普物复习

1. math

1.1. 基础

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}$$

$$a = \ddot{x} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

1.2. cross product:

$$A \times (B + C) = A \times B + A \times C$$

$$\frac{\mathrm{d}(A \times B)}{\mathrm{d}t} = A\frac{\mathrm{d}B}{\mathrm{d}t} + B\frac{\mathrm{d}A}{\mathrm{d}t}$$

$$a \times b = |a||b| \sin(\theta)\hat{n}$$

could also be written as:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

if

$$a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

while the same to b

1.3. 极坐标:

$$\hat{r}, \hat{\phi}$$

1.4. 三维平面

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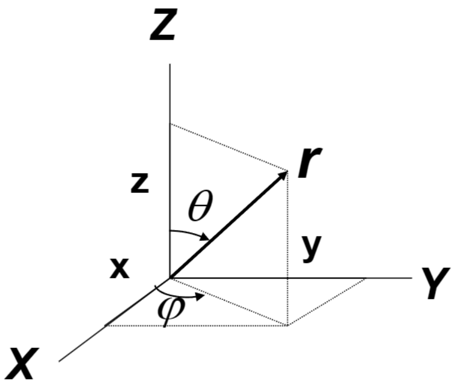

Figure 1: 三维平面

$$r = (x, y, z)$$

$$x = r\sin(\theta)\cos(\varphi)$$
$$y = r\sin(\theta)\sin(\varphi)$$
$$z = r\cos(\theta)$$

2. 各种运动

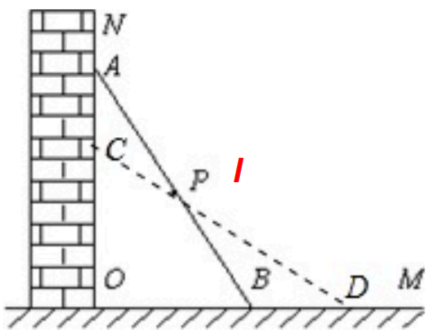

Figure 2: PPT 示例

$$l^2 = x^2 + y^2$$

因为 l 是常量，所以有

$$2x\,\mathrm{d}x + 2y\,\mathrm{d}y = 0$$
$$\frac{\mathrm{d}x}{\mathrm{d}t}x + \frac{\mathrm{d}y}{\mathrm{d}t}y = 0$$
$$\frac{v_x}{v_y} = -\frac{y}{x} = -\tan(\theta)$$

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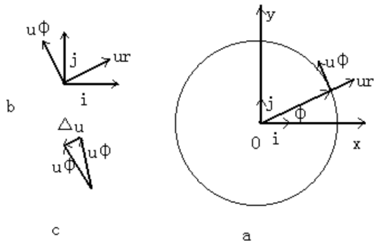
$$\hat{u}_r = \cos\phi\hat{i} + \sin\phi\hat{j}$$

$$\hat{u}_\phi = -\sin\phi\hat{i} + \cos\phi\hat{j}$$

$$\bar{r} = x\hat{i} + y\hat{j}$$

$$\frac{\bar{r}}{r} = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j}$$

$$\hat{u}_r = \cos\phi\hat{i} + \sin\phi\hat{j}$$



$$\frac{d\hat{u}_r}{dt} = \frac{d\phi}{dt}(-\sin\phi\hat{i} + \cos\phi\hat{j}) = \frac{d\phi}{dt}\hat{u}_\phi$$

$$\frac{d\hat{u}_\phi}{dt} = -\frac{d\phi}{dt}(\cos\phi\hat{i} + \sin\phi\hat{j}) = -\frac{d\phi}{dt}\hat{u}_r$$

