TILING DEFICIENT BOARDS WITH PENTOMINOS

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Theorem 1. *For all integers* N ≥ 0, *a* 4^N *x* 4^N *board with one tile removed can be tiled completely by the two pentominos shown in Figure 1.*

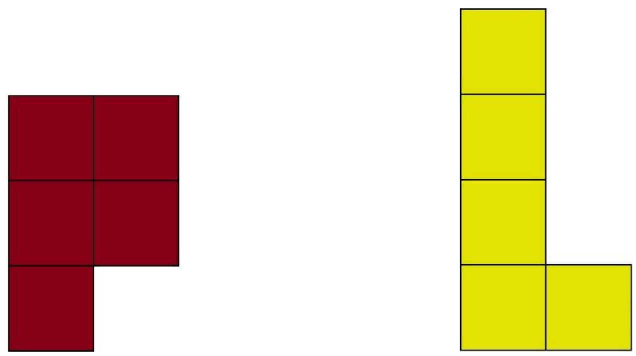

Figure 1. The P (left) and L (right) pentominos.

Proof. For N $= 0$, we have a board consisting of a single tile. After removing this tile there is nothing left to cover, and so Theorem 1 holds for N $= 0$. For N = 1, after accounting for rotations and reflections, there are three ways to remove a tile from a 4 x 4 board, forming three possible 4 x 4 boards to cover. Each of these three boards can be tiled in multiple ways. Figure 2 shows one of these ways for each of the three boards, and so Theorem 1 holds for N $= 1$ as well.

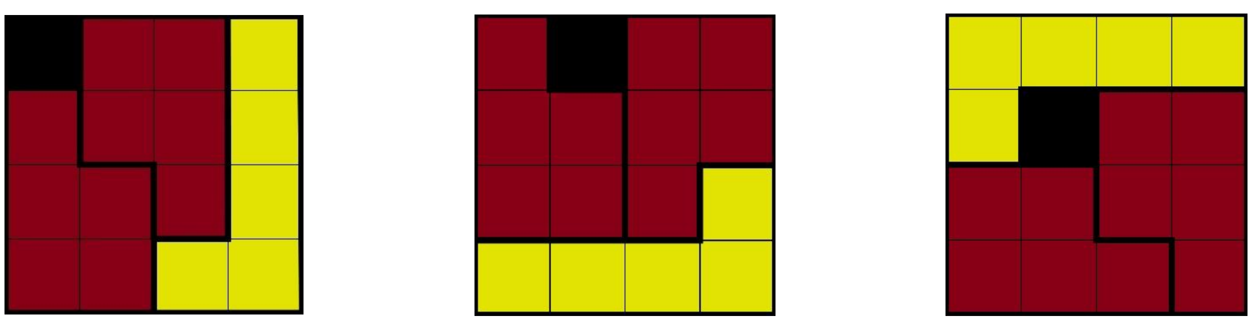

Figure 2. The three 4 x 4 boards covered by P and L pentominos.

Before proving Theorem 1 for all N > 1, we turn now to proving that both P and L pentominos can be tiled to generate arbitrarily large versions of both figures, deemed "scaled pentominos".

Theorem 2. *P and L pentominos can be tiled to produce scaled versions of their corresponding pentominos, comprised of five* 4^M *x* 4^M *boards, for all integers* $M \geq 0$.

Proof. The case where M = 0 is simply the case of the pentominos themselves, and so Theorem 2 holds for M = 0. For M = 1, there are multiple ways to generate scaled versions of the P and L pentominos. Figure 3 shows one of these ways for each of the pentominos, and so Theorem 2 holds for M = 1 as well. Now let k be an integer, where k > 1. Assume as our inductive hypothesis that scaled P and L pentominos comprised of five 4^{k-1} x 4^{k-1} boards can be tiled completely. Now consider a scaled pentomino of either shape comprised of five 4^k x 4^k boards. Using our 4^{k-1} x 4^{k-1} pentominos, we can tile the 4^k x 4^k pentominos by placing the 4^{k-1} x 4^{k-1} pentominos on the 4^k x 4^k pentomino in the same way as in the case where M = 1. Therefore, Theorem 2 is true by induction.

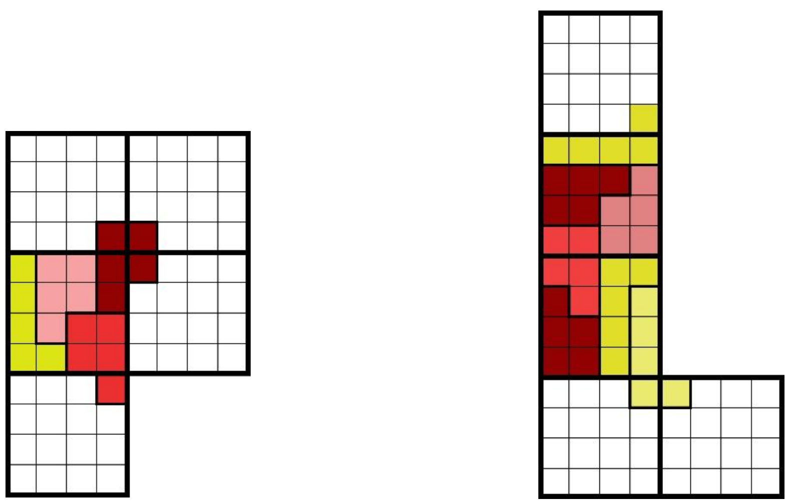

Figure 3. The case M = 1. Note the remaining white squares are able to covered as shown in Figure 2.

Now we return to proving Theorem 1 for N > 1. Let q be an integer, where q > 1. Assume as our inductive hypothesis that a 4^{q-1} x 4^{q-1} board with one square removed can be tiled by P and L pentominos or their scaled versions. Given a 4^q x 4^q board with one square removed, we can divide it into 16 4^{q-1} x 4^{q-1} boards. The removed square must fall into one of these 16 4^{q-1} x 4^{q-1} boards, rendering it deficient.

This deficient 4^{q-1} x 4^{q-1} board can be tiled by our inductive hypothesis. Using scaled pentominos comprised of five 4^{q-1} x 4^{q-1} boards, we can tile the remaining 4^q x 4^q board in the same way as in the case N = 1. Therefore, Theorem 1 is true by induction.