a pair (u, v) is determined by the stabilizer orbit of $t_u^{-1}(v)$ under a BFS transversal t_u. The resulting 5x5 left-multiplication matrices L_i commute; their common eigenvectors yield the eigenmatrix P. Multiplicities are solved from orthogonality equations $\sum_r m_r P_{r,i}^2 = v k_i$. (Bundle: `W33_q7_2800_association_scheme_harmonics_bundle.zip`.)

Key Result

The $q = 7$ projected root shell not only reproduces the "small commutant" signature (dimension 5) but yields an explicit, integer-valued harmonic spectrum with a fiber eigen-split matching $400 \times K_7$. This is the direct $q = 7$ analog of the $q = 5$ 780-cycle harmonic decomposition.

4.8 A closed-form conjectural ladder for odd q (validated at $q = 5, 7$)

Theorem 4.12 (Projectivized root-shell 5-orbital scheme: closed eigenvalue formulas) *Let* q *be an odd prime power and consider the symplectic* $W(3, q)$ *kernel. Suppose the lift-layer produces a projectivized root-shell quotient of size*

$$N = q(q^3 + q^2 + q + 1) = q^4 + q^3 + q^2 + q,$$

with a canonical fiber relation decomposing into $(q^3 + q^2 + q + 1)$ *disjoint* K_q *cliques (degree* $q - 1$*). Then the induced commutant algebra is 5-dimensional (five orbitals), with relation valencies*

$$1, \quad q - 1, \quad q^3, \quad q^2(q + 1), \quad q^3(q - 1),$$

and a five-mode harmonic decomposition with multiplicities

$$1, \quad \frac{q(q+1)^2}{2}, \quad \frac{q(q^2+1)}{2}, \quad q(q^3 - q^2 + q - 1), \quad (q^3 - q^2 + q - 1).$$

Moreover, in the corresponding eigenmatrix (ordering relations by valency as above), the eigenvalues are:

mode mult.	A_{q-1}	A_{q^3}	$A_{q^2(q+1)}$	$A_{q^3(q-1)}$
1	$q-1$	q^3	$q^2(q+1)$	$q^3(q-1)$
$\frac{q(q+1)^2}{2}$	$q-1$	$-q$	$q(q-1)$	$-q(q-1)$
$\frac{q(q^2+1)}{2}$	$q-1$	q	$-q(q+1)$	$q(q-1)$
$q(q^3-q^2+q-1)$	-1	q	0	$-q$
(q^3-q^2+q-1)	-1	$-q^2$	0	q^2

Remark

Status and evidence. The $q = 5$ (780 vertices) and $q = 7$ (2800 vertices) projectivized root shells computed in this work realize this pattern exactly: the five relation degrees match $(q-1, q^3, q^2(q+1), q^3(q-1))$, and the harmonic mode multiplicities and eigenvalues match the above table. (Bundles: `W33_q5_780_association_scheme_harmonics_bundle.zip`, `W33_q7_2800_association_scheme_harmonics_bundle.zip`.)

Proof sketch / audit trail

Given five orbitals, the intersection numbers p_{ij}^k define commuting 5×5 multiplication matrices. The above eigenvalues and multiplicities are uniquely determined by: (i) the valencies, (ii) trace constraints $\sum_r m_r P_{r,i} = 0$ for loopless relations, and (iii) orthogonality $\sum_r m_r P_{r,i}^2 = N k_i$. Solving these equations yields the closed forms above; agreement with $q = 5, 7$ verifies consistency.

Key Result

This theorem isolates the "cheeky" universality: once the lift-layer produces a $v \times K_q$ fibered projectivized root shell, the entire 5-mode harmonic spectrum appears to be forced by symmetry and counting, and depends only on q through simple polynomials. This is the first closed-form candidate for a genuine q-ladder behind the W33 kernel.

4.9 $q = 3$: 120 root-shell scheme confirms the q-ladder

Theorem 4.13 (Five-orbital scheme on the 120 nonsingular orbit) *The 120-element nonsingular orbit (the $q = 3$ root shell) carries a symmetric 5-orbital association scheme with relation valencies*

$$1,\ 2,\ 27,\ 36,\ 54$$

summing to 120.

Theorem 4.14 (Fiber relation equals the 40 flat triples) *The degree-2 relation decomposes into 40 disjoint K_3 cliques, partitioning the 120 roots as*

$$120 = 40 \times 3.$$

This degree-2 fiber relation is exactly the "flat triple" partition produced by the global gauge fix in the $q = 3$ closure step.

Theorem 4.15 (q=3 harmonic spectrum matches the closed-form ladder) *The five primitive mode multiplicities are*

$$1,\ 24,\ 60,\ 15,\ 20,$$

and the eigenvalues on the four nontrivial relations $(2, 27, 36, 54)$ match the closed-form table in Theorem **??** *specialized at $q = 3$:*

$$(2,-3,6,-6)^{24},\quad (-1,3,0,-3)^{60},\quad (2,3,-12,6)^{15},\quad (-1,-9,0,9)^{20}.$$

Proof sketch / audit trail

We induce the Aut(W33) action on the 120 nonsingular H_8 orbit using the 8x8 generator matrices over $\mathbb{F}_2$, compute ordered-pair orbitals (five orbitals), and compute intersection numbers and the eigenmatrix via the 5x5 multiplication matrices. The degree-2 relation is verified to split into 40 disjoint triangles. (Bundle: `W33_q3_120_root_shell_association_scheme_harmonics_bundle.zip`.)

Key Result

This closes the "q-ladder" loop: the same 5-orbital / 5-mode spectral template that appears at $q = 5$ (780) and $q = 7$ (2800) already holds at $q = 3$ on the 120 root shell, and its degree-$(q-1)$ fiber relation *is* the 40-flat-triple partition used in the $q = 3$ closure theorem. Thus $q = 3$ is not an exception; it is the first nontrivial rung of the universal ladder.

4.10 Derivation of the q-ladder spectrum (proof outline)

Remark

Goal. This subsection explains why the closed-form eigenvalue/multiplicity table in Theorem **??** is (essentially) forced once three structural inputs hold: (i) a $v \times K_q$ fiber relation, (ii) five orbitals (commutant dimension 5), and (iii) symmetry/orthogonality constraints of association schemes. The remaining conjectural step is the existence of the projectivized root-shell quotient for all odd q.

Definition

Assume a symmetric 5-class association scheme on $N = qv$ points with relations

$$A_0 = I, \quad A_1 \text{ (fiber)}, \quad A_2, \quad A_3, \quad A_4,$$

with valencies

$$k_0 = 1, \quad k_1 = q - 1, \quad k_2 = q^3, \quad k_3 = q^2(q + 1), \quad k_4 = q^3(q - 1),$$

so that $\sum_i k_i = N$. Assume further that A_1 is a disjoint union of v cliques K_q (equivalently, A_1 has spectrum $(q - 1)^{(v)}$ and $(-1)^{(N-v)}$).

Lemma 4.16 (Two forced eigenvalues for the fiber relation) *The fiber adjacency A_1 has eigenvalues $q - 1$ and -1 only. The multiplicity of eigenvalue $q - 1$ is exactly v (constant-on-fiber vectors), and the multiplicity of eigenvalue -1 is $N - v$ (sum-zero-on-fiber vectors).*

Proof sketch / audit trail

This is immediate from the block-diagonal structure: each fiber contributes one $(q-1)$ eigenvector and $q - 1$ eigenvectors of -1.

Lemma 4.17 (Reduction to four unknown multiplicities) *Let the primitive idempotents (harmonic modes) be $E_0, \ldots, E_4$ with multiplicities $m_r = \operatorname{rank}(E_r)$, with $m_0 = 1$. Then:*

$$\sum_{r=0}^{4} m_r = N, \qquad \sum_{r=0}^{4} m_r\, P_{r,i}^2 = N k_i \quad (i = 0, 1, 2, 3, 4),$$

*where P is the eigenmatrix and $P_{r,i}$ is the eigenvalue of A_i on mode r. Moreover, Lemma **??** forces the A_1 column of P to take only values $q-1$ or -1, with total multiplicities v and $N-v$ respectively.*

Proof sketch / audit trail

These are standard orthogonality relations for symmetric association schemes: $P^\top \operatorname{diag}(m) P = N \operatorname{diag}(k)$ and the fiber eigen-split from Lemma **??**.

Lemma 4.18 (A closed polynomial ansatz with three parameters) *If the scheme arises from a symplectic/projective ladder, then (empirically at $q = 3, 5, 7$) the remaining relations act with eigenvalues in the small set*

$$\{\pm q,\ \pm q^2,\ q(q \pm 1),\ \pm q(q-1),\ 0\},$$

*and the nontrivial modes refine the fiber split into four blocks. Under this ansatz, the unknown entries of P reduce to finitely many sign choices, and the orthogonality system in Lemma **??** becomes a determined linear system in the multiplicities.*

Proof sketch / audit trail

For $q = 3, 5, 7$ the computed eigenmatrices have exactly this shape. The values are natural from representation theory: they match the expected character values of small-rank constituents induced by the polar-space action and its endomorphism-ring reductions.

Theorem 4.19 (Forcing of the closed-form table (conditional)) *Assume (i) the valencies above, (ii) the $v \times K_q$ fiber condition for A_1, (iii) five orbitals, and (iv) the small polynomial ansatz of Lemma **??**. Then the eigenvalue/multiplicity table of Theorem **??** is uniquely determined (up to permuting the last four modes).*

Proof sketch / audit trail

Fix the trivial row of P to be the valency vector (k_i). The fiber column is forced to take values $q-1$ or -1 with multiplicities v and $N-v$. Under the ansatz, choose representatives for the remaining eigenvalues in columns $i = 2, 3, 4$. Plug into $P^\top \operatorname{diag}(m) P = N \operatorname{diag}(k)$; this yields five quadratic equations in the five unknown multiplicities, which reduce to a nonsingular linear system in $(m_1, \ldots, m_4)$ after eliminating $m_0 = 1$. Solving produces:

$$m_1 = \frac{q(q+1)^2}{2}, \quad m_2 = \frac{q(q^2+1)}{2}, \quad m_3 = q(q^3 - q^2 + q - 1), \quad m_4 = q^3 - q^2 + q - 1,$$

and the corresponding eigenvalues are the closed polynomials recorded in Theorem **??**. Validation at $q = 3, 5, 7$ fixes the remaining sign/permutation ambiguities.

Key Result

The "q-ladder spectrum" is not an arbitrary fit: once the fiber relation and small-commutant hypothesis hold, orthogonality and polynomial-size eigenvalues force essentially a single consistent spectrum. The remaining hard problem is therefore structural (existence and canonicality of the projectivized root-shell quotient for all odd q), not spectral.

4.11 Commutant-Type Conjecture (field-extension vs idempotent splitting)

Remark

Motivation. Across $q = 3, 5, 7$ we observe that the local line-pair generators produce a large orbit in H_q, and a small commutant (endomorphism algebra) then produces a canonical quotient of that orbit to size $vq = q(q^3 + q^2 + q + 1)$. The mechanism is "cheeky": for $q = 5$ the commutant contributes an order-3 projectivization, while for $q = 7$ it contributes an idempotent 24+24 splitting with projected orbit size $2800 = 7 \cdot 400$. This suggests the lift mechanism is controlled by the *type* of the commutant algebra.

Theorem 4.20 (Empirical formulas for the code/homology layer (odd q)) *For $q \in \{3, 5, 7\}$ we computed*

$$\dim_{\mathbb{F}_2} H_q = q^2 - 1,$$

and the number of local line-pair generators is

$$\#\mathcal{G}_q = v\binom{q+1}{2} = v \cdot \frac{q(q+1)}{2}.$$

Moreover, in each case the line-pair generators map injectively into H_q and form a single orbit under a symplectic subgroup action.

Proof sketch / audit trail

The values for $q = 3, 5, 7$ were computed explicitly by building the $W(3, q)$ point graph, reducing the adjacency mod 2 to obtain $H_q = \ker(A_q)/\mathrm{im}(A_q)$, and mapping all line-pair XOR generators into H_q coordinates, verifying injectivity and orbit transitivity. (Bundles: `W33_q5_lift_layer_first_pass_bundle.zip`, `W33_q7_lift_layer_first_pass_bundle.zip`.)

Conjecture 4.21 (Canonical quotient size and commutant mechanism) *For each odd prime power q, there exists a canonical commutant action on the line-pair generator orbit of size*

$$\#\mathcal{G}_q = v \cdot \frac{q(q+1)}{2}$$

whose orbits all have size $(q+1)/2$, yielding a canonical quotient set of size

$$\frac{\#\mathcal{G}_q}{(q+1)/2} = vq.$$

The commutant type depends on $q \bmod 4$:

- *If* $q \equiv 1 \pmod 4$*, the commutant contributes an odd-order cyclic projectivization of size* $(q+1)/2$ *(e.g.,* $q = 5$ *gives order 3 / GF(4)*-type).*
- *If* $q \equiv 3 \pmod 4$*, the commutant contributes a 2-group mechanism of size* $(q+1)/2$ *realized via idempotent splittings (e.g.,* $q = 3$ *gives a sign quotient of size 2;* $q = 7$ *yields an idempotent 24+24 split and a size-4 reduction to* 2800 *per half).*

In both cases, the resulting quotient set carries the 5-orbital / 5-harmonic q-ladder association scheme of Theorem **??**.

Remark

Cheeky takeaway. The "lift" is not universally a $\pm$ sign; it is the minimal commutant action required to collapse the local-generator count $v\binom{q+1}{2}$ to the universal ladder size vq. This reframes the remaining general-q existence problem as a commutant classification problem.

Key Result

The data for $q = 3, 5, 7$ suggest a single meta-law: *(i)* H_q has dimension $q^2 - 1$, *(ii)* local generators form one orbit of size $v\binom{q+1}{2}$, and *(iii)* a commutant action of size $(q+1)/2$ produces the canonical vq projectivized root shell whose 5-orbital spectrum is forced. This is the precise place where the TOE kernel is "cheeky".

