

A Finite-Geometric Theory Kernel from W33

Toward a Unified Algebra–Topology–Quantum Computation–Cryptography Framework

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January 20, 2026

Abstract

This document consolidates the W33 tower into a single, self-contained theory kernel. Starting from the symplectic phase space $V = \mathbb{F}_3^4$, we construct the symplectic generalized quadrangle $W(3, 3)$ and its point graph $W33 = \text{SRG}(40, 12, 2, 4)$. Over \mathbb{F}_2 , the adjacency satisfies $A^2 \equiv 0$, producing a canonical code [40, 24, 6] and an intrinsic homology space $H = \ker(A)/\text{im}(A) \cong \mathbb{F}_2^8$. The nonsingular orbit in H yields a 120-element “root shell” with $\text{SRG}(120, 56, 28, 24)$ adjacency, a 240 signed lift admitting global gauge fixing, and a quotient closure back to 40 points as $Q = \overline{W33}$. The quotient carries a canonical \mathbb{Z}_3 holonomy, with flat faces classified exactly by the 90 non-isotropic projective lines. Over \mathbb{Z}_3 , the clique complex of Q has $H^3 \cong (\mathbb{Z}_3)^{89}$, whose 88D core is identified (up to a canonical sign character) with the augmentation quotient on the 90 non-isotropic lines. Finally, the holonomy field F is sourced: $J = dF$ is a 3-cochain supported on 3008 tetrahedra, and explicit sparse transfer operators map J to observed vacuum line responses.

Remark

What is meant by “theory of everything” here. This manuscript presents a mathematically closed kernel in which geometry, algebra, topology, computation, and cryptography are realized as different functorial views of the same finite symplectic/projective object. Claims about physical constants require an additional scaling/continuum layer and are not asserted as part of the kernel.

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Master Equation Summary

Key Result

Discrete gauge kernel (minimal equations). Let $Q = \overline{W33}$ be the quotient graph and $\text{Cl}(Q)$ its clique complex.

Field strength (holonomy). $F \in C^2(\text{Cl}(Q); \mathbb{Z}_3)$ is the computed triangle holonomy.

Sources. $J := dF \in C^3(\text{Cl}(Q); \mathbb{Z}_3)$ is the sourced 3-cochain (supported on 3008 tetrahedra).

Vacuum response (exact constitutive laws). There exist explicit sparse operators

$$M, Z : \mathbb{Z}_3^{9450} \rightarrow \mathbb{Z}_3^{90}$$

such that the observed line fields satisfy

$$m_{\text{line}} = MJ, \quad z_{\text{line}} = ZJ$$

exactly.

Vacuum harmonics. The 90-line sector admits five canonical joint modes under the involution S and meet adjacency A_{meet} :

$$(+, 32)^1, (+, 2)^{24}, (+, -4)^{20}, (-, 8)^{15}, (-, -4)^{30}.$$

Bulk and boundary source classes inject into different harmonic mixtures (mode-response tables).

1 Master Equations and Couplings

Definition

Field variables. On the clique complex $\text{Cl}(Q)$ of the quotient graph $Q = \overline{W33}$:

- $F \in C^2(\text{Cl}(Q); \mathbb{Z}_3)$ is the triangle holonomy field (field strength).
- $J := dF \in C^3(\text{Cl}(Q); \mathbb{Z}_3)$ is the sourced 3-cochain (charge/current).

On the vacuum line set \mathcal{L} (the 90 non-isotropic lines):

- $m_{\text{line}} \in \mathbb{Z}_3^{90}$ is the *boundary moment* observable.
- $z_{\text{line}} \in \mathbb{Z}_3^{90}$ is the *bulk shadow* observable.

Theorem 1.1 (Master operator equations) *The W33 kernel closes as the following exact operator pipeline over \mathbb{Z}_3 :*

$$F \xrightarrow{d} J \xrightarrow{(M,Z)} (m_{\text{line}}, z_{\text{line}}),$$

where d is the simplicial coboundary on $\text{Cl}(Q)$ and $M, Z : \mathbb{Z}_3^{9450} \rightarrow \mathbb{Z}_3^{90}$ are explicit sparse transfer operators. Concretely,

$$J = dF, \quad m_{\text{line}} = MJ, \quad z_{\text{line}} = ZJ,$$

and these identities hold entrywise with no residual error.

Proof sketch / audit trail

F and $J = dF$ are computed from the quotient holonomy. The operators M and Z are constructed canonically from incidence: M routes tetra flux to the unique vacuum line of the tetra's flat face (when present), while Z routes tetra flux to vacuum lines via edge-incidence of curved faces. Exactness was verified against independently computed line observables. (Audit bundle: `W33_transfer_operators_J_to_lines_and_mode_injection_bundle.zip`.)

Definition

Vacuum harmonics. Let S be the canonical involution on \mathcal{L} (45 disjoint transpositions) and A_{meet} the meet adjacency on \mathcal{L} (degree 32). The vacuum line sector decomposes into five joint modes:

$$(+, 32)^1, (+, 2)^{24}, (+, -4)^{20}, (-, 8)^{15}, (-, -4)^{30}.$$

Theorem 1.2 (Coupling selection rules (mode response)) *Bulk sources (tetrahedra with zero flat faces) inject into z_{line} but not m_{line} . Boundary sources (tetrahedra with one flat face) inject into both m_{line} and z_{line} , with mode weights shifted toward $(+, 2)$ and $(-, 8)$ for m_{line} . These couplings are quantified by the mode-response tables.*

Proof sketch / audit trail

Apply M and Z to class-restricted source vectors and project the resulting 90-line fields into the five joint modes using the association-scheme harmonic bases. (Audit bundle: `W33_mode_response_table_bulk_to_vacuum_bundle.zip`.)

Key Result

The equations $J = dF$ and $(m, z) = (MJ, ZJ)$ are the minimal “field equations” of the kernel. Together with the five vacuum harmonics, they provide a complete, symmetry-respecting description of how sourced curvature produces observable vacuum response in the 90-line sector.

2 Closure Principle

Definition

Closure. We say the W33 tower is *closed* if the following hold simultaneously:

1. **(Lift)** The 240 minimal code generators project 2-to-1 onto a 120-element nonsingular orbit in H (the “root shell”).
2. **(Gauge fix)** There exists a global sign section eliminating all weight-16 defects, producing 40 disjoint flat triples.
3. **(Collapse)** Collapsing the 40 triples yields a 40-vertex quotient graph Q with canonical edge transport and \mathbb{Z}_3 holonomy.
4. **(Recursion)** The quotient graph is exactly $Q = \overline{\text{W33}}$.
5. **(Vacuum/matter coincidence)** The 90 non-isotropic lines simultaneously (i) classify flat holonomy faces and (ii) support the 88D core module of H^3 via the 90-line augmentation quotient.

Theorem 2.1 (Closure Theorem) *The W33 tower is closed in the above sense. In particular:*

1. *The globally gauge-fixed signed lift partitions the 120 roots into 40 flat triples.*
2. *The induced quotient is $Q = \overline{\text{W33}}$ and carries canonical \mathbb{Z}_3 triangle holonomy.*
3. *Flat holonomy triangles are exactly the triples lying on the 90 non-isotropic lines of $\text{PG}(3,3)$.*
4. *The 88D core of $H^3(\text{Cl}(Q); \mathbb{Z}_3)$ is (up to the similitude sign twist) the augmentation quotient on these same 90 non-isotropic lines.*

Proof sketch / audit trail

Items (1)–(3) are verified by the explicit gauge-fix computation and quotient construction: the defect-0 edges form 40 disjoint triangles partitioning the 120 roots, and the quotient adjacency equals the complement of W33 with a \mathbb{Z}_3 holonomy classified by non-isotropic line triples. Item (4) is established by comparing the 88D core module of H^3 with the 90-line augmentation quotient: after the canonical similitude sign twist, traces and characteristic-polynomial factor patterns match and an explicit intertwiner exists. (Audit bundles: `W33_global_gaugefix_no16_bundle.zip`, `W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip`, `W33_H3_Aut_action_89Z3_bundle.zip`, `W33_perm_module_vs_H3_match_report_bundle.zip`.)

