A terminating, non-cyclic* path towards the Collatz conjecture

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Abstract

"*Mathematics may not be ready for such problems*" - Paul Erdős on the Collatz conjecture.

It has long been known that the following identity applies to all paths of the Collatz sequence.

$$2^n x_n - 3^m x_0 = k$$

where x_0 is the initial term and x_n is the final term and n, m and k are path dependent parameters.

This paper derives a formula for k derived from 3 parameter sequences m_n, o_n, e_n.

$$m_n = x_n \pmod 2$$

$$o_n = \sum_{k=0}^{k=n} m_k$$

$$e_n = n + 1 - o_n$$

$$k = 2^{e_{n-1}} x_n - 3^{o_{n-1}} x_0 = \sum_{i=0}^{i=n-1} 2^{e_i} 3^{o_{n-1} - o_i} m_i$$

The resultant identity is not novel, for example see [1], but perhaps the technique by which it was derived may be interesting to some.

[1] First odd term of the sequence lower odd number n related to the $3\cdot n+1 (https://mathoverflow.net/questions/448397/first-odd-term-of-the-sequence-lower-odd-number-n-related-to-the-3-cdot-n1?_gl=1*wy9ddp*_ga*MTAyNzkyMTgxNy4xNjYyNzY2MzQw*_ga_S812YQPLT2*MTY5NzI4ND on Maths Overflow Net (https://mathoverflow.net/)

A useful recurrence relation

This paper examines a recurrence relation of the form:

$$x_{n+1} = \frac{x_n + m_n(5x_n + 2)}{2}, m_n = x_n \pmod 2$$

This sequence is identical to the known Collatz sequence, more traditionally defined as:

$x_{n+1} = 3x_n + 1, x \equiv 1 \pmod 2$

$x_{n+1} = \frac{x_n}{2}, x \equiv 0 \pmod 2$

That the relation is equivalent to the Collatz sequence is evident by re-arranging and simplifying this equation:

$x_{n+1} = m_n(3x_n + 1) + (1 - m_n)\frac{x}{2}$

Alternative forms of the recurrence relation

Using the identity:

$5m + 1 = 6^m = 2^m 3^m \iff m \in \{0, 1\}$

the same recurrence relation can also be expressed in all of the following ways:

$x_{n+1} = \frac{(5m_n+1)x_n+2m_n}{2}$

$x_{n+1} = \frac{6^{m_n} x_n+2m_n}{2}$

$x_{n+1} = \frac{2^{m_n} 3^{m_n} x_n+2m_n}{2}$

$x_{n+1} = \frac{2^{m_n} (3^{m_n} x_n+1)}{2}$

$x_{n+1} = 2^{m_n - 1}(3^{m_n} x_n + 1)$

and, finally, this:

$x_{n+1} = 2^{m_n - 1}((2 + 1)^{m_n} x_n + 1)$

which has certain important properties which may be expanded upon in a future paper should ongoing research into these prove fruitful.

In [ ]:
```
from fractions import Fraction
import math

def m(x):
    return x%2

def succ(x):
    succ=(x+m(x)*(5*x+2))//2
    assert not(x%2 == 0) or succ == x//2
    assert not(x%2 == 1) or succ == 3*x+1
    return succ

def collatz(x0, n):
    x=x0
    for k in range(0, n):
        yield {"k": k, "x": x, "x0": x0, "m": m(x), "succ": succ(x)
        x=succ(x)

result=[e for e in collatz(27, 10)]
#result
```

The sequences o_n, e_n

Let $o_0 = m_0, o_{n+1} = o_n + m_{n+1}, n > 0.$

Let $e_n = n + 1 - o_n$

Clearly:

$o_n = \sum_{k=0}^{k=n} m_k$

$e_n = \sum_{k=0}^{k=n} (1 - m_k)$

Interpretation

The series o_n and e_n is that they represent the number of odd and even terms in the path between x_0 and x_n.

