Polyhedral Symmetry from Ribbons and Tubes

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Introduction

Figure 1. The *sepak takraw*. Left: A handmade example. Right: Our modeled interpretation.

A *sepak takraw* is a sports ball with icosahedral symmetry. Our research focused on how this shape can be modeled, and what other possible generalizations exist to create shapes with *polyhedral symmetry* from a collection of bands arranged in space. First, we explain what we mean by polyhedral symmetry in the context of group theory. We then mention *the recipe*, our primary result: an algebraic theorem that suggests a two-step process to create these shapes. Next, we walk through this process in greater detail. Finally, we end with comments on our artistic design process, areas for further research, and a gallery of examples to enjoy.

Polyhedral Symmetry

The symmetries of the five Platonic solids are associated with three *polyhedral groups* – tetrahedral, octahedral, and icosahedral. Each group is a set of transformations on a geometric shape that leave the shape invariant. We *represent* these groups as sets of rotation matrices acting on three-dimensional space.

The Recipe

Theorem. First, create a single band that is invariant under a *subgroup* of a given polyhedral group, called the primary band. Next, reproduce that band using a set of *coset representatives* for that subgroup. The shape produced by the union of all such bands is then guaranteed to be invariant under the entire group.

Creating the Primary Invariant Band

The subgroups of the polyhedral groups are either *cyclic* or *dihedral*. To make a band invariant under a cyclic subgroup, we create a ribbon or tube on a periodic curve with a given order of rotational symmetry. For a dihedral subgroup, an extra half-turn symmetry about a fixed axis must be present.

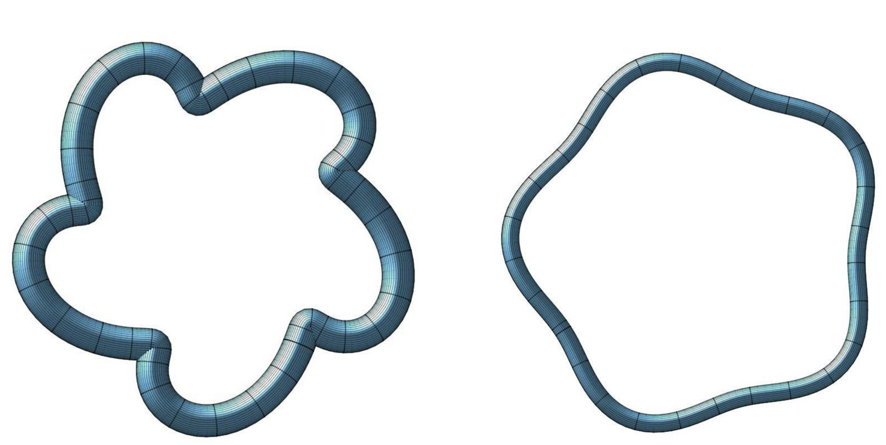


Figure 2. Example invariant bands. Left: Cyclic. Right: Dihedral. Can you spot the half-turn axis?

Placing and Reproducing the Band

Our choice of group representation determines the axes of rotational symmetry we desire in our resulting shapes. Therefore, to guarantee our primary band is invariant, we place it in the *invariant plane* of the axis of symmetry of its subgroup. Note that for dihedral subgroups, our primary band must also be oriented in the invariant plane so that its half-turn axis coincides with its corresponding group element.

Next, we generate a set of coset representatives for our given subgroup. The elements in this set are the rotations which move the primary band to a different location in space, not merely rotate it back onto itself.

Each of these reproduced bands captures some of the group symmetry, but in forming their union, we produce a shape that is invariant under the entire group.