Key Result

Closure is the central “TOE hinge” of the kernel: the same finite geometry simultaneously generates (i) constraints/codes ($A^2 = 0$ over \mathbb{F}_2), (ii) a root shell and gauge-fixed signed lift (120/240), (iii) a recursive quotient $Q = \overline{\text{W33}}$ with \mathbb{Z}_3 holonomy, and (iv) a vacuum line sector that is also the carrier of the nontrivial 88D matter/flux module. This is precisely the structure needed for a self-contained theory kernel.

3 Functorial Field Theory View

Definition

Clique category. Let $Q = \overline{W33}$ and $\text{Cl}(Q)$ its clique (flag) complex. Define a small category $\mathcal{C}(Q)$ as follows:

- Objects are cliques $\sigma \subseteq V(Q)$ (equivalently simplices of $\text{Cl}(Q)$), including vertices, edges, triangles, tetrahedra, etc.
- Morphisms are inclusions $\tau \hookrightarrow \sigma$ (face maps).

Thus $\mathcal{C}(Q)$ encodes the full incidence/facial structure of the quotient geometry.

Definition

Cochain functors. Fix a coefficient ring R (typically $R = \mathbb{Z}_3$). For each $k \geq 0$, define a functor

$$C_R^k : \mathcal{C}(Q)^{\text{op}} \rightarrow \text{Mod}_R$$

by assigning to each k -simplex σ the free rank-one R -module generated by σ , and to each face inclusion the corresponding restriction map. The usual coboundary $d : C_R^k \rightarrow C_R^{k+1}$ is a natural transformation determined by alternating sums of face restrictions (with orientation conventions).

Definition

Vacuum line functor. Let \mathcal{L} be the 90 non-isotropic lines in $PG(3, 3)$, which are also the 90 flat K_4 cells in Q . Define the vacuum sector as the permutation module

$$V := \mathbb{Z}_3^{\mathcal{L}},$$

together with its canonical 88D augmentation quotient $V_{88} = \text{Aug}(\mathcal{L})/\langle 1 \rangle$ (up to the similitude sign twist).

Theorem 3.1 (Kernel as a functorial gauge system) *The W33 tower admits a functorial formulation in which geometry, topology, computation, and quantum structure are different functorial shadows of the same underlying incidence data:*

1. (**Geometry→Topology**) *The clique category $\mathcal{C}(Q)$ determines cochain functors $C_{\mathbb{Z}_3}^k$ and a natural coboundary d . The holonomy field F is an element of $C_{\mathbb{Z}_3}^2$ and the source field is $J = dF \in C_{\mathbb{Z}_3}^3$.*
2. (**Topology→Vacuum response**) *The transfer operators M and Z are natural, $\text{Aut}(W33)$ -equivariant linear maps from the tetra-source module to the vacuum module:*

$$M, Z : \mathbb{Z}_3^{\{\text{tetrahedra}\}} \rightarrow \mathbb{Z}_3^{\mathcal{L}},$$

giving exact observables $(m_{\text{line}}, z_{\text{line}}) = (MJ, ZZ)$.

3. (**Computation**) *Over \mathbb{F}_2 , the W33 adjacency defines a square-zero differential on \mathbb{F}_2^{40} , yielding the intrinsic code $\ker(A)$ and homology $H = \ker(A)/\text{im}(A)$; these are functorial with respect to the $\text{Aut}(W33)$ action.*

4. (**Quantum**) The phase space axiom $V = \mathbb{F}_3^4$ defines the 2-qutrit Weyl functor (Weyl labels and commutator phase) and a projectivized Clifford action by $\mathrm{PGSp}(4, 3)$ on $\mathrm{PG}(3, 3)$; isotropic lines correspond to maximal commuting contexts.

Moreover, the representation-theoretic identification $H^3(\mathrm{Cl}(Q); \mathbb{Z}_3)_{88} \cong \mathbf{V}_{88}$ (up to twist) provides an explicit equivalence between the flux-lattice core and the vacuum line module.

Proof sketch / audit trail

Each item is backed by explicit constructions: (1) and (2) follow from the computed holonomy F , sources $J = dF$, and the sparse transfer operators M, Z built from incidence (Section 11 and associated bundles). (3) follows from the SRG identity implying $A^2 \equiv 0$ over \mathbb{F}_2 and the explicit kernel-code computation (Section 4). (4) follows from the standard Weyl/Clifford construction on V and the identification of W33 points/lines with projective points/isotropic lines in $\mathrm{PG}(3, 3)$ (Section 10). The module equivalence is established by comparing the $\mathrm{Aut}(\mathrm{W33})$ actions and constructing an explicit intertwiner (Section 9).

Key Result

This functorial view is the cleanest “TOE statement” available at the kernel level: a single finite incidence object induces, via natural functors, (i) a sourced gauge field (F, J) , (ii) exact response laws (M, Z) into the vacuum line sector, (iii) an intrinsic error-correcting code over \mathbb{F}_2 , and (iv) a 2-qutrit Weyl/Clifford quantum structure over \mathbb{F}_3 . These are not separate theories but compatible projections of the same kernel.

4 Axioms and kernel construction chain

Definition

Axiom A0 (Phase space). Let $V = \mathbb{F}_3^4$ equipped with a fixed nondegenerate alternating (symplectic) form ω .

Axiom A1 (Isotropy geometry). Let $W(3, 3)$ denote the symplectic generalized quadrangle realized by totally isotropic points and lines in $\mathrm{PG}(3, 3)$ with respect to ω .

Axiom A2 (Point graph). Let $\mathrm{W33}$ be the point graph of $W(3, 3)$: vertices are the 40 isotropic points, and edges represent collinearity.

Remark

These axioms fix the entire tower. Everything below is forced from the adjacency matrix A of $\mathrm{W33}$, its induced actions, and the canonical quotients and lifts defined from it.

Key Result

The W33 tower can be viewed as a closed pipeline:

$$\begin{aligned}\mathbb{F}_3^4 &\Rightarrow W(3,3) \Rightarrow W33 \Rightarrow (A^2 \equiv 0 \text{ over } \mathbb{F}_2) \Rightarrow H \\ &\Rightarrow (120, 240) \text{ signed roots} \Rightarrow Q = \overline{W33} \Rightarrow (\mathbb{Z}_3 \text{ holonomy}) \\ &\Rightarrow H^3(\text{Cl}(Q); \mathbb{Z}_3) \cong (\mathbb{Z}_3)^{89} \Rightarrow 90\text{-line field model.}\end{aligned}$$

5 Master theorems and dictionary

Theorem 5.1 (Master Theorem I: square-zero differential and code) *Over \mathbb{F}_2 , the adjacency matrix A of W33 satisfies $A^2 \equiv 0$. Hence $d(x) = Ax$ defines a differential on \mathbb{F}_2^{40} , producing a canonical code $C = \ker(A)$ with parameters $[40, 24, 6]$ and a homology state space $H = \ker(A)/\text{im}(A) \cong \mathbb{F}_2^8$.*

Theorem 5.2 (Master Theorem II: 120-root shell and 240 signed lift) *The induced action on H preserves a quadratic form of minus type. The nonsingular orbit has size 120 and carries SRG(120, 56, 28, 24) adjacency via the associated bilinear form. The 240 canonical weight-6 generators project 2-to-1 onto this 120-set, yielding a signed lift with a defect cocycle valued in $\text{im}(A)$.*

Theorem 5.3 (Master Theorem III: quotient closure and \mathbb{Z}_3 connection) *There exists a global gauge fix eliminating all weight-16 defects. In that gauge, the 120 roots partition into 40 flat triples (one per W33 point). Collapsing these triples yields a quotient graph Q equal to the complement $\overline{W33}$, equipped with a canonical edge transport rule whose triangle holonomy lies in \mathbb{Z}_3 . Flat holonomy triangles are classified exactly by the 90 non-isotropic projective lines in $PG(3, 3)$.*

Theorem 5.4 (Master Theorem IV: sourced curvature and transfer operators) *Let $F \in C^2(\text{Cl}(Q); \mathbb{Z}_3)$ be the triangle holonomy field and $J = dF \in C^3(\text{Cl}(Q); \mathbb{Z}_3)$ its source. Then J is supported on exactly 3008 tetrahedra. There exist explicit sparse operators*

$$M, Z : \mathbb{Z}_3^{9450} \rightarrow \mathbb{Z}_3^{90}$$

such that the observed vacuum line fields satisfy the exact identities $m_{\text{line}} = MJ$ and $z_{\text{line}} = ZJ$. Vacuum responses decompose into five canonical harmonics determined by the Aut-invariant 90-line association scheme.

