A Finite-Geometric Theory Kernel from W33

Toward a Unified Algebra–Topology–Quantum Computation–Cryptography Framework

Wil Dahn & Sage

January 20, 2026

Abstract

This document consolidates the W33 tower into a single, self-contained theory kernel. Starting from the symplectic phase space $V = \mathbb{F}_3^4$, we construct the symplectic generalized quadrangle $W(3,3)$ and its point graph $\mathrm{W33} = \mathrm{SRG}(40, 12, 2, 4)$. Over $\mathbb{F}_2$, the adjacency satisfies $A^2 \equiv 0$, producing a canonical code $[40, 24, 6]$ and an intrinsic homology space $H = \ker(A)/\mathrm{im}(A) \cong \mathbb{F}_2^8$. The nonsingular orbit in H yields a 120-element "root shell" with $\mathrm{SRG}(120, 56, 28, 24)$ adjacency, a 240 signed lift admitting global gauge fixing, and a quotient closure back to 40 points as $Q = \overline{\mathrm{W33}}$. The quotient carries a canonical $\mathbb{Z}_3$ holonomy, with flat faces classified exactly by the 90 non-isotropic projective lines. Over $\mathbb{Z}_3$, the clique complex of Q has $H^3 \cong (\mathbb{Z}_3)^{89}$, whose 88D core is identified (up to a canonical sign character) with the augmentation quotient on the 90 non-isotropic lines. Finally, the holonomy field F is sourced: $J = dF$ is a 3-cochain supported on 3008 tetrahedra, and explicit sparse transfer operators map J to observed vacuum line responses.

Remark

What is meant by "theory of everything" here. This manuscript presents a mathematically closed kernel in which geometry, algebra, topology, computation, and cryptography are realized as different functorial views of the same finite symplectic/projective object. Claims about physical constants require an additional scaling/continuum layer and are not asserted as part of the kernel.

Contents

Master Equation Summary

Key Result

Discrete gauge kernel (minimal equations). Let $Q = \overline{\mathrm{W33}}$ be the quotient graph and $\mathrm{Cl}(Q)$ its clique complex.

Field strength (holonomy). $F \in C^2(\mathrm{Cl}(Q); \mathbb{Z}_3)$ is the computed triangle holonomy.

Sources. $J := dF \in C^3(\mathrm{Cl}(Q); \mathbb{Z}_3)$ is the sourced 3-cochain (supported on 3008 tetrahedra).

Vacuum response (exact constitutive laws). There exist explicit sparse operators

$$M, Z : \mathbb{Z}_3^{9450} \to \mathbb{Z}_3^{90}$$

such that the observed line fields satisfy

$$m_{\mathrm{line}} = MJ, \qquad z_{\mathrm{line}} = ZJ$$

exactly.

Vacuum harmonics. The 90-line sector admits five canonical joint modes under the involution S and meet adjacency A_{meet}:

$$(+, 32)^1,\ (+, 2)^{24},\ (+, -4)^{20},\ (-, 8)^{15},\ (-, -4)^{30}.$$

Bulk and boundary source classes inject into different harmonic mixtures (mode-response tables).

1 Master Equations and Couplings

Definition

Field variables. On the clique complex $\mathrm{Cl}(Q)$ of the quotient graph $Q = \overline{\mathrm{W33}}$:

- $F \in C^2(\mathrm{Cl}(Q); \mathbb{Z}_3)$ is the triangle holonomy field (field strength).
- $J := dF \in C^3(\mathrm{Cl}(Q); \mathbb{Z}_3)$ is the sourced 3-cochain (charge/current).

On the vacuum line set $\mathcal{L}$ (the 90 non-isotropic lines):

- $m_{\mathrm{line}} \in \mathbb{Z}_3^{90}$ is the *boundary moment* observable.
- $z_{\mathrm{line}} \in \mathbb{Z}_3^{90}$ is the *bulk shadow* observable.

Theorem 1.1 (Master operator equations) *The W33 kernel closes as the following exact operator pipeline over $\mathbb{Z}_3$:*

$$F \xrightarrow{d} J \xrightarrow{(M,Z)} (m_{\mathrm{line}}, z_{\mathrm{line}}),$$

where d is the simplicial coboundary on $\mathrm{Cl}(Q)$ and $M, Z : \mathbb{Z}_3^{9450} \to \mathbb{Z}_3^{90}$ are explicit sparse transfer operators. Concretely,

$$J = dF, \qquad m_{\mathrm{line}} = MJ, \qquad z_{\mathrm{line}} = ZJ,$$

and these identities hold entrywise with no residual error.

Proof sketch / audit trail

F and $J = dF$ are computed from the quotient holonomy. The operators M and Z are constructed canonically from incidence: M routes tetra flux to the unique vacuum line of the tetra's flat face (when present), while Z routes tetra flux to vacuum lines via edge-incidence of curved faces. Exactness was verified against independently computed line observables. (Audit bundle: `W33_transfer_operators_J_to_lines_and_mode_injection_bundle.zip`.)

Definition

Vacuum harmonics. Let S be the canonical involution on $\mathcal{L}$ (45 disjoint transpositions) and A_{meet} the meet adjacency on $\mathcal{L}$ (degree 32). The vacuum line sector decomposes into five joint modes:

$$(+, 32)^1,\ (+, 2)^{24},\ (+, -4)^{20},\ (-, 8)^{15},\ (-, -4)^{30}.$$

Theorem 1.2 (Coupling selection rules (mode response)) *Bulk sources (tetrahedra with zero flat faces) inject into z_{line} but not m_{line}. Boundary sources (tetrahedra with one flat face) inject into both m_{line} and z_{line}, with mode weights shifted toward $(+, 2)$ and $(-, 8)$ for m_{line}. These couplings are quantified by the mode-response tables.*

Proof sketch / audit trail

Apply M and Z to class-restricted source vectors and project the resulting 90-line fields into the five joint modes using the association-scheme harmonic bases. (Audit bundle: `W33_mode_response_table_bulk_to_vacuum_bundle.zip`.)

Key Result

The equations $J = dF$ and $(m, z) = (MJ, ZJ)$ are the minimal "field equations" of the kernel. Together with the five vacuum harmonics, they provide a complete, symmetry-respecting description of how sourced curvature produces observable vacuum response in the 90-line sector.

2 Closure Principle

Definition

Closure. We say the W33 tower is *closed* if the following hold simultaneously:

1. (**Lift**) The 240 minimal code generators project 2-to-1 onto a 120-element nonsingular orbit in H (the "root shell").
2. (**Gauge fix**) There exists a global sign section eliminating all weight-16 defects, producing 40 disjoint flat triples.
3. (**Collapse**) Collapsing the 40 triples yields a 40-vertex quotient graph Q with canonical edge transport and $\mathbb{Z}_3$ holonomy.
4. (**Recursion**) The quotient graph is exactly $Q = \overline{\mathrm{W33}}$.
5. (**Vacuum/matter coincidence**) The 90 non-isotropic lines simultaneously (i) classify flat holonomy faces and (ii) support the 88D core module of H^3 via the 90-line augmentation quotient.

Theorem 2.1 (Closure Theorem) *The W33 tower is closed in the above sense. In particular:*

1. *The globally gauge-fixed signed lift partitions the 120 roots into 40 flat triples.*
2. *The induced quotient is $Q = \overline{\mathrm{W33}}$ and carries canonical $\mathbb{Z}_3$ triangle holonomy.*
3. *Flat holonomy triangles are exactly the triples lying on the 90 non-isotropic lines of $PG(3,3)$.*
4. *The 88D core of $H^3(\mathrm{Cl}(Q);\mathbb{Z}_3)$ is (up to the similitude sign twist) the augmentation quotient on these same 90 non-isotropic lines.*

Proof sketch / audit trail

Items (1)–(3) are verified by the explicit gauge-fix computation and quotient construction: the defect-0 edges form 40 disjoint triangles partitioning the 120 roots, and the quotient adjacency equals the complement of W33 with a $\mathbb{Z}_3$ holonomy classified by non-isotropic line triples. Item (4) is established by comparing the 88D core module of H^3 with the 90-line augmentation quotient: after the canonical similitude sign twist, traces and characteristic-polynomial factor patterns match and an explicit intertwiner exists. (Audit bundles: `W33_global_gaugefix_no16_bundle.zip`, `W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip`, `W33_H3_Aut_action_89Z3_bundle.zip`, `W33_perm_module_vs_H3_match_report_bundle.zip`.)

Key Result

Closure is the central "TOE hinge" of the kernel: the same finite geometry simultaneously generates (i) constraints/codes ($A^2 = 0$ over $\mathbb{F}_2$), (ii) a root shell and gauge-fixed signed lift (120/240), (iii) a recursive quotient $Q = \overline{\mathrm{W33}}$ with $\mathbb{Z}_3$ holonomy, and (iv) a vacuum line sector that is also the carrier of the nontrivial 88D matter/flux module. This is precisely the structure needed for a self-contained theory kernel.