4.12 $q = 11$: prime-field test and Pascal identity

Theorem 4.22 (Kernel layer at $q = 11$ (prime field)) *For the symplectic* $W(3, 11)$ *point graph, we have:*

$$v = 11^3 + 11^2 + 11 + 1 = 1464, \qquad k = 11 \cdot 12 = 132, \qquad (\lambda, \mu) = (10, 12),$$

and $A^2 \equiv 0 \pmod 2$*. Over* $\mathbb{F}_2$ *the computed rank and homology dimensions are:*

$$\operatorname{rank}(A) = 672, \qquad \dim H_{11} = 120 = q^2 - 1.$$

Theorem 4.23 (Injective local generators and a binomial collapse factor) *Let* $\mathcal{G}_{11}$ *be the local line-pair generators (XOR of two isotropic lines through a point). Then*

$$\#\mathcal{G}_{11} = v\binom{12}{2} = 1464 \cdot 66 = 96624,$$

all generators have weight $2(q+1) - 2 = 22$*, and the map* $\mathcal{G}_{11} \to H_{11}$ *is injective (96624 distinct* H_{11} *classes). Moreover, the ladder target size is* $vq = 1464 \cdot 11 = 16104$*, so the required collapse factor is:*

$$\frac{\#\mathcal{G}_{11}}{vq} = \frac{\binom{q+1}{2}}{q} = \frac{q+1}{2} = 6,$$

a Pascal-like binomial identity that continues the pattern $q = 5$ *(factor 3) and* $q = 7$ *(factor 4).*

Proof sketch / audit trail

We enumerate projective points of $PG(3, 11)$, build adjacency via symplectic orthogonality, extract isotropic lines by neighbor partition (12 lines per point), compute H_{11} via mod-2 rank reduction, and map all $v\binom{12}{2}$ line-pair generators into H_{11} classes by reduction modulo $\mathrm{im}(A)$. (Bundle: `W33_q11_prime_field_lift_layer_bundle_v2.zip`.)

Remark

Pascal-like combinatorics. The tower counts repeatedly involve

$$v = \frac{q^4 - 1}{q - 1} = q^3 + q^2 + q + 1, \qquad \#\mathcal{G}_q = v\binom{q+1}{2}.$$

The identity

$$v\binom{q+1}{2} = (vq) \cdot \frac{q+1}{2}$$

is exactly the collapse ratio predicted by the commutant-type conjecture: the commutant must supply a canonical action of size $(q + 1)/2$ to descend from local generators to the universal ladder size vq.

4.13 $q = 11$: commutant search (first pass) and a local Pascal hint

Remark

Goal. The commutant-type conjecture predicts a canonical collapse of the $96624 = v\binom{12}{2}$ local generators to the ladder size $vq = 16104$ by a commutant action of size $(q+1)/2 = 6$. For $q = 5$ this appears as an order-3 projectivization; for $q = 7$ as an idempotent split + size-4 reduction. Here we begin the $q = 11$ commutant search by constructing an explicit symplectic subgroup action on H_{11}.

Theorem 4.24 (A 120D H_{11} action with order-11 elements) *Using 10 explicit generators in $Sp(4, 11)$ (swap, two shears, and transvections), we induce 10 invertible 120×120 matrices over $\mathbb{F}_2$ acting on H_{11}. In this generating set, nine elements have order 11 and one has order 2 (in the induced H_{11} action).*

Remark

First-pass commutant probe. Restricting to the polynomial algebra $\mathbb{F}_2[G]$ generated by an order-11 element G, we tested all 2^{11} polynomials $\sum_{t=0}^{10} c_t G^t$ and found that only the identity (and zero) commute with the full 10-generator set. Thus the predicted size-6 commutant action is not visible as a polynomial in a single order-11 element; it likely arises either from a larger commutant algebra or from an orbit-level commutant acting on the 96624-element generator orbit rather than the full 120D module.

Remark

Pascal-like local hint. Each point has $q+1 = 12$ isotropic lines through it, and local generators correspond to unordered pairs (edges) of K_{12}: $\binom{12}{2} = 66$. The collapse factor $\frac{q+1}{2} = 6$ suggests a canonical partition of these 66 pairs into 11 classes of size 6 at each point. One candidate "cheeky" mechanism is to view the 12 lines through a point as $PG(1, 11)$ and search for a locally natural 6-to-1 invariant (e.g., a cross-ratio or polarity class) that is Aut-equivariant globally.

Proof sketch / audit trail

We construct the induced H_{11} action by permuting point coordinates under $Sp(4, 11)$ matrices, reducing modulo $\mathrm{im}(A)$, and expressing results in the computed 120D H basis. The commutant probe enumerates the polynomial algebra in an order-11 generator. (Bundle: `W33_q11_commutant_search_first_pass_bundle.zip`.)

4.14 $q = 11$: local Pascal factorization of K_{12} (constructive collapse candidate)

Remark

Outside-the-box construction. The collapse factor $(q + 1)/2 = 6$ suggests a local edge factorization of the K_{12} on the 12 isotropic lines through a point into 11 disjoint perfect matchings of size 6 (a 1-factorization). Such 1-factorizations are classical "Pascal-like" objects: they can be generated by a cyclic order-11 action on 11 vertices together with a fixed vertex, giving the round-robin schedule.

Theorem 4.25 (Order-11 stabilizer element induces a cyclic labeling of local lines) *For $q = 11$ and a fixed base point p, there exists an element in the tested point-stabilizer subgroup whose induced action on the 12 isotropic lines through p has order 11, fixing exactly one line and cycling the other 11. This provides a labeling of the 12 lines by $\{\infty\} \cup \mathbb{F}_{11}$.*

Definition

Round-robin / reflection factorization. Given labels $\{\infty\} \cup \mathbb{F}_{11}$, define for each $a \in \mathbb{F}_{11}$ a perfect matching of K_{12}:

$$M_a := \{(\infty, a)\} \ \cup \ \{(x, 2a - x) : x \in \mathbb{F}_{11} \setminus \{a\}\}/2,$$

yielding 11 disjoint matchings that partition all $\binom{12}{2} = 66$ edges into 11 classes of size 6.

Theorem 4.26 (Local 6-to-1 collapse with vanishing XOR checksum) *Let $g_{ij} \in H_{11}$ denote the H_{11} class of the line-pair generator associated to an edge (i, j) of K_{12} (two lines through the same point). For the above 1-factorization, each matching class has vanishing XOR checksum:*

$$\bigoplus_{(i,j) \in M_a} g_{ij} \ = \ 0 \qquad \text{for all } a \in \mathbb{F}_{11}.$$

Thus the local generators admit a canonical 6-to-1 bucketing compatible with the predicted collapse factor.

Remark

Non-uniqueness and canonicity. Different order-11 elements in the point stabilizer can induce different 1-factorizations. Selecting a globally canonical collapse therefore requires an additional normalization rule (e.g., a choice of "distinguished" order-11 element in the stabilizer, or a Clifford/Weyl phase criterion). Nonetheless, this construction demonstrates that the Pascal-like combinatorics required by the commutant conjecture is concretely realizable inside the $q = 11$ local geometry.

Proof sketch / audit trail

We compute an explicit point stabilizer element of order 11 on the 12 local lines, build the 1-factorization, and evaluate XOR sums in H_{11} for the corresponding 6-element buckets. (Bundle: `W33_q11_local_pascal_partition_bundle.zip`.)

Remark

This extends the "square-zero calculus" beyond $q = 3$: the code/homology layer is a stable feature of the entire odd-q family $W(3, q)$.

q	v	#lines	k	(λ, μ)	$A^2 \equiv 0 \pmod 2$
2	15	15	6	(1,3)	no
3	40	40	12	(2,4)	yes
5	156	156	30	(4,6)	yes
7	400	400	56	(6,8)	yes

4.15 Spectral diagnostics on finite approximants

Definition

For a d-regular graph G with adjacency eigenvalues λ_i, the normalized Laplacian eigenvalues are

$$\mu_i = 1 - \frac{\lambda_i}{d}.$$

A standard continuum diagnostic is the heat kernel trace

$$P(t) := \frac{1}{|V(G)|} \sum_i e^{-t\mu_i},$$

whose intermediate-time scaling can be used to define an effective spectral dimension. On finite strongly symmetric graphs, $P(t)$ is often a small sum of exponentials, yielding a multi-scale (non-classical) behavior.

Theorem 4.27 (Exact normalized Laplacian spectra for the kernel graphs) *Let* W33 *be* $SRG(40, 12, 2, 4)$ *and* $Q = \overline{\text{W33}}$ *its quotient graph (degree 27). Then the normalized Laplacian spectrum of* Q *is*

$$0^{(1)}, \qquad \left(\frac{8}{9}\right)^{(15)}, \qquad \left(\frac{10}{9}\right)^{(24)}.$$

Let A_{meet} be the meet adjacency on the 90 non-isotropic lines (degree 32). Then its normalized Laplacian spectrum is

$$0^{(1)}, \qquad \left(\frac{3}{4}\right)^{(15)}, \qquad \left(\frac{15}{16}\right)^{(24)}, \qquad \left(\frac{9}{8}\right)^{(50)}.$$

Proof sketch / audit trail

The adjacency eigenvalues of Q follow from SRG complement eigenvalue relations: if W33 has eigenvalues $12, 2, -4$ with multiplicities $1, 24, 15$, then Q has eigenvalues $27, -3, 3$ with multiplicities $1, 24, 15$. The normalized Laplacian eigenvalues are $1 - \lambda/27$. The meet-graph eigenvalues were computed in the association scheme analysis: $32^{(1)}, 8^{(15)}, 2^{(24)}, (-4)^{(50)}$, yielding normalized Laplacian eigenvalues $1 - \lambda/32$.

Remark

Interpretation. These spectra show the kernel graphs are "two/three-scale" rather than approximations of a smooth manifold in the naive sense: the heat kernel trace is a small mixture of exponentials. In a scaling program, one expects richer spectra to emerge only when the kernel is embedded into a family (e.g., varying q, increasing rank, or taking covers), and the vacuum harmonics (Section 12) provide the correct basis for coarse-grained dynamics.

4.16 Renormalization as module projection

Definition

Mode-space coarse graining. The vacuum association scheme decomposes $\mathbb{Z}_3^{90}$ into five canonical harmonic subspaces (Section 12). A natural renormalization step is projection onto a selected subset of these modes (or onto the 88D core module), followed by rescaling of the transfer operators (M, Z) and the sourced field $J = dF$.

Protocol (testable)

Program (testable).

1. Choose a scaling family (field size q, rank n, or covers).
2. For each instance, compute: (i) closure/gauge fix, (ii) quotient Q, (iii) holonomy F, (iv) sources $J = dF$, (v) transfer operators (M, Z), (vi) vacuum association scheme and mode decomposition.
3. Track invariants across scale: H^3 dimension, module decompositions (e.g., 88+1 analogs), and spectral signatures of meet graphs.
4. Identify fixed points in the induced operator calculus (e.g., stable ratios of mode injection weights under coarse graining).

Key Result

The kernel already provides the correct *renormalization coordinates*: vacuum harmonics (five modes) and the 88D core module. A genuine continuum limit, if it exists, should be formulated as stability of these module-level observables across a scaling family (not as ad hoc constant matching).

5 Axioms and kernel construction chain

Definition

Axiom A0 (Phase space). Let $V = \mathbb{F}_3^4$ equipped with a fixed nondegenerate alternating (symplectic) form ω.

Axiom A1 (Isotropy geometry). Let $W(3,3)$ denote the symplectic generalized quadrangle realized by totally isotropic points and lines in $PG(3,3)$ with respect to ω.

Axiom A2 (Point graph). Let W33 be the point graph of $W(3,3)$: vertices are the 40 isotropic points, and edges represent collinearity.

Remark

These axioms fix the entire tower. Everything below is forced from the adjacency matrix A of W33, its induced actions, and the canonical quotients and lifts defined from it.

Key Result

The W33 tower can be viewed as a closed pipeline:

$$\begin{aligned} &\mathbb{F}_3^4 \Rightarrow W(3,3) \Rightarrow \text{W33} \Rightarrow (A^2 \equiv 0 \text{ over } \mathbb{F}_2) \Rightarrow H \\ &\quad \Rightarrow (120, 240) \text{ signed roots} \Rightarrow Q = \overline{\text{W33}} \Rightarrow (\mathbb{Z}_3 \text{ holonomy}) \\ &\quad \Rightarrow H^3(\mathrm{Cl}(Q); \mathbb{Z}_3) \cong (\mathbb{Z}_3)^{89} \Rightarrow \text{90-line field model.} \end{aligned}$$

6 Master theorems and dictionary

Theorem 6.1 (Master Theorem I: square-zero differential and code) *Over $\mathbb{F}_2$, the adjacency matrix A of* W33 *satisfies $A^2 \equiv 0$. Hence $d(x) = Ax$ defines a differential on $\mathbb{F}_2^{40}$, producing a canonical code $C = \ker(A)$ with parameters $[40, 24, 6]$ and a homology state space $H = \ker(A)/\mathrm{im}(A) \cong \mathbb{F}_2^8$.*

Theorem 6.2 (Master Theorem II: 120-root shell and 240 signed lift) *The induced action on H preserves a quadratic form of minus type. The nonsingular orbit has size 120 and carries* $\mathrm{SRG}(120, 56, 28, 24)$ *adjacency via the associated bilinear form. The 240 canonical weight-6 generators project 2-to-1 onto this 120-set, yielding a signed lift with a defect cocycle valued in* $\mathrm{im}(A)$.

Theorem 6.3 (Master Theorem III: quotient closure and $\mathbb{Z}_3$ connection) *There exists a global gauge fix eliminating all weight-16 defects. In that gauge, the 120 roots partition into 40 flat triples (one per W33 point). Collapsing these triples yields a quotient graph Q equal to the* *complement $\overline{\mathrm{W33}}$, equipped with a canonical edge transport rule whose triangle holonomy lies in $\mathbb{Z}_3$. Flat holonomy triangles are classified exactly by the 90 non-isotropic projective lines in $PG(3,3)$.*

Theorem 6.4 (Master Theorem IV: sourced curvature and transfer operators) *Let $F \in C^2(\mathrm{Cl}(Q);\mathbb{Z}_3)$ be the triangle holonomy field and $J = dF \in C^3(\mathrm{Cl}(Q);\mathbb{Z}_3)$ its source. Then J is supported on exactly 3008 tetrahedra. There exist explicit sparse operators*

$$M, Z : \mathbb{Z}_3^{9450} \to \mathbb{Z}_3^{90}$$

such that the observed vacuum line fields satisfy the exact identities $m_{\mathrm{line}} = MJ$ and $z_{\mathrm{line}} = ZJ$. Vacuum responses decompose into five canonical harmonics determined by the Aut-invariant 90-line association scheme.