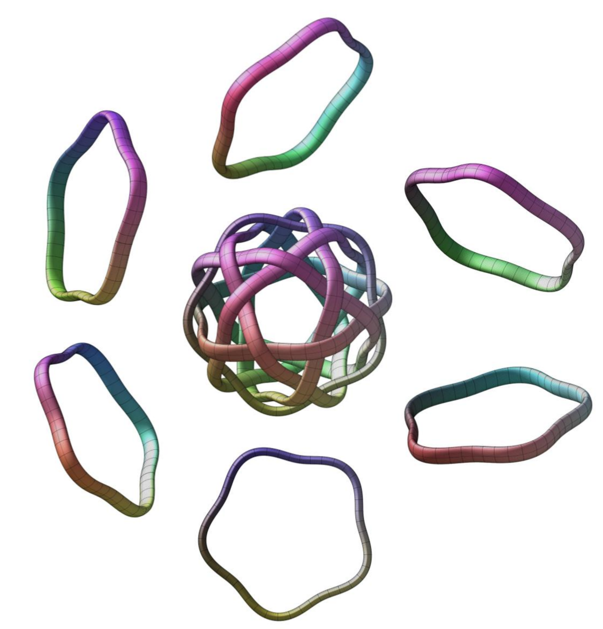


Figure 3. An example shape, extruded for illustration.

Artistic Process

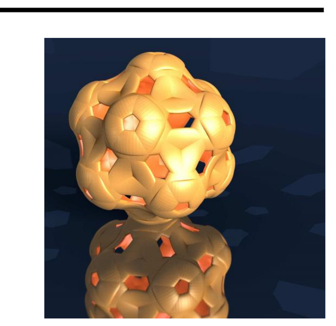
To produce our examples, the shapes are first created in *Maple*. Next, the raw shapes are exported as an STL, and opened in *Photoshop*. From there, they are decorated with a pattern or texture, artificially lit, and brought to life with Photoshop's ray-tracing algorithm. We chose to display the shapes as if sitting on mirrored surfaces to provide a second view.

Conclusions

The numerous subgroups of the polyhedral groups, combined with the vast multi-parameter design space used in constructing individual bands, give rise to an enormous range of possibilities to create polyhedral symmetry from bands in space. We took delight in creating a near-complete set of examples, and encourage the reader to experiment on their own, perhaps finding examples far more beautiful than ours.

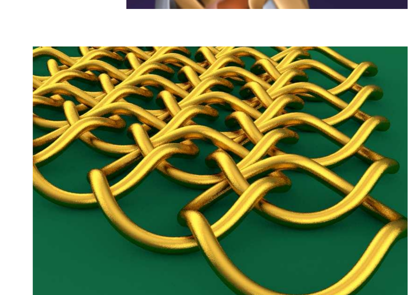
Further Research

In addition to *rotational* symmetries, the Platonic solids possess *mirror* symmetries as well. In using the *full* polyhedral groups instead of the *chiral* groups, we can produce shapes with full icosahedral symmetry as in the example right.



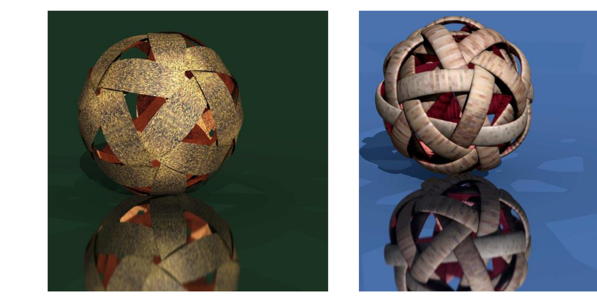
Ignoring the different colored bands, the example right has octahedral symmetry, but in their consideration has only tetrahedral symmetry. We say that this shape belongs to the octahedral *color group* and the tetrahedral symmetry group. What else can be said of color symmetry?

The polyhedral groups are just a few of the many *point groups* in three-dimensional space. By creating bands invariant under other point groups, we can create *mathematical chain mail*, a rich area for further experimentation.

