# Quantum Realization of the W33 Tower

Section 10 Draft (Weyl/Clifford, Contexts, Holonomy-Phase Test)

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**Abstract**

This section drafts the quantum computation layer of the W33 tower in a theorem-forward manner. Starting from the same finite symplectic phase space $V = \mathbb{F}_3^4$, we construct the 2-qutrit Heisenberg–Weyl operators, identify isotropic projective lines with maximal commuting Pauli contexts, interpret the automorphism action as a projectivized Clifford normalizer action, and state a concrete, falsifiable conjecture: that the $\mathbb{Z}_3$ holonomy on the 40-vertex quotient $Q = \overline{\mathrm{W33}}$ equals a symplectic triangle phase built from the Weyl commutator form.

**Remark**

**Scope.** This section is intentionally algebraic and test-driven: it provides a precise quantum dictionary for the already-established W33 kernel and specifies a computation that either confirms or falsifies the "holonomy equals commutator phase" identification. No claims about physical constants are made here.

## Contents

## 1 2-qutrit Weyl operators and the symplectic commutator

**Definition**

Let $\omega := e^{2\pi i/3}$. On $\mathbb{C}^3$ with computational basis $\{|j\rangle : j \in \mathbb{Z}_3\}$ define

$$X|j\rangle = |j+1\rangle, \qquad Z|j\rangle = \omega^j |j\rangle,$$

so that $ZX = \omega XZ$. On two qutrits, for $(a,b,c,d) \in \mathbb{F}_3^4$, define the (unnormalized) Weyl operator

$$W(a,b,c,d) := X^a Z^c \otimes X^b Z^d.$$

**Definition**

Define the standard symplectic form on $V = \mathbb{F}_3^{2n}$ with $n = 2$ by writing $v = (p \mid q)$ with $p, q \in \mathbb{F}_3^2$ and

$$\langle (p \mid q), (p' \mid q') \rangle := p \cdot q' - q \cdot p' \ \in \ \mathbb{F}_3.$$

In coordinates $v = (a,b,c,d)$ and $w = (a',b',c',d')$, this is

$$\langle v, w \rangle = ac' + bd' - ca' - db'.$$

**Theorem 1.1 (Weyl commutator phase)** *For all $v, w \in \mathbb{F}_3^4$,*

$$W(v)\,W(w) = \omega^{\langle v,w \rangle}\,W(w)\,W(v).$$

*Equivalently, $W(v)$ and $W(w)$ commute if and only if $\langle v, w \rangle = 0$.*

**Proof sketch / audit trail**

This is the standard Heisenberg–Weyl relation for odd prime dimension. For the above unnormalized convention, it follows from $ZX = \omega XZ$ on each tensor factor and bilinearity of the commutator exponent.

**Key Result**

The same symplectic form used to build $W(3,3)$ is exactly the commutator phase form in the 2-qutrit Weyl group. This is the first canonical bridge from W33 geometry to quantum operator algebra.

## 2 Projective points as Weyl directions

**Definition**

Let $\mathbb{P}(V) = PG(3,3)$ denote projective 1D subspaces of $V = \mathbb{F}_3^4$. A projective point $[v]$ is the equivalence class $\{v, 2v\}$ for any nonzero $v \in V$.

**Theorem 2.1 (Projective points correspond to cyclic Weyl subgroups)** *Each projective point $[v] \in PG(3,3)$ determines a cyclic order-3 Weyl subgroup*

$$\langle W(v) \rangle = \{I,\, W(v),\, W(2v)\}.$$

*Moreover, $W(2v) = W(v)^{-1}$ and the subgroup depends only on $[v]$ (not the representative).*

**Proof sketch / audit trail**

In $\mathbb{F}_3$, $2 \equiv -1$ and $W(2v) = W(-v) = W(v)^{-1}$ (up to global phase, fixed by convention). Thus $\langle W(v) \rangle$ depends only on the projective class $\{v, -v\}$.

**Remark**

In the W33 tower, the 40 vertices are precisely the 40 projective points of $PG(3, 3)$. Thus W33 vertices can be read as 40 "Pauli directions" (cyclic order-3 Weyl subgroups) for two qutrits.

## 3 Isotropic lines as maximal commuting contexts

**Definition**

A 2D subspace $U \leq V$ is *totally isotropic* if $\langle u, u' \rangle = 0$ for all $u, u' \in U$. Its projectivization is a projective line containing 4 projective points.

**Theorem 3.1 (Isotropic lines give commuting Pauli contexts)** *If $U \leq V$ is a totally isotropic 2D subspace, then $\{W(u) : u \in U\}$ is an abelian subgroup of the 2-qutrit Weyl group of order $3^2 = 9$ (including identity). Equivalently, the 4 projective points on the line correspond to 4 nontrivial cyclic subgroups whose nontrivial elements pairwise commute.*

**Proof sketch / audit trail**

If $U$ is totally isotropic, then $\langle u, u' \rangle = 0$ for all $u, u' \in U$, so $W(u)$ commutes with $W(u')$ by Theorem **??**. Since $U \cong \mathbb{F}_3^2$, the set $\{W(u) : u \in U\}$ has 9 elements.

**Remark**

The symplectic generalized quadrangle $W(3, 3)$ consists precisely of 40 points and 40 totally isotropic projective lines. Thus the GQ lines are canonical maximal commuting Pauli contexts in the 2-qutrit Weyl group.