Definition

Dictionary (high level). Within the exact finite theory:

- **Geometry**: isotropic vs non-isotropic incidence in $PG(3,3)$; the graphs W33 and $Q = \overline{\mathrm{W33}}$.
- **Algebra**: Aut(W33) actions and induced modules on H, the 120-root shell, the 90-line sector, and H^3.
- **Topology**: cochains/coboundaries on $\mathrm{Cl}(Q)$; $J = dF$ as sources; H^3 as flux lattice.
- **Quantum computation**: Weyl/Clifford realization on V; contexts from isotropic lines; holonomy as discrete phase transport.
- **Cryptography**: gauge/coset ambiguity and large symmetry action as secrecy; error correction as intrinsic stability (the $[40, 24, 6]$ code).

3 The W33 Object

Definition

Let $V = \mathbb{F}_3^4$ equipped with a nondegenerate alternating (symplectic) form ω. Let $W(3,3)$ denote the symplectic generalized quadrangle arising from totally isotropic points and lines in $PG(3,3)$ with respect to ω. The *W33 point graph* is the graph whose vertices are the 40 isotropic points and whose edges connect collinear pairs (i.e., pairs lying on a common isotropic line). We denote its adjacency matrix by A and the graph by W33.

Theorem 3.1 (SRG parameters) W33 *is a strongly regular graph with parameters*

$$(v, k, \lambda, \mu) = (40, 12, 2, 4).$$

Equivalently, each vertex has degree 12; adjacent pairs have exactly 2 common neighbors; non-adjacent pairs have exactly 4 common neighbors.

Proof sketch / audit trail

This is a standard property of the point graph of the symplectic generalized quadrangle $W(3,3)$. It was also verified computationally by explicit incidence construction of $W(3,3)$ and counting common neighbors in the point graph (audit bundle: `W33_symplectic_audit_bundle.zip`).

Theorem 3.2 (Adjacency spectrum) *The adjacency spectrum of* W33 *is*

$$\mathrm{spec}(A) = 12^{(1)}, \quad 2^{(24)}, \quad (-4)^{(15)}.$$

Equivalently, the characteristic polynomial is

$$P(x) = (x-12)(x-2)^{24}(x+4)^{15}.$$

Proof sketch / audit trail

For $\mathrm{SRG}(v, k, \lambda, \mu)$, the nontrivial eigenvalues are roots of a quadratic determined by (k, λ, μ), with multiplicities forced by trace identities. Here this yields eigenvalues 2 and -4 with multiplicities 24 and 15. Verified directly by eigen-computation on the explicit adjacency matrix (audit bundle: `W33_symplectic_audit_bundle.zip`).

Theorem 3.3 (Automorphism group order) $|\mathrm{Aut}(\mathrm{W33})| = 51840$.

Proof sketch / audit trail

In the symplectic model, Aut(W33) is realized as the projective symplectic similitude group acting on isotropic points. A concrete generating set (symplectic transvections, a block-swap, and a multiplier-2 similitude) was used to generate the full permutation group on the 40 vertices, yielding order 51840. (Audit bundle: `W33_orbits_squarezero_bundle.zip`.)

Key Result

The W33 point graph is not merely a convenient combinatorial object; it is the *canonical* SRG arising from the symplectic quadrangle $W(3,3)$. The entire tower below is forced from $(40, 12, 2, 4)$ together with the induced group action.

4 Differential Structure over $\mathbb{F}_2$

Theorem 4.1 (Square-zero adjacency over $\mathbb{F}_2$) *Let A be the adjacency matrix of* W33*. Over $\mathbb{F}_2$, one has*

$$A^2 \equiv 0 \pmod 2.$$

Proof sketch / audit trail

For any $\mathrm{SRG}(v, k, \lambda, \mu)$ with adjacency A and all-ones matrix J,

$$A^2 = kI + \lambda A + \mu(J - I - A).$$

Plugging $(k, \lambda, \mu) = (12, 2, 4)$ yields $A^2 = 8I - 2A + 4J$. Reducing mod 2 gives $A^2 \equiv 0$. Verified directly by matrix multiplication mod 2 in the audit bundle.

Definition

Define a differential $d : \mathbb{F}_2^{40} \to \mathbb{F}_2^{40}$ by $d(x) = Ax \pmod 2$. Since $d^2 = 0$, we can form:

$$C := \ker(d) \subset \mathbb{F}_2^{40}, \qquad H := \ker(d)/\mathrm{im}(d).$$

Theorem 4.2 (Dimensions) *Over* $\mathbb{F}_2$*,*

$$\operatorname{rank}(A) = 16, \qquad \dim\ker(A) = 24, \qquad \dim H = 8.$$

Proof sketch / audit trail

Rank was computed by mod-2 row reduction on the explicit 40×40 adjacency matrix. Nullity follows by rank-nullity. Since $\operatorname{im}(A) \subseteq \ker(A)$ (square-zero), $\dim H = \dim\ker(A) - \dim\operatorname{im}(A) = 24 - 16 = 8$.

Theorem 4.3 (Canonical local generators and code distance) *The kernel* $C = \ker(A) \subset \mathbb{F}_2^{40}$ *is a* $[40, 24, 6]$ *linear code. Moreover, there are exactly 240 canonical weight-6 codewords obtained as XORs of pairs of isotropic lines through a common point, and these 240 codewords generate* C*.*

Proof sketch / audit trail

Each point lies on 4 isotropic lines; choosing 2 lines yields $\binom{4}{2} = 6$ line-pairs per point, hence $40\cdot6 = 240$ codewords. Each is weight 6 and lies in $\ker(A)$; exhaustive search up to weight 5 found none in $\ker(A)$, so $d_{\min} = 6$. A row-reduced basis extracted from the 240 generators spans a 24-dimensional space, matching $\dim\ker(A)$. (Audit bundle: `W33_GF2_kernel_code_bundle.zip`.)

Key Result

The identity $A^2 \equiv 0$ is the first "TOE hinge": it turns a finite SRG into a genuine chain complex, producing (i) a stabilizer-like code and (ii) an 8-dimensional homology state space H.

5 Orthogonal Geometry on H and the 120-Root Structure

Theorem 5.1 (Quadratic form and orbit split) *The induced action of* $\operatorname{Aut}(\mathrm{W33})$ *on* H *preserves a nontrivial quadratic form* $q : H \to \mathbb{F}_2$ *of minus type. Consequently, the nonzero vectors in* H *split into exactly two orbits:*

$$\{x \in H \setminus \{0\} : q(x) = 0\} \text{ of size } 135, \qquad \{x \in H \setminus \{0\} : q(x) = 1\} \text{ of size } 120.$$

Proof sketch / audit trail

A concrete basis of H was chosen by splitting $\ker(A) = \operatorname{im}(A) \oplus K$ with $\dim K = 8$. The group action on points induces an action on H, from which an invariant quadratic polynomial of degree 2 was solved. Enumerating values of q gives the $(135, 120)$ split, and orbit computation confirms exactly two nonzero orbits. (Audit bundle: `W33_H8_quadratic_form_bundle.zip`.)

Theorem 5.2 (240 → 120 projection) *Projecting the 240 canonical weight-6 code generators (Theorem* **??***) from* $\ker(A)$ *to* $H = \ker(A)/\operatorname{im}(A)$ *yields exactly 120 distinct nonzero elements, each appearing with multiplicity 2. All 120 satisfy* $q = 1$ *(the nonsingular orbit).*

Proof sketch / audit trail

Each of the 240 generators was mapped to an 8-bit H coordinate; 120 distinct values occur, each exactly twice. All map to the $q = 1$ orbit. (Audit bundle: `W33_to_H_to_120root_SRG_bundle.zip` and `W33_root_preimage_pairing_bundle.zip`.)

Definition

Define the associated bilinear form

$$b(x, y) = q(x + y) + q(x) + q(y) \in \mathbb{F}_2.$$

On the 120-element nonsingular orbit, define adjacency by $b(x, y) = 1$.

Theorem 5.3 (The 120-root SRG) *The graph on the 120 nonsingular elements with adjacency $b = 1$ is strongly regular:*

$$\mathrm{SRG}(120, 56, 28, 24).$$

Proof sketch / audit trail

Adjacency counts were computed directly from the bilinear form on the explicit 120-root list; all vertices have degree 56, adjacent pairs have 28 common neighbors, and nonadjacent pairs have 24. (Audit bundle: `W33_to_H_to_120root_SRG_bundle.zip`.)

Theorem 5.4 (An E_8 Dynkin subgraph and reflection generation) *Inside* $\mathrm{SRG}(120, 56, 28, 24)$ *there exists an induced subgraph isomorphic to the E_8 Dynkin diagram. The corresponding 8 nonsingular elements $\{r_i\}$ define involutions*

$$s_r(x) = x + b(x, r)\, r,$$

and the group generated by these involutions acts transitively on the 120-root set.

Proof sketch / audit trail

An induced E_8 configuration was found and canonically chosen (lexicographically minimal under a fixed branching constraint). Coxeter relations were verified on H (order 3 on adjacent nodes, order 2 otherwise), and orbit generation under reflections yields the full 120-root orbit. (Audit bundle: `W33_E8_simple_root_system_bundle.zip`.)

Key Result

The nonsingular orbit of the intrinsic homology H behaves as a finite "root shell" with $\mathrm{SRG}(120, 56, 28, 24)$ adjacency and an embedded E_8 Dynkin skeleton. This is the precise point where Lie-type structure emerges from the W33 tower.

6 Signed Lift, Cocycle, and Global Gauge Fixing

Definition

Each of the 120 roots has two preimages among the 240 generators. A *section* s selects one lift for each root. For adjacent roots h_1, h_2 (so $b(h_1, h_2) = 1$), define $h_3 = h_1 \oplus h_2$ and the defect (cocycle candidate)

$$g(h_1, h_2) := s(h_1) + s(h_2) + s(h_3) \;\in\; \mathrm{im}(A) \subset \mathbb{F}_2^{40},$$

where addition is XOR of the corresponding 40-bit supports.

Theorem 6.1 (Two-weight defect) *For the canonical section (choosing the smaller preimage index), the defect $g(h_1, h_2)$ takes only two Hamming weights:*

$$|g(h_1, h_2)| \in \{12, 16\}.$$

Across all 3360 edges of $\mathrm{SRG}(120, 56, 28, 24)$*, weight 12 occurs 1560 times and weight 16 occurs 1800 times.*

Proof sketch / audit trail

Computed exhaustively over all edges using the explicit 240 generator supports and the canonical section. Verified that $g(h_1, h_2)$ always projects to 0 in H, hence lies in $\mathrm{im}(A)$. (Audit bundle: `W33_signed_root_cocycle_and_lift_bundle.zip`.)

Theorem 6.2 (Steiner triples) *Edges of* $\mathrm{SRG}(120, 56, 28, 24)$ *partition into 1120 Steiner triples* $\{a, b, a \oplus b\}$*, and for a fixed section s, the defect value is constant on the three edges of each triple.*

Proof sketch / audit trail

If $b(a, b) = 1$ then $q(a \oplus b) = 1$; hence $a \oplus b$ is again a root. Each edge (a, b) has a unique third root $a \oplus b$, and the unordered triple partitions edges into 1120 groups. The defect $s(a) + s(b) + s(a \oplus b)$ is symmetric in $(a, b, a \oplus b)$, hence constant on the triple edges. Verified by enumeration.

Theorem 6.3 (Global gauge fix (no-16)) *There exists a global choice of signs (i.e., a section s selecting one of the two lifts at every root) such that all defects of weight 16 are eliminated. In this gauge-fixed section, all edge defects have weight in* $\{0, 12\}$*, with exactly 120 edges of weight 0 and 3240 edges of weight 12.*

Proof sketch / audit trail

A greedy local-flip optimization over the 120 root vertices (flipping lift choice at a vertex updates the defects on incident edges) yields a configuration with no 16-weight defects. This configuration was reproduced across random restarts. (Audit bundle: `W33_global_gaugefix_no16_bundle.zip`.)

Theorem 6.4 (40 flat triples) *The 120 roots partition into 40 disjoint triples (one per original W33 point) such that exactly those 40 triples have defect weight 0 under the globally gauge-fixed section. Equivalently, the 120 weight-0 edges form 40 disjoint triangles that partition the root set.*

Proof sketch / audit trail

From the gauge-fixed edge list, the weight-0 edges were found to group into 40 triangles. Each triangle's three vertices share the same base point in the original 40-point geometry, yielding a partition of the 120 roots into 40 fibers of size 3. (Audit bundle: `W33_global_gaugefix_no16_bundle.zip`.)

7 Quotient Closure and $\mathbb{Z}_3$ Holonomy

Definition

Collapse each of the 40 flat triples (Theorem **??**) to a meta-vertex labeled by its base point $p \in \{0, \dots, 39\}$. Define the quotient graph Q on these 40 meta-vertices by connecting $p \neq q$ if there exists a defect-12 edge between the fibers over p and q.

Theorem 7.1 (Quotient graph is the complement) *The quotient graph Q is regular of degree 27 on 40 vertices and is exactly the complement of the original W33 point graph:*

$$Q = \overline{\mathrm{W33}}.$$

Proof sketch / audit trail

For each pair of base points (p, q), the number of defect-12 edges between the 3-element fibers is either 0 or 6. Adjacency in Q occurs exactly for multiplicity 6. The resulting 40-vertex graph is 27-regular; direct comparison of neighbor sets confirms Q equals the complement of the W33 adjacency. (Audit bundle: `W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip`.)

Theorem 7.2 (Edge decoration is a 6-cycle) *For every edge $p \sim q$ in Q, the induced bipartite graph between the 3 roots over p and the 3 roots over q has exactly 6 edges and is 2-regular on each side. Equivalently, it is $K_{3,3}$ minus a perfect matching, i.e. a 6-cycle. The missing perfect matching defines a canonical transport bijection between the two 3-element fibers.*

Proof sketch / audit trail

Verified by explicit enumeration for all 540 quotient edges: the 3×3 adjacency matrix always has three zeros (a perfect matching) and six ones, with row and column sums all equal to 2. Connectivity check confirms a single 6-cycle.

Definition

Define the holonomy of a quotient triangle (p, q, r) as the permutation of the fiber over p obtained by composing the three transport bijections along $p \to q \to r \to p$. This holonomy lies in $A_3 \cong \mathbb{Z}_3$.