3 Functorial Field Theory View

Definition

Clique category. Let $Q = \overline{\mathrm{W33}}$ and $\mathrm{Cl}(Q)$ its clique (flag) complex. Define a small category $\mathcal{C}(Q)$ as follows:

- Objects are cliques $\sigma \subseteq V(Q)$ (equivalently simplices of $\mathrm{Cl}(Q)$), including vertices, edges, triangles, tetrahedra, etc.
- Morphisms are inclusions $\tau \hookrightarrow \sigma$ (face maps).

Thus $\mathcal{C}(Q)$ encodes the full incidence/facial structure of the quotient geometry.

Definition

Cochain functors. Fix a coefficient ring R (typically $R = \mathbb{Z}_3$). For each $k \geq 0$, define a functor

$$\mathsf{C}^k_R : \mathcal{C}(Q)^{\mathrm{op}} \to \mathsf{Mod}_R$$

by assigning to each k-simplex σ the free rank-one R-module generated by σ, and to each face inclusion the corresponding restriction map. The usual coboundary $d : \mathsf{C}^k_R \to \mathsf{C}^{k+1}_R$ is a natural transformation determined by alternating sums of face restrictions (with orientation conventions).

Definition

Vacuum line functor. Let $\mathcal{L}$ be the 90 non-isotropic lines in $PG(3,3)$, which are also the 90 flat K_4 cells in Q. Define the vacuum sector as the permutation module

$$\mathsf{V} := \mathbb{Z}_3^{\mathcal{L}},$$

together with its canonical 88D augmentation quotient $\mathsf{V}_{88} = \mathrm{Aug}(\mathcal{L})/\langle \mathbf{1} \rangle$ (up to the similitude sign twist).

Theorem 3.1 (Kernel as a functorial gauge system) *The W33 tower admits a functorial formulation in which geometry, topology, computation, and quantum structure are different functorial shadows of the same underlying incidence data:*

1. *(**Geometry→Topology**) The clique category $\mathcal{C}(Q)$ determines cochain functors $\mathsf{C}^k_{\mathbb{Z}_3}$ and a natural coboundary d. The holonomy field F is an element of $\mathsf{C}^2_{\mathbb{Z}_3}$ and the source field is $J = dF \in \mathsf{C}^3_{\mathbb{Z}_3}$.*

2. *(**Topology→Vacuum response**) The transfer operators M and Z are natural, Aut(W33)-equivariant linear maps from the tetra-source module to the vacuum module:*

$$M, Z : \mathbb{Z}_3^{\{tetrahedra\}} \to \mathbb{Z}_3^{\mathcal{L}},$$

giving exact observables $(m_{\mathrm{line}}, z_{\mathrm{line}}) = (MJ, ZJ)$.

3. *(**Computation**) Over $\mathbb{F}_2$, the W33 adjacency defines a square-zero differential on $\mathbb{F}_2^{40}$, yielding the intrinsic code $\ker(A)$ and homology $H = \ker(A)/\mathrm{im}(A)$; these are functorial with respect to the Aut(W33) action.*

4. (***Quantum***) *The phase space axiom* $V = \mathbb{F}_3^4$ *defines the 2-qutrit Weyl functor (Weyl labels and commutator phase) and a projectivized Clifford action by* $PGSp(4,3)$ *on* $PG(3,3)$*; isotropic lines correspond to maximal commuting contexts.*

Moreover, the representation-theoretic identification $H^3(\mathrm{Cl}(Q);\mathbb{Z}_3)_{88} \cong \mathsf{V}_{88}$ *(up to twist) provides an explicit equivalence between the flux-lattice core and the vacuum line module.*

Proof sketch / audit trail

Each item is backed by explicit constructions: (1) and (2) follow from the computed holonomy F, sources $J = dF$, and the sparse transfer operators M, Z built from incidence (Section 11 and associated bundles). (3) follows from the SRG identity implying A^2
*equiv*0 over
mathbbF$_2$ and the explicit kernel-code computation (Section 4). (4) follows from the standard Weyl/Clifford construction on V and the identification of W33 points/lines with projective points/isotropic lines in $PG(3,3)$ (Section 10). The module equivalence is established by comparing the Aut(W33) actions and constructing an explicit intertwiner (Section 9).

Key Result

This functorial view is the cleanest "TOE statement" available at the kernel level: a single finite incidence object induces, via natural functors, (i) a sourced gauge field (F, J), (ii) exact response laws (M, Z) into the vacuum line sector, (iii) an intrinsic error-correcting code over
mathbbF$_2$, and (iv) a 2-qutrit Weyl/Clifford quantum structure over
mathbbF$_3$. These are not separate theories but compatible projections of the same kernel.

4 Continuum and Scaling Layer (Optional Program)

Remark

Status. Everything in Sections 1–12 is a finite, exact kernel. This section is explicitly labeled optional: it proposes principled scaling routes that could connect the finite kernel to effective continuum physics, without asserting any numerical "constant matching" as part of the kernel.

4.1 Three natural scaling parameters

Definition

Scaling routes. The W33 kernel suggests three canonical families:

1. (**Field size**) Replace $\mathbb{F}_3$ by $\mathbb{F}_q$ and study $V = \mathbb{F}_q^4$ with symplectic form, yielding $W(3, q)$ and its point graph.
2. (**Rank**) Replace $V = \mathbb{F}_q^4$ by $V = \mathbb{F}_q^{2n}$, studying $W(2n-1, q)$ and the resulting tower as n grows.
3. (**Covers / coarse graining**) Use regular covers of the quotient connection (e.g., minimal regular covers of transport/holonomy data) as lattice refinements, and study renormalization via pushforward/pullback of cochains.

4.2 Spectral diagnostics on finite approximants

Definition

For a d-regular graph G with adjacency eigenvalues λ_i, the normalized Laplacian eigenvalues are

$$\mu_i = 1 - \frac{\lambda_i}{d}.$$

A standard continuum diagnostic is the heat kernel trace

$$P(t) := \frac{1}{|V(G)|} \sum_i e^{-t\mu_i},$$

whose intermediate-time scaling can be used to define an effective spectral dimension. On finite strongly symmetric graphs, $P(t)$ is often a small sum of exponentials, yielding a multi-scale (non-classical) behavior.

Theorem 4.1 (Exact normalized Laplacian spectra for the kernel graphs) *Let* W33 *be* $SRG(40, 12, 2, 4)$ *and* $Q = \overline{\mathrm{W33}}$ *its quotient graph (degree 27). Then the normalized Laplacian spectrum of* Q *is*

$$0^{(1)}, \qquad \left(\frac{8}{9}\right)^{(15)}, \qquad \left(\frac{10}{9}\right)^{(24)}.$$

Let A_{meet} *be the meet adjacency on the 90 non-isotropic lines (degree 32). Then its normalized Laplacian spectrum is*

$$0^{(1)}, \qquad \left(\frac{3}{4}\right)^{(15)}, \qquad \left(\frac{15}{16}\right)^{(24)}, \qquad \left(\frac{9}{8}\right)^{(50)}.$$

Proof sketch / audit trail

The adjacency eigenvalues of Q follow from SRG complement eigenvalue relations: if W33 has eigenvalues $12, 2, -4$ with multiplicities $1, 24, 15$, then Q has eigenvalues $27, -3, 3$ with multiplicities $1, 24, 15$. The normalized Laplacian eigenvalues are $1 - \lambda/27$. The meet-graph eigenvalues were computed in the association scheme analysis: $32^{(1)}, 8^{(15)}, 2^{(24)}, (-4)^{(50)}$, yielding normalized Laplacian eigenvalues $1 - \lambda/32$.

Remark

Interpretation. These spectra show the kernel graphs are "two/three-scale" rather than approximations of a smooth manifold in the naive sense: the heat kernel trace is a small mixture of exponentials. In a scaling program, one expects richer spectra to emerge only when the kernel is embedded into a family (e.g., varying q, increasing rank, or taking covers), and the vacuum harmonics (Section 12) provide the correct basis for coarse-grained dynamics.

4.3 Renormalization as module projection

Definition

Mode-space coarse graining. The vacuum association scheme decomposes $\mathbb{Z}_3^{90}$ into five canonical harmonic subspaces (Section 12). A natural renormalization step is projection onto a selected subset of these modes (or onto the 88D core module), followed by rescaling of the transfer operators (M, Z) and the sourced field $J = dF$.

Protocol (testable)

Program (testable).

