Othello Board Analogy for Collatz-Type Cycles

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1 Introduction

We study the realization of cycles in two-parameter Collatz-type maps using an Othello board analogy. Let

$$T(x) = \begin{cases} gx + q, & x \text{ odd}, \\ \dfrac{x}{h}, & x \equiv 0 \pmod{h}, \end{cases}$$

where g is odd, $h \geq 2$, and $q \in \mathbb{Z}$. For a cycle with o odd steps and e even steps, define

$$d(g, h) = h^e - g^o$$

and let $k(g, h)$ encode the full parity structure of the cycle.

The fundamental *cycle-element identity* is

$$q\, k(g, h) - x_0\, d(g, h) = 0.$$

The polynomial $p(g, h)$

Define

$$p(g, h) = q\, k(g, h) - x_0\, d(g, h),$$

regarded as a bivariate polynomial in g and h with integer coefficients determined by q, x_0, and the cycle exponents o, e. Then

$$p(g, h) = 0 \iff (g, h, q) \text{ admit a cycle with structure } k(g, h).$$

2 Othello Board Representation

2.1 Active Region

The *active region* of the board spans $o + 1$ columns and $e + 1$ rows. The horizontal coordinate j runs from the starting column -1 to $o - 1$, and the vertical coordinate i runs from 0 to e. This ensures the entire set of monomials in $p(g, h)$ is represented, including the starting state at $j = -1$.

2.2 Pebble Placement

For each monomial $c_{j,i} g^{o-1-j} h^i$ in $p(g, h)$:

- Place a stack of $c_{j,i}$ white pebbles if the coefficient is positive, black pebbles if negative.
- Each pebble sits in grid square (j, i).

3 Conservation Laws

1. **Cancellation:** A black and white pebble on the same square may be removed simultaneously.
2. **Column multiplication by g:** A pebble may be exchanged for g pebbles of the same color on the right, or vice versa.
3. **Row multiplication by h:** A pebble may be exchanged for h pebbles of the same color below, or vice versa.
4. **Special Collatz relations:**
   - For 3x+1 systems ($g = h^2 - 1$), a white(/black) pebble at (j, i) may be replaced by a white(/black) pebble at $(j + 1, i + 2)$ and a black(/white) pebble at $(j + 1, i)$.
   - For 5x+1 systems ($g = h^2 + h - 1$), a white(/black) pebble at (j, i) may be replaced by two white(/black) pebbles and one black(/white) pebble according to the three monomials in the identity.

4 Worked Example: $x = 17, g = 5, h = 2, q = 1$

4.1 Polynomial

The exact polynomial is

$$p_{1093}(g, h) = 17g^3 + g^2q + ghq - 17h^7 + h^4q.$$

4.2 Othello Board Diagram

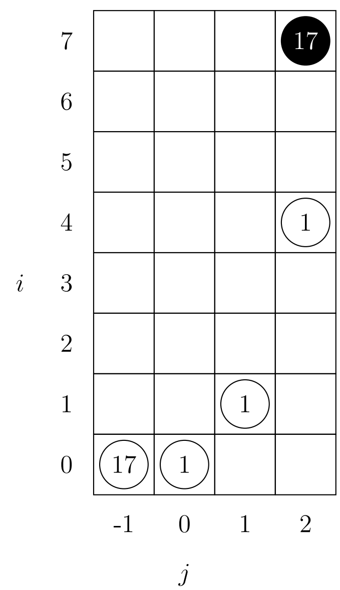

Figure 1: Othello board representation of $p_{1093}(g, h)$. Horizontal axis indexed by j, vertical axis by i. White pebbles with black numerals correspond to positive coefficients; black pebbles with white numerals correspond to negative coefficients.

4.3 Gameplay

Gameplay proceeds by applying the conservation laws until:

- All pebbles disappear (zero pebble state), corresponding to a cycle of the related Collatz system, or

- All remaining pebbles are of the same color (no further cancellation possible).

5 Constraints on Initial State

- For all $j \in [0, o-1]$, $i \in [0, -1]$, any non-zero $c_{j,i}$ cannot "see" any other pebbles in its lower-right quadrant (including its own column and row).
- Inside this region, all stacks are of identical height and color.
- Every column in $[0, o-1]$ has the same non-zero number of pebbles.

6 Conclusion

The Othello board analogy provides a concrete visualization of the polynomial $p(g, h)$ associated with Collatz-type cycles. Pebbles correspond to monomials, and conservation laws encode allowed transformations. The zeroing game then directly mirrors cycle realization in the underlying map.