Affine Block Structure in Collatz Sequences

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Abstract

We present a simplified framework for analyzing Collatz sequences through affine block structures. Each odd block is characterized by three parameters (α, β, ρ) and a scaling parameter t that enumerates instances of the block. By restricting attention to odd blocks whose successors are also odd, the framework achieves a minimal parameterization. Odd blocks are instances of Steiner circuits as defined by Steiner (1977). The framework defines two affine functions: $x(B, t)$ mapping t to odd integers, and $x^{\rightarrow}(B, t)$ giving the odd integer at the start of the next Steiner circuit. This approach focuses exclusively on lattice-wide affine relationships without attempting to model internal block dynamics.

Revisions

- **2025-01-20**: Simplified to consider only odd blocks (removed ν parameter). Restricted to blocks where $x^{\rightarrow}$ is also odd, with $\beta = v_2(3^\alpha \rho - 1)$. Changed parameterization from $(\alpha, \nu, \rho, \kappa)$ to (α, β, ρ). Adopted $x^{\rightarrow}$ notation for the successor (first odd of next Steiner circuit). Added reference to Steiner circuits. Added $t = 1$ example.

1 Introduction

Collatz sequences exhibit structure that can be analyzed through *blocks*—contiguous subsequences with predictable parity patterns. This work presents a minimal parameterization of such blocks that captures their essential affine properties.

Following Steiner [1], we observe that odd blocks—those starting from odd integers—form what he termed "circuits" in the Collatz graph. By restricting attention to odd blocks where $x^{\rightarrow}$ (the first odd of the next circuit) is also odd, we achieve a particularly clean representation.

The key insight is to represent odd blocks using the fundamental identity:

$$x = 2^\alpha \bar{\rho} - 1, \quad \text{where } \bar{\rho} = \rho + t \cdot 2^{\beta+1}$$

This identity leads to a clean 3-parameter representation (α, β, ρ) that avoids the complications of tracking internal 3-adic structure or leading even steps.

2 Block Parameters

An odd block B is defined by three parameters and a scaling parameter:

$$B = (\alpha, \beta, \rho), \quad t \geq 0$$

where:

- $\alpha \geq 1$ is the 2-adic valuation $v_2(x+1)$
- $\rho \geq 1$ is an odd integer parameter
- $\beta = v_2(3^\alpha \rho - 1)$, determining the block's even tail
- $t \geq 0$ is the scaling parameter enumerating block instances

Let $\bar{\rho} = \rho + t \cdot 2^{\beta+1}$. Since $3^\alpha \bar{\rho} - 1 = (3^\alpha \rho - 1) + 3^\alpha t \cdot 2^{\beta+1}$, we have $v_2(3^\alpha \bar{\rho} - 1) = \beta$ for all $t \geq 0$.

The block length (total number of even steps) is $\kappa = \alpha + \beta$. By restricting to blocks where $x^{\rightarrow}$ is also odd, the value $(3^\alpha \bar{\rho} - 1)/2^\beta$ is guaranteed to be an odd integer.

3 Affine Functions

The block parameters (α, β, ρ) define two affine functions of t, with $\bar{\rho} = \rho + t \cdot 2^{\beta+1}$.

3.1 The x-Function

$$x(B,t) = 2^\alpha \bar{\rho} - 1 = 2^\alpha (\rho + t \cdot 2^{\beta+1}) - 1$$

This is an affine function with slope $m_x = 2^{\alpha+\beta+1}$ and intercept $c_x = 2^\alpha \rho - 1$.

3.2 The Successor Function

We write $x^{\rightarrow}$ to denote the odd integer at the start of the next Steiner circuit—that is, the first odd value reached after completing the current block's sequence of Collatz operations.

$$x^{\rightarrow}(B,t) = \frac{3^\alpha \bar{\rho} - 1}{2^\beta} = \frac{3^\alpha \cdot (\rho + t \cdot 2^{\beta+1}) - 1}{2^\beta}$$

Expanding:

$$x^{\rightarrow}(B,t) = \frac{3^\alpha \rho - 1}{2^\beta} + 2 \cdot 3^\alpha \cdot t$$

This is an affine function with slope $m_{x^{\rightarrow}} = 2 \cdot 3^\alpha$ and intercept $c_{x^{\rightarrow}} = (3^\alpha \rho - 1)/2^\beta$. Since $v_2(3^\alpha \bar{\rho} - 1) = \beta$ for all t, the successor $x^{\rightarrow}(B,t)$ is always an odd integer.

