Block Decomposition of Collatz Trajectories

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Abstract

We show that the Collatz conjecture is equivalent to the statement that every odd integer $x > 1$ can be described by a single block with parameters $(\alpha, \beta, \rho, \varphi, t)$. Given an odd x, we extract a natural block for the first Steiner circuit and recursively compose it with the block describing the successor's trajectory. The resulting composite block encodes the complete path from x to 1 in five parameters. The decomposition is unique and provides a canonical representation of Collatz trajectories as nested block compositions.

1 Introduction

In previous work [1], we introduced the affine block framework: every odd integer x determines a *natural block* (α, β, ρ) with scaling parameter t describing the Steiner circuit from x to the next odd value $x^{\rightarrow}$. The present paper extends that framework by defining *composite blocks* with parameters $(\alpha, \beta, \rho, \varphi, t)$ where $\varphi > 0$, obtained by composing adjacent natural blocks.

We make the following observation: if the Collatz trajectory of x reaches 1, then the entire trajectory can be encoded as a single composite block. Conversely, if every odd $x > 1$ admits such a block description, then every trajectory reaches 1. This establishes an equivalence between the Collatz conjecture and the completeness of block decomposition.

2 Background

2.1 Blocks

A *block* $B = (\alpha, \beta, \rho, \varphi, t)$ describes a segment of a Collatz trajectory from an odd integer x to an odd successor $x^{\rightarrow}$. The original framework [1] uses the 3-parameter form (α, β, ρ) with scaling parameter t for natural blocks (where $\varphi = 0$). We extend this to a 5-parameter form that accommodates composite blocks. The parameters $\alpha \geq 1$, $\beta \geq 1$, ρ, $\varphi \geq 0$, and $t \geq 0$ determine x and $x^{\rightarrow}$ via the affine equations:

$$x = 2^{\alpha}(\rho + t \cdot 2^{\beta+1}) - 1 - \varphi \tag{1}$$

$$x^{\rightarrow} = \frac{3^{\alpha}(\rho + t \cdot 2^{\beta+1}) - 1}{2^{\beta}} \tag{2}$$

Every block satisfies the invariant $2^{\alpha+\beta}x^{\to} - 3^\alpha x = k$. We define the *natural invariant* $\hat{k} = 3^\alpha - 2^\alpha$ (the value of k when $\varphi = 0$) and the *deviation* $\Delta k = k - \hat{k}$. The perturbation φ is then:

$$\varphi = \frac{\Delta k}{3^\alpha} = \frac{k - \hat{k}}{3^\alpha} \tag{3}$$

Equivalently, $k = \hat{k} + \varphi \cdot 3^\alpha$. For natural blocks $\Delta k = 0$ and $\varphi = 0$; for composite blocks $\Delta k > 0$ and $\varphi > 0$.

These equations hold uniformly for both natural and composite blocks.

2.2 Natural Blocks

A *natural block* is a block with $\varphi = 0$. Every odd integer x determines a unique natural block $B = (\alpha, \beta, \rho, 0, t)$ via:

$$\alpha = v_2(x+1) \tag{4}$$

$$\bar{\rho} = (x+1)/2^\alpha \tag{5}$$

$$\beta = v_2(3^\alpha \bar{\rho} - 1) \tag{6}$$

$$\rho = \bar{\rho} \bmod 2^{\beta+1} \tag{7}$$

$$t = \lfloor \bar{\rho}/2^{\beta+1} \rfloor \tag{8}$$

For natural blocks, ρ is an odd integer and the affine equations (1)–(2) reduce to $x = 2^\alpha(\rho + t \cdot 2^{\beta+1}) - 1$ and $x^{\to} = (3^\alpha(\rho + t \cdot 2^{\beta+1}) - 1)/2^\beta$.