1. Choose a scaling family (field size q, rank n, or covers).
2. For each instance, compute: (i) closure/gauge fix, (ii) quotient Q, (iii) holonomy F, (iv) sources $J = dF$, (v) transfer operators (M, Z), (vi) vacuum association scheme and mode decomposition.
3. Track invariants across scale: H^3 dimension, module decompositions (e.g., 88+1 analogs), and spectral signatures of meet graphs.
4. Identify fixed points in the induced operator calculus (e.g., stable ratios of mode injection weights under coarse graining).

Key Result

The kernel already provides the correct *renormalization coordinates*: vacuum harmonics (five modes) and the 88D core module. A genuine continuum limit, if it exists, should be formulated as stability of these module-level observables across a scaling family (not as ad hoc constant matching).

5 Axioms and kernel construction chain

Definition

Axiom A0 (Phase space). Let $V = \mathbb{F}_3^4$ equipped with a fixed nondegenerate alternating (symplectic) form ω.

Axiom A1 (Isotropy geometry). Let $W(3, 3)$ denote the symplectic generalized quadrangle realized by totally isotropic points and lines in $PG(3, 3)$ with respect to ω.

Axiom A2 (Point graph). Let W33 be the point graph of $W(3, 3)$: vertices are the 40 isotropic points, and edges represent collinearity.

Remark

These axioms fix the entire tower. Everything below is forced from the adjacency matrix A of W33, its induced actions, and the canonical quotients and lifts defined from it.

Key Result

The W33 tower can be viewed as a closed pipeline:

$$\begin{aligned}\mathbb{F}_3^4 \;\Rightarrow\; W(3,3) \;\Rightarrow\; \text{W33} \;\Rightarrow\; (A^2 \equiv 0 \text{ over } \mathbb{F}_2) \;\Rightarrow\; H \\ \Rightarrow\; (120, 240) \text{ signed roots} \;\Rightarrow\; Q = \overline{\text{W33}} \;\Rightarrow\; (\mathbb{Z}_3 \text{ holonomy}) \\ \Rightarrow\; H^3(\text{Cl}(Q); \mathbb{Z}_3) \cong (\mathbb{Z}_3)^{89} \;\Rightarrow\; \text{90-line field model.}\end{aligned}$$

6 Master theorems and dictionary

Theorem 6.1 (Master Theorem I: square-zero differential and code) *Over $\mathbb{F}_2$, the adjacency matrix A of W33 satisfies $A^2 \equiv 0$. Hence $d(x) = Ax$ defines a differential on $\mathbb{F}_2^{40}$, producing a canonical code $C = \ker(A)$ with parameters $[40, 24, 6]$ and a homology state space $H = \ker(A)/\mathrm{im}(A) \cong \mathbb{F}_2^8$.*

Theorem 6.2 (Master Theorem II: 120-root shell and 240 signed lift) *The induced action on H preserves a quadratic form of minus type. The nonsingular orbit has size 120 and carries $\mathrm{SRG}(120, 56, 28, 24)$ adjacency via the associated bilinear form. The 240 canonical weight-6 generators project 2-to-1 onto this 120-set, yielding a signed lift with a defect cocycle valued in $\mathrm{im}(A)$.*

Theorem 6.3 (Master Theorem III: quotient closure and $\mathbb{Z}_3$ connection) *There exists a global gauge fix eliminating all weight-16 defects. In that gauge, the 120 roots partition into 40 flat triples (one per W33 point). Collapsing these triples yields a quotient graph Q equal to the complement $\overline{\text{W33}}$, equipped with a canonical edge transport rule whose triangle holonomy lies in $\mathbb{Z}_3$. Flat holonomy triangles are classified exactly by the 90 non-isotropic projective lines in $PG(3,3)$.*

Theorem 6.4 (Master Theorem IV: sourced curvature and transfer operators) *Let $F \in C^2(\text{Cl}(Q); \mathbb{Z}_3)$ be the triangle holonomy field and $J = dF \in C^3(\text{Cl}(Q); \mathbb{Z}_3)$ its source. Then J is supported on exactly 3008 tetrahedra. There exist explicit sparse operators*

$$M, Z : \mathbb{Z}_3^{9450} \to \mathbb{Z}_3^{90}$$

such that the observed vacuum line fields satisfy the exact identities $m_{\text{line}} = MJ$ and $z_{\text{line}} = ZJ$. Vacuum responses decompose into five canonical harmonics determined by the Aut-invariant 90-line association scheme.

Definition

Dictionary (high level). Within the exact finite theory:

- **Geometry**: isotropic vs non-isotropic incidence in $PG(3,3)$; the graphs W33 and $Q = \overline{\mathrm{W33}}$.
- **Algebra**: Aut(W33) actions and induced modules on H, the 120-root shell, the 90-line sector, and H^3.
- **Topology**: cochains/coboundaries on $\mathrm{Cl}(Q)$; $J = dF$ as sources; H^3 as flux lattice.
- **Quantum computation**: Weyl/Clifford realization on V; contexts from isotropic lines; holonomy as discrete phase transport.
- **Cryptography**: gauge/coset ambiguity and large symmetry action as secrecy; error correction as intrinsic stability (the $[40, 24, 6]$ code).

3 The W33 Object

Definition

Let $V = \mathbb{F}_3^4$ equipped with a nondegenerate alternating (symplectic) form ω. Let $W(3,3)$ denote the symplectic generalized quadrangle arising from totally isotropic points and lines in $PG(3,3)$ with respect to ω. The *W33 point graph* is the graph whose vertices are the 40 isotropic points and whose edges connect collinear pairs (i.e., pairs lying on a common isotropic line). We denote its adjacency matrix by A and the graph by W33.

Theorem 3.1 (SRG parameters) W33 *is a strongly regular graph with parameters*

$$(v, k, \lambda, \mu) = (40, 12, 2, 4).$$

Equivalently, each vertex has degree 12; adjacent pairs have exactly 2 common neighbors; non-adjacent pairs have exactly 4 common neighbors.

Proof sketch / audit trail

This is a standard property of the point graph of the symplectic generalized quadrangle $W(3,3)$. It was also verified computationally by explicit incidence construction of $W(3,3)$ and counting common neighbors in the point graph (audit bundle: `W33_symplectic_audit_bundle.zip`).

Theorem 3.2 (Adjacency spectrum) *The adjacency spectrum of* W33 *is*

$$\mathrm{spec}(A) = 12^{(1)}, \quad 2^{(24)}, \quad (-4)^{(15)}.$$

Equivalently, the characteristic polynomial is

$$P(x) = (x - 12)(x - 2)^{24}(x + 4)^{15}.$$

Proof sketch / audit trail

For SRG(v, k, λ, μ), the nontrivial eigenvalues are roots of a quadratic determined by (k, λ, μ), with multiplicities forced by trace identities. Here this yields eigenvalues 2 and -4 with multiplicities 24 and 15. Verified directly by eigen-computation on the explicit adjacency matrix (audit bundle: `W33_symplectic_audit_bundle.zip`).

Theorem 3.3 (Automorphism group order) $|\mathrm{Aut}(\mathrm{W33})| = 51840$.

Proof sketch / audit trail

In the symplectic model, Aut(W33) is realized as the projective symplectic similitude group acting on isotropic points. A concrete generating set (symplectic transvections, a block-swap, and a multiplier-2 similitude) was used to generate the full permutation group on the 40 vertices, yielding order 51840. (Audit bundle: `W33_orbits_squarezero_bundle.zip`.)

Key Result

The W33 point graph is not merely a convenient combinatorial object; it is the *canonical* SRG arising from the symplectic quadrangle $W(3,3)$. The entire tower below is forced from $(40, 12, 2, 4)$ together with the induced group action.

4 Differential Structure over $\mathbb{F}_2$

Theorem 4.1 (Square-zero adjacency over $\mathbb{F}_2$) *Let A be the adjacency matrix of* W33. *Over $\mathbb{F}_2$, one has*

$$A^2 \equiv 0 \pmod 2.$$

Proof sketch / audit trail

For any SRG(v, k, λ, μ) with adjacency A and all-ones matrix J,

$$A^2 = kI + \lambda A + \mu(J - I - A).$$

Plugging $(k, \lambda, \mu) = (12, 2, 4)$ yields $A^2 = 8I - 2A + 4J$. Reducing mod 2 gives $A^2 \equiv 0$. Verified directly by matrix multiplication mod 2 in the audit bundle.

Definition

Define a differential $d : \mathbb{F}_2^{40} \to \mathbb{F}_2^{40}$ by $d(x) = Ax \pmod 2$. Since $d^2 = 0$, we can form:

$$C := \ker(d) \subset \mathbb{F}_2^{40}, \qquad H := \ker(d)/\mathrm{im}(d).$$

Theorem 4.2 (Dimensions) *Over $\mathbb{F}_2$,*

$$\mathrm{rank}(A) = 16, \qquad \dim\ker(A) = 24, \qquad \dim H = 8.$$

Proof sketch / audit trail

Rank was computed by mod-2 row reduction on the explicit 40×40 adjacency matrix. Nullity follows by rank-nullity. Since $\mathrm{im}(A) \subseteq \ker(A)$ (square-zero), $\dim H = \dim\ker(A) - \dim\mathrm{im}(A) = 24 - 16 = 8$.

