The W33 Tower as a Kernel for Algebra, Topology, and Computation

Sections 3–7 (Theorem-Forward Draft)

Wil Dahn & Sage

January 19, 2026

Abstract

This document is a theorem-forward draft of Sections 3–7 of the W33 "tower" program. The objective here is not speculative physics claims but a rigorous kernel: a finite symplectic phase space over $\mathbb{F}_3$, the symplectic generalized quadrangle $W(3,3)$, its point graph $\mathrm{W33} = \mathrm{SRG}(40, 12, 2, 4)$, and the derived structures that are forced from it—a square-zero adjacency differential over $\mathbb{F}_2$, a canonical code and homology space $H = \ker(A)/\mathrm{im}(A)$ of dimension 8, a nonsingular orbit of size 120 inducing $\mathrm{SRG}(120, 56, 28, 24)$ with an E_8 Dynkin subgraph, and a signed lift admitting global gauge fixing. Collapsing the globally gauge-fixed signed lift yields a 40-vertex quotient equal to $\overline{\mathrm{W33}}$ whose triangular holonomy is $\mathbb{Z}_3$-valued, with the flat faces classified exactly by the 90 non-isotropic projective lines in $PG(3,3)$.

Remark

Computation provenance. Each theorem below is either a standard fact from finite geometry / SRG theory or was verified by direct computation from explicit data files and bundles produced in the accompanying work session. When a statement is computationally certified, we include a brief audit note and refer to a named artifact bundle.

Contents

1 The W33 Object

Definition

Let $V = \mathbb{F}_3^4$ equipped with a nondegenerate alternating (symplectic) form ω. Let $W(3,3)$ denote the symplectic generalized quadrangle arising from totally isotropic points and lines in $PG(3,3)$ with respect to ω. The *W33 point graph* is the graph whose vertices are the 40 isotropic points and whose edges connect collinear pairs (i.e., pairs lying on a common isotropic line). We denote its adjacency matrix by A and the graph by W33.

Theorem 1.1 (SRG parameters) W33 *is a strongly regular graph with parameters*

$$(v, k, \lambda, \mu) = (40, 12, 2, 4).$$

Equivalently, each vertex has degree 12; adjacent pairs have exactly 2 common neighbors; non-adjacent pairs have exactly 4 common neighbors.

Proof sketch / audit trail

This is a standard property of the point graph of the symplectic generalized quadrangle $W(3,3)$. It was also verified computationally by explicit incidence construction of $W(3,3)$ and counting common neighbors in the point graph (audit bundle: `W33_symplectic_audit_bundle.zip`).

Theorem 1.2 (Adjacency spectrum) *The adjacency spectrum of* W33 *is*

$$\mathrm{spec}(A) = 12^{(1)}, \quad 2^{(24)}, \quad (-4)^{(15)}.$$

Equivalently, the characteristic polynomial is

$$P(x) = (x-12)(x-2)^{24}(x+4)^{15}.$$

Proof sketch / audit trail

For SRG(v, k, λ, μ), the nontrivial eigenvalues are roots of a quadratic determined by (k, λ, μ), with multiplicities forced by trace identities. Here this yields eigenvalues 2 and -4 with multiplicities 24 and 15. Verified directly by eigen-computation on the explicit adjacency matrix (audit bundle: `W33_symplectic_audit_bundle.zip`).

Theorem 1.3 (Automorphism group order) $|\mathrm{Aut}(\mathrm{W33})| = 51840$.

Proof sketch / audit trail

In the symplectic model, Aut(W33) is realized as the projective symplectic similitude group acting on isotropic points. A concrete generating set (symplectic transvections, a block-swap, and a multiplier-2 similitude) was used to generate the full permutation group on the 40 vertices, yielding order 51840. (Audit bundle: `W33_orbits_squarezero_bundle.zip`.)

Key Result

The W33 point graph is not merely a convenient combinatorial object; it is the *canonical* SRG arising from the symplectic quadrangle $W(3,3)$. The entire tower below is forced from $(40, 12, 2, 4)$ together with the induced group action.

2 Differential Structure over $\mathbb{F}_2$

Theorem 2.1 (Square-zero adjacency over $\mathbb{F}_2$) *Let A be the adjacency matrix of* W33*. Over $\mathbb{F}_2$, one has*

$$A^2 \equiv 0 \pmod 2.$$

Proof sketch / audit trail

For any $\mathrm{SRG}(v, k, \lambda, \mu)$ with adjacency A and all-ones matrix J,

$$A^2 = kI + \lambda A + \mu(J - I - A).$$

Plugging $(k, \lambda, \mu) = (12, 2, 4)$ yields $A^2 = 8I - 2A + 4J$. Reducing mod 2 gives $A^2 \equiv 0$. Verified directly by matrix multiplication mod 2 in the audit bundle.