Theorem 7.3 (90 non-isotropic lines classify flat holonomy) *Among the 3240 triangles of Q, exactly 360 have identity holonomy and 2880 have 3-cycle holonomy. Moreover, the identity-holonomy triangles are* exactly *the triples of points lying on the 90 non-isotropic projective lines in $PG(3,3)$ (each such line contains 4 points and contributes $\binom{4}{3} = 4$ triples, hence $90 \cdot 4 = 360$).*

Proof sketch / audit trail

Holonomy was computed for all quotient triangles from the edge matchings. Independently, all non-isotropic lines in $PG(3,3)$ were enumerated (90 lines), and the set of their 3-subsets was computed (360 triples). These match exactly the identity-holonomy triangle set. (Audit bundle: `W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip`.)

Key Result

The W33 tower closes: after global gauge fixing and collapsing flat triples, the induced 40-vertex quotient is $\overline{\mathrm{W33}}$ with a canonical $\mathbb{Z}_3$ connection. The set of flat faces is classified precisely by the 90 non-isotropic projective lines in $PG(3,3)$.

Artifact Index (computational)

Bundle	Contents / Purpose

8 Cohomology and flux lattice (summary of computed results)

Theorem 8.1 (Clique-complex cohomology over $\mathbb{Z}_3$) *Let $\mathrm{Cl}(Q)$ be the clique complex of $Q = \overline{\mathrm{W33}}$. Over $\mathbb{Z}_3$, its cohomology dimensions are:*

$$H^0 = 1, \quad H^1 = 0, \quad H^2 = 0, \quad H^3 = 89, \quad H^4 = 1, \quad H^5 = 0, \quad H^6 = 1.$$

In particular, the flux lattice is $H^3(\mathrm{Cl}(Q);\mathbb{Z}_3) \cong (\mathbb{Z}_3)^{89}$, and an explicit 89-element basis can be constructed.

Remark

The vanishing $H^2 = 0$ on the full clique complex explains why 2-skeleton obstructions disappear once tetrahedra are included: closed 2-forms are exact in the full flag complex, while the physically relevant sourced curvature is encoded by $J = dF$ (a 3-cochain).

9 Representation theory of the flux lattice and the 90-line module

Definition

Let $Q = \overline{\mathrm{W33}}$ be the 40-vertex quotient graph and $\mathrm{Cl}(Q)$ its clique (flag) complex. The flux lattice is

$$H^3(\mathrm{Cl}(Q);\mathbb{Z}_3) \cong (\mathbb{Z}_3)^{89}.$$

The Aut(W33) action on the 40 base points induces an action on all cliques of Q and hence on cochains, coboundaries, and cohomology.

Theorem 9.1 (An explicit basis for H^3) *There exists an explicit basis of 89 cocycles in $C^3(\mathrm{Cl}(Q);\mathbb{Z}_3)$ representing a basis of $H^3(\mathrm{Cl}(Q);\mathbb{Z}_3)$. Each basis element is given in sparse form as a $\mathbb{Z}_3$-valued cochain supported on tetrahedra (K_4 cliques) of Q.*

Proof sketch / audit trail

We compute $\ker(\delta_3) \subset C^3$ from the K_5 constraints and quotient by $\mathrm{im}(\delta_2)$ coming from triangles. In free coordinates for $\ker(\delta_3)$, the image of δ_2 has rank 2739, leaving dimension 89. We select 89 nonpivot free coordinates and back-substitute to construct cocycles. (Audit bundle: `W33_H3_basis_89_Z3_on_clique_complex_bundle.zip`.)

Theorem 9.2 (88+1 module structure and similitude character) *The 89-dimensional $\mathbb{Z}_3$-module $H^3(\mathrm{Cl}(Q);\mathbb{Z}_3)$ admits an invariant 88-dimensional submodule W_{88} such that the quotient is 1-dimensional. The 1-dimensional quotient carries the canonical "similitude sign" character: an index-2 subgroup acts trivially, while a distinguished multiplier-2 element acts by $-1 \equiv 2 \pmod 3$.*

Proof sketch / audit trail

Using the explicit Aut(W33) generators on points, we compute the induced action on tetrahedra, incorporate the orientation sign for 3-cochains, and build the resulting 89×89 matrices over $\mathbb{Z}_3$ on the computed H^3 basis. Empirically, the module has an invariant 88D submodule and a 1D quotient; the quotient character is detected by a dual functional w transforming by ± 1. (Audit bundle: `W33_H3_Aut_action_89Z3_bundle.zip`.)

Definition

Let $\mathcal{L}$ be the set of 90 non-isotropic projective lines in $PG(3,3)$. Consider the permutation module $\mathbb{Z}_3^{\mathcal{L}}$ and its augmentation submodule

$$\mathrm{Aug}(\mathcal{L}) := \Big\{x \in \mathbb{Z}_3^{\mathcal{L}} :\ \sum_{\ell\in\mathcal{L}} x_\ell = 0\Big\}.$$

Since $90 \equiv 0 \pmod 3$, the all-ones vector lies in $\mathrm{Aug}(\mathcal{L})$; quotienting by this trivial line yields an 88D module.

Theorem 9.3 (Geometric identification with 90-line augmentation quotient) *The 88D core module W_{88} is isomorphic (up to the similitude sign twist) to the augmentation quotient of the 90-line permutation module:*

$$W_{88} \ \cong\ \mathrm{Aug}(\mathcal{L})/\langle \mathbf{1}\rangle \ \otimes\ \chi,$$

where χ is the 1D similitude sign character. Moreover, an explicit intertwiner T between these modules can be computed.

Proof sketch / audit trail

We compute the Aut(W33) action on 90 non-isotropic lines, form the augmentation quotient, and compare with the H^3 88D core via traces and characteristic polynomial factor patterns. After twisting by the similitude sign (multiplying the multiplier-2 generator by -1), the modules match; an explicit 88×88 intertwiner T is constructed. (Audit bundles: `W33_perm_module_vs_H3_match_report_bundle.zip`, `W33_H3_to_noniso_line_weights_intertwiner_bundle.zip`.)

Theorem 9.4 (Explicit lift to labeled 90-line weights) *There is an explicit linear lift from 88D core coordinates to a labeled 90-entry non-isotropic line field (defined up to adding a constant all-ones vector). Concretely, there exists a 90×88 matrix $M_{H3\to 90}$ over $\mathbb{Z}_3$ such that*

$$w_{90} \equiv M_{H3\to 90}\, x_{88} \pmod{\langle \mathbf{1}\rangle},$$

and the 90 coordinates are indexed by the 4-point line-sets in $\mathcal{L}$.

Proof sketch / audit trail

A section $L_{88\to 90}$ of the augmentation quotient is constructed and composed with the 88D intertwiner T to yield $M_{H3\to 90}$. The resulting 90-vector is unique up to addition of a constant, reflecting the quotient by $\langle \mathbf{1} \rangle$. Line labeling is provided by the explicit 90 line list. (Audit bundle: `W33_lift_to_90_line_weights_with_labels_bundle.zip`.)

Key Result

This section fixes the representation-theoretic meaning of the flux lattice: the nontrivial 88D core of H^3 is (up to the canonical similitude sign) the augmentation quotient on the 90 non-isotropic lines. In particular, the "vacuum cells" that classify flat holonomy also carry the matter/flux degrees of freedom.

10 2-qutrit Weyl operators and the symplectic commutator

Definition

Let $\omega := e^{2\pi i/3}$. On $\mathbb{C}^3$ with computational basis $\{|j\rangle : j \in \mathbb{Z}_3\}$ define

$$X|j\rangle = |j+1\rangle, \qquad Z|j\rangle = \omega^j |j\rangle,$$

so that $ZX = \omega XZ$. On two qutrits, for $(a, b, c, d) \in \mathbb{F}_3^4$, define the (unnormalized) Weyl operator

$$W(a, b, c, d) := X^a Z^c \ \otimes \ X^b Z^d.$$

Definition

Define the standard symplectic form on $V = \mathbb{F}_3^{2n}$ with $n = 2$ by writing $v = (p \mid q)$ with $p, q \in \mathbb{F}_3^2$ and

$$\langle (p \mid q), (p' \mid q') \rangle := p \cdot q' - q \cdot p' \ \in \ \mathbb{F}_3.$$

In coordinates $v = (a, b, c, d)$ and $w = (a', b', c', d')$, this is

$$\langle v, w \rangle = ac' + bd' - ca' - db'.$$

Theorem 10.1 (Weyl commutator phase) *For all $v, w \in \mathbb{F}_3^4$,*

$$W(v)\, W(w) = \omega^{\langle v,w \rangle}\, W(w)\, W(v).$$

Equivalently, $W(v)$ and $W(w)$ commute if and only if $\langle v, w \rangle = 0$.

Proof sketch / audit trail

This is the standard Heisenberg–Weyl relation for odd prime dimension. For the above unnormalized convention, it follows from $ZX = \omega XZ$ on each tensor factor and bilinearity of the commutator exponent.

Key Result

The same symplectic form used to build $W(3,3)$ is exactly the commutator phase form in the 2-qutrit Weyl group. This is the first canonical bridge from W33 geometry to quantum operator algebra.

11 Projective points as Weyl directions

Definition

Let $\mathbb{P}(V) = PG(3,3)$ denote projective 1D subspaces of $V = \mathbb{F}_3^4$. A projective point $[v]$ is the equivalence class $\{v, 2v\}$ for any nonzero $v \in V$.

Theorem 11.1 (Projective points correspond to cyclic Weyl subgroups) *Each projective point $[v] \in PG(3,3)$ determines a cyclic order-3 Weyl subgroup*

$$\langle W(v) \rangle = \{I,\, W(v),\, W(2v)\}.$$

Moreover, $W(2v) = W(v)^{-1}$ and the subgroup depends only on $[v]$ (not the representative).

Proof sketch / audit trail

In $\mathbb{F}_3$, $2 \equiv -1$ and $W(2v) = W(-v) = W(v)^{-1}$ (up to global phase, fixed by convention). Thus $\langle W(v) \rangle$ depends only on the projective class $\{v, -v\}$.

Remark

In the W33 tower, the 40 vertices are precisely the 40 projective points of $PG(3,3)$. Thus W33 vertices can be read as 40 "Pauli directions" (cyclic order-3 Weyl subgroups) for two qutrits.

12 Isotropic lines as maximal commuting contexts

Definition

A 2D subspace $U \leq V$ is *totally isotropic* if $\langle u, u' \rangle = 0$ for all $u, u' \in U$. Its projectivization is a projective line containing 4 projective points.

Theorem 12.1 (Isotropic lines give commuting Pauli contexts) *If $U \leq V$ is a totally isotropic 2D subspace, then $\{W(u) : u \in U\}$ is an abelian subgroup of the 2-qutrit Weyl group of order $3^2 = 9$ (including identity). Equivalently, the 4 projective points on the line correspond to 4 nontrivial cyclic subgroups whose nontrivial elements pairwise commute.*

Proof sketch / audit trail

If U is totally isotropic, then $\langle u, u' \rangle = 0$ for all $u, u' \in U$, so $W(u)$ commutes with $W(u')$ by Theorem **??**. Since $U \cong \mathbb{F}_3^2$, the set $\{W(u) : u \in U\}$ has 9 elements.

Remark

The symplectic generalized quadrangle $W(3,3)$ consists precisely of 40 points and 40 totally isotropic projective lines. Thus the GQ lines are canonical maximal commuting Pauli contexts in the 2-qutrit Weyl group.

13 Non-isotropic lines as canonical phase cells

Definition

A projective line (2D subspace) U is *non-isotropic* if $\langle\cdot,\cdot\rangle|_U$ is nondegenerate. In this case, there exist $u, u' \in U$ with $\langle u, u'\rangle = 1$, generating a Heisenberg pair.

Theorem 13.1 (Non-isotropic lines contain conjugate pairs) *Let $U \leq V$ be a non-isotropic 2D subspace. Then there exist $u, u' \in U$ such that $\langle u, u'\rangle = 1$, and hence*

$$W(u)\,W(u') = \omega\,W(u')\,W(u).$$

Proof sketch / audit trail

Nondegeneracy of $\langle\cdot,\cdot\rangle|_U$ implies there exists a basis with symplectic form matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ on U. Choosing u, u' as basis vectors yields $\langle u, u'\rangle = 1$.

Remark

In the W33 tower, $PG(3,3)$ has 130 lines total: 40 isotropic (GQ) and 90 non-isotropic. The "90" distinguished by the quotient holonomy are exactly these non-isotropic lines.

14 Clifford normalizer and the W33 automorphism action

Theorem 14.1 (Clifford induces symplectic action) *Let $\mathcal{C}$ denote the 2-qutrit Clifford group (normalizer of the Weyl group in $U(9)$). Then conjugation by any $U \in \mathcal{C}$ induces a linear transformation $M \in Sp(4,3)$ on phase space such that*

$$UW(v)U^\dagger \;=\; \omega^{\kappa(v)}\,W(Mv).$$

Conversely, each $M \in Sp(4,3)$ is induced by some Clifford up to phase.

Proof sketch / audit trail

Standard result for odd prime-power dimension: the Clifford group projects onto the symplectic group acting on discrete phase space, with kernel the Heisenberg–Weyl phases.

15 Holonomy equals commutator phase: a falsifiable conjecture

Definition

Define the symplectic "triangle phase" functional on three phase points $u, v, w \in V$ by

$$\Phi(u, v, w) := \langle u, v\rangle + \langle v, w\rangle + \langle w, u\rangle \;\in\; \mathbb{F}_3.$$

Theorem 15.1 (Closed-loop phase identity) *For any $u, v, w \in V$ with $u+v+w=0$, the triple Weyl product has the form*

$$W(u)\,W(v)\,W(w) \;=\; \omega^{\Phi(u,v,w)}\,I$$

up to a global convention factor (which can be fixed by choosing standard displacement operators).

Proof sketch / audit trail

Use the Weyl multiplication law and bilinearity: $W(u)W(v)$ equals a scalar times $W(u+v)$. If $u+v+w=0$, then $W(u+v)W(w)$ is scalar times identity. Exponents combine to the cyclic sum Φ (mod 3).

Theorem 15.2 (Holonomy-phase conjecture (testable)) *Let $Q = \overline{\mathrm{W33}}$ be the 40-vertex quotient graph produced by the globally gauge-fixed signed lift, with each triangle (p, q, r) assigned a holonomy value $F(p, q, r) \in \mathbb{Z}_3$ (identity vs 3-cycle orientation). There exists a projective representative assignment $p \mapsto [v_p] \in PG(3, 3)$, and representative choices $v_p \in V$, such that for every triangle,*

$$F(p, q, r) \;\equiv\; \Phi(v_p, v_q, v_r) \pmod 3,$$

up to the standard gauge ambiguity corresponding to adding a constant all-ones vector in the 90-line weight model.