Definition

Dictionary (high level). Within the exact finite theory:

- **Geometry:** isotropic vs non-isotropic incidence in $PG(3, 3)$; the graphs $W33$ and $Q = \overline{W33}$.
- **Algebra:** $\text{Aut}(W33)$ actions and induced modules on H , the 120-root shell, the 90-line sector, and H^3 .
- **Topology:** cochains/coboundaries on $\text{Cl}(Q)$; $J = dF$ as sources; H^3 as flux lattice.
- **Quantum computation:** Weyl/Clifford realization on V ; contexts from isotropic lines; holonomy as discrete phase transport.
- **Cryptography:** gauge/co-set ambiguity and large symmetry action as secrecy; error correction as intrinsic stability (the $[40, 24, 6]$ code).

3 The $W33$ Object

Definition

Let $V = \mathbb{F}_3^4$ equipped with a nondegenerate alternating (symplectic) form ω . Let $W(3, 3)$ denote the symplectic generalized quadrangle arising from totally isotropic points and lines in $PG(3, 3)$ with respect to ω . The $W33$ point graph is the graph whose vertices are the 40 isotropic points and whose edges connect collinear pairs (i.e., pairs lying on a common isotropic line). We denote its adjacency matrix by A and the graph by $W33$.

Theorem 3.1 (SRG parameters) $W33$ is a strongly regular graph with parameters

$$(v, k, \lambda, \mu) = (40, 12, 2, 4).$$

Equivalently, each vertex has degree 12; adjacent pairs have exactly 2 common neighbors; non-adjacent pairs have exactly 4 common neighbors.

Proof sketch / audit trail

This is a standard property of the point graph of the symplectic generalized quadrangle $W(3, 3)$. It was also verified computationally by explicit incidence construction of $W(3, 3)$ and counting common neighbors in the point graph (audit bundle: `W33_symplectic_audit_bundle.zip`).

Theorem 3.2 (Adjacency spectrum) The adjacency spectrum of $W33$ is

$$\text{spec}(A) = 12^{(1)}, \quad 2^{(24)}, \quad (-4)^{(15)}.$$

Equivalently, the characteristic polynomial is

$$P(x) = (x - 12)(x - 2)^{24}(x + 4)^{15}.$$

Proof sketch / audit trail

For SRG(v, k, λ, μ), the nontrivial eigenvalues are roots of a quadratic determined by (k, λ, μ) , with multiplicities forced by trace identities. Here this yields eigenvalues 2 and -4 with multiplicities 24 and 15. Verified directly by eigen-computation on the explicit adjacency matrix (audit bundle: `W33_symplectic_audit_bundle.zip`).

Theorem 3.3 (Automorphism group order) $|\text{Aut}(\text{W33})| = 51840$.

Proof sketch / audit trail

In the symplectic model, $\text{Aut}(\text{W33})$ is realized as the projective symplectic similitude group acting on isotropic points. A concrete generating set (symplectic transvections, a block-swap, and a multiplier-2 similitude) was used to generate the full permutation group on the 40 vertices, yielding order 51840. (Audit bundle: `W33_orbits_squarezero_bundle.zip`.)

Key Result

The W33 point graph is not merely a convenient combinatorial object; it is the *canonical* SRG arising from the symplectic quadrangle $W(3, 3)$. The entire tower below is forced from $(40, 12, 2, 4)$ together with the induced group action.

4 Differential Structure over \mathbb{F}_2

Theorem 4.1 (Square-zero adjacency over \mathbb{F}_2) *Let A be the adjacency matrix of W33. Over \mathbb{F}_2 , one has*

$$A^2 \equiv 0 \pmod{2}.$$

Proof sketch / audit trail

For any SRG(v, k, λ, μ) with adjacency A and all-ones matrix J ,

$$A^2 = kI + \lambda A + \mu(J - I - A).$$

Plugging $(k, \lambda, \mu) = (12, 2, 4)$ yields $A^2 = 8I - 2A + 4J$. Reducing mod 2 gives $A^2 \equiv 0$. Verified directly by matrix multiplication mod 2 in the audit bundle.

Definition

Define a differential $d : \mathbb{F}_2^{40} \rightarrow \mathbb{F}_2^{40}$ by $d(x) = Ax \pmod{2}$. Since $d^2 = 0$, we can form:

$$C := \ker(d) \subset \mathbb{F}_2^{40}, \quad H := \ker(d)/\text{im}(d).$$

Theorem 4.2 (Dimensions) *Over \mathbb{F}_2 ,*

$$\text{rank}(A) = 16, \quad \dim \ker(A) = 24, \quad \dim H = 8.$$

Proof sketch / audit trail

Rank was computed by mod-2 row reduction on the explicit 40×40 adjacency matrix. Nullity follows by rank-nullity. Since $\text{im}(A) \subseteq \ker(A)$ (square-zero), $\dim H = \dim \ker(A) - \dim \text{im}(A) = 24 - 16 = 8$.

Theorem 4.3 (Canonical local generators and code distance) *The kernel $C = \ker(A) \subset \mathbb{F}_2^{40}$ is a $[40, 24, 6]$ linear code. Moreover, there are exactly 240 canonical weight-6 codewords obtained as XORs of pairs of isotropic lines through a common point, and these 240 codewords generate C .*

Proof sketch / audit trail

Each point lies on 4 isotropic lines; choosing 2 lines yields $\binom{4}{2} = 6$ line-pairs per point, hence $40 \cdot 6 = 240$ codewords. Each is weight 6 and lies in $\ker(A)$; exhaustive search up to weight 5 found none in $\ker(A)$, so $d_{\min} = 6$. A row-reduced basis extracted from the 240 generators spans a 24-dimensional space, matching $\dim \ker(A)$. (Audit bundle: `W33_GF2_kernel_code_bundle.zip`.)

Key Result

The identity $A^2 \equiv 0$ is the first “TOE hinge”: it turns a finite SRG into a genuine chain complex, producing (i) a stabilizer-like code and (ii) an 8-dimensional homology state space H .

5 Orthogonal Geometry on H and the 120-Root Structure

Theorem 5.1 (Quadratic form and orbit split) *The induced action of $\text{Aut}(W33)$ on H preserves a nontrivial quadratic form $q : H \rightarrow \mathbb{F}_2$ of minus type. Consequently, the nonzero vectors in H split into exactly two orbits:*

$$\{x \in H \setminus \{0\} : q(x) = 0\} \text{ of size } 135, \quad \{x \in H \setminus \{0\} : q(x) = 1\} \text{ of size } 120.$$

Proof sketch / audit trail

A concrete basis of H was chosen by splitting $\ker(A) = \text{im}(A) \oplus K$ with $\dim K = 8$. The group action on points induces an action on H , from which an invariant quadratic polynomial of degree 2 was solved. Enumerating values of q gives the $(135, 120)$ split, and orbit computation confirms exactly two nonzero orbits. (Audit bundle: `W33_H8_quadratic_form_bundle.zip`.)

Theorem 5.2 (240 \rightarrow 120 projection) *Projecting the 240 canonical weight-6 code generators (Theorem ??) from $\ker(A)$ to $H = \ker(A)/\text{im}(A)$ yields exactly 120 distinct nonzero elements, each appearing with multiplicity 2. All 120 satisfy $q = 1$ (the nonsingular orbit).*

Proof sketch / audit trail

Each of the 240 generators was mapped to an 8-bit H coordinate; 120 distinct values occur, each exactly twice. All map to the $q = 1$ orbit. (Audit bundle: `W33_to_H_to_120root_SRG_bundle.zip` and `W33_root_preimage_pairing_bundle.zip`.)

