Soal dan Solusi UAS Persamaan Diferensial Biasa 2023

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Matematika 2022

Question 1

Selesaikan persamaan diferensial berikut dengan menggunakan deret pangkat

$$2y' + y = 1.$$

Penyelesaian.

Misalkan $y(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ di mana $a_0, a_1, a_2, \cdots$ suatu konstan. Dari sini diperoleh bahwa $y' = \sum\limits_{n=1}^{\infty} a_n n x^{n-1} = \sum\limits_{n=0}^{\infty} a_{n+1}(n+1)x^n$, maka

$$1 = 2y' + y = 2\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\Big(2a_{n+1}(n+1) + a_n\Big)x^n.$$

Ini berarti $2a_1 + a_0 = 1 \iff a_1 = \frac{1-a_0}{2}$ dan $2(n+1)a_{n+1} + a_n = 0 \iff \frac{a_{n+1}}{a_n} = -\frac{1}{2(n+1)}$ untuk setiap bilangan asli n. Untuk setiap bilangan asli n,

$$\begin{aligned} a_{n+1} &= \frac{a_{n+1}}{a_n} \cdot \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdot \ldots \cdot \frac{a_2}{a_1} \cdot a_1 \\ &= \left(-\frac{1}{2(n+1)}\right)\left(-\frac{1}{2n}\right)\left(-\frac{1}{2(n-1)}\right)\cdots\left(-\frac{1}{2(2)}\right) a_1 \\ &= (-1)^n \cdot \frac{1}{2^n(n+1)!} \cdot \frac{1-a_0}{2} \\ &= (-1)^n \frac{1-a_0}{2^{n+1}(n+1)!}. \end{aligned}$$

Tinjau $a_1 = \frac{1-a_0}{2} = (-1)^0 \cdot \frac{1-a_0}{2^1 \cdot 1!}$, dari sini dapat disimpulkan $a_n = (-1)^{n-1} \cdot \frac{1-a_0}{2^n \cdot n!}$. Maka

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1-a_0}{2^n \cdot n!} x^n = a_0 + \sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n \cdot \frac{1}{-1} \cdot \frac{1-a_0}{n!} \\ &= a_0 + \sum_{n=1}^{\infty}\left(-\frac{x}{2}\right)^n \frac{a_0-1}{n!} = a_0 + \sum_{n=1}^{\infty}\left(-\frac{x}{2}\right)^n \frac{a_0}{n!} - \sum_{n=1}^{\infty}\left(-\frac{x}{2}\right)^n \frac{1}{n!} \\ &= a_0 \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{x}{2}\right)^n - \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{x}{2}\right)^n + 1 \\ &= a_0 e^{-\frac{x}{2}} - e^{-\frac{x}{2}} + 1 \\ &= (1-a_0)e^{-\frac{x}{2}} + 1 \\ &= Ce^{-\frac{x}{2}} + 1 \end{aligned}$$

di mana $C = 1 - a_0$ suatu konstan.

Catatan. Penentuan $\sum\limits_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{x}{2}\right)^n = e^{-\frac{x}{2}}$ diperoleh berdasarkan Deret Mclaurin. Deret Mclaurin dari $f(x)$ dinyatakan sebagai

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots .$$

Secara khusus, untuk $f(x) = e^{-\frac{x}{2}}$, dapat diperoleh bahwa $f^{(n)}(x) = \left(-\frac{1}{2}\right)^n e^{-\frac{x}{2}} \implies f^{(n)}(0) = \left(-\frac{1}{2}\right)^n$, diperoleh

$$e^{-\frac{x}{2}} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)^n}{n!}x^n = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{x}{2}\right)^n .$$

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Question 2

Tentukan $f(t)$ bila diketahui
$$F(s) = \frac{s+4}{s^2+4s+5} + \frac{e^{-2s}}{s+1}.$$

Penyelesaian.