## 4 Non-isotropic lines as canonical phase cells

**Definition**

A projective line (2D subspace) $U$ is *non-isotropic* if $\langle \cdot, \cdot \rangle|_U$ is nondegenerate. In this case, there exist $u, u' \in U$ with $\langle u, u' \rangle = 1$, generating a Heisenberg pair.

**Theorem 4.1 (Non-isotropic lines contain conjugate pairs)** *Let $U \leq V$ be a non-isotropic 2D subspace. Then there exist $u, u' \in U$ such that $\langle u, u' \rangle = 1$, and hence*

$$W(u)\, W(u') = \omega\, W(u')\, W(u).$$

**Proof sketch / audit trail**

Nondegeneracy of $\langle\cdot,\cdot\rangle|_U$ implies there exists a basis with symplectic form matrix $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ on $U$. Choosing $u, u'$ as basis vectors yields $\langle u, u'\rangle = 1$.

**Remark**

In the W33 tower, $PG(3,3)$ has 130 lines total: 40 isotropic (GQ) and 90 non-isotropic. The "90" distinguished by the quotient holonomy are exactly these non-isotropic lines.

## 5 Clifford normalizer and the W33 automorphism action

**Theorem 5.1 (Clifford induces symplectic action)** *Let $\mathcal{C}$ denote the 2-qutrit Clifford group (normalizer of the Weyl group in $U(9)$). Then conjugation by any $U \in \mathcal{C}$ induces a linear transformation $M \in Sp(4,3)$ on phase space such that*

$$UW(v)U^\dagger \;=\; \omega^{\kappa(v)}\, W(Mv).$$

*Conversely, each $M \in Sp(4,3)$ is induced by some Clifford up to phase.*

**Proof sketch / audit trail**

Standard result for odd prime-power dimension: the Clifford group projects onto the symplectic group acting on discrete phase space, with kernel the Heisenberg–Weyl phases.

## 6 Holonomy equals commutator phase: a falsifiable conjecture

**Definition**

Define the symplectic "triangle phase" functional on three phase points $u, v, w \in V$ by

$$\Phi(u,v,w) := \langle u,v\rangle + \langle v,w\rangle + \langle w,u\rangle \;\in\; \mathbb{F}_3.$$

**Theorem 6.1 (Closed-loop phase identity)** *For any $u, v, w \in V$ with $u+v+w=0$, the triple Weyl product has the form*

$$W(u)\,W(v)\,W(w) \;=\; \omega^{\Phi(u,v,w)}\, I$$

*up to a global convention factor (which can be fixed by choosing standard displacement operators).*

**Proof sketch / audit trail**

Use the Weyl multiplication law and bilinearity: $W(u)W(v)$ equals a scalar times $W(u+v)$. If $u+v+w=0$, then $W(u+v)W(w)$ is scalar times identity. Exponents combine to the cyclic sum $\Phi$ (mod 3).

**Theorem 6.2 (Holonomy-phase conjecture (testable))** *Let $Q = \overline{\mathrm{W33}}$ be the 40-vertex quotient graph produced by the globally gauge-fixed signed lift, with each triangle $(p,q,r)$ assigned a* *holonomy value $F(p,q,r) \in \mathbb{Z}_3$ (identity vs 3-cycle orientation). There exists a projective representative assignment $p \mapsto [v_p] \in PG(3,3)$, and representative choices $v_p \in V$, such that for every triangle,*

$$F(p,q,r) \;\equiv\; \Phi(v_p, v_q, v_r) \pmod{3},$$

*up to the standard gauge ambiguity corresponding to adding a constant all-ones vector in the 90-line weight model.*

**Protocol (testable)**

**Protocol: verifying Theorem ??.**

1. Use the explicit projective representatives for the 40 points in $PG(3,3)$ (present in the symplectic audit bundle).
2. Compute $\Phi(v_p, v_q, v_r)$ for all 3240 triangles of $Q$.
3. Compare to the computed holonomy values (identity/3-cycle with orientation) on the same triangle list.
4. If a mismatch occurs only by a constant shift (global gauge), quotient out by the all-ones line and recompare.
5. If mismatches persist with nonconstant residuals, the conjecture fails and the representative assignment must be refined (or the holonomy is not a pure symplectic cocycle).

## Artifact Index (quantum layer)

| Bundle | Contents / Purpose |
|---|---|
| `W33_symplectic_audit_bundle.zip` | Projective representatives for $PG(3,3)$ points and all 130 lines; isotropic vs nonisotropic split. |
| `W33_quotient_closure_complement_and_noniso_lines_curvature_bundle.zip` | Quotient $Q = \overline{W33}$, edge matching, triangle holonomy values, and the proof that flat triangles equal nonisotropic line triples. |
| `W33_Z3_curvature_cohomology_on_quotient_bundle.zip` | $\mathbb{Z}_3$ curvature cochain on triangles and non-exactness on the 2-skeleton. |
| `W33_H3_basis_89_Z3_on_clique_complex_bundle.zip` | Explicit basis for $H^3$ over $\mathbb{Z}_3$ (flux lattice). |
| `W33_lift_to_90_line_weights_with_labels_bundle.zip` | Explicit lift from $H^3$ 88D core to a 90-entry nonisotropic line-weight field (mod all-ones gauge). |