Definition

Define the associated bilinear form

$$b(x, y) = q(x + y) + q(x) + q(y) \in \mathbb{F}_2.$$

On the 120-element nonsingular orbit, define adjacency by $b(x, y) = 1$.

Theorem 5.3 (The 120-root SRG) *The graph on the 120 nonsingular elements with adjacency $b = 1$ is strongly regular:*

$$\text{SRG}(120, 56, 28, 24).$$

Proof sketch / audit trail

Adjacency counts were computed directly from the bilinear form on the explicit 120-root list; all vertices have degree 56, adjacent pairs have 28 common neighbors, and nonadjacent pairs have 24. (Audit bundle: `W33_to_H_to_120root_SRG_bundle.zip`.)

Theorem 5.4 (An E_8 Dynkin subgraph and reflection generation) *Inside $\text{SRG}(120, 56, 28, 24)$ there exists an induced subgraph isomorphic to the E_8 Dynkin diagram. The corresponding 8 nonsingular elements $\{r_i\}$ define involutions*

$$s_r(x) = x + b(x, r) r,$$

and the group generated by these involutions acts transitively on the 120-root set.

Proof sketch / audit trail

An induced E_8 configuration was found and canonically chosen (lexicographically minimal under a fixed branching constraint). Coxeter relations were verified on H (order 3 on adjacent nodes, order 2 otherwise), and orbit generation under reflections yields the full 120-root orbit. (Audit bundle: `W33_E8_simple_root_system_bundle.zip`.)

Key Result

The nonsingular orbit of the intrinsic homology H behaves as a finite “root shell” with $\text{SRG}(120, 56, 28, 24)$ adjacency and an embedded E_8 Dynkin skeleton. This is the precise point where Lie-type structure emerges from the W33 tower.

6 Signed Lift, Cocycle, and Global Gauge Fixing

Definition

Each of the 120 roots has two preimages among the 240 generators. A section s selects one lift for each root. For adjacent roots h_1, h_2 (so $b(h_1, h_2) = 1$), define $h_3 = h_1 \oplus h_2$ and the defect (cocycle candidate)

$$g(h_1, h_2) := s(h_1) + s(h_2) + s(h_3) \in \text{im}(A) \subset \mathbb{F}_2^{40},$$

where addition is XOR of the corresponding 40-bit supports.

Theorem 6.1 (Two-weight defect) *For the canonical section (choosing the smaller preimage index), the defect $g(h_1, h_2)$ takes only two Hamming weights:*

$$|g(h_1, h_2)| \in \{12, 16\}.$$

Across all 3360 edges of SRG(120, 56, 28, 24), weight 12 occurs 1560 times and weight 16 occurs 1800 times.

Proof sketch / audit trail

Computed exhaustively over all edges using the explicit 240 generator supports and the canonical section. Verified that $g(h_1, h_2)$ always projects to 0 in H , hence lies in $\text{im}(A)$. (Audit bundle: `W33_signed_root_cocycle_and_lift_bundle.zip`.)

Theorem 6.2 (Steiner triples) *Edges of SRG(120, 56, 28, 24) partition into 1120 Steiner triples $\{a, b, a \oplus b\}$, and for a fixed section s , the defect value is constant on the three edges of each triple.*

Proof sketch / audit trail

If $b(a, b) = 1$ then $q(a \oplus b) = 1$; hence $a \oplus b$ is again a root. Each edge (a, b) has a unique third root $a \oplus b$, and the unordered triple partitions edges into 1120 groups. The defect $s(a) + s(b) + s(a \oplus b)$ is symmetric in $(a, b, a \oplus b)$, hence constant on the triple edges. Verified by enumeration.

Theorem 6.3 (Global gauge fix (no-16)) *There exists a global choice of signs (i.e., a section s selecting one of the two lifts at every root) such that all defects of weight 16 are eliminated. In this gauge-fixed section, all edge defects have weight in $\{0, 12\}$, with exactly 120 edges of weight 0 and 3240 edges of weight 12.*

Proof sketch / audit trail

A greedy local-flip optimization over the 120 root vertices (flipping lift choice at a vertex updates the defects on incident edges) yields a configuration with no 16-weight defects. This configuration was reproduced across random restarts. (Audit bundle: `W33_global_gaugefix_no16_bundle.zip`.)

Theorem 6.4 (40 flat triples) *The 120 roots partition into 40 disjoint triples (one per original W33 point) such that exactly those 40 triples have defect weight 0 under the globally gauge-fixed section. Equivalently, the 120 weight-0 edges form 40 disjoint triangles that partition the root set.*

Proof sketch / audit trail

From the gauge-fixed edge list, the weight-0 edges were found to group into 40 triangles. Each triangle's three vertices share the same base point in the original 40-point geometry, yielding a partition of the 120 roots into 40 fibers of size 3. (Audit bundle: `W33_global_gaugefix_no16_bundle.zip`.)

7 Quotient Closure and \mathbb{Z}_3 Holonomy

Definition

Collapse each of the 40 flat triples (Theorem ??) to a meta-vertex labeled by its base point $p \in \{0, \dots, 39\}$. Define the quotient graph Q on these 40 meta-vertices by connecting $p \neq q$ if there exists a defect-12 edge between the fibers over p and q .

Theorem 7.1 (Quotient graph is the complement) *The quotient graph Q is regular of degree 27 on 40 vertices and is exactly the complement of the original W33 point graph:*

$$Q = \overline{\text{W33}}.$$

Proof sketch / audit trail

For each pair of base points (p, q) , the number of defect-12 edges between the 3-element fibers is either 0 or 6. Adjacency in Q occurs exactly for multiplicity 6. The resulting 40-vertex graph is 27-regular; direct comparison of neighbor sets confirms Q equals the complement of the W33 adjacency. (Audit bundle: `W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip`.)

Theorem 7.2 (Edge decoration is a 6-cycle) *For every edge $p \sim q$ in Q , the induced bipartite graph between the 3 roots over p and the 3 roots over q has exactly 6 edges and is 2-regular on each side. Equivalently, it is $K_{3,3}$ minus a perfect matching, i.e. a 6-cycle. The missing perfect matching defines a canonical transport bijection between the two 3-element fibers.*

Proof sketch / audit trail

Verified by explicit enumeration for all 540 quotient edges: the 3×3 adjacency matrix always has three zeros (a perfect matching) and six ones, with row and column sums all equal to 2. Connectivity check confirms a single 6-cycle.

Definition

Define the holonomy of a quotient triangle (p, q, r) as the permutation of the fiber over p obtained by composing the three transport bijections along $p \rightarrow q \rightarrow r \rightarrow p$. This holonomy lies in $A_3 \cong \mathbb{Z}_3$.

Theorem 7.3 (90 non-isotropic lines classify flat holonomy) *Among the 3240 triangles of Q , exactly 360 have identity holonomy and 2880 have 3-cycle holonomy. Moreover, the identity-holonomy triangles are exactly the triples of points lying on the 90 non-isotropic projective lines in $PG(3, 3)$ (each such line contains 4 points and contributes $\binom{4}{3} = 4$ triples, hence $90 \cdot 4 = 360$).*

Proof sketch / audit trail

Holonomy was computed for all quotient triangles from the edge matchings. Independently, all non-isotropic lines in $PG(3, 3)$ were enumerated (90 lines), and the set of their 3-subsets was computed (360 triples). These match exactly the identity-holonomy triangle set. (Audit bundle: `W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip`.)

Key Result

The W33 tower closes: after global gauge fixing and collapsing flat triples, the induced 40-vertex quotient is $\overline{W33}$ with a canonical \mathbb{Z}_3 connection. The set of flat faces is classified precisely by the 90 non-isotropic projective lines in $PG(3, 3)$.