Perhatikan bahwa

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left(\frac{s+4}{s^2+4s+5} + \frac{e^{-2s}}{s+1}\right) \\
&= \mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+1} + \frac{2}{(s+2)^2+1} + \frac{e^{-2s}}{s+1}\right) \\
&= \mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+1}\right) + \mathcal{L}^{-1}\left(\frac{2}{(s+2)^2+1}\right) + \mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+1}\right) \\
&= \mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+1}\right) + 2\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2+1}\right) + \mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+1}\right).
\end{aligned}$$

Perhatikan bahwa $A(s) = \frac{s}{s^2+1} \iff a(t) = \mathcal{L}^{-1}(A(s)) = \cos(t)$, tinjau

$$\frac{s+2}{(s+2)^2+1} = A(s+2) = \mathcal{L}\left(e^{-2t}a(t)\right) \iff \mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+1}\right) = e^{-2t}a(t) = e^{-2t}\cos(t).$$

Perhatikan bahwa $B(s) = \frac{1}{s^2+1} \iff b(t) = \mathcal{L}^{-1}(B(s)) = \sin(t)$, tinjau

$$\frac{1}{(s+2)^2+1} = B(s+2) = \mathcal{L}\left(e^{-2t}b(t)\right) \iff \mathcal{L}^{-1}\left(\frac{1}{(s+2)^2+1}\right) = e^{-2t}b(t) = e^{-2t}\sin(t).$$

Perhatikan bahwa $C(s) = \frac{1}{s} \iff c(t) = \mathcal{L}^{-1}(C(s)) = 1$, tinjau

$$\frac{e^{-2s}}{s} = e^{-2s}C(s) = \mathcal{L}\left(u_2(t)c(t-2)\right).$$

Dengan memisalkan $d(t) = u_2(t)c(t-2) = u_2(t)\cdot 1 = u_2(t)$, ini berarti $\mathcal{L}\left(e^{-t}d(t)\right) = D(t+1)$ di mana $D(s) = \mathcal{L}(d(t)) = e^{-2s}C(s)$. Ini berarti

$$\mathcal{L}\left(e^{-t}d(t)\right) = D(s+1) = e^{-2(s+1)}C(s+1) = \frac{e^{-2s-2}}{s+1} = e^{-2}\cdot\frac{e^{-2s}}{s+1}.$$

Diperoleh

$$e^{-t}d(t) = \mathcal{L}^{-1}\left(e^{-2}\cdot\frac{e^{-2s}}{s+1}\right) = e^{-2}\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+1}\right) \implies \mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+1}\right) = e^2e^{-t}d(t) = e^{2-t}u_2(t).$$

Jadi,

$$f(t) = e^{-2t}\cos(t) + 2e^{-2t}\sin(t) + e^{2-t}u_2(t) = \begin{cases} e^{-2t}\cos(t) + 2e^{-2t}\sin(t), & t < 2 \\ e^{-2t}\cos(t) + 2e^{-2t}\sin(t) + e^{2-t}, & t \geq 2 \end{cases}.$$

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Question 3

Selesaikan masalah nilai awal berikut dengan mengunakan transformasi Laplace

$$y''' - 2y'' + 9y' - 18y = 4e^t, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1.$$

Penyelesaian.

Misalkan $\mathcal{L}(y(t)) = Y(s)$. Ambil transformasi Laplace di kedua ruas,

$$\begin{aligned}
\mathcal{L}\left(4e^t\right) &= \mathcal{L}\left(y''' - 2y'' + 9y' - 18y\right) \\
4\mathcal{L}\left(e^t\right) &= \mathcal{L}\left(y'''\right) - \mathcal{L}\left(2y''\right) + \mathcal{L}\left(9y'\right) - \mathcal{L}(18y) \\
4 \cdot \frac{1}{s-1} &= \mathcal{L}\left(y'''\right) - 2\mathcal{L}\left(y''\right) + 9\mathcal{L}\left(y'\right) - 18\mathcal{L}(y) \\
\frac{4}{s-1} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) - 2\left(s^2Y(s) - sy(0) - y'(0)\right) + 9\left(sY(s) - y(0)\right) - 18Y(s) \\
\frac{4}{s-1} &= s^3Y(s) - 0 - 0 - 1 - 2\left(s^2Y(s) - 0 - 0\right) + 9\left(sY(s) - 0\right) - 18Y(s) \\
&= s^3Y(s) - 1 - 2s^2Y(s) + 9sY(s) - 18Y(s) \\
&= \left(s^3 - 2s^2 + 9s - 18\right)Y(s) - 1.
\end{aligned}$$