2.3 Block Composition

Given two blocks $B_1 = (\alpha_1, \beta_1, \rho_1, \varphi_1, t_1)$ and $B_2 = (\alpha_2, \beta_2, \rho_2, \varphi_2, t_2)$ with $x_1^{\to} = x_2$, their composition $B_c = B_1 \circ B_2$ is a block satisfying $x_c = x_1$ and $x_c^{\to} = x_2^{\to}$ (see Appendix A for verification). The composite parameters are:

$$\alpha_c = \alpha_1 + \alpha_2 \tag{9}$$

$$\beta_c = \beta_1 + \beta_2 \tag{10}$$

$$k_c = 3^{\alpha_2} k_1 + 2^{\alpha_1+\beta_1} k_2 \tag{11}$$

The composite perturbation is:

$$\varphi_c = \varphi_1 + \frac{2^{\alpha_1+\beta_1}}{3^{\alpha_1}}\varphi_2 + \frac{(2^{\alpha_1+\beta_1} - 2^{\alpha_1})(3^{\alpha_2} - 2^{\alpha_2})}{3^{\alpha_c}} \tag{12}$$

The adjacency condition $x_1^{\to} = x_2$ constrains t_1 to a residue class modulo $2^{\alpha_2+\beta_2}$. The canonical offset $\hat{t}_1 = t_1 \bmod 2^{\alpha_2+\beta_2}$ determines the composite ρ:

$$\rho_c = \frac{2^{\alpha_1}(\rho_1 + \hat{t}_1 \cdot 2^{\beta_1+1}) + \varphi_c - \varphi_1}{2^{\alpha_c}} \tag{13}$$

The composite scaling parameter is $t_c = (t_1 - \hat{t}_1)/2^{\alpha_2+\beta_2}$.

Since the composite is a block, it satisfies the same affine equations (1)–(2) with its own parameters $(\alpha_c, \beta_c, \rho_c, \varphi_c, t_c)$. Composition of two or more natural blocks always yields $\varphi_c > 0$; for composite blocks, ρ_c is generally rational with denominator a power of 3.

3 Recursive Decomposition

3.1 The Decomposition Algorithm

Definition 3.1 (Block decomposition)**.** For an odd integer $x > 1$, the *block decomposition* $\mathcal{B}(x)$ is defined recursively:

1. Extract the natural block $B = (\alpha, \beta, \rho, 0, t)$ from x.
2. Compute the successor $x^{\to}$ using (2).
3. If $x^{\to} = 1$, return B.
4. Otherwise, return $B \circ \mathcal{B}(x^{\to})$.

Remark 3.2. Step 4 composes B with the block describing the remainder of the trajectory. Since $x^{\to}$ is odd (by construction of β), $\mathcal{B}(x^{\to})$ is well-defined provided the recursion terminates. We address termination in Section 5.

3.2 Properties of the Decomposition

When $\mathcal{B}(x)$ terminates, it produces a block $(\alpha_x, \beta_x, \rho_x, \varphi_x, t_x)$ with the following properties.

Proposition 3.3 (Boundary values)**.** *If $\mathcal{B}(x)$ terminates, the resulting block B_x satisfies $x(B_x) = x$ and $x^{\to}(B_x) = 1$.*

Proof. By induction on the recursion depth. In the base case ($x^{\to} = 1$), the natural block B has $x(B) = x$ and $x^{\to}(B) = 1$ by construction.

For the inductive case, let B be the natural block for x with successor $x^{\to} > 1$, and suppose $\mathcal{B}(x^{\to}) = B'$ satisfies $x(B') = x^{\to}$ and $x^{\to}(B') = 1$. Then $B \circ B'$ satisfies $x(B \circ B') = x(B) = x$ and $x^{\to}(B \circ B') = x^{\to}(B') = 1$. □

Proposition 3.4 (Accumulated parameters)**.** *If x reaches 1 through n Steiner circuits with natural blocks $B_1, B_2, \ldots, B_n$, then $\mathcal{B}(x)$ has:*

$$\alpha_x = \sum_{i=1}^{n} \alpha_i \quad \textit{(total odd steps)} \tag{14}$$

$$\beta_x = \sum_{i=1}^{n} \beta_i \quad \textit{(total excess even steps)} \tag{15}$$

$$\alpha_x + \beta_x = \textit{total even steps from } x \textit{ to } 1 \tag{16}$$