Theorem 4.3 (Canonical local generators and code distance) *The kernel $C = \ker(A) \subset \mathbb{F}_2^{40}$ is a $[40, 24, 6]$ linear code. Moreover, there are exactly 240 canonical weight-6 codewords obtained as XORs of pairs of isotropic lines through a common point, and these 240 codewords generate C.*

Proof sketch / audit trail

Each point lies on 4 isotropic lines; choosing 2 lines yields $\binom{4}{2} = 6$ line-pairs per point, hence $40\cdot 6 = 240$ codewords. Each is weight 6 and lies in $\ker(A)$; exhaustive search up to weight 5 found none in $\ker(A)$, so $d_{\min} = 6$. A row-reduced basis extracted from the 240 generators spans a 24-dimensional space, matching $\dim\ker(A)$. (Audit bundle: `W33_GF2_kernel_code_bundle.zip`.)

Key Result

The identity $A^2 \equiv 0$ is the first "TOE hinge": it turns a finite SRG into a genuine chain complex, producing (i) a stabilizer-like code and (ii) an 8-dimensional homology state space H.

5 Orthogonal Geometry on H and the 120-Root Structure

Theorem 5.1 (Quadratic form and orbit split) *The induced action of* $\mathrm{Aut}(\mathrm{W33})$ *on H preserves a nontrivial quadratic form $q : H \to \mathbb{F}_2$ of minus type. Consequently, the nonzero vectors in H split into exactly two orbits:*

$$\{x \in H \setminus \{0\} : q(x) = 0\} \text{ of size } 135, \qquad \{x \in H \setminus \{0\} : q(x) = 1\} \text{ of size } 120.$$

Proof sketch / audit trail

A concrete basis of H was chosen by splitting $\ker(A) = \mathrm{im}(A) \oplus K$ with $\dim K = 8$. The group action on points induces an action on H, from which an invariant quadratic polynomial of degree 2 was solved. Enumerating values of q gives the $(135, 120)$ split, and orbit computation confirms exactly two nonzero orbits. (Audit bundle: `W33_H8_quadratic_form_bundle.zip`.)

Theorem 5.2 (240 → 120 projection) *Projecting the 240 canonical weight-6 code generators (Theorem* **??***) from $\ker(A)$ to $H = \ker(A)/\mathrm{im}(A)$ yields exactly 120 distinct nonzero elements, each appearing with multiplicity 2. All 120 satisfy $q = 1$ (the nonsingular orbit).*

Proof sketch / audit trail

Each of the 240 generators was mapped to an 8-bit H coordinate; 120 distinct values occur, each exactly twice. All map to the $q = 1$ orbit. (Audit bundle: `W33_to_H_to_120root_SRG_bundle.zip` and `W33_root_preimage_pairing_bundle.zip`.)

Definition

Define the associated bilinear form

$$b(x, y) = q(x + y) + q(x) + q(y) \in \mathbb{F}_2.$$

On the 120-element nonsingular orbit, define adjacency by $b(x, y) = 1$.

Theorem 5.3 (The 120-root SRG) *The graph on the 120 nonsingular elements with adjacency $b = 1$ is strongly regular:*

$$\mathrm{SRG}(120, 56, 28, 24).$$

Proof sketch / audit trail

Adjacency counts were computed directly from the bilinear form on the explicit 120-root list; all vertices have degree 56, adjacent pairs have 28 common neighbors, and nonadjacent pairs have 24. (Audit bundle: `W33_to_H_to_120root_SRG_bundle.zip`.)

Theorem 5.4 (An E_8 Dynkin subgraph and reflection generation) *Inside* $\mathrm{SRG}(120, 56, 28, 24)$ *there exists an induced subgraph isomorphic to the E_8 Dynkin diagram. The corresponding 8 nonsingular elements $\{r_i\}$ define involutions*

$$s_r(x) = x + b(x, r)\, r,$$

and the group generated by these involutions acts transitively on the 120-root set.

Proof sketch / audit trail

An induced E_8 configuration was found and canonically chosen (lexicographically minimal under a fixed branching constraint). Coxeter relations were verified on H (order 3 on adjacent nodes, order 2 otherwise), and orbit generation under reflections yields the full 120-root orbit. (Audit bundle: `W33_E8_simple_root_system_bundle.zip`.)

Key Result

The nonsingular orbit of the intrinsic homology H behaves as a finite "root shell" with $\mathrm{SRG}(120, 56, 28, 24)$ adjacency and an embedded E_8 Dynkin skeleton. This is the precise point where Lie-type structure emerges from the W33 tower.

6 Signed Lift, Cocycle, and Global Gauge Fixing

Definition

Each of the 120 roots has two preimages among the 240 generators. A *section* s selects one lift for each root. For adjacent roots h_1, h_2 (so $b(h_1, h_2) = 1$), define $h_3 = h_1 \oplus h_2$ and the defect (cocycle candidate)

$$g(h_1, h_2) := s(h_1) + s(h_2) + s(h_3) \;\in\; \mathrm{im}(A) \subset \mathbb{F}_2^{40},$$

where addition is XOR of the corresponding 40-bit supports.

Theorem 6.1 (Two-weight defect) *For the canonical section (choosing the smaller preimage index), the defect $g(h_1, h_2)$ takes only two Hamming weights:*

$$|g(h_1, h_2)| \in \{12, 16\}.$$

Across all 3360 edges of $\mathrm{SRG}(120, 56, 28, 24)$*, weight 12 occurs 1560 times and weight 16 occurs 1800 times.*

Proof sketch / audit trail

Computed exhaustively over all edges using the explicit 240 generator supports and the canonical section. Verified that $g(h_1, h_2)$ always projects to 0 in H, hence lies in $\mathrm{im}(A)$. (Audit bundle: `W33_signed_root_cocycle_and_lift_bundle.zip`.)

Theorem 6.2 (Steiner triples) *Edges of* $\mathrm{SRG}(120, 56, 28, 24)$ *partition into 1120 Steiner triples* $\{a, b, a \oplus b\}$*, and for a fixed section s, the defect value is constant on the three edges of each triple.*

Proof sketch / audit trail

If $b(a, b) = 1$ then $q(a \oplus b) = 1$; hence $a \oplus b$ is again a root. Each edge (a, b) has a unique third root $a \oplus b$, and the unordered triple partitions edges into 1120 groups. The defect $s(a) + s(b) + s(a \oplus b)$ is symmetric in $(a, b, a \oplus b)$, hence constant on the triple edges. Verified by enumeration.

Theorem 6.3 (Global gauge fix (no-16)) *There exists a global choice of signs (i.e., a section s selecting one of the two lifts at every root) such that all defects of weight 16 are eliminated. In this gauge-fixed section, all edge defects have weight in* $\{0, 12\}$*, with exactly 120 edges of weight 0 and 3240 edges of weight 12.*

Proof sketch / audit trail

A greedy local-flip optimization over the 120 root vertices (flipping lift choice at a vertex updates the defects on incident edges) yields a configuration with no 16-weight defects. This configuration was reproduced across random restarts. (Audit bundle: `W33_global_gaugefix_no16_bundle.zip`.)

Theorem 6.4 (40 flat triples) *The 120 roots partition into 40 disjoint triples (one per original W33 point) such that exactly those 40 triples have defect weight 0 under the globally gauge-fixed section. Equivalently, the 120 weight-0 edges form 40 disjoint triangles that partition the root set.*

Proof sketch / audit trail

From the gauge-fixed edge list, the weight-0 edges were found to group into 40 triangles. Each triangle's three vertices share the same base point in the original 40-point geometry, yielding a partition of the 120 roots into 40 fibers of size 3. (Audit bundle: `W33_global_gaugefix_no16_bundle.zip`.)

7 Quotient Closure and $\mathbb{Z}_3$ Holonomy

Definition

Collapse each of the 40 flat triples (Theorem **??**) to a meta-vertex labeled by its base point $p \in \{0, \ldots, 39\}$. Define the quotient graph Q on these 40 meta-vertices by connecting $p \neq q$ if there exists a defect-12 edge between the fibers over p and q.

Theorem 7.1 (Quotient graph is the complement) *The quotient graph Q is regular of degree 27 on 40 vertices and is exactly the complement of the original W33 point graph:*

$$Q = \overline{\mathrm{W33}}.$$

Proof sketch / audit trail

For each pair of base points (p, q), the number of defect-12 edges between the 3-element fibers is either 0 or 6. Adjacency in Q occurs exactly for multiplicity 6. The resulting 40-vertex graph is 27-regular; direct comparison of neighbor sets confirms Q equals the complement of the W33 adjacency. (Audit bundle: `W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip`.)