Artifact Index (computational)

Bundle	Contents / Purpose
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8 Cohomology and flux lattice (summary of computed results)

Theorem 8.1 (Clique-complex cohomology over \mathbb{Z}_3) *Let $Cl(Q)$ be the clique complex of $Q = \overline{W33}$. Over \mathbb{Z}_3 , its cohomology dimensions are:*

$$H^0 = 1, \quad H^1 = 0, \quad H^2 = 0, \quad H^3 = 89, \quad H^4 = 1, \quad H^5 = 0, \quad H^6 = 1.$$

In particular, the flux lattice is $H^3(Cl(Q); \mathbb{Z}_3) \cong (\mathbb{Z}_3)^{89}$, and an explicit 89-element basis can be constructed.

Remark

The vanishing $H^2 = 0$ on the full clique complex explains why 2-skeleton obstructions disappear once tetrahedra are included: closed 2-forms are exact in the full flag complex, while the physically relevant sourced curvature is encoded by $J = dF$ (a 3-cochain).

9 Representation theory of the flux lattice and the 90-line module

Definition

Let $Q = \overline{W33}$ be the 40-vertex quotient graph and $Cl(Q)$ its clique (flag) complex. The flux lattice is

$$H^3(Cl(Q); \mathbb{Z}_3) \cong (\mathbb{Z}_3)^{89}.$$

The $\text{Aut}(W33)$ action on the 40 base points induces an action on all cliques of Q and hence on cochains, coboundaries, and cohomology.

Theorem 9.1 (An explicit basis for H^3) *There exists an explicit basis of 89 cocycles in $C^3(Cl(Q); \mathbb{Z}_3)$ representing a basis of $H^3(Cl(Q); \mathbb{Z}_3)$. Each basis element is given in sparse form as a \mathbb{Z}_3 -valued cochain supported on tetrahedra (K_4 cliques) of Q .*

Proof sketch / audit trail

We compute $\ker(\delta_3) \subset C^3$ from the K_5 constraints and quotient by $\text{im}(\delta_2)$ coming from triangles. In free coordinates for $\ker(\delta_3)$, the image of δ_2 has rank 2739, leaving dimension 89. We select 89 nonpivot free coordinates and back-substitute to construct cocycles. (Audit bundle: `W33_H3_basis_89_Z3_on_clique_complex_bundle.zip`.)

Theorem 9.2 (88+1 module structure and similitude character) *The 89-dimensional \mathbb{Z}_3 -module $H^3(\text{Cl}(Q); \mathbb{Z}_3)$ admits an invariant 88-dimensional submodule W_{88} such that the quotient is 1-dimensional. The 1-dimensional quotient carries the canonical “similitude sign” character: an index-2 subgroup acts trivially, while a distinguished multiplier-2 element acts by $-1 \equiv 2 \pmod{3}$.*

Proof sketch / audit trail

Using the explicit $\text{Aut}(W_{33})$ generators on points, we compute the induced action on tetrahedra, incorporate the orientation sign for 3-cochains, and build the resulting 89×89 matrices over \mathbb{Z}_3 on the computed H^3 basis. Empirically, the module has an invariant 88D submodule and a 1D quotient; the quotient character is detected by a dual functional w transforming by ± 1 . (Audit bundle: `W33_H3_Aut_action_89Z3_bundle.zip`.)

Definition

Let \mathcal{L} be the set of 90 non-isotropic projective lines in $PG(3, 3)$. Consider the permutation module $\mathbb{Z}_3^{\mathcal{L}}$ and its augmentation submodule

$$\text{Aug}(\mathcal{L}) := \left\{ x \in \mathbb{Z}_3^{\mathcal{L}} : \sum_{\ell \in \mathcal{L}} x_{\ell} = 0 \right\}.$$

Since $90 \equiv 0 \pmod{3}$, the all-ones vector lies in $\text{Aug}(\mathcal{L})$; quotienting by this trivial line yields an 88D module.

Theorem 9.3 (Geometric identification with 90-line augmentation quotient) *The 88D core module W_{88} is isomorphic (up to the similitude sign twist) to the augmentation quotient of the 90-line permutation module:*

$$W_{88} \cong \text{Aug}(\mathcal{L})/\langle \mathbf{1} \rangle \otimes \chi,$$

where χ is the 1D similitude sign character. Moreover, an explicit intertwiner T between these modules can be computed.

Proof sketch / audit trail

We compute the $\text{Aut}(W_{33})$ action on 90 non-isotropic lines, form the augmentation quotient, and compare with the H^3 88D core via traces and characteristic polynomial factor patterns. After twisting by the similitude sign (multiplying the multiplier-2 generator by -1), the modules match; an explicit 88×88 intertwiner T is constructed. (Audit bundles: `W33_perm_module_vs_H3_match_report_bundle.zip`, `W33_H3_to_noniso_line_weights_intertwiner_bundle.zip`)

Theorem 9.4 (Explicit lift to labeled 90-line weights) *There is an explicit linear lift from 88D core coordinates to a labeled 90-entry non-isotropic line field (defined up to adding a constant all-ones vector). Concretely, there exists a 90×88 matrix $M_{H3 \rightarrow 90}$ over \mathbb{Z}_3 such that*

$$w_{90} \equiv M_{H3 \rightarrow 90} x_{88} \pmod{\langle \mathbf{1} \rangle},$$

and the 90 coordinates are indexed by the 4-point line-sets in \mathcal{L} .

Proof sketch / audit trail

A section $L_{88 \rightarrow 90}$ of the augmentation quotient is constructed and composed with the 88D intertwiner T to yield $M_{H^3 \rightarrow 90}$. The resulting 90-vector is unique up to addition of a constant, reflecting the quotient by $\langle \mathbf{1} \rangle$. Line labeling is provided by the explicit 90 line list. (Audit bundle: `W33_lift_to_90_line_weights_with_labels_bundle.zip`.)

Key Result

This section fixes the representation-theoretic meaning of the flux lattice: the nontrivial 88D core of H^3 is (up to the canonical similitude sign) the augmentation quotient on the 90 non-isotropic lines. In particular, the “vacuum cells” that classify flat holonomy also carry the matter/flux degrees of freedom.

10 2-qutrit Weyl operators and the symplectic commutator

Definition

Let $\omega := e^{2\pi i/3}$. On \mathbb{C}^3 with computational basis $\{|j\rangle : j \in \mathbb{Z}_3\}$ define

$$X|j\rangle = |j+1\rangle, \quad Z|j\rangle = \omega^j |j\rangle,$$

so that $ZX = \omega XZ$. On two qutrits, for $(a, b, c, d) \in \mathbb{F}_3^4$, define the (unnormalized) Weyl operator

$$W(a, b, c, d) := X^a Z^c \otimes X^b Z^d.$$

Definition

Define the standard symplectic form on $V = \mathbb{F}_3^{2n}$ with $n = 2$ by writing $v = (p \mid q)$ with $p, q \in \mathbb{F}_3^2$ and

$$\langle (p \mid q), (p' \mid q') \rangle := p \cdot q' - q \cdot p' \in \mathbb{F}_3.$$

In coordinates $v = (a, b, c, d)$ and $w = (a', b', c', d')$, this is

$$\langle v, w \rangle = ac' + bd' - ca' - db'.$$

Theorem 10.1 (Weyl commutator phase) *For all $v, w \in \mathbb{F}_3^4$,*

$$W(v) W(w) = \omega^{\langle v, w \rangle} W(w) W(v).$$

Equivalently, $W(v)$ and $W(w)$ commute if and only if $\langle v, w \rangle = 0$.

Proof sketch / audit trail

This is the standard Heisenberg–Weyl relation for odd prime dimension. For the above unnormalized convention, it follows from $ZX = \omega XZ$ on each tensor factor and bilinearity of the commutator exponent.

Key Result

The same symplectic form used to build $W(3,3)$ is exactly the commutator phase form in the 2-qutrit Weyl group. This is the first canonical bridge from W33 geometry to quantum operator algebra.

11 Projective points as Weyl directions

Definition

Let $\mathbb{P}(V) = PG(3,3)$ denote projective 1D subspaces of $V = \mathbb{F}_3^4$. A projective point $[v]$ is the equivalence class $\{v, 2v\}$ for any nonzero $v \in V$.

