Soal dan Solusi UTS Fungsi Khusus 2024

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Question 1

Hitunglah fungsi Gamma:

(a). $\Gamma(-2.3)$.

(b). $\int\limits_0^\infty x^5 e^{-4x}\, dx$.

Penyelesaian.

(a). Menggunakan sifat $\Gamma(n+1) = n\Gamma(n) \iff \Gamma(n) = \frac{\Gamma(n+1)}{n}$, diperoleh

$$\Gamma(-2.3) = \frac{\Gamma(-1.3)}{(-2.3)} = \frac{\Gamma(-0.3)}{(-2.3)(-1.3)} = \frac{\Gamma(0.7)}{(-2.3)(-1.3)(-0.3)} = \frac{\Gamma(1.7)}{(-2.3)(-1.3)(-0.3)(0.7)}.$$

Berdasarkan $\Gamma(1.7) = 0.90864$, diperoleh $\Gamma(-2.3) = -1.44711$.

(b). Misalkan $u = 4x$, maka $du = 4\, dx$. Diperoleh

$$\int\limits_0^\infty x^5 e^{-4x}\, dx = \int\limits_0^\infty \left(\frac{u}{4}\right)^5 e^{-u}\, \frac{du}{4} = \frac{1}{4^6}\int\limits_0^\infty u^5 e^{-u}\, du = \frac{\Gamma(6)}{4^6} = \frac{5!}{4^6} = \frac{30}{4^5}.$$

Question 2

Hitunglah fungsi Beta:

(a). $\int\limits_0^{\pi/2} \sqrt{\cot(x)}\, dx$.

(b). $\int\limits_2^5 \frac{dx}{\sqrt{(x-2)(5-x)}}$.

Penyelesaian.

Akan digunakan fakta $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

(a). Perhatikan bahwa

$$\int\limits_0^{\pi/2} \sqrt{\cot(x)}\, dx = \int\limits_0^{\pi/2} \Big(\cos(x)\Big)^{1/2} \Big(\sin(x)\Big)^{-1/2} dx.$$

Perhatikan bahwa

$$\int\limits_0^{\pi/2} \Big(\cos(x)\Big)^{2m-1} \Big(\sin(x)\Big)^{2n-1} dx = \frac{1}{2} B(m,n).$$

Dengan memerhatikan nilai m dan n yang memenuhi $2m-1 = \frac{1}{2}$ dan $2n-1 = -\frac{1}{2}$, diperoleh $m = \frac{3}{4}$ dan $n = \frac{1}{4}$. Jadi,

$$\int\limits_0^{\pi/2} \Big(\cos(x)\Big)^{1/2} \Big(\sin(x)\Big)^{-1/2} dx = \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)} = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2}$$

mengingat $\Gamma(1) = 1$. Menurut Euler Reflection Formula, yaitu $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ untuk $z \in \mathbb{C}$ dengan $\text{Re}(z) > 0$, dapat diperoleh bahwa untuk $z = \frac{1}{4}$ berlaku

$$\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\frac{1}{2}\sqrt{2}} = \frac{2\pi}{\sqrt{2}} = \pi\sqrt{2}.$$

Jadi, $\int\limits_0^{\pi/2} \sqrt{\cot(x)}\, dx = \frac{1}{2} \cdot \pi\sqrt{2} = \frac{\pi}{2}\sqrt{2}$.

(b). Misalkan $x = 3u + 2$, maka $dx = 3\, du$ dan diperoleh

$$\int\limits_2^5 \frac{dx}{\sqrt{(x-2)(5-x)}} = \int\limits_0^1 \frac{3\, du}{\sqrt{3u(3-3u)}} = \int\limits_0^1 \frac{du}{\sqrt{u(1-u)}} = \int\limits_0^1 u^{1/2}(1-u)^{1/2}\, du.$$

Tinjau fakta bahwa $\int\limits_0^1 x^{m-1}(1-x)^{n-1} = B(m,n)$. Dengan memerhatikan nilai m dan n yang memenuhi $m-1 = \frac{1}{2}$ dan $n-1 = \frac{1}{2}$, diperoleh $m = n = \frac{3}{2}$. Ini berarti

$$\int\limits_0^1 u^{1/2}(1-u)^{1/2}\, du = B\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{3}{2}\right)} = \frac{\left(\Gamma\left(\frac{3}{2}\right)\right)^2}{\Gamma(3)} = \frac{\left(\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right)^2}{2!} = \frac{\frac{1}{4}\pi}{2} = \frac{\pi}{8}$$

mengingat $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

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Question 3

Menggunakan relasi rekurensi fungsi Mittag-Leffler,

$$E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)},$$

tentukan $\frac{d}{dz}\left(z^2 E_{1,3}(z)\right)$.

Penyelesaian.

Perhatikan bahwa

$$E_{1,2}(z) = zE_{1,3}(z) + \frac{1}{\Gamma(2)} = zE_{1,3}(z) + 1 \implies z^2 E_{1,3}(z) = zE_{1,2}(z) - z.$$

Diperoleh pula

$$E_{1,1}(z) = zE_{1,2}(z) + \frac{1}{\Gamma(1)} = zE_{1,2}(z) + 1 \implies zE_{1,2}(z) = E_{1,1}(z) - 1.$$

Dari sini diperoleh

$$z^2 E_{1,3}(z) = zE_{1,2}(z) - z = E_{1,1}(z) - z - 1.$$

Perhatikan bahwa

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

Dari sini diperoleh $z^2 E_{1,3}(z) = e^z - z - 1$ yang memberikan

$$\frac{d}{dz}\left(z^2 E_{1,3}(z)\right) = \frac{d}{dz}\left(e^z - z - 1\right) = e^z - 1.$$

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Question 4

(a). Menggunakan rumus Rodrigues, hitunglah polinom Legendre $P_3(x)$.

(b). Hitung $P_3^1(0.5)$ jika polinom Legendre associate derajat n dan orde m didefinisikan dengan

$$P_n^m(x) = (-1)^m \left(1-x^2\right)^{m/2} \frac{d^m}{dx^m} P_n(x).$$

Penyelesaian.

(a). Akan digunakan $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2-1\right)^n$ sehingga diperoleh $P_3(x) = \frac{1}{8 \cdot 6} \frac{d^3}{dx^3} \left(x^2-1\right)^3$. Maka

$$\frac{d^3}{dx^3}\left(x^6-3x^4+3x^2+1\right) = \frac{d^2}{dx^2}\left(6x^5-12x^3+6x\right) = \frac{d}{dx}\left(30x^4-36x^2+6\right) = 120x^3-72x.$$

Ini berarti $P_3(x) = \frac{1}{48}\left(120x^3-72x\right) = \frac{5x^3-3x}{2}$.

(b). Diperoleh $P_3^1(x) = (-1)^1 \left(1-x^2\right)^{1/2} \frac{d}{dx}\left(\frac{5x^3-3x}{2}\right)$ yang memberikan

$$P_3^1(x) = -\left(1-x^2\right)^{1/2} \cdot \frac{15x^2-3}{2} = \frac{3}{2}\left(1-5x^2\right)\left(1-x^2\right)^{1/2}.$$

Diperoleh pula

$$P_3^1(0.5) = \frac{3}{2}\left(1-\frac{5}{4}\right)\sqrt{1-\frac{1}{4}} = \frac{3}{2}\left(-\frac{1}{4}\right) \cdot \frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{16}.$$

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