Theorem 7.2 (Edge decoration is a 6-cycle) *For every edge $p \sim q$ in Q, the induced bipartite graph between the 3 roots over p and the 3 roots over q has exactly 6 edges and is 2-regular on each side. Equivalently, it is $K_{3,3}$ minus a perfect matching, i.e. a 6-cycle. The missing perfect matching defines a canonical transport bijection between the two 3-element fibers.*

Proof sketch / audit trail

Verified by explicit enumeration for all 540 quotient edges: the 3×3 adjacency matrix always has three zeros (a perfect matching) and six ones, with row and column sums all equal to 2. Connectivity check confirms a single 6-cycle.

Definition

Define the holonomy of a quotient triangle (p, q, r) as the permutation of the fiber over p obtained by composing the three transport bijections along $p \to q \to r \to p$. This holonomy lies in $A_3 \cong \mathbb{Z}_3$.

Theorem 7.3 (90 non-isotropic lines classify flat holonomy) *Among the 3240 triangles of Q, exactly 360 have identity holonomy and 2880 have 3-cycle holonomy. Moreover, the identity-holonomy triangles are* exactly *the triples of points lying on the 90 non-isotropic projective lines in $PG(3,3)$ (each such line contains 4 points and contributes $\binom{4}{3} = 4$ triples, hence $90 \cdot 4 = 360$).*

Proof sketch / audit trail

Holonomy was computed for all quotient triangles from the edge matchings. Independently, all non-isotropic lines in $PG(3,3)$ were enumerated (90 lines), and the set of their 3-subsets was computed (360 triples). These match exactly the identity-holonomy triangle set. (Audit bundle: `W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip`.)

Key Result

The W33 tower closes: after global gauge fixing and collapsing flat triples, the induced 40-vertex quotient is $\overline{\mathrm{W33}}$ with a canonical $\mathbb{Z}_3$ connection. The set of flat faces is classified precisely by the 90 non-isotropic projective lines in $PG(3,3)$.

Artifact Index (computational)

`Bundle`	**Contents / Purpose**

8 Cohomology and flux lattice (summary of computed results)

Theorem 8.1 (Clique-complex cohomology over $\mathbb{Z}_3$) *Let $\mathrm{Cl}(Q)$ be the clique complex of $Q = \overline{\mathrm{W33}}$. Over $\mathbb{Z}_3$, its cohomology dimensions are:*

$$H^0 = 1, \quad H^1 = 0, \quad H^2 = 0, \quad H^3 = 89, \quad H^4 = 1, \quad H^5 = 0, \quad H^6 = 1.$$

In particular, the flux lattice is $H^3(\mathrm{Cl}(Q); \mathbb{Z}_3) \cong (\mathbb{Z}_3)^{89}$, and an explicit 89-element basis can be constructed.

Remark

The vanishing $H^2 = 0$ on the full clique complex explains why 2-skeleton obstructions disappear once tetrahedra are included: closed 2-forms are exact in the full flag complex, while the physically relevant sourced curvature is encoded by $J = dF$ (a 3-cochain).

9 Representation theory of the flux lattice and the 90-line module

Definition

Let $Q = \overline{\mathrm{W33}}$ be the 40-vertex quotient graph and $\mathrm{Cl}(Q)$ its clique (flag) complex. The flux lattice is

$$H^3(\mathrm{Cl}(Q); \mathbb{Z}_3) \cong (\mathbb{Z}_3)^{89}.$$

The $\mathrm{Aut}(\mathrm{W33})$ action on the 40 base points induces an action on all cliques of Q and hence on cochains, coboundaries, and cohomology.

Theorem 9.1 (An explicit basis for H^3) *There exists an explicit basis of 89 cocycles in $C^3(\mathrm{Cl}(Q); \mathbb{Z}_3)$ representing a basis of $H^3(\mathrm{Cl}(Q); \mathbb{Z}_3)$. Each basis element is given in sparse form as a $\mathbb{Z}_3$-valued cochain supported on tetrahedra (K_4 cliques) of Q.*

Proof sketch / audit trail

We compute $\ker(\delta_3) \subset C^3$ from the K_5 constraints and quotient by $\mathrm{im}(\delta_2)$ coming from triangles. In free coordinates for $\ker(\delta_3)$, the image of δ_2 has rank 2739, leaving dimension 89. We select 89 nonpivot free coordinates and back-substitute to construct cocycles. (Audit bundle: `W33_H3_basis_89_Z3_on_clique_complex_bundle.zip`.)

Theorem 9.2 (88+1 module structure and similitude character) *The 89-dimensional $\mathbb{Z}_3$-module $H^3(\mathrm{Cl}(Q);\mathbb{Z}_3)$ admits an invariant 88-dimensional submodule W_{88} such that the quotient is 1-dimensional. The 1-dimensional quotient carries the canonical "similitude sign" character: an index-2 subgroup acts trivially, while a distinguished multiplier-2 element acts by $-1 \equiv 2 \pmod 3$.*

Proof sketch / audit trail

Using the explicit Aut(W33) generators on points, we compute the induced action on tetrahedra, incorporate the orientation sign for 3-cochains, and build the resulting 89×89 matrices over $\mathbb{Z}_3$ on the computed H^3 basis. Empirically, the module has an invariant 88D submodule and a 1D quotient; the quotient character is detected by a dual functional w transforming by ± 1. (Audit bundle: `W33_H3_Aut_action_89Z3_bundle.zip`.)

Definition

Let $\mathcal{L}$ be the set of 90 non-isotropic projective lines in $PG(3,3)$. Consider the permutation module $\mathbb{Z}_3^{\mathcal{L}}$ and its augmentation submodule

$$\mathrm{Aug}(\mathcal{L}) := \Big\{ x \in \mathbb{Z}_3^{\mathcal{L}} : \sum_{\ell\in\mathcal{L}} x_\ell = 0 \Big\}.$$

Since $90 \equiv 0 \pmod 3$, the all-ones vector lies in $\mathrm{Aug}(\mathcal{L})$; quotienting by this trivial line yields an 88D module.

Theorem 9.3 (Geometric identification with 90-line augmentation quotient) *The 88D core module W_{88} is isomorphic (up to the similitude sign twist) to the augmentation quotient of the 90-line permutation module:*

$$W_{88} \;\cong\; \mathrm{Aug}(\mathcal{L})/\langle \mathbf{1}\rangle \;\otimes\; \chi,$$

where χ is the 1D similitude sign character. Moreover, an explicit intertwiner T between these modules can be computed.

Proof sketch / audit trail

We compute the Aut(W33) action on 90 non-isotropic lines, form the augmentation quotient, and compare with the H^3 88D core via traces and characteristic polynomial factor patterns. After twisting by the similitude sign (multiplying the multiplier-2 generator by -1), the modules match; an explicit 88×88 intertwiner T is constructed. (Audit bundles: `W33_perm_module_vs_H3_match_report_bundle.zip`, `W33_H3_to_noniso_line_weights_intertwiner_bundle.zip`.)

Theorem 9.4 (Explicit lift to labeled 90-line weights) *There is an explicit linear lift from 88D core coordinates to a labeled 90-entry non-isotropic line field (defined up to adding a constant all-ones vector). Concretely, there exists a 90×88 matrix $M_{H3\to 90}$ over $\mathbb{Z}_3$ such that*

$$w_{90} \equiv M_{H3\to 90}\, x_{88} \pmod{\langle \mathbf{1}\rangle},$$

and the 90 coordinates are indexed by the 4-point line-sets in $\mathcal{L}$.

Proof sketch / audit trail

A section $L_{88\to90}$ of the augmentation quotient is constructed and composed with the 88D intertwiner T to yield $M_{H3\to90}$. The resulting 90-vector is unique up to addition of a constant, reflecting the quotient by $\langle \mathbf{1} \rangle$. Line labeling is provided by the explicit 90 line list. (Audit bundle: `W33_lift_to_90_line_weights_with_labels_bundle.zip`.)

Key Result

This section fixes the representation-theoretic meaning of the flux lattice: the nontrivial 88D core of H^3 is (up to the canonical similitude sign) the augmentation quotient on the 90 non-isotropic lines. In particular, the "vacuum cells" that classify flat holonomy also carry the matter/flux degrees of freedom.