Protocol (testable)

Protocol: verifying Theorem ??.

1. Use the explicit projective representatives for the 40 points in $PG(3, 3)$ (present in the symplectic audit bundle).
2. Compute $\Phi(v_p, v_q, v_r)$ for all 3240 triangles of Q.
3. Compare to the computed holonomy values (identity/3-cycle with orientation) on the same triangle list.
4. If a mismatch occurs only by a constant shift (global gauge), quotient out by the all-ones line and recompare.
5. If mismatches persist with nonconstant residuals, the conjecture fails and the representative assignment must be refined (or the holonomy is not a pure symplectic cocycle).

Artifact Index (quantum layer)

`Bundle`	**Contents / Purpose**

11 The quotient as a simplicial gauge system

Definition

Let $Q = \overline{\mathrm{W33}}$ be the 40-vertex quotient graph obtained by collapsing the 40 flat triples in the globally gauge-fixed 240→120 lift. Let $\mathrm{Cl}(Q)$ denote the clique (flag) complex of Q. Then:

$$C^2 := \mathbb{Z}_3^{\{\text{triangles of } Q\}} \cong \mathbb{Z}_3^{3240}, \qquad C^3 := \mathbb{Z}_3^{\{\text{tetrahedra } (K_4) \text{ of } Q\}} \cong \mathbb{Z}_3^{9450}.$$

Let $d : C^2 \to C^3$ be the simplicial coboundary map.

Definition

The quotient construction assigns to each triangle (p, q, r) a holonomy value $F(p, q, r) \in \mathbb{Z}_3$ (identity vs 3-cycle orientation). We view this as a 2-cochain

$$F \in C^2(\mathrm{Cl}(Q); \mathbb{Z}_3).$$

Define the sourced 3-cochain

$$J := dF \in C^3(\mathrm{Cl}(Q); \mathbb{Z}_3),$$

which assigns a flux/charge value to each tetrahedron.

Theorem 11.1 (Sourced curvature) *$J = dF$ is supported on exactly 3008 tetrahedra:*

$$\#\{t : J(t) \neq 0\} = 3008,$$

with flux distribution $J = 1$ on 1512 tetrahedra and $J = 2$ on 1496 tetrahedra. Moreover, the 90 tetrahedra corresponding to the 90 non-isotropic projective lines (vacuum cells) all satisfy $J = 0$.

Proof sketch / audit trail

This was computed by exhaustive enumeration of all 9450 tetrahedra in Q and evaluation of the simplicial coboundary formula

$$(dF)(a, b, c, d) = F(b, c, d) - F(a, c, d) + F(a, b, d) - F(a, b, c) \pmod 3.$$

The 90 non-isotropic line tetrahedra were identified as the unique K_4 cliques whose 4 triangular faces are all flat (holonomy 0). All have $J = 0$. (Audit bundles: `W33_minimal_Z3_flux_cycles_tetrahedra_bundle.zip`, `W33_charge_decomposition_and_line_moments_bundle.zip`.)

Key Result

The quotient holonomy F is a genuine *sourced* field strength: its 3-coboundary $J = dF$ is the discrete charge/current, with vacuum cells (non-isotropic lines) exactly flux-free.

12 Vacuum sector: the 90 non-isotropic lines

Definition

Let $\mathcal{L}$ denote the 90 non-isotropic projective lines in $PG(3,3)$, each a 4-point set in the 40-point geometry. These 90 lines are in bijection with:

- the 90 K_4 cliques in Q whose four triangular faces are flat,
- the Aut(W33)-distinguished vacuum cells for the quotient connection.

We identify the vacuum line field space with $\mathbb{Z}_3^{\mathcal{L}} \cong \mathbb{Z}_3^{90}$.

Remark

Because $90 \equiv 0 \pmod 3$, the constant all-ones vector lies in the $\mathbb{Z}_3$ augmentation subspace. Thus quotienting by the all-ones line produces the canonical 88-dimensional vacuum/matter module used in the H^3 identification.

13 Transfer operators from sources to vacuum observables

Definition

Partition tetrahedra in Q into three Aut(W33)-orbits by the number of flat faces:

bulk: #flat faces = 0 (6480), boundary: #flat faces = 1 (2880), vacuum: #flat faces = 4 (90).

In the boundary orbit, each tetrahedron has a *unique* flat face, hence a unique attached vacuum line $\ell \in \mathcal{L}$.

Definition

Define two linear maps over $\mathbb{Z}_3$:

$$M : \mathbb{Z}_3^{9450} \to \mathbb{Z}_3^{90}, \qquad Z : \mathbb{Z}_3^{9450} \to \mathbb{Z}_3^{90}.$$

They are defined on a tetrahedron t as follows:

1. (**Boundary moment** M) If t has exactly one flat face, let $\ell(t)$ be its unique attached non-isotropic line. Then M adds the tetra flux $J(t)$ to coordinate $\ell(t)$. Otherwise t contributes 0.
2. (**Bulk shadow** Z) For each *curved* triangular face of t, push $J(t)$ along the three edges of that face. Each edge of Q belongs to a unique non-isotropic line in $\mathcal{L}$ (since $540 = 90 \cdot 6$). Summing these contributions defines $Z(J)$ on $\mathcal{L}$.

Theorem 13.1 (Exact transfer identities) *Let $J = dF$ be the sourced 3-cochain. Then the two observed vacuum line fields*

$$m_{\text{line}} \in \mathbb{Z}_3^{90}, \qquad z_{\text{line}} \in \mathbb{Z}_3^{90}$$

satisfy the exact *operator identities*

$$m_{\text{line}} = M\,J, \qquad z_{\text{line}} = Z\,J,$$

with no residual error.

Proof sketch / audit trail

Both operators were constructed explicitly in sparse COO form and applied to the computed J. The resulting 90-vectors agree entrywise with the independently computed line observables from the earlier operator chains:

$$m_{\text{line}} = C_{\text{lineface}}\,J, \qquad z_{\text{line}} = R\,(K_0 + K_1)\,J.$$

(Audit bundle: `W33_transfer_operators_J_to_lines_and_mode_injection_bundle.zip`.)

Key Result

The W33 quotient admits explicit, Aut(W33)-equivariant transfer operators from sources J to vacuum line observables. This is the discrete analog of a constitutive relation (sources $\to$ observed vacuum response).

14 Vacuum harmonics and mode-resolved response

Definition

The Aut(W33) commutant algebra acting on $\mathbb{Z}_3^{\mathcal{L}}$ has dimension 5 (an association scheme). Equivalently, the 90-line sector admits a canonical decomposition into 5 joint harmonic modes under the commuting operators:

- S: the Aut-invariant fixed-point-free involution pairing on the 90 lines (45 disjoint transpositions),
- A_{meet}: line meet adjacency (two lines adjacent iff they intersect in a point), degree 32.

Joint modes are indexed by $(\text{sign}(S), \lambda(A_{\text{meet}}))$:

$$(+, 32)^1, \quad (+, 2)^{24}, \quad (+, -4)^{20}, \quad (-, 8)^{15}, \quad (-, -4)^{30}.$$

Theorem 14.1 (Mode-resolved injection table) *For each tetra orbit class (bulk vs boundary) and each flux sign $J \in \{1, 2\}$, the induced vacuum responses $M(J)$ and $Z(J)$ decompose into the above 5 modes with explicit energy fractions. In particular:*

- *Bulk sources (flat-face count 0) inject only into z_{line} (never into m_{line}).*
- *Boundary sources (flat-face count 1) inject into both m_{line} and z_{line}, with mode weights shifted toward $(+, 2)$ and $(-, 8)$ for m_{line}.*

Proof sketch / audit trail

This was computed by restricting J to each class+flux, applying the exact transfer operators M and Z, mapping $\mathbb{Z}_3$ entries to real values $\{-1,0,1\}$ (with $2 \mapsto -1$), removing the mean, and projecting onto the orthonormal joint-mode bases. The resulting mode-energy fractions are tabulated. (Audit bundle: `W33_mode_response_table_bulk_to_vacuum_bundle.zip`.)

Key Result

The vacuum sector is not a single "channel": bulk and boundary sources excite different vacuum harmonics. This explains why no Aut-equivariant line-only operator can strongly predict m from z (they are distinct projections of the same bulk source field).

Artifact Index (field-equation layer)

`Bundle`	**Contents / Purpose**

15 Vacuum association scheme and canonical harmonics

Theorem 15.1 (90-line association scheme and involution) *The Aut(W33) action on the 90 non-isotropic lines induces an association scheme with commutant dimension 5 (five orbitals on ordered pairs). One orbital is the diagonal; another is a fixed-point-free involution σ pairing the 90 lines into 45 disjoint transpositions, with each paired lineset disjoint (skew).*

Theorem 15.2 (Five canonical harmonics) *Let S be the permutation matrix of σ and let A_{meet} be the adjacency of the line-meet graph (degree 32). Then S and A_{meet} commute and admit a joint decomposition into five modes:*

$$(+,32)^1, \quad (+,2)^{24}, \quad (+,-4)^{20}, \quad (-,8)^{15}, \quad (-,-4)^{30}.$$

These modes provide the canonical "vacuum harmonics" for line fields.

Remark

This harmonic analysis explains why distinct vacuum observables (e.g., boundary moment m vs bulk shadow z) are not related by a single Aut-equivariant line-only operator: they occupy different mixtures of the canonical modes. The correct dynamics closes only when bulk source variables $J = dF$ and the transfer operators M, Z are included.

15.1 $q = 11$: global vq construction from local 1-factorizations (first pass)

Theorem 15.3 (Constructive vq quotient map at $q = 11$) *Assume that for each point p one selects an order-11 element in the point stabilizer whose induced action on the 12 isotropic lines through p fixes one line and cycles the other 11. This yields a labeling of the local lines by $\{\infty\} \cup \mathbb{F}_{11}$. Using the round-robin/reflection 1-factorization $\{M_a\}_{a\in\mathbb{F}_{11}}$ of K_{12}, the 66 local line-pair generators at p partition into 11 buckets of size 6. Hence there is a canonical 6-to-1 map*

$$\{\textit{local generators}\} \longrightarrow \{(p,a) : p \in PG(3,11),\ a \in \mathbb{F}_{11}\},$$

whose target has size $vq = 1464 \cdot 11 = 16104$ *(the universal ladder size).*

Theorem 15.4 (Checksum constraint) *For the above partition, each bucket has vanishing XOR checksum in* H_{11}*:*

$$\bigoplus_{(i,j)\in M_a} g_{ij} = 0,$$

so the 6-to-1 quotient respects the additive structure of H_{11} *at the local level.*

Remark

Status. We verified this construction and checksum constraint for a random sample of points using a deterministic per-point search for an order-11 stabilizer element in the tested subgroup. The remaining step is global canonicity: selecting a stabilizer element consistently (or proving independence of choice) so that the induced vq quotient carries the 5-orbital q-ladder association scheme.

Proof sketch / audit trail

We construct per-point order-11 actions on the 12 local lines, define the 11 matchings M_a, compute the corresponding 6-element buckets, and evaluate XOR sums in H_{11} for each bucket. (Bundle: `W33_q11_vq_projectivized_root_shell_construction_bundle.zip`.)

15.2 $q = 11$: $PGL(2, 11)$ label transport on local $PG(1, 11)$ charts (first pass)

Theorem 15.5 (Label transport is projective linear fractional) *Fix a canonical local labeling of the 12 isotropic lines through each point* p *by* $PG(1, 11) = \{\infty\} \cup \mathbb{F}_{11}$ *via an order-11 stabilizer element (one fixed line + one 11-cycle). For the tested symplectic subgroup generators, the induced transport between local labelings is always representable by a* $PGL(2, 11)$ *Möbius transformation on* $PG(1, 11)$*.*

Remark

Consequence. The natural transport group on local charts is projective (fractional linear), not merely affine. This explains why only a minority of transports preserve the specific reflection 1-factorization family $\{M_a\}$ in an index-wise way: most transports send ∞ to a finite label and hence conjugate the factorization to another, projectively equivalent 1-factorization. The correct global vq quotient should therefore be formulated as a projective-covariant structure: either (i) a canonical choice of the distinguished "∞ line" that is equivariant, or (ii) an equivalence class of factorization charts modulo $PGL(2, 11)$.

Proof sketch / audit trail

For a sample of points, we computed canonical local labelings and, for each generator mapping $p \mapsto p'$, computed the induced permutation of the 12 labels by pushing local lines forward under the generator and reading their labels at p'. Each such permutation was fit exactly by a $PGL(2, 11)$ fractional linear map. (Bundle: `W33_q11_label_transport_PGL2_bundle.zip`.)

15.3 $q = 11$: obstruction to a global infinity section (projective-bundle viewpoint)

Remark

Problem. The constructive vq map at $q = 11$ (local K_{12} factorization into 11 buckets of size 6) uses a local affine chart on $PG(1, 11)$: it distinguishes a local "infinity" line among the 12 isotropic lines through each point. Empirically, however, the transport group on these local charts is projective ($PGL(2, 11)$), and typically sends ∞ to a finite label. This means a naïve choice of $\infty(p)$ at each point is not equivariant under symmetry transport.

Theorem 15.6 (Empirical non-equivariance of $\infty(p)$) *Let $\infty(p)$ be the "fixed local line" extracted from a per-point order-11 chart selection rule. On a random sample of points and subgroup generators, the transported line $g(\infty(p))$ agrees with $\infty(gp)$ only rarely (about 10–15% in our tests). Thus $\infty(p)$ is not a globally equivariant section for the tested subgroup.*

Remark

Interpretation. The local label sets form a principal $PGL(2, 11)$ bundle of $PG(1, 11)$ charts. The K_{12} 1-factorization used for the 6-to-1 collapse is an *affine* reduction of structure group (a choice of Borel subgroup / point at infinity). The observed non-equivariance indicates this affine reduction is obstructed without extra gauge-fixing data. In kernel terms, this is the precise location where a "connection/holonomy" degree of freedom enters: the obstruction class is a candidate source of the emergent constants in the ladder.

Proof sketch / audit trail

We compute per-point $\infty(p)$ from canonical order-11 local charts and test equivariance under the generator-induced transport on local lines. (Bundle: `W33_q11_global_section_obstruction_bundle.zip`.)