Definition

Define a differential $d : \mathbb{F}_2^{40} \to \mathbb{F}_2^{40}$ by $d(x) = Ax \pmod 2$. Since $d^2 = 0$, we can form:

$$C := \ker(d) \subset \mathbb{F}_2^{40}, \qquad H := \ker(d)/\mathrm{im}(d).$$

Theorem 2.2 (Dimensions) *Over $\mathbb{F}_2$,*

$$\mathrm{rank}(A) = 16, \qquad \dim\ker(A) = 24, \qquad \dim H = 8.$$

Proof sketch / audit trail

Rank was computed by mod-2 row reduction on the explicit 40×40 adjacency matrix. Nullity follows by rank-nullity. Since $\mathrm{im}(A) \subseteq \ker(A)$ (square-zero), $\dim H = \dim\ker(A) - \dim\mathrm{im}(A) = 24 - 16 = 8$.

Theorem 2.3 (Canonical local generators and code distance) *The kernel $C = \ker(A) \subset \mathbb{F}_2^{40}$ is a $[40, 24, 6]$ linear code. Moreover, there are exactly 240 canonical weight-6 codewords obtained as XORs of pairs of isotropic lines through a common point, and these 240 codewords generate C.*

Proof sketch / audit trail

Each point lies on 4 isotropic lines; choosing 2 lines yields $\binom{4}{2} = 6$ line-pairs per point, hence $40\cdot6 = 240$ codewords. Each is weight 6 and lies in $\ker(A)$; exhaustive search up to weight 5 found none in $\ker(A)$, so $d_{\min} = 6$. A row-reduced basis extracted from the 240 generators spans a 24-dimensional space, matching $\dim\ker(A)$. (Audit bundle: `W33_GF2_kernel_code_bundle.zip`.)

Key Result

The identity $A^2 \equiv 0$ is the first "TOE hinge": it turns a finite SRG into a genuine chain complex, producing (i) a stabilizer-like code and (ii) an 8-dimensional homology state space H.

3 Orthogonal Geometry on H and the 120-Root Structure

Theorem 3.1 (Quadratic form and orbit split) *The induced action of* Aut(W33) *on* H *preserves a nontrivial quadratic form* $q : H \to \mathbb{F}_2$ *of minus type. Consequently, the nonzero vectors in* H *split into exactly two orbits:*

$$\{x \in H \setminus \{0\} : q(x) = 0\} \text{ of size } 135, \qquad \{x \in H \setminus \{0\} : q(x) = 1\} \text{ of size } 120.$$

Proof sketch / audit trail

A concrete basis of H was chosen by splitting $\ker(A) = \mathrm{im}(A) \oplus K$ with $\dim K = 8$. The group action on points induces an action on H, from which an invariant quadratic polynomial of degree 2 was solved. Enumerating values of q gives the $(135, 120)$ split, and orbit computation confirms exactly two nonzero orbits. (Audit bundle: `W33_H8_quadratic_form_bundle.zip`.)

Theorem 3.2 (240 → 120 projection) *Projecting the 240 canonical weight-6 code generators (Theorem 2.3) from* $\ker(A)$ *to* $H = \ker(A)/\mathrm{im}(A)$ *yields exactly 120 distinct nonzero elements, each appearing with multiplicity 2. All 120 satisfy* $q = 1$ *(the nonsingular orbit).*

Proof sketch / audit trail

Each of the 240 generators was mapped to an 8-bit H coordinate; 120 distinct values occur, each exactly twice. All map to the $q = 1$ orbit. (Audit bundle: `W33_to_H_to_120root_SRG_bundle.zip` and `W33_root_preimage_pairing_bundle.zip`.)

Definition

Define the associated bilinear form

$$b(x, y) = q(x + y) + q(x) + q(y) \in \mathbb{F}_2.$$

On the 120-element nonsingular orbit, define adjacency by $b(x, y) = 1$.

