PROJECT EULER PROBLEM 129

GVHL

1. Mathematics

Let a so called repunit be defined as follows:

$$R(k) := \underbrace{11\ldots1}_{k}$$

Given some $n \in \mathbb{N}$ with $\gcd(10, n) = 1$, we can find a least value of k such that n divides $R(k)$. Let $A(n) := \min\{k \in \mathbb{N} \mid R(k) \equiv 0 \mod n\}$ be that value of k. We will first find $f(n)$, for some n with $\gcd(10, n) = 1$.

1.1. Finding $A(n)$.

First of all, note that:

$$R(k) = \sum_{l=0}^{k-1} 10^l = \frac{10^k - 1}{9}$$

For n to divide $R(k)$, it must also divide the right hand side. Multiplying by 9 yields that we are looking for k such that $9n$ divides $10^k - 1$. This is equivalent with:

$$10^k \equiv 1 \mod 9n$$

Because of 10 and $9n$ being coprime, by Euler's Theorem such k always exists. It now makes sense to work in $(\mathbb{Z}/9n\mathbb{Z})^*$, that is, the multiplicative group of integers modulo n.
It may be evident that the smallest $f(k)$ must be equal to $\text{ord}(10)$, as the order of an element is per definition minimal.

Recall that the order of an element divides the order of the group. The latter here is $\varphi(9n)$, where φ is Euler's totient function.
For small n, we can compute $\varphi(n)$ fast. In order to find $\text{ord}(9n)$, we can divide $\varphi(9n)$ by its prime factors, as long as it is still a multiple of $\text{ord}(10)$.

1.2. $A(n)$ exceeding 10^6.

Recall that we're working in $(\mathbb{Z}/9n\mathbb{Z})^*$.

Claim: $\text{ord}(10) \leq n$.

Although there might be theorems for this I am not familiar with at this time, this can for instance be proven using the Chinese remainder theorem and that $10^{3^{m-2}} \equiv 1 \mod 3^m$ for $m \in \mathbb{N}_{\geq 2}$. I will not be doing this here.

Using the above claim, we can skip any numbers lower that 10^6. See code for implementation.

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