In [ ]:
```
def o(seq):
    for e in seq:
        if e["k"] == 0:
            e["o"]=0
        else:
            e["o"]=pred["o"]+e["m"]
        assert e["o"] <= e["k"]
        yield e
        pred=e

result=[e for e in o(collatz(27, 10))]
#result
```

An expression for x_n in terms of x_0.

the sequences a_n, c_n and d_n

In this section we rewrite each term x_n as combination of terms from three simpler monotonically incresing sequences:

- a_n (*the accumulator sequence*)
- c_n (*the coefficient sequence*)
- d_n (*the divisor sequence*).

Let the first term of the sequence, x_n be named x_0 with subsequent terms being x_1, x_2 et cetera.

Then we can define:

$$x_n = \frac{c_0 x_0 + a_n}{d_n}$$

with $a_0 = 0, c_0 = 1$ and $d_0 = 1$.

Now, let's derive an expression for x_{n+1} in terms of x_0, a_n and d_n

Recall that:

$$x_{n+1} = \frac{x_n + m_n(5x_n + 2)}{2}$$

Substituting $\frac{c_n x_n + a_n}{d_n}$ for x_n yields:

$$x_{n+1} = \frac{\frac{c_n x_n + a_n}{d_n} + m_n(5(\frac{c_n x_n + a_n}{d_n}) + 2)}{2}$$

Simplifying:

$$x_{n+1} = \frac{c_n x_n + a_n + m_n(5(c_n x_n + a_n) + 2d_n)}{2d_n}$$

Redistributing terms:

$$x_{n+1} = \frac{c_n(5m_n+1)x_n + a_n(5m_n+1) + m_n 2d_n}{2d_n}$$

Noting the identity:

$$5m_n + 1 = 6^{m_n}$$

a_n can be further simplified to:

$$x_{n+1} = \frac{6^{m_n} c_n x_n + 6^{m_n} a_n + m_n 2d_n}{2d_n}$$

This is of the form:

$$x_{n+1} = \frac{c_{n+1} x_{n+1} + a_{n+1}}{d_{n+1}}$$

with:

$$a_{n+1} = 6^{m_n} a_n + m_n 2d_n$$

$$c_{n+1} = 6^{m_n} c_n$$

$$d_{n+1} = 2d_n$$

Rewriting a_{n+1}, c_{n+1}, d_{n+1}, x_{n+1} in terms of x_0

Case: $x_0 \rightarrow x_1$

We can check this for n=0 as follows:

$$a_1 = 6^{m_0} a_0 + 2m_0 d_0 = 2m_0 d_0 = 2m_0$$

$$c_1 = 6^{m_0} c_0 = 6^{m_0}$$

$$d_1 = 2d_0 = 2$$

$$x_1 = \frac{6^{m_0} x_0 + 2m_0}{2} = \frac{(5m_0+2)x_0 + m_0}{2} = \frac{x_0 + m_0(5x_0+2)}{2}$$

Case: $x_0 \rightarrow x_1 \rightarrow x_2$

$$a_2 = 6^{m_1} a_1 + 2d_1 m_1 = 6^{m_1} m_0 2 + m_1 4$$

$$c_2 = 6^{m_1} c_1 = 6^{m_1} 6^{m_0}$$

$$d_2 = 2d_1 = 4$$

$$x_2 = \frac{6^{m_1} 6^{m_0} x_0 + 6^{m_1} m_0 2 + m_1 4}{4}$$

Case: $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow x_3$

$$a_3 = 6^{m_2} a_2 + 2d_2 m_2 = 6^{m_2}(6^{m_1} 2m_0 + 4m_1) + m_2 8 = 6^{m_2} 6^{m_1} m_0 2 + 6^{m_2} m_1 4 + m_2 8$$

$$a_3 = 6^{o_2}(\sum_{i=0}^{i=2} \frac{2^{i+1} m_i}{6^{o_i}})$$

$$c_3 = 6^{m_2} 6^{m_1} 6^{m_0} = 6^{o_2}$$

$$d_3 = 8 = 2^3$$

$$x_3 = \frac{6^{o_2} x_0 + 6^{o_2}(\sum_{i=0}^{i=2} \frac{2^{i+1} m_i}{6^{o_i}})}{2^3}$$

$$x_3 = 6^{o_2}(\frac{x_0 + \sum_{i=0}^{i=2} \frac{2^{i+1} m_i}{6^{o_i}})}{2^3})$$