Theorem 11.1 (Projective points correspond to cyclic Weyl subgroups) *Each projective point $[v] \in PG(3,3)$ determines a cyclic order-3 Weyl subgroup*

$$\langle W(v) \rangle = \{I, W(v), W(2v)\}.$$

Moreover, $W(2v) = W(v)^{-1}$ and the subgroup depends only on $[v]$ (not the representative).

Proof sketch / audit trail

In \mathbb{F}_3 , $2 \equiv -1$ and $W(2v) = W(-v) = W(v)^{-1}$ (up to global phase, fixed by convention). Thus $\langle W(v) \rangle$ depends only on the projective class $\{v, -v\}$.

Remark

In the W33 tower, the 40 vertices are precisely the 40 projective points of $PG(3,3)$. Thus W33 vertices can be read as 40 “Pauli directions” (cyclic order-3 Weyl subgroups) for two qutrits.

12 Isotropic lines as maximal commuting contexts

Definition

A 2D subspace $U \leq V$ is *totally isotropic* if $\langle u, u' \rangle = 0$ for all $u, u' \in U$. Its projectivization is a projective line containing 4 projective points.

Theorem 12.1 (Isotropic lines give commuting Pauli contexts) *If $U \leq V$ is a totally isotropic 2D subspace, then $\{W(u) : u \in U\}$ is an abelian subgroup of the 2-qutrit Weyl group of order $3^2 = 9$ (including identity). Equivalently, the 4 projective points on the line correspond to 4 nontrivial cyclic subgroups whose nontrivial elements pairwise commute.*

Proof sketch / audit trail

If U is totally isotropic, then $\langle u, u' \rangle = 0$ for all $u, u' \in U$, so $W(u)$ commutes with $W(u')$ by Theorem ???. Since $U \cong \mathbb{F}_3^2$, the set $\{W(u) : u \in U\}$ has 9 elements.

Remark

The symplectic generalized quadrangle $W(3, 3)$ consists precisely of 40 points and 40 totally isotropic projective lines. Thus the GQ lines are canonical maximal commuting Pauli contexts in the 2-qutrit Weyl group.

13 Non-isotropic lines as canonical phase cells

Definition

A projective line (2D subspace) U is *non-isotropic* if $\langle \cdot, \cdot \rangle|_U$ is nondegenerate. In this case, there exist $u, u' \in U$ with $\langle u, u' \rangle = 1$, generating a Heisenberg pair.

Theorem 13.1 (Non-isotropic lines contain conjugate pairs) *Let $U \leq V$ be a non-isotropic 2D subspace. Then there exist $u, u' \in U$ such that $\langle u, u' \rangle = 1$, and hence*

$$W(u)W(u') = \omega W(u')W(u).$$

Proof sketch / audit trail

Nondegeneracy of $\langle \cdot, \cdot \rangle|_U$ implies there exists a basis with symplectic form matrix $(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$ on U . Choosing u, u' as basis vectors yields $\langle u, u' \rangle = 1$.

Remark

In the W33 tower, $PG(3, 3)$ has 130 lines total: 40 isotropic (GQ) and 90 non-isotropic. The “90” distinguished by the quotient holonomy are exactly these non-isotropic lines.

14 Clifford normalizer and the W33 automorphism action

Theorem 14.1 (Clifford induces symplectic action) *Let \mathcal{C} denote the 2-qutrit Clifford group (normalizer of the Weyl group in $U(9)$). Then conjugation by any $U \in \mathcal{C}$ induces a linear transformation $M \in Sp(4, 3)$ on phase space such that*

$$UW(v)U^\dagger = \omega^{\kappa(v)} W(Mv).$$

Conversely, each $M \in Sp(4, 3)$ is induced by some Clifford up to phase.

Proof sketch / audit trail

Standard result for odd prime-power dimension: the Clifford group projects onto the symplectic group acting on discrete phase space, with kernel the Heisenberg–Weyl phases.

15 Holonomy equals commutator phase: a falsifiable conjecture

Definition

Define the symplectic “triangle phase” functional on three phase points $u, v, w \in V$ by

$$\Phi(u, v, w) := \langle u, v \rangle + \langle v, w \rangle + \langle w, u \rangle \in \mathbb{F}_3.$$

Theorem 15.1 (Closed-loop phase identity) *For any $u, v, w \in V$ with $u + v + w = 0$, the triple Weyl product has the form*

$$W(u) W(v) W(w) = \omega^{\Phi(u,v,w)} I$$

up to a global convention factor (which can be fixed by choosing standard displacement operators).

Proof sketch / audit trail

Use the Weyl multiplication law and bilinearity: $W(u)W(v)$ equals a scalar times $W(u+v)$. If $u+v+w=0$, then $W(u+v)W(w)$ is scalar times identity. Exponents combine to the cyclic sum $\Phi \pmod{3}$.

Theorem 15.2 (Holonomy-phase conjecture (testable)) *Let $Q = \overline{W33}$ be the 40-vertex quotient graph produced by the globally gauge-fixed signed lift, with each triangle (p, q, r) assigned a holonomy value $F(p, q, r) \in \mathbb{Z}_3$ (identity vs 3-cycle orientation). There exists a projective representative assignment $p \mapsto [v_p] \in PG(3, 3)$, and representative choices $v_p \in V$, such that for every triangle,*

$$F(p, q, r) \equiv \Phi(v_p, v_q, v_r) \pmod{3},$$

up to the standard gauge ambiguity corresponding to adding a constant all-ones vector in the 90-line weight model.

Protocol (testable)

Protocol: verifying Theorem ??.

1. Use the explicit projective representatives for the 40 points in $PG(3, 3)$ (present in the symplectic audit bundle).
2. Compute $\Phi(v_p, v_q, v_r)$ for all 3240 triangles of Q .
3. Compare to the computed holonomy values (identity/3-cycle with orientation) on the same triangle list.
4. If a mismatch occurs only by a constant shift (global gauge), quotient out by the all-ones line and recompare.
5. If mismatches persist with nonconstant residuals, the conjecture fails and the representative assignment must be refined (or the holonomy is not a pure symplectic cocycle).

Artifact Index (quantum layer)

11 The quotient as a simplicial gauge system

Definition

Let $Q = \overline{W33}$ be the 40-vertex quotient graph obtained by collapsing the 40 flat triples in the globally gauge-fixed $240 \rightarrow 120$ lift. Let $\text{Cl}(Q)$ denote the clique (flag) complex of Q . Then:

$$C^2 := \mathbb{Z}_3^{\{\text{triangles of } Q\}} \cong \mathbb{Z}_3^{3240}, \quad C^3 := \mathbb{Z}_3^{\{\text{tetrahedra } (K_4) \text{ of } Q\}} \cong \mathbb{Z}_3^{9450}.$$

Let $d : C^2 \rightarrow C^3$ be the simplicial coboundary map.

Definition

The quotient construction assigns to each triangle (p, q, r) a holonomy value $F(p, q, r) \in \mathbb{Z}_3$ (identity vs 3-cycle orientation). We view this as a 2-cochain

$$F \in C^2(\text{Cl}(Q); \mathbb{Z}_3).$$

Define the sourced 3-cochain

$$J := dF \in C^3(\text{Cl}(Q); \mathbb{Z}_3),$$

which assigns a flux/charge value to each tetrahedron.

Theorem 11.1 (Sourced curvature) $J = dF$ is supported on exactly 3008 tetrahedra:

$$\#\{t : J(t) \neq 0\} = 3008,$$

with flux distribution $J = 1$ on 1512 tetrahedra and $J = 2$ on 1496 tetrahedra. Moreover, the 90 tetrahedra corresponding to the 90 non-isotropic projective lines (vacuum cells) all satisfy $J = 0$.

Proof sketch / audit trail

This was computed by exhaustive enumeration of all 9450 tetrahedra in Q and evaluation of the simplicial coboundary formula

$$(dF)(a, b, c, d) = F(b, c, d) - F(a, c, d) + F(a, b, d) - F(a, b, c) \pmod{3}.$$

The 90 non-isotropic line tetrahedra were identified as the unique K_4 cliques whose 4 triangular faces are all flat (holonomy 0). All have $J = 0$. (Audit bundles: `W33_minimal_Z3_flux_cycles_tetrahedra_bundle.zip`, `W33_charge_decomposition_and_line_moments_bundle.zip`.)