15.4 $q = 11$: holonomy of the $PGL(2, 11)$ cocycle (first evidence of a $PSL(2, 11)$ obstruction class)

Remark

Setup. From the canonical local $PG(1, 11)$ labelings, each generator move $p \mapsto g(p)$ induces a $PGL(2, 11)$ Möbius transform on labels, giving a 1-cocycle on the Cayley graph of the tested subgroup. The obstruction to a global affine section is measured by the holonomy of this cocycle around loops.

Theorem 15.7 (Nontrivial holonomy and a 660-element closure) *In a sample of short loops in the Cayley graph (words of length ≤ 6 returning to the same point), the induced holonomy is frequently non-identity. The subgroup generated by the observed loop holonomies closes to a group of size 660, consistent with $PSL(2, 11)$ (the index-2 subgroup of $PGL(2, 11)$).*

Remark

Interpretation. This identifies the precise "gauge field" at the $q = 11$ rung: a projective connection with holonomy in $PSL(2, 11)$. The local K_{12} 1-factorization is an affine reduction of structure group, and the nontrivial $PSL(2, 11)$ holonomy is the discrete obstruction class preventing a global choice of infinity. In the TOE narrative, this is the exact analog of curvature: the field is not the matching itself but the projective transport cocycle.

Proof sketch / audit trail

We fit each generator step to an exact $PGL(2, 11)$ matrix and compute products along loop words. The closure size is measured by multiplying the observed holonomies and their inverses until no new elements appear. (Bundle: `W33_q11_PGL2_holonomy_PSL2_11_bundle.zip`.)

15.5 $q = 11$: vq orbital degrees confirmed (5-orbital q-ladder at 16104)

Theorem 15.8 (Stabilizer orbit sizes match the q-ladder valencies) *Using the gauge-free vq construction at $q = 11$ (16104 objects) with fiber $PSL(2, 11)/A_5$ and cocycle transport, the stabilizer of a base object has exactly five orbits on the 16104-object set, with sizes*

$$1,\ 10,\ 1331,\ 1452,\ 13310,$$

which coincide with the closed-form q-ladder valencies

$$1,\ q-1,\ q^3,\ q^2(q+1),\ q^3(q-1)$$

specialized at $q = 11$.

Proof sketch / audit trail

We build the induced 16104-object action from the point permutation generators and the $PSL(2, 11)/A_5$ coset fiber, using the computed cocycle matrices for label transport. We then compute stabilizer orbits via Schreier generators compiled into explicit stabilizer permutations, and obtain the orbit-size multiset above. (Bundle: `W33_q11_vq_orbital_degrees_confirmed_bundle.zip`.)

Key Result

This is the missing q=11 rung: the projective holonomy (PSL(2,11)) does not destroy the ladder; it *implements* it. The 16104-object quotient carries the forced 5-orbital structure, with fiber relation degree $q - 1 = 10$ and the remaining three valencies matching the universal polynomials.

15.6 $q = 11$: 16104-cycle harmonics (five primitive modes)

Theorem 15.9 (Closed-form q-ladder spectrum at $q = 11$) *The 16104-object vq scheme at $q = 11$ has the five valencies*

$$1,\ 10,\ 1331,\ 1452,\ 13310,$$

and the five primitive harmonic multiplicities

$$1,\ 792,\ 671,\ 13420,\ 1220,$$

summing to 16104. The eigenvalues of the four nontrivial relations on these five modes are:

mode mult.	A_{10}	A_{1331}	A_{1452}	A_{13310}
1	10	1331	1452	13310
792	10	-11	110	-110
671	10	11	-132	110
13420	-1	11	0	-11
1220	-1	-121	0	121

Proof sketch / audit trail

This is the q-ladder eigenmatrix of Theorem **??** specialized at $q = 11$. Orthogonality is verified exactly:

$$P^{\top}\mathrm{diag}(m)\,P = N\,\mathrm{diag}(k),$$

with $N = 16104$, k the valencies, and m the multiplicities above. (Bundle: `W33_q11_16104_association_scheme_harmonics_bundle.zip`.)

Key Result

With the q=11 orbital degrees confirmed and the full five-mode spectrum fixed by orthogonality, the $q = 11$ rung is now on equal footing with $q = 3, 5, 7$: it carries the same 5-orbital / 5-harmonic q-ladder structure, realized gauge-freely through $PSL(2, 11)$ holonomy.

15.7 $q = 11$: full intersection numbers (Bose–Mesner multiplication table)

Theorem 15.10 (All intersection numbers p^k_{ij} at $q = 11$) *For the 16104-object q-ladder scheme at $q = 11$, the full set of structure constants (intersection numbers) p^k_{ij} defined by*

$$A_i A_j = \sum_{k=0}^{4} p^k_{ij} A_k$$

are determined uniquely from the eigenmatrices. Concretely, with eigenmatrix P and dual eigenmatrix Q, we have

$$p^k_{ij} = \frac{1}{N}\sum_{r=0}^{4} P_{r,i}\,P_{r,j}\,Q_{r,k},$$

and every p^k_{ij} is a nonnegative integer. Moreover, the consistency identity

$$\sum_{k=0}^{4} p^k_{ij}\,k_k = k_i k_j$$

holds for all i, j, where k_k are the relation valencies.

Remark

Deliverable. We export the full 5×5×5 table both as a flat list and as left-multiplication matrices L_i (5×5 each), which together fully specify the Bose–Mesner algebra at $q = 11$. (Bundle: `W33_q11_16104_intersection_numbers_bundle.zip`.)

15.8 Closed-form intersection numbers for the q-ladder (polynomial Bose–Mesner algebra)

Theorem 15.11 (Intersection numbers are polynomials in q) *For the 5-class q-ladder association scheme (relations ordered as A_0, A_1, A_2, A_3, A_4 with valencies $1, q-1, q^3, q^2(q+1), q^3(q-1)$), every structure constant*

$$A_iA_j = \sum_{k=0}^{4} p_{ij}^k A_k$$

is an integer polynomial in q. Equivalently, the left-multiplication matrices L_i defined by $(L_i)_{j,k} = p_{ij}^k$ have entries in $\mathbb{Z}[q]$.

Remark

Constructive formula. Using the eigenmatrix $P(q)$ from Theorem **??** and dual eigenmatrix $Q(q) = \operatorname{diag}(m)\,P\,\operatorname{diag}(k)^{-1}$, we have

$$p_{ij}^k = \frac{1}{N}\sum_{r=0}^{4} P_{r,i}(q)\,P_{r,j}(q)\,Q_{r,k}(q), \qquad N = q(q^3+q^2+q+1).$$

Symbolic evaluation yields $p_{ij}^k \in \mathbb{Z}[q]$ for all i, j, k (no denominators appear).

15.9 Uniqueness of the q-ladder scheme (spectral determination of the Bose–Mesner algebra)

Theorem 15.12 (Eigenmatrices determine the intersection numbers uniquely) *Fix q and suppose a symmetric 5-class association scheme has valencies k_i and eigenmatrix P with multiplicities m_r satisfying the standard orthogonality*

$$P^\top \operatorname{diag}(m) P = N\operatorname{diag}(k),$$

with $N = \sum_i k_i = \sum_r m_r$. Then the dual eigenmatrix Q is uniquely determined by

$$Q = \operatorname{diag}(m)\,P\,\operatorname{diag}(k)^{-1}, \qquad \text{so that} \qquad P^\top Q = NI.$$

Consequently the intersection numbers are uniquely determined by the Fourier inversion formula

$$p_{ij}^k = \frac{1}{N}\sum_r P_{r,i}\,P_{r,j}\,Q_{r,k}.$$

*In particular, for the q-ladder eigenmatrices $P(q)$ and $m(q)$ in Theorem **??**, the entire Bose–Mesner algebra (hence the scheme multiplication table) is uniquely forced.*

Proof sketch / audit trail

The orthogonality equations imply P is invertible and fix Q uniquely by $Q = \operatorname{diag}(m)\,P\,\operatorname{diag}(k)^{-1}$; equivalently $P^{-1} = \frac{1}{N}\operatorname{diag}(k)^{-1}P^\top\operatorname{diag}(m)$. The adjacency algebra is commutative semisimple; the primitive idempotents are recovered from (P, Q), and multiplication in the adjacency basis is recovered by expanding pointwise products of characters, yielding the stated inversion formula for p_{ij}^k. No additional combinatorial input is required once (k, P, m) are fixed.

Corollary 15.13 (Polynomial family is uniquely specified) *The polynomial intersection table $p_{ij}^k(q) \in \mathbb{Z}[q]$ computed in the previous subsection is the unique 5-class Bose–Mesner algebra compatible with the q-ladder spectrum. Therefore any realization of the q-ladder at a given odd prime q necessarily has the same intersection numbers and hence the same orbital degrees and harmonic spectrum.*

Remark

Fiber relation multiplication. The fiber relation A_1 (a disjoint union of v cliques K_q) forces the simple multiplication rules:

$$A_1^2 = (q-1)A_0 + (q-2)A_1, \qquad A_1A_3 = (q-1)A_3, \qquad A_1A_2 = A_4, \qquad A_1A_4 = (q-1)A_2 + (q-2)A_4.$$

These are exactly the first row/column blocks of the polynomial left-multiplication matrix $L_1(q)$.

Remark

Full tables. We export the complete polynomial table $p_{ij}^k(q)$ and all $L_i(q)$ to CSV in the bundle `W33_q_ladder_intersection_polynomials_bundle.zip`.

15.10 $q = 11$: a gauge-free 11-label fiber via $PSL(2, 11)/A_5$

Remark

Key idea. The affine labels $a \in \mathbb{F}_{11}$ are not globally well-defined because transport is projective ($PGL(2, 11)$) and holonomy is nontrivial. A gauge-free replacement is to use an *associated-bundle* fiber that has size 11 intrinsically.

Theorem 15.14 (An intrinsic 11-element fiber from the holonomy group) *Let $G_{\text{hol}} \le PGL(2, 11)$ be the $q = 11$ holonomy group. Empirically $|G_{\text{hol}}| = 660$ and its element order spectrum matches $PSL(2, 11)$. Inside G_{hol} there exists a subgroup $H \cong A_5$ of order 60 (generated by a $(2, 3, 5)$ triangle presentation). Hence the left coset space*

$$G_{\text{hol}}/H$$

has size 11 and provides a canonical 11-element label fiber with a natural G_{hol} action by left multiplication.

Remark

Why 11 is not an accident. The local Pascal/K12 collapse partitions 66 line-pair edges into 11 buckets of size 6. In the holonomy picture, the *same 11* appears as the index of A_5 in $PSL(2, 11)$:

$$660/60 = 11.$$

This identifies the q=11 ladder label set with a finite-group coset geometry rather than an affine coordinate choice, removing the need for a global infinity section.

Proof sketch / audit trail

We close the holonomy subgroup from observed loop products, then search for a $(2,3,5)$ generating pair producing a subgroup of size 60, and compute its 11 cosets. (Bundle: `W33_q11_PSL2_11_coset_fiber_A5_bundle.zip`.)

16 Császár–Szilassi as the toroidal gate: K_7, Heawood, and the $K_{12}/66$ hinge

Remark

Why this matters for the kernel. The Császár and Szilassi polyhedra are the unique toroidal analogs of the tetrahedron with "complete adjacency" properties: Császár has *complete vertex adjacency* (skeleton K_7) and Szilassi has *complete face adjacency* (7 faces, each adjacent to all others). Both satisfy the torus hole equations

$$h = \frac{(v-3)(v-4)}{12} \quad \text{and} \quad h = \frac{(f-4)(f-3)}{12},$$

and the next solution $(v, f) = (12, 12)$ predicts the combinatorial hinge $66 = \binom{12}{2}$ that reappears in the $q = 11$ lift-layer as the K_{12} edge set of local line-pairs.

16.1 Combinatorial extraction from the uploaded edge data

Theorem 16.1 (Császár and Szilassi skeletons in the uploaded data) *The uploaded Császár edge lists `Cs_v#_edges_and_forms.csv` define a 7-vertex, 21-edge graph in which every vertex has degree 6; hence the skeleton is K_7. The uploaded Szilassi edge lists `Sz_v#_edges_and_forms.csv` define a 14-vertex, 21-edge 3-regular graph; hence the skeleton matches the Heawood graph embedding.*

Proof sketch / audit trail

We read the endpoints (i, j) in the CSV edge lists and compute graph degree sequences: Császár gives degree sequence 6^7 (complete graph K_7), while Szilassi gives degree sequence 3^{14} (Heawood). Edge-length and Hodge-star columns vary by realization version but do not affect the combinatorial skeleton.

16.2 Where K_7 and $66 = \binom{12}{2}$ sit in the q-ladder

Key Result

The Császár–Szilassi "torus gate" sits exactly on the ladder rungs we have computed:

- At $q = 7$, the projectivized root-shell scheme contains a canonical fiber relation that is a disjoint union of 400 cliques K_7 (degree 6), matching the Császár skeleton degree and the toroidal "complete adjacency" motif.
- At $q = 11$, the local geometry at a point has $q + 1 = 12$ isotropic lines; the 66 line-pair generators are exactly the 66 edges of K_{12}. The required collapse factor is $(q + 1)/2 = 6$, and we constructed a Pascal/round-robin 1-factorization of K_{12} into 11 buckets of size 6 with vanishing XOR checksum in H_{11}.

Thus the celebrated "next hole-equation solution" $(v, f) = (12, 12)$ reappears as a *local* projective-line object at $q = 11$ (12 lines through a point), and the 66 hinge is literally the local generator set that must be quotiented to reach the universal ladder size vq.

Remark

Cheeky interpretation. In the finite kernel, the non-realizability of a Euclidean polyhedron with $(v, h) = (12, 6)$ does not kill the combinatorics; it relocates it: K_{12} appears as the local "vertex figure" (lines-through-a-point) in the $q = 11$ symplectic geometry, and the toroidal K_7/Heawood pair appear as the $q = 7$ fiber relation and its dual adjacency motif.

16.3 Existence theorem (conditional) and final classification statement

Remark

What remains. The previous subsections establish that once a 5-class q-ladder scheme exists with the q-ladder spectrum, its full Bose–Mesner algebra is uniquely determined (intersection numbers in $\mathbb{Z}[q]$). The remaining content is *existence*: producing, for each odd prime q, an actual combinatorial realization of the vq quotient and its 5 relations. Our computations provide explicit realizations for $q = 3, 5, 7, 11$.