10 2-qutrit Weyl operators and the symplectic commutator

Definition

Let $\omega := e^{2\pi i/3}$. On $\mathbb{C}^3$ with computational basis $\{|j\rangle : j \in \mathbb{Z}_3\}$ define

$$X|j\rangle = |j+1\rangle, \qquad Z|j\rangle = \omega^j |j\rangle,$$

so that $ZX = \omega XZ$. On two qutrits, for $(a,b,c,d) \in \mathbb{F}_3^4$, define the (unnormalized) Weyl operator

$$W(a,b,c,d) := X^a Z^c \;\otimes\; X^b Z^d.$$

Definition

Define the standard symplectic form on $V = \mathbb{F}_3^{2n}$ with $n = 2$ by writing $v = (p \mid q)$ with $p, q \in \mathbb{F}_3^2$ and

$$\langle (p \mid q), (p' \mid q') \rangle := p \cdot q' - q \cdot p' \;\in\; \mathbb{F}_3.$$

In coordinates $v = (a,b,c,d)$ and $w = (a',b',c',d')$, this is

$$\langle v, w \rangle = ac' + bd' - ca' - db'.$$

Theorem 10.1 (Weyl commutator phase) *For all $v, w \in \mathbb{F}_3^4$,*

$$W(v)\,W(w) = \omega^{\langle v,w \rangle}\, W(w)\,W(v).$$

Equivalently, $W(v)$ and $W(w)$ commute if and only if $\langle v, w \rangle = 0$.

Proof sketch / audit trail

This is the standard Heisenberg–Weyl relation for odd prime dimension. For the above unnormalized convention, it follows from $ZX = \omega XZ$ on each tensor factor and bilinearity of the commutator exponent.

Key Result

The same symplectic form used to build $W(3,3)$ is exactly the commutator phase form in the 2-qutrit Weyl group. This is the first canonical bridge from W33 geometry to quantum operator algebra.

11 Projective points as Weyl directions

Definition

Let $\mathbb{P}(V) = PG(3,3)$ denote projective 1D subspaces of $V = \mathbb{F}_3^4$. A projective point $[v]$ is the equivalence class $\{v, 2v\}$ for any nonzero $v \in V$.

Theorem 11.1 (Projective points correspond to cyclic Weyl subgroups) *Each projective point $[v] \in PG(3,3)$ determines a cyclic order-3 Weyl subgroup*

$$\langle W(v)\rangle = \{I,\, W(v),\, W(2v)\}.$$

Moreover, $W(2v) = W(v)^{-1}$ and the subgroup depends only on $[v]$ (not the representative).

Proof sketch / audit trail

In $\mathbb{F}_3$, $2 \equiv -1$ and $W(2v) = W(-v) = W(v)^{-1}$ (up to global phase, fixed by convention). Thus $\langle W(v)\rangle$ depends only on the projective class $\{v, -v\}$.

Remark

In the W33 tower, the 40 vertices are precisely the 40 projective points of $PG(3,3)$. Thus W33 vertices can be read as 40 “Pauli directions” (cyclic order-3 Weyl subgroups) for two qutrits.

12 Isotropic lines as maximal commuting contexts

Definition

A 2D subspace $U \leq V$ is *totally isotropic* if $\langle u, u'\rangle = 0$ for all $u, u' \in U$. Its projectivization is a projective line containing 4 projective points.

Theorem 12.1 (Isotropic lines give commuting Pauli contexts) *If $U \leq V$ is a totally isotropic 2D subspace, then $\{W(u) : u \in U\}$ is an abelian subgroup of the 2-qutrit Weyl group of order $3^2 = 9$ (including identity). Equivalently, the 4 projective points on the line correspond to 4 nontrivial cyclic subgroups whose nontrivial elements pairwise commute.*

Proof sketch / audit trail

If U is totally isotropic, then $\langle u, u'\rangle = 0$ for all $u, u' \in U$, so $W(u)$ commutes with $W(u')$ by Theorem **??**. Since $U \cong \mathbb{F}_3^2$, the set $\{W(u) : u \in U\}$ has 9 elements.

Remark

The symplectic generalized quadrangle $W(3,3)$ consists precisely of 40 points and 40 totally isotropic projective lines. Thus the GQ lines are canonical maximal commuting Pauli contexts in the 2-qutrit Weyl group.

13 Non-isotropic lines as canonical phase cells

Definition

A projective line (2D subspace) U is *non-isotropic* if $\langle\cdot,\cdot\rangle|_U$ is nondegenerate. In this case, there exist $u, u' \in U$ with $\langle u, u'\rangle = 1$, generating a Heisenberg pair.

Theorem 13.1 (Non-isotropic lines contain conjugate pairs) *Let $U \leq V$ be a non-isotropic 2D subspace. Then there exist $u, u' \in U$ such that $\langle u, u'\rangle = 1$, and hence*

$$W(u)\,W(u') = \omega\,W(u')\,W(u).$$

Proof sketch / audit trail

Nondegeneracy of $\langle\cdot,\cdot\rangle|_U$ implies there exists a basis with symplectic form matrix $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ on U. Choosing u, u' as basis vectors yields $\langle u, u'\rangle = 1$.

Remark

In the W33 tower, $PG(3,3)$ has 130 lines total: 40 isotropic (GQ) and 90 non-isotropic. The "90" distinguished by the quotient holonomy are exactly these non-isotropic lines.

14 Clifford normalizer and the W33 automorphism action

Theorem 14.1 (Clifford induces symplectic action) *Let $\mathcal{C}$ denote the 2-qutrit Clifford group (normalizer of the Weyl group in $U(9)$). Then conjugation by any $U \in \mathcal{C}$ induces a linear transformation $M \in Sp(4,3)$ on phase space such that*

$$UW(v)U^{\dagger} \;=\; \omega^{\kappa(v)}\,W(Mv).$$

Conversely, each $M \in Sp(4,3)$ is induced by some Clifford up to phase.

Proof sketch / audit trail

Standard result for odd prime-power dimension: the Clifford group projects onto the symplectic group acting on discrete phase space, with kernel the Heisenberg–Weyl phases.

15 Holonomy equals commutator phase: a falsifiable conjecture

Definition

Define the symplectic "triangle phase" functional on three phase points $u, v, w \in V$ by

$$\Phi(u, v, w) := \langle u, v\rangle + \langle v, w\rangle + \langle w, u\rangle \ \in \ \mathbb{F}_3.$$

Theorem 15.1 (Closed-loop phase identity) *For any $u, v, w \in V$ with $u+v+w = 0$, the triple Weyl product has the form*

$$W(u)\, W(v)\, W(w) \ = \ \omega^{\Phi(u,v,w)}\, I$$

up to a global convention factor (which can be fixed by choosing standard displacement operators).

Proof sketch / audit trail

Use the Weyl multiplication law and bilinearity: $W(u)W(v)$ equals a scalar times $W(u+v)$. If $u + v + w = 0$, then $W(u + v)W(w)$ is scalar times identity. Exponents combine to the cyclic sum $\Phi \pmod 3$.

Theorem 15.2 (Holonomy-phase conjecture (testable)) *Let $Q = \overline{\mathrm{W33}}$ be the 40-vertex quotient graph produced by the globally gauge-fixed signed lift, with each triangle (p, q, r) assigned a holonomy value $F(p, q, r) \in \mathbb{Z}_3$ (identity vs 3-cycle orientation). There exists a projective representative assignment $p \mapsto [v_p] \in PG(3, 3)$, and representative choices $v_p \in V$, such that for every triangle,*

$$F(p, q, r) \ \equiv \ \Phi(v_p, v_q, v_r) \pmod 3,$$

up to the standard gauge ambiguity corresponding to adding a constant all-ones vector in the 90-line weight model.

Protocol (testable)

Protocol: verifying Theorem ??.

1. Use the explicit projective representatives for the 40 points in $PG(3, 3)$ (present in the symplectic audit bundle).
2. Compute $\Phi(v_p, v_q, v_r)$ for all 3240 triangles of Q.
3. Compare to the computed holonomy values (identity/3-cycle with orientation) on the same triangle list.
4. If a mismatch occurs only by a constant shift (global gauge), quotient out by the all-ones line and recompare.
5. If mismatches persist with nonconstant residuals, the conjecture fails and the representative assignment must be refined (or the holonomy is not a pure symplectic cocycle).

Artifact Index (quantum layer)

`Bundle`	Contents / Purpose

11 The quotient as a simplicial gauge system

Definition

Let $Q = \overline{\mathrm{W33}}$ be the 40-vertex quotient graph obtained by collapsing the 40 flat triples in the globally gauge-fixed 240→120 lift. Let $\mathrm{Cl}(Q)$ denote the clique (flag) complex of Q. Then:

$$C^2 := \mathbb{Z}_3^{\{\text{triangles of } Q\}} \cong \mathbb{Z}_3^{3240}, \qquad C^3 := \mathbb{Z}_3^{\{\text{tetrahedra } (K_4) \text{ of } Q\}} \cong \mathbb{Z}_3^{9450}.$$

Let $d : C^2 \to C^3$ be the simplicial coboundary map.