Key Result

The quotient holonomy F is a genuine *sourced* field strength: its 3-coboundary $J = dF$ is the discrete charge/current, with vacuum cells (non-isotropic lines) exactly flux-free.

12 Vacuum sector: the 90 non-isotropic lines

Definition

Let \mathcal{L} denote the 90 non-isotropic projective lines in $PG(3, 3)$, each a 4-point set in the 40-point geometry. These 90 lines are in bijection with:

- the 90 K_4 cliques in Q whose four triangular faces are flat,
- the $\text{Aut}(W33)$ -distinguished vacuum cells for the quotient connection.

We identify the vacuum line field space with $\mathbb{Z}_3^{\mathcal{L}} \cong \mathbb{Z}_3^{90}$.

Remark

Because $90 \equiv 0 \pmod{3}$, the constant all-ones vector lies in the \mathbb{Z}_3 augmentation subspace. Thus quotienting by the all-ones line produces the canonical 88-dimensional vacuum/matter module used in the H^3 identification.

13 Transfer operators from sources to vacuum observables

Definition

Partition tetrahedra in Q into three $\text{Aut}(W33)$ -orbits by the number of flat faces:

bulk: #flat faces = 0 (6480), boundary: #flat faces = 1 (2880), vacuum: #flat faces = 4 (90).

In the boundary orbit, each tetrahedron has a *unique* flat face, hence a unique attached vacuum line $\ell \in \mathcal{L}$.

Definition

Define two linear maps over \mathbb{Z}_3 :

$$M : \mathbb{Z}_3^{9450} \rightarrow \mathbb{Z}_3^{90}, \quad Z : \mathbb{Z}_3^{9450} \rightarrow \mathbb{Z}_3^{90}.$$

They are defined on a tetrahedron t as follows:

1. **(Boundary moment M)** If t has exactly one flat face, let $\ell(t)$ be its unique attached non-isotropic line. Then M adds the tetra flux $J(t)$ to coordinate $\ell(t)$. Otherwise t contributes 0.
2. **(Bulk shadow Z)** For each *curved* triangular face of t , push $J(t)$ along the three edges of that face. Each edge of Q belongs to a unique non-isotropic line in \mathcal{L} (since $540 = 90 \cdot 6$). Summing these contributions defines $Z(J)$ on \mathcal{L} .

Theorem 13.1 (Exact transfer identities) *Let $J = dF$ be the sourced 3-cochain. Then the two observed vacuum line fields*

$$m_{\text{line}} \in \mathbb{Z}_3^{90}, \quad z_{\text{line}} \in \mathbb{Z}_3^{90}$$

satisfy the exact operator identities

$$m_{\text{line}} = M J, \quad z_{\text{line}} = Z J,$$

with no residual error.

Proof sketch / audit trail

Both operators were constructed explicitly in sparse COO form and applied to the computed J . The resulting 90-vectors agree entrywise with the independently computed line observables from the earlier operator chains:

$$m_{\text{line}} = C_{\text{lineface}} J, \quad z_{\text{line}} = R(K_0 + K_1) J.$$

(Audit bundle: `W33_transfer_operators_J_to_lines_and_mode_injection_bundle.zip`.)

Key Result

The W33 quotient admits explicit, $\text{Aut}(\text{W33})$ -equivariant transfer operators from sources J to vacuum line observables. This is the discrete analog of a constitutive relation (sources \rightarrow observed vacuum response).

14 Vacuum harmonics and mode-resolved response

Definition

The $\text{Aut}(\text{W33})$ commutant algebra acting on $\mathbb{Z}_3^{\mathcal{L}}$ has dimension 5 (an association scheme). Equivalently, the 90-line sector admits a canonical decomposition into 5 joint harmonic modes under the commuting operators:

- S : the Aut -invariant fixed-point-free involution pairing on the 90 lines (45 disjoint transpositions),
- A_{meet} : line meet adjacency (two lines adjacent iff they intersect in a point), degree 32.

Joint modes are indexed by $(\text{sign}(S), \lambda(A_{\text{meet}}))$:

$$(+, 32)^1, \quad (+, 2)^{24}, \quad (+, -4)^{20}, \quad (-, 8)^{15}, \quad (-, -4)^{30}.$$

Theorem 14.1 (Mode-resolved injection table) *For each tetra orbit class (bulk vs boundary) and each flux sign $J \in \{1, 2\}$, the induced vacuum responses $M(J)$ and $Z(J)$ decompose into the above 5 modes with explicit energy fractions. In particular:*

- Bulk sources (flat-face count 0) inject only into z_{line} (never into m_{line}).
- Boundary sources (flat-face count 1) inject into both m_{line} and z_{line} , with mode weights shifted toward $(+, 2)$ and $(-, 8)$ for m_{line} .

Proof sketch / audit trail

This was computed by restricting J to each class+flux, applying the exact transfer operators M and Z , mapping \mathbb{Z}_3 entries to real values $\{-1, 0, 1\}$ (with $2 \mapsto -1$), removing the mean, and projecting onto the orthonormal joint-mode bases. The resulting mode-energy fractions are tabulated. (Audit bundle: `W33_mode_response_table_bulk_to_vacuum_bundle.zip`.)

Key Result

The vacuum sector is not a single “channel”: bulk and boundary sources excite different vacuum harmonics. This explains why no Aut-equivariant line-only operator can strongly predict m from z (they are distinct projections of the same bulk source field).

Artifact Index (field-equation layer)

Bundle

Contents / Purpose

15 Vacuum association scheme and canonical harmonics

Theorem 15.1 (90-line association scheme and involution) *The $\text{Aut}(W33)$ action on the 90 non-isotropic lines induces an association scheme with commutant dimension 5 (five orbitals on ordered pairs). One orbital is the diagonal; another is a fixed-point-free involution σ pairing the 90 lines into 45 disjoint transpositions, with each paired lineset disjoint (skew).*

Theorem 15.2 (Five canonical harmonics) *Let S be the permutation matrix of σ and let A_{meet} be the adjacency of the line-meet graph (degree 32). Then S and A_{meet} commute and admit a joint decomposition into five modes:*

$$(+, 32)^1, \quad (+, 2)^{24}, \quad (+, -4)^{20}, \quad (-, 8)^{15}, \quad (-, -4)^{30}.$$

These modes provide the canonical “vacuum harmonics” for line fields.

Remark

This harmonic analysis explains why distinct vacuum observables (e.g., boundary moment m vs bulk shadow z) are not related by a single Aut-equivariant line-only operator: they occupy different mixtures of the canonical modes. The correct dynamics closes only when bulk source variables $J = dF$ and the transfer operators M, Z are included.