Proof. Immediate from the additivity of α and β under composition (9)–(10). □

Proposition 3.5 (Uniqueness)**.** *The block decomposition $\mathcal{B}(x)$ is unique: the parameters $(\alpha_x, \beta_x, \rho_x, \varphi_x, t_x)$ are uniquely determined by x.*

Proof. Each step of the recursion is deterministic: the natural block extraction from an odd x is unique (the parameters α, β, ρ, t are computed by formulas), and the successor $x^{\to}$ is uniquely determined. The composition formulas (9)–(12) then determine the composite parameters uniquely. Since the recursion builds the composition in a fixed right-recursive order $B_1 \circ (B_2 \circ (\cdots \circ B_n))$, the result is unique. □

4 Structure of the Block Parameters

The decomposition reveals how the five parameters encode trajectory information.

4.1 The Parameters α and β

The total α counts the number of odd steps in the entire trajectory, and $\alpha + \beta$ counts the total even steps. These satisfy:

$$2^{\alpha+\beta} \cdot 1 = 3^{\alpha} \cdot x + k \tag{17}$$

giving:

$$2^{\alpha+\beta} = 3^{\alpha} x + k \tag{18}$$

Since $x^{\rightarrow} = 1$, the block invariant yields k directly.

4.2 The Parameter ρ

For a natural block, ρ is an odd integer. For a composite block arising from decomposition, ρ is generally rational with denominator a power of 3:

$$\rho = \frac{p}{3^m} \tag{19}$$

for some integers p and $m \leq \alpha$. This follows from the composition formula for ρ_c, which introduces factors of $3^{-\alpha_1}$ through the φ terms.

4.3 The Perturbation φ

For trajectories of length $n > 1$ (more than one Steiner circuit), $\varphi > 0$. The perturbation satisfies:

$$\varphi = \frac{k - (3^{\alpha} - 2^{\alpha})}{3^{\alpha}} = \frac{\Delta k}{3^{\alpha}} \tag{20}$$

where $\Delta k = k - \hat{k}$ measures the deviation from natural block structure. For a trajectory through n circuits, Δk accumulates contributions from each composition step.

4.4 The Parameter t

The scaling parameter t determines where x sits within its affine family. Two odd integers sharing the same trajectory structure (same sequence of $(\alpha_i, \beta_i, \rho_i)$ values) differ only in t.

5 Equivalence with the Collatz Conjecture

Theorem 5.1 (Equivalence)**.** *The following are equivalent:*

(C) Every Collatz trajectory starting from a positive integer reaches 1*.*

(B) Every odd integer $x > 1$ *has a block decomposition* $\mathcal{B}(x) = (\alpha, \beta, \rho, \varphi, t)$ *with* $x^{\rightarrow} = 1$*.*

Proof. $(C) \Rightarrow (B)$: If the trajectory of x reaches 1, it passes through finitely many odd values $x = x_1, x_2, \ldots, x_n$ with $x_n^{\rightarrow} = 1$. Each x_i determines a natural block B_i, and the recursion in $\mathcal{B}$ terminates after n steps, producing $B_1 \circ B_2 \circ \cdots \circ B_n$.

$(B) \Rightarrow (C)$: If $\mathcal{B}(x)$ exists with $x^{\rightarrow} = 1$, then by Proposition 3.3, the composite block maps x to 1. Since the composite encodes a valid sequence of Collatz operations (each natural block corresponds to a Steiner circuit), the trajectory from x reaches 1. Every even positive integer $2^k m$ with m odd reaches m by repeated halving, so all positive integers reach 1. □

6 Example: $x = 7$

The trajectory of 7 is: $7 \to 11 \to 17 \to 13 \to 5 \to 1$.

Step 1: $x = 7$. Natural block: $\alpha = 3$, $\bar{\rho} = 1$, $\beta = v_2(3^3 \cdot 1 - 1) = v_2(26) = 1$, $\rho = 1$, $t = 0$. Successor: $x^{\rightarrow} = (27 - 1)/2 = 13$.