Case: $x_0 \rightarrow \ldots \rightarrow x_n \rightarrow x_{n+1}$

It can be shown that this generalises to:

$$x_{n+1} = 6^{o_n}(\frac{x_0 + \sum_{i=0}^{i=n} \frac{2^{i+1} m_i}{6^{o_i}})}{2^{n+1}})$$

Replacing n with $n - 1$ everywhere yields:

$$x_n = 6^{o_{n-1}}(\frac{x_0 + \sum_{i=0}^{i=n-1} \frac{2^{i+1} m_i}{6^{o_i}})}{2^n})$$

A simplifcation

As beautiful identities go, this one needs some work:

$$x_n = 6^{o_{n-1}}(\frac{x_0 + \sum_{i=0}^{i=n-1} \frac{2^{i+1} m_i}{6^{o_i}})}{2^n})$$

Recognising that $n - o_{n-1} = e_{n-1}$ we get:

$$x_n = 2^{o_{n-1}} 3^{o_{n-1}}(\frac{x_0 + \sum_{i=0}^{i=n-1} \frac{2^{i+1} m_i}{6^{o_i}})}{2^{o_{n-1}+e_{n-1}}})$$

$$x_n = \frac{3^{o_{n-1}} x_0 + \sum_{i=0}^{i=n-1} \frac{2^{i+1} 3^{o_{n-1}} m_i}{2^{o_i} 3^{o_i}}}{2^{e_{n-1}}}$$

$$x_n = \frac{3^{o_{n-1}} x_0 + \sum_{i=0}^{i=n-1} 2^{e_i} 3^{o_{n-1}-o_i} m_i}{2^{e_{n-1}}}$$

Rearranged, this yields:

$$k = 2^{e_{n-1}} x_n - 3^{o_{n-1}} x_0 = \sum_{i=0}^{i=n-1} 2^{e_i} 3^{o_{n-1}-o_i} m_i$$

which describes an identity that applies to all Collatz sequences of finite length.

A discussion about the structure of this identity and how this structure relates to the paths that are described is deferred to a future paper. For now, it is interesting to note, that k is independent of x_0 and that all such paths of length $n+1$ which share the same sequence of odd-even transitions will have the same value of k.

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* *whatever the importance of this contribution, the title is technically not wrong - these adjectives also describe paths that fail to reach their final objective*

Appendix A: an example: $x_0 = 3 \to x_7 = 1$

Consider $x_0 = 3$:

$\{x_k\} = \{3, 10, 5, 16, 8, 4, 2, 1\}$

$n = 7$

$\{m_k\} = \{1, 0, 1, 0, 0, 0, 0, 1\}$

$\{o_k\} = \{1, 1, 2, 2, 2, 2, 2, 3\}$

$\{e_k\} = \{0, 1, 1, 2, 3, 4, 5, 5\}$

The 8th term of $\{x_k\}$ is x_7=1.

Substituting into the identity we have:

$2^{e_6} x_7 - 3^{o_6} x_0 = 2^{e_0} 3^{o_6 - o_0} + 2^{e_2} 3^{o_6 - o_2}$

$2^{e_6} x_7 - 3^{o_6} x_0 = 2^{e_0} 3^{o_6 - o_0} + 2^{e_2} 3^{o_6 - o_2}$

$2^5 x_7 - 3^2 x_0 = 2^0 3^{2-1} + 2^1 3^{2-2}$

$2^5 x_7 - 3^2 x_0 = 2^0 3^1 + 2^1 3^0$

$32x_7 - 9x_0 = 3 + 2$

$32 \times 1 - 9 \times 3 = 3 + 2$

$5 = 5$

Revision History

version	date	notes
1.0.4	2023-10-14	reworded abstract to reflect the fact the identity is already well known
1.0.3	2023-10-14	cleanup some typos and added acknowledgments and dedications
1.0.2	2023-10-14	a beautiful k formulation
1.0.1	2023-10-12	slight cleanups of formulae.
1.0	12023-10-10	initial version - no revision history