Definition

The quotient construction assigns to each triangle (p, q, r) a holonomy value $F(p, q, r) \in \mathbb{Z}_3$ (identity vs 3-cycle orientation). We view this as a 2-cochain

$$F \in C^2(\mathrm{Cl}(Q); \mathbb{Z}_3).$$

Define the sourced 3-cochain

$$J := dF \in C^3(\mathrm{Cl}(Q); \mathbb{Z}_3),$$

which assigns a flux/charge value to each tetrahedron.

Theorem 11.1 (Sourced curvature) *$J = dF$ is supported on exactly 3008 tetrahedra:*

$$\#\{t : J(t) \neq 0\} = 3008,$$

with flux distribution $J = 1$ on 1512 tetrahedra and $J = 2$ on 1496 tetrahedra. Moreover, the 90 tetrahedra corresponding to the 90 non-isotropic projective lines (vacuum cells) all satisfy $J = 0$.

Proof sketch / audit trail

This was computed by exhaustive enumeration of all 9450 tetrahedra in Q and evaluation of the simplicial coboundary formula

$$(dF)(a, b, c, d) = F(b, c, d) - F(a, c, d) + F(a, b, d) - F(a, b, c) \pmod 3.$$

The 90 non-isotropic line tetrahedra were identified as the unique K_4 cliques whose 4 triangular faces are all flat (holonomy 0). All have $J = 0$. (Audit bundles: `W33_minimal_Z3_flux_cycles_tetrahedra_bundle.zip`, `W33_charge_decomposition_and_line_moments_bundle.zip`.)

Key Result

The quotient holonomy F is a genuine *sourced* field strength: its 3-coboundary $J = dF$ is the discrete charge/current, with vacuum cells (non-isotropic lines) exactly flux-free.

12 Vacuum sector: the 90 non-isotropic lines

Definition

Let $\mathcal{L}$ denote the 90 non-isotropic projective lines in $PG(3,3)$, each a 4-point set in the 40-point geometry. These 90 lines are in bijection with:

- the 90 K_4 cliques in Q whose four triangular faces are flat,
- the Aut(W33)-distinguished vacuum cells for the quotient connection.

We identify the vacuum line field space with $\mathbb{Z}_3^{\mathcal{L}} \cong \mathbb{Z}_3^{90}$.

Remark

Because $90 \equiv 0 \pmod 3$, the constant all-ones vector lies in the $\mathbb{Z}_3$ augmentation subspace. Thus quotienting by the all-ones line produces the canonical 88-dimensional vacuum/matter module used in the H^3 identification.

13 Transfer operators from sources to vacuum observables

Definition

Partition tetrahedra in Q into three Aut(W33)-orbits by the number of flat faces:

$$\text{bulk: \#flat faces} = 0\ (6480), \qquad \text{boundary: \#flat faces} = 1\ (2880), \qquad \text{vacuum: \#flat faces} = 4\ (90).$$

In the boundary orbit, each tetrahedron has a *unique* flat face, hence a unique attached vacuum line $\ell \in \mathcal{L}$.

Definition

Define two linear maps over $\mathbb{Z}_3$:

$$M : \mathbb{Z}_3^{9450} \to \mathbb{Z}_3^{90}, \qquad Z : \mathbb{Z}_3^{9450} \to \mathbb{Z}_3^{90}.$$

They are defined on a tetrahedron t as follows:

1. (**Boundary moment** M) If t has exactly one flat face, let $\ell(t)$ be its unique attached non-isotropic line. Then M adds the tetra flux $J(t)$ to coordinate $\ell(t)$. Otherwise t contributes 0.
2. (**Bulk shadow** Z) For each *curved* triangular face of t, push $J(t)$ along the three edges of that face. Each edge of Q belongs to a unique non-isotropic line in $\mathcal{L}$ (since $540 = 90 \cdot 6$). Summing these contributions defines $Z(J)$ on $\mathcal{L}$.

Theorem 13.1 (Exact transfer identities) *Let $J = dF$ be the sourced 3-cochain. Then the two observed vacuum line fields*

$$m_{\text{line}} \in \mathbb{Z}_3^{90}, \qquad z_{\text{line}} \in \mathbb{Z}_3^{90}$$

satisfy the exact *operator identities*

$$m_{\text{line}} = M\,J, \qquad z_{\text{line}} = Z\,J,$$

with no residual error.

Proof sketch / audit trail

Both operators were constructed explicitly in sparse COO form and applied to the computed J. The resulting 90-vectors agree entrywise with the independently computed line observables from the earlier operator chains:

$$m_{\text{line}} = C_{\text{lineface}}\,J, \qquad z_{\text{line}} = R\,(K_0 + K_1)\,J.$$

(Audit bundle: `W33_transfer_operators_J_to_lines_and_mode_injection_bundle.zip`.)

Key Result

The W33 quotient admits explicit, Aut(W33)-equivariant transfer operators from sources J to vacuum line observables. This is the discrete analog of a constitutive relation (sources $\to$ observed vacuum response).

14 Vacuum harmonics and mode-resolved response

Definition

The Aut(W33) commutant algebra acting on $\mathbb{Z}_3^{\mathcal{L}}$ has dimension 5 (an association scheme). Equivalently, the 90-line sector admits a canonical decomposition into 5 joint harmonic modes under the commuting operators:

- S: the Aut-invariant fixed-point-free involution pairing on the 90 lines (45 disjoint transpositions),
- A_{meet}: line meet adjacency (two lines adjacent iff they intersect in a point), degree 32.

Joint modes are indexed by $(\text{sign}(S), \lambda(A_{\text{meet}}))$:

$$(+, 32)^1, \quad (+, 2)^{24}, \quad (+, -4)^{20}, \quad (-, 8)^{15}, \quad (-, -4)^{30}.$$

Theorem 14.1 (Mode-resolved injection table) *For each tetra orbit class (bulk vs boundary) and each flux sign $J \in \{1, 2\}$, the induced vacuum responses $M(J)$ and $Z(J)$ decompose into the above 5 modes with explicit energy fractions. In particular:*

- *Bulk sources (flat-face count 0) inject only into z_{line} (never into m_{line}).*
- *Boundary sources (flat-face count 1) inject into both m_{line} and z_{line}, with mode weights shifted toward $(+, 2)$ and $(-, 8)$ for m_{line}.*

Proof sketch / audit trail

This was computed by restricting J to each class+flux, applying the exact transfer operators M and Z, mapping $\mathbb{Z}_3$ entries to real values $\{-1, 0, 1\}$ (with $2 \mapsto -1$), removing the mean, and projecting onto the orthonormal joint-mode bases. The resulting mode-energy fractions are tabulated. (Audit bundle: `W33_mode_response_table_bulk_to_vacuum_bundle.zip`.)

Key Result

The vacuum sector is not a single "channel": bulk and boundary sources excite different vacuum harmonics. This explains why no Aut-equivariant line-only operator can strongly predict m from z (they are distinct projections of the same bulk source field).

Artifact Index (field-equation layer)

`Bundle`	Contents / Purpose

15 Vacuum association scheme and canonical harmonics

Theorem 15.1 (90-line association scheme and involution) *The Aut(W33) action on the 90 non-isotropic lines induces an association scheme with commutant dimension 5 (five orbitals on ordered pairs). One orbital is the diagonal; another is a fixed-point-free involution σ pairing the 90 lines into 45 disjoint transpositions, with each paired lineset disjoint (skew).*

Theorem 15.2 (Five canonical harmonics) *Let S be the permutation matrix of σ and let A_{meet} be the adjacency of the line-meet graph (degree 32). Then S and A_{meet} commute and admit a joint decomposition into five modes:*

$$(+, 32)^1, \quad (+, 2)^{24}, \quad (+, -4)^{20}, \quad (-, 8)^{15}, \quad (-, -4)^{30}.$$

These modes provide the canonical "vacuum harmonics" for line fields.

Remark

This harmonic analysis explains why distinct vacuum observables (e.g., boundary moment m vs bulk shadow z) are not related by a single Aut-equivariant line-only operator: they occupy different mixtures of the canonical modes. The correct dynamics closes only when bulk source variables $J = dF$ and the transfer operators M, Z are included.