Diperoleh

$$Y(s) = \frac{1}{s^3 - 2s^2 + 9s - 18}\left(\frac{4}{s-1} + 1\right) = \frac{1}{(s-2)\left(s^2+9\right)} \cdot \frac{4+s-1}{s-1} = \frac{s+3}{(s-1)(s-2)\left(s^2+9\right)}.$$

Misalkan

$$\frac{s+3}{(s-1)(s-2)\left(s^2+9\right)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{Cs+D}{s^2+9}$$

di mana A, B, C, D suatu konstan. Dengan menyamakan penyebut, diperoleh

$$s + 3 = A(s-2)\left(s^2+9\right) + B(s-1)\left(s^2+9\right) + (s-1)(s-2)(Cs+D)$$

untuk setiap s. Untuk $s = 1$ diperoleh

$$4 = A(-1)(10) + B(0)(10) + (0)(-1)(C+D) = -10A \iff A = -\frac{4}{10} = -\frac{2}{5}.$$

Untuk $s = 2$,

$$5 = A(0)(13) + B(1)(13) + (1)(0)(2s+D) = 13B \iff B = \frac{5}{13}.$$

Untuk $s = 3$,

$$6 = A(1)(18) + B(2)(18) + (2)(1)(3C+D) = 18\left(-\frac{2}{5}\right) + 36\left(\frac{5}{13}\right) + 2(3C+D)$$

sehingga diperoleh $3C + D = -\frac{21}{65}$. Untuk $s = 0$,

$$3 = A(-2)(9) + B(-1)(9) + (-1)(-2)D = -18\left(-\frac{2}{5}\right) - 9\left(\frac{5}{13}\right) + 2D$$

sehingga $D = -\frac{24}{65}$. Diperoleh

$$C = \frac{1}{3}\left(-\frac{21}{65} - D\right) = \frac{1}{3}\left(-\frac{21}{65} + \frac{24}{65}\right) = \frac{1}{3} \cdot \frac{3}{65} = \frac{1}{65}.$$

Jadi,

$$Y(s) = -\frac{2}{5}\cdot\frac{1}{s-1}+\frac{5}{13}\cdot\frac{1}{s-2}+\frac{\frac{1}{65}s-\frac{24}{65}}{s^2+9} = -\frac{2}{5}\left(\frac{1}{s-1}\right)+\frac{5}{13}\left(\frac{1}{s-2}\right)+\frac{1}{65}\left(\frac{s}{s^2+9}\right)-\frac{8}{65}\left(\frac{3}{s^2+9}\right).$$

Ini berarti

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left[-\frac{2}{5}\left(\frac{1}{s-1}\right)+\frac{5}{13}\left(\frac{1}{s-2}\right)+\frac{1}{65}\left(\frac{s}{s^2+9}\right)-\frac{8}{65}\left(\frac{3}{s^2+9}\right)\right] \\
&= \mathcal{L}^{-1}\left(-\frac{2}{5}\cdot\frac{1}{s-1}\right)+\mathcal{L}^{-1}\left(\frac{5}{13}\cdot\frac{1}{s-2}\right)+\mathcal{L}^{-1}\left(\frac{1}{65}\cdot\frac{s}{s^2+9}\right)-\mathcal{L}^{-1}\left(\frac{8}{65}\cdot\frac{3}{s^2+9}\right) \\
&= -\frac{2}{5}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)+\frac{5}{13}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)+\frac{1}{65}\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right)-\frac{8}{65}\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right) \\
&= -\frac{2}{5}e^t+\frac{5}{13}e^{2t}+\frac{1}{65}\cos(3t)-\frac{8}{65}\sin(3t).
\end{aligned}$$