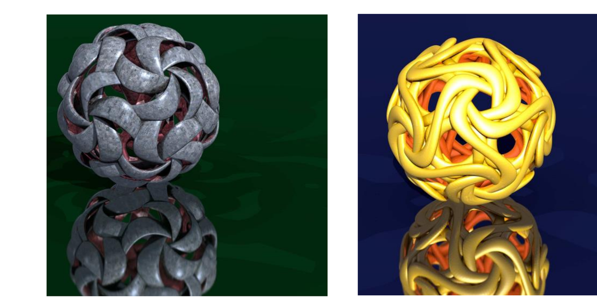


Gallery

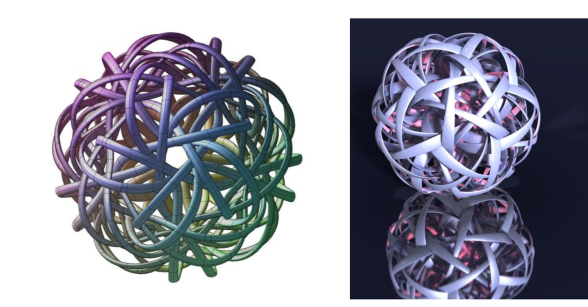
Two 10-band icosahedra, one made from ribbons and another from tubes



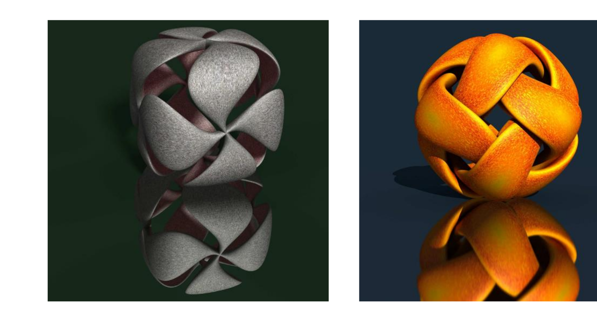
12- and 15-band icosahedra



A 20-band icosahedron, as seen in Maple, and its well-dressed 30-band relative

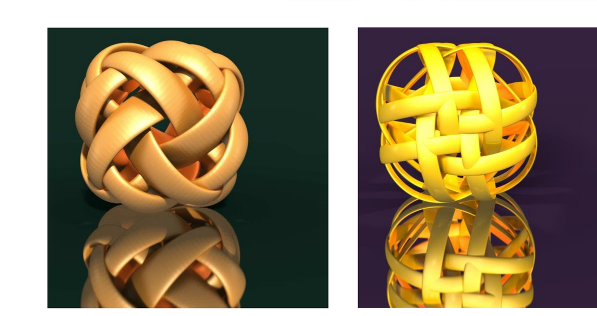


3- and 4-band octahedra

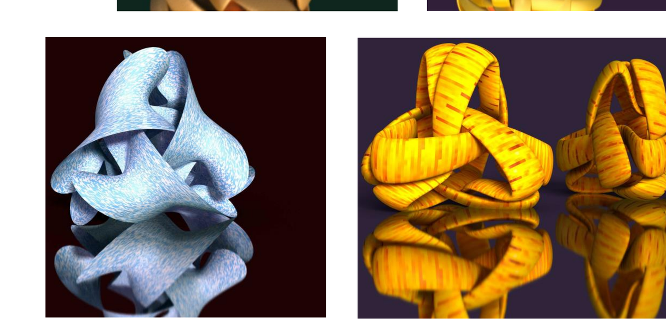


Two 6-band octahedra, one from a cyclic subgroup and the other from a dihedral subgroup. This is the only occurrence of distinct shapes with the same number of bands.

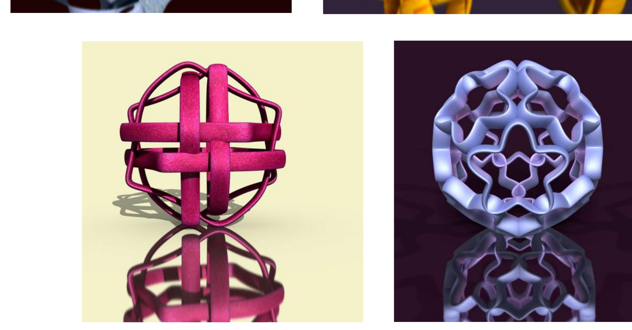
8- and 12-band octahedra



3- and 4-band tetrahedra



6- and 12-band tetrahedra. Despite the recurring five-pointed star, no fivefold symmetry is present here.



Acknowledgements

We dedicate this research to the memory of Jean Pedersen, who shared her love for polyhedra through numerous publications. We inherited from her office the *sepak takraw* that inspired this research.

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