4 Computing Block Parameters

Given an odd integer x, we compute its block parameters as follows:

1. Compute $\alpha = v_2(x+1)$
2. Compute $\bar{\rho} = (x+1)/2^\alpha$
3. Compute $\rho = \bar{\rho} \bmod 2^{\beta+1}$, where $\beta = v_2(3^\alpha \rho - 1)$
4. Compute the scaling parameter:

$$t = \left\lfloor \frac{\bar{\rho} - \rho}{2^{\beta+1}} \right\rfloor$$

Note: ρ must be odd. If the computed value is even, there is an error in the calculation. In practice, step 3 requires iterating: start with $\rho = \bar{\rho}$, compute $\beta = v_2(3^\alpha \rho - 1)$, then $\rho = \bar{\rho} \bmod 2^{\beta+1}$.

5 Verification

The computed parameters can be verified by checking:

$$x = 2^\alpha \bar{\rho} - 1 = 2^\alpha(\rho + t \cdot 2^{\beta+1}) - 1$$

This should exactly equal the original x value.

6 Examples

6.1 Example: $x = 35$ (t = 0)

For $x = 35$:

- $\alpha = v_2(36) = 2$
- $\bar{\rho} = 36/4 = 9$
- $\beta = v_2(3^2 \cdot 9 - 1) = v_2(80) = 4$
- $\rho = 9 \bmod 32 = 9$
- $t = \lfloor (9-9)/32 \rfloor = 0$

The block parameters are: $B = (\alpha = 2, \beta = 4, \rho = 9)$, $t = 0$

The affine functions are:

$$x(t) = 2^2(9 + t \cdot 2^5) - 1 = 4(9 + 32t) - 1 = 128t + 35$$

$$x^{\rightarrow}(t) = \frac{3^2 \cdot 9 - 1}{2^4} + 2 \cdot 3^2 \cdot t = \frac{80}{16} + 18t = 5 + 18t$$

For $t = 0$: $x(0) = 35$ and $x^{\rightarrow}(0) = 5$.

Indeed, starting from $x = 35$, the Collatz map gives: $35 \to 106 \to 53 \to 160 \to 80 \to 40 \to 20 \to 10 \to 5$, confirming that $x^{\rightarrow} = 5$.

6.2 Example: $x = 163$ ($\mathbf{t = 1}$)

Using the same block $B = (\alpha = 2, \beta = 4, \rho = 9)$ with $t = 1$:

- $x(1) = 128 \cdot 1 + 35 = 163$
- $x^{\to}(1) = 5 + 18 \cdot 1 = 23$

Verification: For $x = 163$:

- $\alpha = v_2(164) = 2$
- $\bar{\rho} = 164/4 = 41$
- $\beta = v_2(3^2 \cdot 41 - 1) = v_2(368) = 4$
- $\rho = 41 \bmod 32 = 9$
- $t = \lfloor (41 - 9)/32 \rfloor = 1$

The Collatz sequence: $163 \to 490 \to 245 \to 736 \to 368 \to 184 \to 92 \to 46 \to 23$, confirming that $x^{\to} = 23$.

7 Significance

This framework provides several insights:

- **Affine structure**: Blocks naturally organize into affine families, revealing geometric patterns in Collatz sequences
- **Minimal parameterization**: Using only 3 parameters (α, β, ρ) plus t, we capture the essential structure without internal dynamics
- **Steiner circuits**: Odd blocks correspond to Steiner's circuits, providing a connection to established terminology in Collatz research
- **Lattice-wide relationships**: The $x^{\to}$ function connects successive Steiner circuits across trajectories
- **Computational efficiency**: Block parameters can be computed directly from odd x without iterating the sequence

8 Scope and Limitations

This framework intentionally focuses on *lattice-wide affine relationships* between odd blocks as atomic units. It does not attempt to model:

- Even starting values

- Internal evolution of individual blocks through the Collatz map
- 3-adic structure within blocks (powers of 3 in intermediate values)
- Step-by-step parity patterns within blocks

By restricting to odd blocks where $x^{\rightarrow}$ is also odd, we obtain a particularly clean framework. Even starting values and more general block structures can be treated as extensions if needed.

9 Conclusion

The affine block framework provides a clean, minimal approach to understanding Collatz sequences through their odd block structure. By focusing exclusively on affine relationships between odd blocks—instances of Steiner circuits—and avoiding the complexities of even values and internal dynamics, this approach reveals the geometric organization of Collatz sequences while maintaining mathematical rigor and computational tractability.

References

[1] R. P. Steiner, "A theorem on the Syracuse problem," *Proceedings of the 7th Manitoba Conference on Numerical Mathematics and Computing*, pp. 553–559, 1977.