Theorem 16.2 (Conditional existence from the vq quotient) *Fix an odd prime q. Assume the following kernel-to-ladder construction data:*

1. *The symplectic point graph G_q of $W(3, q)$ (vertices $v = q^3 + q^2 + q + 1$).*
2. *The mod-2 square-zero differential $A_q^2 \equiv 0$ giving $H_q = \ker(A_q)/\mathrm{im}(A_q)$ of dimension $q^2 - 1$.*
3. *A canonical local generator orbit $\mathcal{G}_q$ (line-pair XORs) that maps injectively into H_q and is transitive under a symplectic subgroup action.*
4. *A canonical commutant/holonomy mechanism producing a gauge-free quotient set Ω_q of size vq with a symmetry action induced from the kernel, together with a distinguished fiber relation that is a disjoint union of v cliques K_q.*

Then Ω_q carries a symmetric 5-class association scheme whose orbital degrees, harmonic spectrum, and intersection numbers coincide with the unique q-ladder polynomials derived in Theorems **??** *and the polynomial Bose–Mesner tables.*

Proof sketch / audit trail

Given Ω_q and the fiber relation A_1, the remaining three relations are determined by symmetry: they are the remaining orbitals in the commutant algebra (dimension 5). The eigenmatrix must match the q-ladder spectrum because (i) A_1 fixes the fiber eigen-split and (ii) the remaining four modes are forced by orthogonality and integrality as shown in the q-ladder derivation/uniqueness subsections. Therefore the multiplication table agrees with the polynomial intersection numbers.

Theorem 16.3 (Final classification statement (prime-field ladder)) *For odd primes q for which the vq quotient exists, the resulting ladder is completely classified:*

- *The 5 relation valencies are* $1,\ q-1,\ q^3,\ q^2(q+1),\ q^3(q-1)$.
- *The 5 primitive multiplicities are* $1,\ \frac{q(q+1)^2}{2},\ \frac{q(q^2+1)}{2},\ q(q^3-q^2+q-1),\ (q^3-q^2+q-1)$.
- *The full intersection numbers $p_{ij}^k(q)$ are integer polynomials and uniquely determined by the spectrum.*

Thus the ladder is a single polynomial association-scheme family whose concrete realizations at $q = 3, 5, 7, 11$ have been constructed and verified in this work.

Key Result

At this point the TOE kernel has two logically distinct parts:

1. **Kernel (exact finite geometry)**: $W(3, q)$, the mod-2 square-zero calculus, the homology module H_q, and the local generator orbit $\mathcal{G}_q$.
2. **Gauge/holonomy lift (cheeky step)**: the commutant/holonomy mechanism producing the vq quotient Ω_q. For $q = 11$ this is explicitly projective with holonomy in $PSL(2, 11)$ and a gauge-free fiber $PSL(2, 11)/A_5$ of size 11; nevertheless it lands exactly on the universal 5-orbital q-ladder scheme.

Once the lift exists, the rest of the structure is forced.

16.4 $q = 13$: prime-field kernel test (next rung)

Theorem 16.4 (Kernel invariants at $q = 13$) *For the symplectic $W(3, 13)$ point graph:*

$$v = 13^3 + 13^2 + 13 + 1 = 2380, \qquad k = 13 \cdot 14 = 182, \qquad (\lambda, \mu) = (12, 14),$$

and $A^2 \equiv 0 \pmod 2$. Over $\mathbb{F}_2$ the computed rank and homology dimensions are:

$$\operatorname{rank}(A) = 1106, \qquad \dim H_{13} = 168 = q^2 - 1.$$

Each point lies on $q + 1 = 14$ isotropic lines of size 14, and the number of isotropic lines is 2380.

Remark

Local generator check (sample). Line-pair XOR generators at a point have weight $2(q+1)-2=26$; this holds in our sample (only weight 26 observed). A 30k-sample of line-pair generators produced 27962 distinct reduced H_{13} classes (2038 collisions), indicating the injectivity/orbit-transitivity phenomena seen at $q=5,7,11$ may require the full commutant/holonomy lift mechanism to recover a canonical orbit/quotient at larger q (to be resolved by deeper testing). (Bundle: `W33_q13_prime_field_kernel_test_bundle.zip`.)

16.5 $q=13$: holonomy of the $PGL(2,13)$ cocycle (first evidence of a $PSL(2,13)$ gauge field)

Theorem 16.5 (Nontrivial holonomy closes to $PSL(2,13)$) *Using a base-point trivialization induced by an order-13 cycle on the 14 local lines (fixed line + 13-cycle), each generator step induces an exact $PGL(2,13)$ Möbius transform on local $PG(1,13)$ labels. Sampling short loop words (length ≤ 6) that return to the same point yields frequent non-identity holonomy. The subgroup generated by observed loop holonomies closes to a group of size 1092, consistent with $PSL(2,13)$.*

Remark

Interpretation. This is the direct $q=13$ analog of the $q=11$ story: the obstruction to a global affine section is measured by a projective connection whose holonomy is (empirically) $PSL(2,q)$. Thus the "cheeky" lift step persists at the next prime rung, and the gauge-field picture scales with q.

Proof sketch / audit trail

We compute canonical local labelings for a BFS sample of points, fit each generator transport exactly by a $PGL(2,13)$ matrix from images of $(0,1,\infty)$, and multiply along loop words. (Bundle: `W33_q13_PGL2_holonomy_PSL2_13_bundle.zip`.)

16.6 $q=13$: local Pascal factorization of K_{14} (13 buckets of size 7)

Theorem 16.6 (Local K_{14} edge partition and checksum) *At a point p in the $q=13$ symplectic geometry, there are $q+1=14$ isotropic lines through p, so local line-pair generators form the 91 edges of K_{14}. Choosing a local order-13 cycle on these 14 lines (one fixed line + one 13-cycle) yields a labeling by $PG(1,13)=\{\infty\}\cup\mathbb{F}_{13}$. The round-robin/reflection 1-factorization partitions the 91 edges into 13 buckets of size 7 (collapse factor $(q+1)/2=7$). Moreover, each bucket has vanishing XOR checksum modulo* $\mathrm{im}(A)$*:*

$$\bigoplus_{(i,j)\in M_a} g_{ij} \equiv 0 \pmod{\mathrm{im}(A)} \qquad (a\in\mathbb{F}_{13}).$$

Remark

This is the direct q=13 generalization of the q=11 K12/66 hinge: $91 = \binom{14}{2} = 13 \cdot 7$ and the required collapse factor is $(q+1)/2 = 7$. Combined with the holonomy result (closing to $PSL(2,13)$), this strongly supports that the prime-field ladder admits the same local Pascal/binomial reduction mechanism at each q. (Bundle: `W33_q13_local_pascal_factorization_bundle.zip`.)

16.7 $q = 13$: vq construction frontier (current obstruction and next move)

Remark

At $q = 11$ we constructed a gauge-free 11-fiber using $PSL(2,11)/A_5$ and confirmed the full vq rung (16104) carries the forced 5-orbital ladder scheme. For $q = 13$ we have already verified (i) kernel invariants ($\dim H_{13} = q^2 - 1$), (ii) projective holonomy closing to $PSL(2,13)$, and (iii) the local Pascal factorization $K_{14} \to 13$ buckets of size 7 with vanishing checksum. The remaining step is to construct a *gauge-free* size-13 fiber that is stable under the $PGL(2,13)$ transport cocycle.

16.8 $q = 13$: matchings and factorization orbits under $PSL(2,13)$ (fiber search)

Theorem 16.7 (Orbit sizes on local combinatorial structures) *Let $G = PSL(2,13)$ act on $PG(1,13)$ (14 points) by fractional linear transformations. Consider: (i) a single perfect matching on K_{14} (7 disjoint edges) and (ii) the standard round-robin 1-factorization (13 perfect matchings). Then under the induced action on edge-sets:*

$$|\mathrm{Orb}(M)| = 91, \qquad |\mathrm{Orb}(\mathcal{F})| = 14.$$

Remark

Interpretation. The 1-factorization orbit size 14 matches the natural projective-line size (choices of "infinity"), confirming that the Pascal/round-robin factorization family is essentially a projective-gauge choice. The perfect-matching orbit size 91 indicates a stabilizer of size 12, consistent with an A_4-type symmetry in the action on matchings. In particular, this suggests that a gauge-free 13-fiber is not obtained simply by taking the PSL-orbit of a single matching or factorization; a different associated object (or an augmented commutant beyond bare $PSL(2,13)$ holonomy) is required. (Bundle: `W33_q13_matching_factorization_orbits_bundle.zip`.)

16.9 $q = 13$: no 13-orbit among k-subsets of $PG(1,13)$ (negative result)

Remark

To obtain a gauge-free 13-fiber by a holonomy-coset mechanism, one might hope for a 13-element orbit of $PSL(2,13)$ acting on some natural derived structure of $PG(1,13)$ (14 points). A first search is the action on k-subsets of the 14 points.

16.10 $q = 13$: a 91-to-13 collapse inside the matching orbit (new fiber candidate)

Theorem 16.8 (Matching orbit partitions into 13 classes of size 7) *Let $G = PSL(2,13)$ act on $PG(1,13)$ (14 points) and hence on perfect matchings of K_{14}. The orbit of a single round-robin* *matching has size 91. Moreover, if one fixes a distinguished vertex (the "infinity" label) and maps each matching to the unique partner of that vertex in the matching, then the 91 matchings partition into 13 classes of size 7:*

$$91 = 13 \cdot 7,$$

and each of the 13 partners occurs exactly 7 times.

Remark

Interpretation. This provides the first concrete 13-fiber-like collapse at $q = 13$ despite the absence of a 13-orbit among k-subsets of $PG(1, 13)$. It suggests that the gauge-free fiber at $q = 13$ may be realized as a *quotient of a 91-element matching orbit* by the internal collapse factor $(q + 1)/2 = 7$, rather than as a subgroup coset $PSL(2, 13)/H$ with $|H| = 84$. This is exactly the kind of "cheeky" combinatorial relocation seen elsewhere in the kernel. (Bundle: `W33_q13_matching_orbit_partition_91_to_13_bundle.zip`.)

16.11 $q = 13$: partner-label partition on $PSL(2, 13)$ elements (84-per-label classes)

Theorem 16.9 (A 13-class partition of $PSL(2, 13)$ compatible with the 91-to-13 collapse) *Fix the base matching M_0 (round-robin matching with ∞ paired to 0) and define*

$$f(g) := \textit{the partner of } \infty \textit{ in the matching } g \cdot M_0, \qquad g \in PSL(2, 13).$$

Then f takes values in $\mathbb{F}_{13}$ and partitions the 1092 group elements into 13 classes of equal size:

$$|f^{-1}(a)| = 84 \quad \textit{for every } a \in \mathbb{F}_{13}.$$

Remark

Right-invariance and non-coset nature. The partition is invariant under right multiplication by the size-12 stabilizer of M_0 (as expected from the matching orbit size 91). However, the 84-element fibers are *not* right cosets of a size-84 subgroup (the only global right-invariance subgroup is size 12). Thus the 13 classes arise as a genuine "cheeky" quotient of the 91 matching orbit (and its 12-element stabilizer), not as a straightforward coset fiber $PSL(2, 13)/H$. (Bundle: `W33_q13_PSL13_partner_label_partition_bundle.zip`.)

16.12 $q = 13$: why the 13-partition does not yet give a transport (failed label-action attempts)

Remark

We identified a canonical 91-to-13 collapse inside the perfect-matching orbit (partitioning 91 matchings into 13 classes of size 7 by the partner of ∞). To build the $vq = 30940$ rung, we would like a *transport rule* making these 13 labels into a genuine fiber acted on by the holonomy group.

Theorem 16.10 (The partner partition is not $PSL(2, 13)$-invariant) *Let $f(M)$ be the partner of ∞ in a matching M. Although the orbit of a matching partitions into 13 classes of size 7 by f, this partition is not preserved by the full $PSL(2, 13)$ action on matchings: for a generic $g \in PSL(2, 13)$, the image of a class $f^{-1}(a)$ is not contained in a single class $f^{-1}(a')$. Equivalently, the partition is not a system of imprimitivity for the PSL action.*

Remark

Interpretation. The 13-partition depends on a choice of distinguished ∞ (a projective gauge). In $q = 11$ this was resolved by a gauge-free coset fiber $PSL(2, 11)/A_5$ of size 11. For $q = 13$, extensive searches indicate no 13-orbit among k-subsets of $PG(1, 13)$ and the 13-partition is not PSL-invariant. Thus a gauge-free 13-fiber likely requires additional structure from the full $W(3, 13)$ kernel (e.g., an associated object built from the $PGL(2, 13)$ cocycle, or a commutant action on the local-generator orbit), not the projective-line action alone. (Bundle: `W33_q13_label_transport_attempts_bundle.zip`.)

Theorem 16.11 (Orbit sizes on k-subsets exclude size 13) *For the natural action of $PSL(2, 13)$ on $PG(1, 13)$, the induced action on k-subsets (for $k = 3, 4, 5, 6, 7$) has orbit sizes among:*

$$k = 3 : \ 182; \quad k = 4 : \ 91, 273, 546; \quad k = 5 : \ 182, 546; \quad k = 6 : \ 91, 546, 1092; \quad k = 7 : \ 78, 182, 364, 546,$$

and in particular no orbit of size 13 occurs.

Remark

Interpretation. This strongly suggests the desired 13-fiber is not obtained from the projective-line action alone; it must come from a richer associated object (e.g., derived from the full $W(3, 13)$ incidence, the homology module H_{13}, or a commutant action on the local-generator orbit), analogous to the $q = 11$ coset fiber $PSL(2, 11)/A_5$ which is not a $PG(1, 11)$ point set. (Bundle: `W33_q13_fiber_orbit_search_bundle.zip`.)

Remark

Current obstruction. A direct "label $a \in \mathbb{F}_{13}$" fiber is not stable under full projective transport (the cocycle typically sends ∞ to a finite label), so bucket indices do not transport canonically. A natural approach is to realize the 13-fiber as an orbit/coset space of the holonomy group. This would require an index-13 subgroup of $PSL(2, 13)$ (order 84), but our exploratory random subgroup searches did not locate such a subgroup among elements of the observed orders $\{2, 3, 6, 7, 13\}$. Standard subgroup classifications for $PSL(2, 13)$ suggest prominent maximal subgroups of orders 78 (index 14 Borel), and dihedral/A4/S4/A5 types, so the size-13 fiber may need to be realized as an orbit on a richer combinatorial object (e.g., factorization or spread structures) rather than simple cosets.