Theorem 3.3 (The 120-root SRG) *The graph on the 120 nonsingular elements with adjacency* $b = 1$ *is strongly regular:*

$$\mathrm{SRG}(120, 56, 28, 24).$$

Proof sketch / audit trail

Adjacency counts were computed directly from the bilinear form on the explicit 120-root list; all vertices have degree 56, adjacent pairs have 28 common neighbors, and nonadjacent pairs have 24. (Audit bundle: `W33_to_H_to_120root_SRG_bundle.zip`.)

Theorem 3.4 (An E_8 Dynkin subgraph and reflection generation) *Inside* $\mathrm{SRG}(120, 56, 28, 24)$ *there exists an induced subgraph isomorphic to the* E_8 *Dynkin diagram. The corresponding 8 nonsingular elements* $\{r_i\}$ *define involutions*

$$s_r(x) = x + b(x, r)\, r,$$

and the group generated by these involutions acts transitively on the 120-root set.

Proof sketch / audit trail

An induced E_8 configuration was found and canonically chosen (lexicographically minimal under a fixed branching constraint). Coxeter relations were verified on H (order 3 on adjacent nodes, order 2 otherwise), and orbit generation under reflections yields the full 120-root orbit. (Audit bundle: `W33_E8_simple_root_system_bundle.zip`.)

Key Result

The nonsingular orbit of the intrinsic homology H behaves as a finite "root shell" with $\mathrm{SRG}(120, 56, 28, 24)$ adjacency and an embedded E_8 Dynkin skeleton. This is the precise point where Lie-type structure emerges from the W33 tower.

4 Signed Lift, Cocycle, and Global Gauge Fixing

Definition

Each of the 120 roots has two preimages among the 240 generators. A *section* s selects one lift for each root. For adjacent roots h_1, h_2 (so $b(h_1, h_2) = 1$), define $h_3 = h_1 \oplus h_2$ and the defect (cocycle candidate)

$$g(h_1, h_2) := s(h_1) + s(h_2) + s(h_3) \ \in \ \mathrm{im}(A) \subset \mathbb{F}_2^{40},$$

where addition is XOR of the corresponding 40-bit supports.

Theorem 4.1 (Two-weight defect) *For the canonical section (choosing the smaller preimage index), the defect $g(h_1, h_2)$ takes only two Hamming weights:*

$$|g(h_1, h_2)| \in \{12, 16\}.$$

Across all 3360 edges of $\mathrm{SRG}(120, 56, 28, 24)$*, weight 12 occurs 1560 times and weight 16 occurs 1800 times.*

Proof sketch / audit trail

Computed exhaustively over all edges using the explicit 240 generator supports and the canonical section. Verified that $g(h_1, h_2)$ always projects to 0 in H, hence lies in $\mathrm{im}(A)$. (Audit bundle: `W33_signed_root_cocycle_and_lift_bundle.zip`.)

Theorem 4.2 (Steiner triples) *Edges of* $\mathrm{SRG}(120, 56, 28, 24)$ *partition into 1120 Steiner triples* $\{a, b, a \oplus b\}$*, and for a fixed section s, the defect value is constant on the three edges of each triple.*

Proof sketch / audit trail

If $b(a, b) = 1$ then $q(a \oplus b) = 1$; hence $a \oplus b$ is again a root. Each edge (a, b) has a unique third root $a \oplus b$, and the unordered triple partitions edges into 1120 groups. The defect $s(a) + s(b) + s(a \oplus b)$ is symmetric in $(a, b, a \oplus b)$, hence constant on the triple edges. Verified by enumeration.

Theorem 4.3 (Global gauge fix (no-16)) *There exists a global choice of signs (i.e., a section s selecting one of the two lifts at every root) such that all defects of weight 16 are eliminated. In this gauge-fixed section, all edge defects have weight in $\{0, 12\}$, with exactly 120 edges of weight 0 and 3240 edges of weight 12.*

Proof sketch / audit trail

A greedy local-flip optimization over the 120 root vertices (flipping lift choice at a vertex updates the defects on incident edges) yields a configuration with no 16-weight defects. This configuration was reproduced across random restarts. (Audit bundle: `W33_global_gaugefix_no16_bundle.zip`.)

Theorem 4.4 (40 flat triples) *The 120 roots partition into 40 disjoint triples (one per original W33 point) such that exactly those 40 triples have defect weight 0 under the globally gauge-fixed section. Equivalently, the 120 weight-0 edges form 40 disjoint triangles that partition the root set.*

Proof sketch / audit trail

From the gauge-fixed edge list, the weight-0 edges were found to group into 40 triangles. Each triangle's three vertices share the same base point in the original 40-point geometry, yielding a partition of the 120 roots into 40 fibers of size 3. (Audit bundle: `W33_global_gaugefix_no16_bundle.zip`.)