Pembaca dapat mengeceknya secara manual bahwa solusi ini memenuhi (saya malas). ▼

Question 4

Tentukan solusi umum sistem persamaan diferensial biasa berikut dengan menggunakan nilai eigen dan vektor eigen matriks koefisien dalam sistem PDB

$$\frac{dx}{dt} = 2x - y \quad \text{dan} \quad \frac{dy}{dt} = x + 2y.$$

Penyelesaian.

Sistem PDB dapat ditulis ulang sebagai

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \implies \frac{d\mathbf{x}}{dt} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\mathbf{x}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Substitusi $\mathbf{x} = e^{\lambda t}\mathbf{u}$ di mana $\mathbf{u} \in \mathbb{C}^2 - \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$, diperoleh $\frac{d\mathbf{x}}{dt} = \lambda e^{\lambda t}\mathbf{u}$. Dari sini diperoleh

$$\lambda e^{\lambda t}\mathbf{u} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} e^{\lambda t}\mathbf{u} \iff \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} e^{\lambda t}\mathbf{u} - \lambda e^{\lambda t}\mathbf{u} = \left(\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} - \lambda I\right) e^{\lambda t}\mathbf{u}$$

yang berarti $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix}\mathbf{u}$ karena $e^{\lambda t} \neq 0$. Apabila $\begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} \neq 0$, maka $\begin{bmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix}^{-1}$ ada. Ini berakibat

$$\begin{aligned}
\mathbf{u} = I\mathbf{u} &= \left(\begin{bmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix}^{-1}\begin{bmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix}\mathbf{u}\right) \\
&= \begin{bmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix}^{-1}\left(\begin{bmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix}\mathbf{u}\right) \\
&= \begin{bmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\end{aligned}$$

kontradiksi. Maka haruslah

$$0 = \begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 1 \implies \lambda = 2 \pm i.$$

Untuk $\lambda = 2 + i$ dan misalkan $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, ini berarti

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix}\mathbf{u} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \xrightarrow{r_2 - ir_1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

yang memberikan $\mathbf{u} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. Ini berarti

$$\mathbf{x} = e^{\lambda t}\mathbf{u} = e^{(2+i)t}\mathbf{u} = e^{2t}e^{it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{2t}\Big(\cos(t) + i\sin(t)\Big)\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} e^{2t}\cos(t) + ie^{2t}\sin(t) \\ -ie^{2t}\cos(t) + e^{2t}\sin(t) \end{bmatrix}$$

yang memberikan $\mathbf{x} = \begin{bmatrix} e^{2t}\cos(t) \\ e^{2t}\sin(t) \end{bmatrix} + i \begin{bmatrix} e^{2t}\sin(t) \\ -e^{2t}\cos(t) \end{bmatrix}$. Jadi, solusi umum untuk $\mathbf{x}$ adalah

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \mathbf{x} = C_1 \begin{bmatrix} e^{2t}\cos(t) \\ e^{2t}\sin(t) \end{bmatrix} + C_2 \begin{bmatrix} e^{2t}\sin(t) \\ -e^{2t}\cos(t) \end{bmatrix} = \begin{bmatrix} C_1e^{2t}\cos(t) + C_2e^{2t}\sin(t) \\ C_1e^{2t}\sin(t) - C_2e^{2t}\cos(t) \end{bmatrix}.$$

Jadi, solusi umum untuk sistem PDB tersebut adalah

$$x(t) = C_1e^{2t}\cos(t) + C_2e^{2t}\sin(t), \quad y(t) = C_1e^{2t}\sin(t) - C_2e^{2t}\cos(t).$$

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