A Global Artifact Index

Bundle	Contents / Purpose
`W33_symplectic_audit_bundle.zip`	Explicit construction of $W(3,3)$ and W33; point/line incidence; $PG(3,3)$ points; isotropic vs nonisotropic line lists; verification of SRG parameters and spectrum.
`W33_orbits_squarezero_bundle.zip`	Aut(W33) generators (permutations and GF(3) matrices); orbit computations; square-zero and symmetry checkpoints.
`W33_GF2_kernel_code_bundle.zip`	The $[40, 24, 6]$ kernel code $\ker(A)$ over $\mathbb{F}$_2; 240 weight-6 generators; code basis and supporting tables.
`W33_H8_quadratic_form_bundle.zip`	Basis of $H = \ker(A)/\mathrm{im}(A)$; invariant quadratic form q; orbit split (135 singular / 120 nonsingular).
`W33_to_H_to_120root_SRG_bundle.zip`	The 120 nonsingular orbit list; SRG(120,56,28,24) edges/adjacency; mappings from code generators to H.
`W33_E8_simple_root_system_bundle.zip`	Canonical induced E_8 Dynkin configuration inside the 120-root SRG; Coxeter checks; reflection orbit generation.
`W33_signed_root_cocycle_and_lift_bundle.zip`	Signed lift/cocycle computations on 120-root edges and Steiner triples; defect weights; gauge studies.
`W33_global_gaugefix_no16_bundle.zip`	Global sign/gauge fix removing all weight-16 defects; resulting 0,12 defect spectrum; 40 flat triples.
`W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip`	Quotient $Q = \overline{\mathrm{W33}}$; edge matchings; triangle holonomy values; proof that flat holonomy triangles are exactly nonisotropic line triples.
`W33_Z3_curvature_cohomology_on_quotient_bundle.zip`	Triangle curvature cochain over $\mathbb{Z}$_3; non-exactness on the 2-skeleton; supporting tables.
`W33_minimal_Z3_flux_cycles_tetrahedra_bundle.zip`	Minimal-support flux cycles (tetrahedron boundaries) and flux statistics for $J = dF$.
`W33_flux_lattice_clique_complex_Z3_cohomology_bundle.zip`	Clique-complex cohomology ranks and dimensions over $\mathbb{Z}$_3; H^3 dimension 89; higher cohomology signature.
`W33_H3_basis_89_Z3_on_clique_complex_bundle.zip`	Explicit 89-element basis for H^3 as sparse tetra-cochains; pivot/free coordinate metadata.
`W33_H3_Aut_action_89Z3_bundle.zip`	Aut(W33) action matrices on H^3; 88+1 decomposition; quotient functional and block form.
`W33_perm_module_vs_H3_match_report_bundle.zip`	Evidence and generators showing the 88D core matches the 90-line augmentation quotient up to the similitude sign twist.
`W33_H3_to_noniso_line_weights_intertwiner_bundle.zip`	Explicit intertwiner between H^3 88D core and the twisted 90-line augmentation quotient.
`W33_lift_to_90_line_weights_with_labels_bundle.zip`	Explicit lift to labeled 90 nonisotropic line weights (mod all-ones gauge); line_id to 4-point set.
`W33_holonomy_phase_test_bundle.zip`	Holonomy vs symplectic triangle phase test; shows background closed 2-form vs sourced curvature.
`W33_current_operator_C_lineface_bundle.zip`	Operator C_lineface and line-moment statistics (source attachments to vacuum cells).
`W33_bulk_operator_K0K1_curved_triangle_current_bundle.zip`	Bulk current operators on curved triangles (K_0, K_1); outputs y on the 2880 curved triangle orbit.
`W33_curved_triangle_to_noniso_line_operator_R_bundle.zip`	Operator R mapping curved-triangle current to 90-line aggregates via edge-incidence.
`W33_charge_decomposition_and_line_moments_bundle.zip`	Charge decomposition $J = dF$; point incidences; preliminary line moments and constraints.

B Global Dictionary Table

Object	**Interpretation**	**Algebra**	**Geometry/-Topology**	**Quantum computation**	**Crypto / security**
$V = \mathbb{F}\_3^4$	Finite phase space; 2-qutrit discrete symplectic phase space.	Vector space over $\mathbb{F}$_3 with symplectic form.	Underlying coordinate domain for projective geometry and Weyl operators.	Pauli/Weyl labels; Clifford acts by $Sp(4,3)$.	Key space for symplectic commutator phase.
$W(3,3)$ / isotropic lines	Maximal commuting contexts.	Incidence geometry of totally isotropic points/lines.	Produces W33 as point graph.	Stabilizer contexts for two qutrits.	Basis for context-based protocols.
W33 = SRG(40,12,2,4)	Base combinatorial geometry.	Adjacency matrix A with SRG identities.	Over $\mathbb{F}$_2, yields differential $A^2 = 0$.	Constraint graph / stabilizer structure.	Public structure; secrecy comes from gauge/coset choices.
$A^2 \equiv 0$ over $\mathbb{F}$_2	Chain-complex calculus.	Defines $d(x) = Ax$ with $d^2 = 0$.	Produces code $\ker(A)$ and homology H.	Error correction / stabilizer relations.	Syndromes / tamper detection.
$H = \ker(A)/\mathrm{im}(A)$ (8D)	Intrinsic state space.	Carries invariant quadratic form; orbit split.	Nonsingular orbit gives 120-root shell.	Finite "root" degrees; phase classes.	Key reduction space for encoding.
120/240 roots	Finite root shell and signed lift.	SRG(120) adjacency via bilinear form; 2-to-1 lift.	Global gauge fixing yields flat triples.	Discrete gauge degrees; lift choices.	Keyed section choices = secrecy.
$Q = \overline{\mathrm{W33}}$	Quotient spacetime / interaction graph.	40 meta-vertices after collapse; edge matchings.	Supports $\mathbb{Z}$_3 holonomy.	Transport/holonomy = topological gate.	Holonomy checks = authentication.
Holonomy $F \in C^2(\mathrm{Cl}(Q); \mathbb{Z}\_3)$	Field strength / curvature.	Triangle cochain valued in $\mathbb{Z}$_3.	Flat set classified by 90 nonisotropic lines.	Discrete phase curvature.	Consistency checks / signatures.
Sources $J = dF \in C^3$	Charge/current.	Supported on 3008 tetrahedra.	Generates vacuum responses via M, Z.	Excitations / particles.	Error/fault injection model.
90 nonisotropic lines	Vacuum cells and matter carrier space.	Association scheme (5-mode harmonic analysis).	Line-weight field model (mod all-ones).	Contextual phase cells.	Share space for schemes; 88D core module.
Transfer operators M, Z	Constitutive laws.	Exact maps $J \mapsto (m, z)$.	Mode-resolved response tables.	Measurement/readout operators.	Encryption/readout operators.

C Reproducibility Checklist

Remark

Short SHA-256 prefixes (first 16 hex characters) for primary bundles in the current workspace.

File	SHA-256 prefix
W33_symplectic_audit_bundle.zip	c8f7547649abdab1
W33_orbits_squarezero_bundle.zip	84835a9889e4380b
W33_GF2_kernel_code_bundle.zip	952858afb5d65007
W33_H8_quadratic_form_bundle.zip	de3a9a9b0afb6a37
W33_to_H_to_120root_SRG_bundle.zip	3257de84a4b9c466
W33_E8_simple_root_system_bundle.zip	d200bec6ff81f00a
W33_signed_root_cocycle_and_lift_bundle.zip	d33146ea2d96104f
W33_global_gaugefix_no16_bundle.zip	8de8d1182056ac00
W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip	8a6cda139ed0a0e6
W33_Z3_curvature_cohomology_on_quotient_bundle.zip	1a7804dd46ccb1b5
W33_minimal_Z3_flux_cycles_tetrahedra_bundle.zip	8d69efdc34b5a0e6
W33_flux_lattice_clique_complex_Z3_cohomology_bundle.zip	17f5bb8490fc2d36
W33_H3_basis_89_Z3_on_clique_complex_bundle.zip	2fa53b14fcd57da9
W33_H3_Aut_action_89Z3_bundle.zip	032be0e14f33c5cc
W33_perm_module_vs_H3_match_report_bundle.zip	535aa4d6b03264d9
W33_H3_to_noniso_line_weights_intertwiner_bundle.zip	da15db795acf478b
W33_lift_to_90_line_weights_with_labels_bundle.zip	81b9f049398d5f93
W33_holonomy_phase_test_bundle.zip	5991ca050359bc4b
W33_current_operator_C_lineface_bundle.zip	02e3566e1869ce07
W33_bulk_operator_K0K1_curved_triangle_current_bundle.zip	5953f1541d2793f1
W33_curved_triangle_to_noniso_line_operator_R_bundle.zip	633e86c28d6433cf
W33_charge_decomposition_and_line_moments_bundle.zip	d9c00f5e46ca2658
W33_nonisotropic_line_association_scheme_bundle.zip	ec4b4b8e10918586
W33_vacuum_line_scheme_mode_decomposition_bundle.zip	d8545a6b843ab310
W33_transfer_operators_J_to_lines_and_mode_injection_bundle.zip	647e18c9a6ac8f7c
W33_best_field_equation_operator_on_lines_bundle.zip	3494bf1e74c08f1b