Remark

Next move. Two promising paths: (i) search for a 13-element orbit of $PSL(2, 13)$ acting on derived structures on $PG(1, 13)$ (perfect matchings / factorization orbits), or (ii) use the natural 14-point $PG(1, 13)$ fiber (index 14) and incorporate the Pascal buckets as a secondary quotient to obtain an effective 13-label gauge-free lift. (Bundle: `W33_q13_vq_construction_frontier_bundle.zip`.)

16.13 $q = 13$: gauge-cocycle action on $\Omega_{14} = PG(3,13) \times PG(1,13)$ (8-orbital super-scheme)

Theorem 16.12 (An 8-orbital scheme on 33320 objects) *Using the exact $PGL(2,13)$ cocycle computed from line transport, we obtain a well-defined action on the 33320-object set*

$$\Omega_{14} = PG(3,13) \times PG(1,13),$$

of size $2380 \cdot 14 = 33320$. The stabilizer of a base object has 8 orbits with sizes:

$$1,\ 13,\ 13,\ 13^2,\ 13^2,\ 13^3,\ 13^3,\ 13^4,$$

summing to 33320.

Remark

Interpretation. This shows that the projective cocycle naturally yields a *super-scheme* whose orbital degrees factor cleanly as powers of q. The desired q-ladder rung at $vq = 2380 \cdot 13 = 30940$ should arise as a quotient/refinement of Ω_{14} that removes the extra projective point (the moving "infinity") and collapses the duplicated $13/13^2/13^3$ orbits appropriately. In geometric language, Ω_{14} is the principal $PGL(2,13)$-bundle-associated object, while the q-ladder fiber seeks an additional reduction (a gauge-fixing functional) to a 13-fiber. (Bundle: `W33_q13_Omega14_orbital_degrees_bundle.zip`.)

Remark

Attempted reduction $\Omega_{14} \to \Omega_{13}$ by gauge fixing. A natural idea is to project the fiber $PG(1,13)$ to $\mathbb{F}_{13}$ by a pointwise gauge fix (e.g., map $\infty \mapsto 0$ after transport). We implemented this as an induced action on $2380 \cdot 13 = 30940$ objects, but the resulting maps are not bijections: multiple labels can map to ∞ under a Möbius transform, causing collisions after projection. Thus the vq reduction at $q = 13$ cannot be achieved by a naive pointwise projection; it must use a holonomy-aware quotient/associated object. (Bundle: `W33_q13_Omega13_gaugefix_attempt_bundle.zip`.)

Remark

Parallel transport projection attempt (fails). We also tried a tree-based gauge-fixed projection: for each point p, compute a transport T_p from the base chart and define $a = \pi(p, x)$ by pulling back x to the base ($y = T_p^{-1}x$) and then collapsing $\infty \mapsto 0$. Even using the parallel-transport section $x = T_p(a)$ for finite a, the induced maps on $vq = 2380 \cdot 13$ objects are not bijections: Möbius transport can send finite labels to ∞, so collapsing ∞ causes unavoidable collisions. This reinforces that the vq reduction at $q = 13$ cannot be done by any pointwise projection $PG(1,13) \to \mathbb{F}_{13}$, even with spanning-tree transport; a non-pointwise quotient is required. (Bundle: `W33_q13_path_transport_projection_attempt_bundle.zip`.)

16.14 $q = 13$: first bijective vq action from cocycle renormalization (partial rung)

Theorem 16.13 (A bijective 30940-object action with 4 orbital degrees) *Define a 13-label update rule by renormalizing each transport to send the pole to infinity: for a cocycle matrix M at a* *step $p \to p'$, let $t = M(\infty)$. For $a \in \mathbb{F}_{13}$ define*

$$a' = \begin{cases} 2\,(M(a) - t)^{-1} & M(a) \neq \infty,\ t \neq \infty, \\ 0 & M(a) = \infty,\ t \neq \infty, \\ 2\,M(a) & t = \infty, \end{cases}$$

with the scalar 2 chosen as a nonsquare to merge the quadratic-residue split. This yields bijections on the 30940-object set $PG(3,13) \times \mathbb{F}_{13}$ for each generator. The stabilizer of a base object has 4 orbits of sizes:

$$1,\ 12,\ 2366,\ 28561.$$

Remark

Interpretation. The bijective vq action confirms that a holonomy-aware, non-pointwise normalization can remove the pole collision obstruction. The orbit sizes match the q-ladder values for 1 and $q-1$ and $q^2(q+1)$, but the expected split $(q^3,\ q^3(q-1))$ appears merged into q^4. Thus this construction is a partial rung: additional structure (likely commutant information from H_{13} or a larger acting subgroup) is needed to split the large orbital and recover the full 5-orbital q-ladder scheme. (Bundle: `W33_q13_vq_action_partial_success_bundle.zip`.)

16.15 $q = 13$: split search update (still merged q^4 orbital)

Remark

We increased the acting subgroup (14 symplectic generators) and compiled a larger stabilizer generator set via Schreier sampling. The resulting bijective $vq = 30940$ action persists under the cocycle-renormalization update rule, but the stabilizer orbit partition remains

$$1,\ 12,\ 2366,\ 28561,$$

with the expected $(q^3,\ q^3(q-1)) = (2197, 26364)$ still merged into $q^4 = 28561$. This indicates the missing split is not an artifact of too-small generator sets; it likely requires an additional commutant/H-module invariant beyond pure projective renormalization. (Bundle: `W33_q13_vq_split_search_attempt_bundle.zip`.)

A Global Artifact Index

`Bundle`	**Contents / Purpose**
`W33_symplectic_audit_bundle.zip`	Explicit construction of $W(3,3)$ and W33; point/line incidence; $PG(3,3)$ points; isotropic vs nonisotropic line lists; verification of SRG parameters and spectrum.
`W33_orbits_squarezero_bundle.zip`	Aut(W33) generators (permutations and GF(3) matrices); orbit computations; square-zero and symmetry checkpoints.
`W33_GF2_kernel_code_bundle.zip`	The $[40,24,6]$ kernel code $\ker(A)$ over $\mathbb{F}$__2; 240 weight-6 generators; code basis and supporting tables.
`W33_H8_quadratic_form_bundle.zip`	Basis of $H = \ker(A)/\mathrm{im}(A)$; invariant quadratic form q; orbit split (135 singular / 120 nonsingular).
`W33_to_H_to_120root_SRG_bundle.zip`	The 120 nonsingular orbit list; SRG(120,56,28,24) edges/adjacency; mappings from code generators to H.
`W33_E8_simple_root_system_bundle.zip`	Canonical induced E__8 Dynkin configuration inside the 120-root SRG; Coxeter checks; reflection orbit generation.
`W33_signed_root_cocycle_and_lift_bundle.zip`	Signed lift/cocycle computations on 120-root edges and Steiner triples; defect weights; gauge studies.
`W33_global_gaugefix_no16_bundle.zip`	Global sign/gauge fix removing all weight-16 defects; resulting 0,12 defect spectrum; 40 flat triples.
`W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip`	Quotient $Q = \overline{\mathrm{W33}}$; edge matchings; triangle holonomy values; proof that flat holonomy triangles are exactly nonisotropic line triples.
`W33_Z3_curvature_cohomology_on_quotient_bundle.zip`	Triangle curvature cochain over $\mathbb{Z}$__3; non-exactness on the 2-skeleton; supporting tables.
`W33_minimal_Z3_flux_cycles_tetrahedra_bundle.zip`	Minimal-support flux cycles (tetrahedron boundaries) and flux statistics for $J = dF$.
`W33_flux_lattice_clique_complex_Z3_cohomology_bundle.zip`	Clique-complex cohomology ranks and dimensions over $\mathbb{Z}$__3; H^3 dimension 89; higher cohomology signature.
`W33_H3_basis_89_Z3_on_clique_complex_bundle.zip`	Explicit 89-element basis for H^3 as sparse tetra-cochains; pivot/free coordinate metadata.
`W33_H3_Aut_action_89Z3_bundle.zip`	Aut(W33) action matrices on H^3; 88+1 decomposition; quotient functional and block form.
`W33_perm_module_vs_H3_match_report_bundle.zip`	Evidence and generators showing the 88D core matches the 90-line augmentation quotient up to the similitude sign twist.
`W33_H3_to_noniso_line_weights_intertwiner_bundle.zip`	Explicit intertwiner between H^3 88D core and the twisted 90-line augmentation quotient.
`W33_lift_to_90_line_weights_with_labels_bundle.zip`	Explicit lift to labeled 90 nonisotropic line weights (mod all-ones gauge); line__id to 4-point set.
`W33_holonomy_phase_test_bundle.zip`	Holonomy vs symplectic triangle phase test; shows background closed 2-form vs sourced curvature.
`W33_current_operator_C_lineface_bundle.zip`	Operator C__lineface and line-moment statistics (source attachments to vacuum cells).
`W33_bulk_operator_K0K1_curved_triangle_current_bundle.zip`	Bulk current operators on curved triangles (K__0, K__1); outputs y on the 2880 curved triangle orbit.
`W33_curved_triangle_to_noniso_line_operator_R_bundle.zip`	Operator R mapping curved-triangle current to 90-line aggregates via edge-incidence.
`W33_charge_decomposition_and_line_moments_bundle.zip`	Charge decomposition $J = dF$; point incidences; preliminary line moments and constraints.

B Global Dictionary Table

Object	Interpretation	Algebra	Geometry/-Topology	Quantum computation	Crypto / security
$V = \mathbb{F}\_3^4$	Finite phase space; 2-qutrit discrete symplectic phase space.	Vector space over $\mathbb{F}\_3$ with symplectic form.	Underlying coordinate domain for projective geometry and Weyl operators.	Pauli/Weyl labels; Clifford acts by $Sp(4,3)$.	Key space for symplectic commutator phase.
$W(3,3)$ / isotropic lines	Maximal commuting contexts.	Incidence geometry of totally isotropic points/lines.	Produces W33 as point graph.	Stabilizer contexts for two qutrits.	Basis for context-based protocols.
W33 = SRG(40,12,2,4)	Base combinatorial geometry.	Adjacency matrix A with SRG identities.	Over $\mathbb{F}\_2$, yields differential $A^2 = 0$.	Constraint graph / stabilizer structure.	Public structure; secrecy comes from gauge/coset choices.
$A^2 \equiv 0$ over $\mathbb{F}\_2$	Chain-complex calculus.	Defines $d(x) = Ax$ with $d^2 = 0$.	Produces code $\ker(A)$ and homology H.	Error correction / stabilizer relations.	Syndromes / tamper detection.
$H = \ker(A)/\mathrm{im}(A)$ (8D)	Intrinsic state space.	Carries invariant quadratic form; orbit split.	Nonsingular orbit gives 120-root shell.	Finite "root" degrees; phase classes.	Key reduction space for encoding.
120/240 roots	Finite root shell and signed lift.	SRG(120) adjacency via bilinear form; 2-to-1 lift.	Global gauge fixing yields flat triples.	Discrete gauge degrees; lift choices.	Keyed section choices = secrecy.
$Q = \overline{\mathrm{W33}}$	Quotient spacetime / interaction graph.	40 meta-vertices after collapse; edge matchings.	Supports $\mathbb{Z}\_3$ holonomy.	Transport/holonomy = topological gate.	Holonomy checks = authentication.
Holonomy $F \in C^2(\mathrm{Cl}(Q); \mathbb{Z}\_3)$	Field strength / curvature.	Triangle cochain valued in $\mathbb{Z}\_3$.	Flat set classified by 90 nonisotropic lines.	Discrete phase curvature.	Consistency checks / signatures.
Sources $J = dF \in C^3$	Charge/current.	Supported on 3008 tetrahedra.	Generates vacuum responses via M, Z.	Excitations / particles.	Error/fault injection model.
90 nonisotropic lines	Vacuum cells and matter carrier space.	Association scheme (5-mode harmonic analysis).	Line-weight field model (mod all-ones).	Contextual phase cells.	Share space for schemes; 88D core module.
Transfer operators M, Z	Constitutive laws.	Exact maps $J \mapsto (m, z)$.	Mode-resolved response tables.	Measurement/readout operators.	Encryption/readout operators.

C Reproducibility Checklist

Remark

Short SHA-256 prefixes (first 16 hex characters) for primary bundles in the current workspace.

File	SHA-256 prefix
W33_symplectic_audit_bundle.zip	c8f7547649abdab1
W33_orbits_squarezero_bundle.zip	84835a9889e4380b
W33_GF2_kernel_code_bundle.zip	952858afb5d65007
W33_H8_quadratic_form_bundle.zip	de3a9a9b0afb6a37
W33_to_H_to_120root_SRG_bundle.zip	3257de84a4b9c466
W33_E8_simple_root_system_bundle.zip	d200bec6ff81f00a
W33_signed_root_cocycle_and_lift_bundle.zip	d33146ea2d96104f
W33_global_gaugefix_no16_bundle.zip	8de8d1182056ac00
W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip	8a6cda139ed0a0e6
W33_Z3_curvature_cohomology_on_quotient_bundle.zip	1a7804dd46ccb1b5
W33_minimal_Z3_flux_cycles_tetrahedra_bundle.zip	8d69efdc34b5a0e6
W33_flux_lattice_clique_complex_Z3_cohomology_bundle.zip	17f5bb8490fc2d36
W33_H3_basis_89_Z3_on_clique_complex_bundle.zip	2fa53b14fcd57da9
W33_H3_Aut_action_89Z3_bundle.zip	032be0e14f33c5cc
W33_perm_module_vs_H3_match_report_bundle.zip	535aa4d6b03264d9
W33_H3_to_noniso_line_weights_intertwiner_bundle.zip	da15db795acf478b
W33_lift_to_90_line_weights_with_labels_bundle.zip	81b9f049398d5f93
W33_holonomy_phase_test_bundle.zip	5991ca050359bc4b
W33_current_operator_C_lineface_bundle.zip	02e3566e1869ce07
W33_bulk_operator_K0K1_curved_triangle_current_bundle.zip	5953f1541d2793f1
W33_curved_triangle_to_noniso_line_operator_R_bundle.zip	633e86c28d6433cf
W33_charge_decomposition_and_line_moments_bundle.zip	d9c00f5e46ca2658
W33_nonisotropic_line_association_scheme_bundle.zip	ec4b4b8e10918586
W33_vacuum_line_scheme_mode_decomposition_bundle.zip	d8545a6b843ab310
W33_transfer_operators_J_to_lines_and_mode_injection_bundle.zip	647e18c9a6ac8f7c
W33_best_field_equation_operator_on_lines_bundle.zip	3494bf1e74c08f1b