Wait—let us verify. $7 \to 22 \to 11$: that is one odd step ($\alpha = 1$?). Let us recompute.

$x = 7$: $\alpha = v_2(8) = 3$, $\bar{\rho} = 8/8 = 1$.

The Steiner circuit from 7: $7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13$. That is $\alpha = 3$ odd steps and $\beta = v_2(3^3 \cdot 1 - 1) = v_2(26) = 1$ extra even step. So $x^{\rightarrow} = 26/2 = 13$. Confirmed.

Step 2: $x = 13$. $\alpha = v_2(14) = 1$, $\bar{\rho} = 7$, $\beta = v_2(3 \cdot 7 - 1) = v_2(20) = 2$, $\rho = 7 \bmod 8 = 7$, $t = 0$. Successor: $x^{\rightarrow} = (21 - 1)/4 = 5$.

Step 3: $x = 5$. $\alpha = v_2(6) = 1$, $\bar{\rho} = 3$, $\beta = v_2(3 \cdot 3 - 1) = v_2(8) = 3$, $\rho = 3 \bmod 16 = 3$, $t = 0$. Successor: $x^{\rightarrow} = (9 - 1)/8 = 1$. Terminate.

The three natural blocks are:

$$
\begin{aligned}
B_1 &= (3, 1, 1, 0, 0) & \qquad & x = 7,\ x^{\rightarrow} = 13 \\
B_2 &= (1, 2, 7, 0, 0) & & x = 13,\ x^{\rightarrow} = 5 \\
B_3 &= (1, 3, 3, 0, 0) & & x = 5,\ x^{\rightarrow} = 1
\end{aligned}
$$

Composing $B_2 \circ B_3$: $\alpha = 2$, $\beta = 5$, and using the composition formulas yields a composite block mapping $13 \to 1$.

Composing $B_1 \circ (B_2 \circ B_3)$: $\alpha = 5$, $\beta = 6$, giving a single block $\mathcal{B}(7)$ with $\alpha + \beta = 11$ total even steps mapping $7 \to 1$.

Verification: $2^{11} = 2048$ and $3^5 \cdot 7 + k = 1701 + k$. So $k = 2048 - 1701 = 347$, and $\hat{k} = 3^5 - 2^5 = 243 - 32 = 211$, giving $\Delta k = 136$ and $\varphi = 136/243$.

7 Discussion

The block decomposition provides a canonical five-parameter encoding of Collatz trajectories. The equivalence with the Collatz conjecture (Theorem 5.1) reframes the conjecture as: *every odd $x > 1$ admits a block $(\alpha, \beta, \rho, \varphi, t)$ with $x^{\rightarrow} = 1$.*

This perspective shifts attention from the dynamics of the Collatz map to the existence of block parameters. The question becomes: for a given x, do there exist (α, β) such that:

1. $2^{\alpha+\beta} - 3^\alpha x = k$ has a solution with k achievable by some composition of natural blocks,
2. the resulting ρ and φ are consistent with the composition formulas.

The first condition is a Diophantine constraint on (α, β) given x. The second condition links the algebraic structure of k (as a weighted sum over the trajectory's natural blocks) to the geometric structure of ρ.

A Composition Preserves Boundary Values

We verify that the composition formulas from Section 2.3 satisfy $x_c = x_1$ and $x_c^{\rightarrow} = x_2^{\rightarrow}$.

Proposition A.1. *If $B_c = B_1 \circ B_2$, then $x(B_c) = x(B_1)$.*

Proof. Write $\bar{\rho}_i = \rho_i + t_i \cdot 2^{\beta_i+1}$ for $i = 1, 2, c$. We need to show:

$$2^{\alpha_c}\bar{\rho}_c - 1 - \varphi_c = 2^{\alpha_1}\bar{\rho}_1 - 1 - \varphi_1$$

i.e., that $2^{\alpha_c}\bar{\rho}_c = 2^{\alpha_1}\bar{\rho}_1 + \varphi_c - \varphi_1$.