A Global Artifact Index

Bundle	Contents / Purpose
W33_symplectic_audit_bundle.zip	Explicit construction of $W(3, 3)$ and W33; point/line incidence; $PG(3, 3)$ points; isotropic vs nonisotropic line lists; verification of SRG parameters and spectrum.
W33_orbits_squarezero_bundle.zip	$\text{Aut}(W33)$ generators (permutations and GF(3) matrices); orbit computations; square-zero and symmetry checkpoints.
W33_GF2_kernel_code_bundle.zip	The [40, 24, 6] kernel code $\ker(A)$ over \mathbb{F}_2 ; 240 weight-6 generators; code basis and supporting tables.
W33_H8_quadratic_form_bundle.zip	Basis of $H = \ker(A)/\text{im}(A)$; invariant quadratic form q ; orbit split (135 singular / 120 nonsingular).
W33_to_H_to_120root_SRG_bundle.zip	The 120 nonsingular orbit list; SRG(120,56,28,24) edges/adjacency; mappings from code generators to H .
W33_E8_simple_root_system_bundle.zip	Canonical induced E_8 Dynkin configuration inside the 120-root SRG; Coxeter checks; reflection orbit generation.
W33_signed_root_cocycle_and_lift_bundle.zip	Signed lift/cocycle computations on 120-root edges and Steiner triples; defect weights; gauge studies.
W33_global_gaugefix_no16_bundle.zip	Global sign/gauge fix removing all weight-16 defects; resulting 0,12 defect spectrum; 40 flat triples.
W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip	Quotient $Q = \overline{W33}$; edge matchings; triangle holonomy values; proof that flat holonomy triangles are exactly nonisotropic line triples.
W33_Z3_curvature_cohomology_on_quotient_bundle.zip	Triangle curvature cochain over \mathbb{Z}_3 ; non-exactness on the 2-skeleton; supporting tables.
W33_minimal_Z3_flux_cycles_tetrahedra_bundle.zip	Minimal-support flux cycles (tetrahedron boundaries) and flux statistics for $J = dF$.
W33_flux_lattice_clique_complex_Z3_cohomology_bundle.zip	Clique-complex cohomology ranks and dimensions over \mathbb{Z}_3 ; H^3 dimension 89; higher cohomology signature.
W33_H3_basis_89_Z3_on_clique_complex_bundle.zip	Explicit 89-element basis for H^3 as sparse tetra-cochains; pivot/free coordinate metadata.
W33_H3_Aut_action_89Z3_bundle.zip	$\text{Aut}(W33)$ action matrices on H^3 ; 88+1 decomposition; quotient functional and block form.
W33_perm_module_vs_H3_match_report_bundle.zip	Evidence and generators showing the 88D core matches the 90-line augmentation quotient up to the similitude sign twist.
W33_H3_to_noniso_line_weights_intertwiner_bundle.zip	Explicit intertwiner between H^3 88D core and the twisted 90-line augmentation quotient.
W33_lift_to_90_line_weights_with_labels_bundle.zip	Explicit lift to labeled 90 nonisotropic line weights (mod all-ones gauge); line_id to 4-point set.
W33_holonomy_phase_test_bundle.zip	Holonomy vs symplectic triangle phase test; shows background closed 2-form vs sourced curvature.
W33_current_operator_C_lineface_bundle.zip	Operator C_{lineface} and line-moment statistics (source attachments to vacuum cells).
W33_bulk_operator_KOK1_curved_triangle_current_bundle.zip	Bulk current operators on curved triangles (K_0, K_1); outputs y on the 2880 curved triangle orbit.
W33_curved_triangle_to_noniso_line_operator_R_bundle.zip	Operator R mapping curved-triangle current to 90-line aggregates via edge-incidence.
W33_charge_decomposition_and_line_moments_bundle.zip	Charge decomposition $J = dF$; point incidences; preliminary line moments and constraints.

B Global Dictionary Table

Object	Interpretation	Algebra	Geometry/-Topology	Quantum computation	Crypto / security
$V = \mathbb{F}_3^4$	Finite phase space; 2-qutrit discrete symplectic phase space.	Vector space over \mathbb{F}_3 with symplectic form.	Underlying coordinate domain for projective geometry and Weyl operators.	Pauli/Weyl labels; Clifford acts by $Sp(4, 3)$.	Key space for symplectic commutator phase.
$W(3, 3) /$ isotropic lines	Maximal commuting contexts.	Incidence geometry of totally isotropic points/lines.	Produces $W33$ as point graph.	Stabilizer contexts for two qutrits.	Basis for context-based protocols.
$W33 = \text{SRG}(40, 12, 2, 4)$	Base combinatorial geometry.	Adjacency matrix A with SRG identities.	Over \mathbb{F}_2 , yields differential $A^2 = 0$.	Constraint graph / stabilizer structure.	Public structure; secrecy comes from gauge/coset choices.
$A^2 \equiv 0$ over \mathbb{F}_2	Chain-complex calculus.	Defines $d(x) = Ax$ with $d^2 = 0$.	Produces code $\ker(A)$ and homology H .	Error correction / stabilizer relations.	Syndromes / tamper detection.
$H = \ker(A)/\text{im}(A)$ (8D)	Intrinsic state space.	Carries invariant quadratic form; orbit split.	Nonsingular orbit gives 120-root shell.	Finite “root” degrees; phase classes.	Key reduction space for encoding.
120/240 roots	Finite root shell and signed lift.	SRG(120) adjacency via bilinear form; 2-to-1 lift.	Global gauge fixing yields flat triples.	Discrete gauge degrees; lift choices.	Keyed section choices = secrecy.
$Q = \overline{W33}$	Quotient spacetime / interaction graph.	40 meta-vertices after collapse; edge matchings.	Supports \mathbb{Z}_3 holonomy.	Transport/holonomy = topological gate.	Holonomy checks = authentication.
Holonomy $F \in C^2(\text{Cl}(Q); \mathbb{Z}_3)$	Field strength / curvature.	Triangle cochain valued in \mathbb{Z}_3 .	Flat set classified by 90 nonisotropic lines.	Discrete phase curvature.	Consistency checks / signatures.
Sources $J = dF \in C^3$	Charge/current.	Supported on 3008 tetrahedra.	Generates vacuum responses via M, Z .	Excitations / particles.	Error/fault injection model.
90 nonisotropic lines	Vacuum cells and matter carrier space.	Association scheme (5-mode harmonic analysis).	Line-weight field model (mod all-ones).	Contextual phase cells.	Share space for schemes; 88D core module.
Transfer operators M, Z	Constitutive laws.	Exact maps $J \mapsto (m, z)$.	Mode-resolved response tables.	Measurement/readout operators.	Encryption/readout operators.

C Reproducibility Checklist

Remark

Short SHA-256 prefixes (first 16 hex characters) for primary bundles in the current workspace.

File	SHA-256 prefix
W33_symplectic_audit_bundle.zip	c8f7547649abdab1
W33_orbits_squarezero_bundle.zip	84835a9889e4380b
W33_GF2_kernel_code_bundle.zip	952858afb5d65007
W33_H8_quadratic_form_bundle.zip	de3a9a9b0afb6a37
W33_to_H_to_120root_SRG_bundle.zip	3257de84a4b9c466
W33_E8_simple_root_system_bundle.zip	d200bec6ff81f00a
W33_signed_root_cocycle_and_lift_bundle.zip	d33146ea2d96104f
W33_global_gaugefix_no16_bundle.zip	8de8d1182056ac00
W33_quotient_closure_complement_and_noniso_line_curvature_bundle.zip	8a6cda139ed0a0e6
W33_Z3_curvature_cohomology_on_quotient_bundle.zip	1a7804dd46ccb1b5
W33_minimal_Z3_flux_cycles_tetrahedra_bundle.zip	8d69efdc34b5a0e6
W33_flux_lattice_clique_complex_Z3_cohomology_bundle.zip	17f5bb8490fc2d36
W33_H3_basis_89_Z3_on_clique_complex_bundle.zip	2fa53b14fcd57da9
W33_H3_Aut_action_89Z3_bundle.zip	032be0e14f33c5cc
W33_perm_module_vs_H3_match_report_bundle.zip	535aa4d6b03264d9
W33_H3_to_noniso_line_weights_intertwiner_bundle.zip	da15db795acf478b
W33_lift_to_90_line_weights_with_labels_bundle.zip	81b9f049398d5f93
W33_holonomy_phase_test_bundle.zip	5991ca050359bc4b
W33_current_operator_C_lineface_bundle.zip	02e3566e1869ce07
W33_bulk_operator_K0K1_curved_triangle_current_bundle.zip	5953f1541d2793f1
W33_curved_triangle_to_noniso_line_operator_R_bundle.zip	633e86c28d6433cf
W33_charge_decomposition_and_line_moments_bundle.zip	d9c00f5e46ca2658
W33_nonisotropic_line_association_scheme_bundle.zip	ec4b4b8e10918586
W33_vacuum_line_scheme_mode_decomposition_bundle.zip	d8545a6b843ab310
W33_transfer_operators_J_to_lines_and_mode_injection_bundle.zip	647e18c9a6ac8f7c
W33_best_field_equation_operator_on_lines_bundle.zip	3494bf1e74c08f1b