5 Quotient Closure and $\mathbb{Z}_3$ Holonomy

Definition

Collapse each of the 40 flat triples (Theorem 4.4) to a meta-vertex labeled by its base point $p \in \{0, \ldots, 39\}$. Define the quotient graph Q on these 40 meta-vertices by connecting $p \neq q$ if there exists a defect-12 edge between the fibers over p and q.

Theorem 5.1 (Quotient graph is the complement) *The quotient graph Q is regular of degree 27 on 40 vertices and is exactly the complement of the original W33 point graph:*

$$Q = \overline{\mathrm{W33}}.$$

Proof sketch / audit trail

For each pair of base points (p, q), the number of defect-12 edges between the 3-element fibers is either 0 or 6. Adjacency in Q occurs exactly for multiplicity 6. The resulting 40-vertex graph is 27-regular; direct comparison of neighbor sets confirms Q equals the complement of the W33 adjacency. (Audit bundle: `W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip`.)

Theorem 5.2 (Edge decoration is a 6-cycle) *For every edge $p \sim q$ in Q, the induced bipartite graph between the 3 roots over p and the 3 roots over q has exactly 6 edges and is 2-regular on each side. Equivalently, it is $K_{3,3}$ minus a perfect matching, i.e. a 6-cycle. The missing perfect matching defines a canonical transport bijection between the two 3-element fibers.*

Proof sketch / audit trail

Verified by explicit enumeration for all 540 quotient edges: the 3×3 adjacency matrix always has three zeros (a perfect matching) and six ones, with row and column sums all equal to 2. Connectivity check confirms a single 6-cycle.

Definition

Define the holonomy of a quotient triangle (p, q, r) as the permutation of the fiber over p obtained by composing the three transport bijections along $p \to q \to r \to p$. This holonomy lies in $A_3 \cong \mathbb{Z}_3$.

Theorem 5.3 (90 non-isotropic lines classify flat holonomy) *Among the 3240 triangles of Q, exactly 360 have identity holonomy and 2880 have 3-cycle holonomy. Moreover, the identity-holonomy triangles are* exactly *the triples of points lying on the 90 non-isotropic projective lines in $PG(3,3)$ (each such line contains 4 points and contributes $\binom{4}{3} = 4$ triples, hence $90 \cdot 4 = 360$).*

Proof sketch / audit trail

Holonomy was computed for all quotient triangles from the edge matchings. Independently, all non-isotropic lines in $PG(3,3)$ were enumerated (90 lines), and the set of their 3-subsets was computed (360 triples). These match exactly the identity-holonomy triangle set. (Audit bundle: `W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip`.)

Key Result

The W33 tower closes: after global gauge fixing and collapsing flat triples, the induced 40-vertex quotient is $\overline{\text{W33}}$ with a canonical $\mathbb{Z}_3$ connection. The set of flat faces is classified precisely by the 90 non-isotropic projective lines in $PG(3,3)$.

Artifact Index (computational)

Bundle	Contents / Purpose
`W33_symplectic_audit_bundle.zip`	Explicit construction of $W(3,3)$, point graph edges, incidence, 8-cycles; verifies SRG parameters and spectrum.
`W33_orbits_squarezero_bundle.zip`	Aut(W33) generators and orbit facts; verifies group order 51840 and transitivity on core objects.
`W33_GF2_kernel_code_bundle.zip`	The $[40, 24, 6]$ kernel code; 240 weight-6 generators; basis extraction.
`W33_H8_quadratic_form_bundle.zip`	$H = \ker(A)/\mathrm{im}(A)$ basis; invariant quadratic form q; orbit split 135/120.
`W33_to_H_to_120root_SRG_bundle.zip`	$240 \to 120$ projection; SRG(120,56,28,24) edges/adjacency.
`W33_E8_simple_root_system_bundle.zip`	Canonical induced E_8 configuration; Coxeter checks; reflection orbit generation.
`W33_signed_root_cocycle_and_lift_bundle.zip`	Defect cocycle on edges and Steiner triples; weights 12/16; Dynkin-edge gauge studies.

W33_global_gaugefix_no16_bundle.zip	Global sign assignment eliminating all 16-weight defects; identifies 40 flat triples partition.
W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip	Quotient graph $Q = \overline{W33}$, edge matchings; triangle holonomy classification; non-isotropic line correspondence.