From (13), $\rho_c = (2^{\alpha_1}(\rho_1 + \hat{t}_1 \cdot 2^{\beta_1+1}) + \varphi_c - \varphi_1)/2^{\alpha_c}$, and $t_c = (t_1 - \hat{t}_1)/2^{\alpha_2+\beta_2}$. Therefore:

$$\begin{aligned} 2^{\alpha_c}\bar{\rho}_c &= 2^{\alpha_c}\rho_c + 2^{\alpha_c} \cdot t_c \cdot 2^{\beta_c+1} \\ &= 2^{\alpha_1}(\rho_1 + \hat{t}_1 \cdot 2^{\beta_1+1}) + \varphi_c - \varphi_1 + \frac{(t_1 - \hat{t}_1) \cdot 2^{\alpha_c+\beta_c+1}}{2^{\alpha_2+\beta_2}} \\ &= 2^{\alpha_1}(\rho_1 + \hat{t}_1 \cdot 2^{\beta_1+1}) + \varphi_c - \varphi_1 + (t_1 - \hat{t}_1) \cdot 2^{\alpha_1+\beta_1+1} \end{aligned}$$

where the last step uses $\alpha_c + \beta_c = \alpha_1 + \alpha_2 + \beta_1 + \beta_2$. Combining the $\hat{t}_1$ terms:

$$\begin{aligned} &= 2^{\alpha_1}\rho_1 + \hat{t}_1 \cdot 2^{\alpha_1+\beta_1+1} + \varphi_c - \varphi_1 + t_1 \cdot 2^{\alpha_1+\beta_1+1} - \hat{t}_1 \cdot 2^{\alpha_1+\beta_1+1} \\ &= 2^{\alpha_1}\rho_1 + t_1 \cdot 2^{\alpha_1+\beta_1+1} + \varphi_c - \varphi_1 \\ &= 2^{\alpha_1}\bar{\rho}_1 + \varphi_c - \varphi_1 \end{aligned}$$

Substituting into the x-equation (1):

$$x_c = 2^{\alpha_c}\bar{\rho}_c - 1 - \varphi_c = 2^{\alpha_1}\bar{\rho}_1 + \varphi_c - \varphi_1 - 1 - \varphi_c = 2^{\alpha_1}\bar{\rho}_1 - 1 - \varphi_1 = x_1.$$

□

Proposition A.2. *If $B_c = B_1 \circ B_2$ and $x_1^{\rightarrow} = x_2$, then $x_c^{\rightarrow} = x_2^{\rightarrow}$.*

Proof. Each block satisfies the invariant $2^{\alpha+\beta}x^{\rightarrow} - 3^\alpha x = k$, so $x^{\rightarrow} = (3^\alpha x + k)/2^{\alpha+\beta}$. Applying this to block 2 with the adjacency condition $x_2 = x_1^{\rightarrow}$:

$$x_2^{\rightarrow} = \frac{3^{\alpha_2}x_1^{\rightarrow} + k_2}{2^{\alpha_2+\beta_2}} = \frac{3^{\alpha_2} \cdot \frac{3^{\alpha_1}x_1+k_1}{2^{\alpha_1+\beta_1}} + k_2}{2^{\alpha_2+\beta_2}} = \frac{3^{\alpha_c}x_1 + 3^{\alpha_2}k_1 + 2^{\alpha_1+\beta_1}k_2}{2^{\alpha_c+\beta_c}}$$

By (11), the numerator is $3^{\alpha_c}x_1 + k_c$. Since $x_c = x_1$ (Proposition A.1), the composite invariant gives:

$$x_c^{\rightarrow} = \frac{3^{\alpha_c}x_c + k_c}{2^{\alpha_c+\beta_c}} = \frac{3^{\alpha_c}x_1 + k_c}{2^{\alpha_c+\beta_c}} = x_2^{\rightarrow}.$$

□

References

[1] Jon Seymour, *Affine Block Structure in Collatz Sequences*, 2025.