Figure 3: 圆周运动坐标系

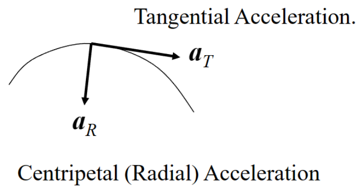
$$\bar{v} = v\hat{u}_\phi$$

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d}{dt}(v\hat{u}_\phi)$$

$$= v\frac{d\hat{u}_\phi}{dt} + \frac{dv}{dt}\hat{u}_\phi$$

$$= -\frac{v^2}{r}\hat{u}_r + \frac{dv}{dt}\hat{u}_\phi$$

$$= a_R\hat{u}_r + a_T\hat{u}_\phi$$



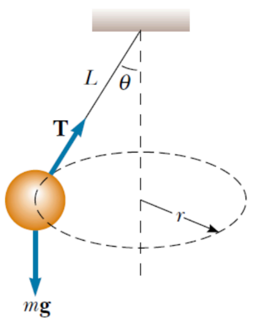
$$\bar{a}_R = -\frac{v^2}{r}\hat{u}_r$$

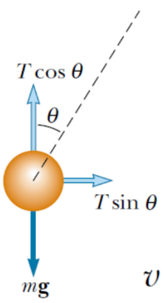
$$\bar{a}_T = \frac{dv}{dt}\hat{u}_T$$

$$a = \sqrt{a_R^2 + a_T^2}$$

Figure 4: 非匀速的圆周运动加速度分析

圆锥摆：





$$\sum F_r = T\sin\theta = ma_r = \frac{mv^2}{r}$$

$$T\cos\theta = mg$$

$$\tan\theta = \frac{v^2}{rg}$$

$$v = \sqrt{rg\tan\theta}$$

$$v = \boxed{\sqrt{Lg\sin\theta\tan\theta}}$$

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$$I = \int r^2 \, \mathrm{d}m$$

此即转动惯量

3. 牛顿定律

3.1. 惯性系(Inertial)与非惯性系 (Non-Inertial)

伽利略变换 :from Inertial to Non-Inertial

$$\vec{r'} = \vec{r} - \vec{v_0} \cdot t$$
$$\vec{v'} = \vec{v} - \vec{v_0}$$
$$\vec{a'} = \vec{a}$$

3.2. 流体阻力

低速下:

$$F = -bv$$

高速下:

$$F = -cv^2$$

低速稳态解:v_t指v_{terminal}

$$v_t = m\frac{g}{b}$$

令

$$\tau = \frac{m}{b} = \frac{v_t}{g}$$

则任意时刻速度为:

$$v = v_t\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = ge^{-\frac{t}{\tau}}$$

高速下:

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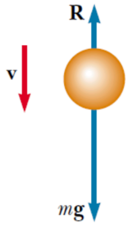
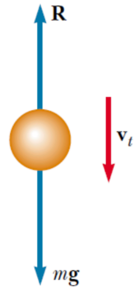
$$R=\frac{1}{2}D\rho Av^2$$

ρ: density of fluid
A: cross-sectional area of the falling object
D: drag coefficient



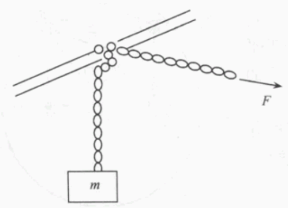
$$\sum F = mg-\frac{1}{2}D\rho Av^2$$

$$g-\left(\frac{D\rho A}{2m}\right)v_t^2=0 \Rightarrow v_t=\sqrt{\frac{2mg}{D\rho A}}$$



例题一：

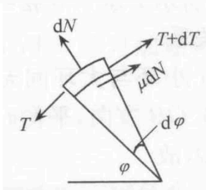
【实际问题研究】 一个质量为 m 的物体通过一根质量可以忽略不计的绳子绕水平棒$1\frac{1}{4}$周后于另一端加一水平力 F，如图所示．若绳子和棒之间的摩擦因素为 μ，要使物体保持静止状态，应施加多大的水平拉力？



$$T+\mathrm{d}T+\mu\mathrm{d}N=T$$

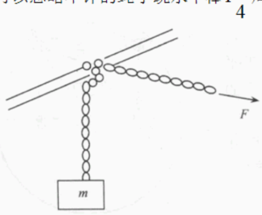
$$\mathrm{d}N=2T\sin\frac{\mathrm{d}\varphi}{2}=T\mathrm{d}\varphi$$

when $\varphi=0,\ T=mg,\ \varphi=\frac{5}{2}\pi,\ T=F_{\max}$



$$\int_{mg}^{F_{\max}}\frac{\mathrm{d}T}{T}=-\int_0^{\frac{5}{2}\pi}\mu\mathrm{d}\varphi \qquad F_{\max}=mge^{-\frac{5}{2}\mu\pi}$$

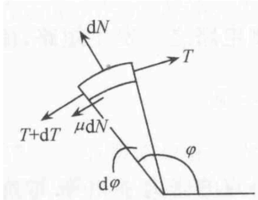
【实际问题研究】 一个质量为 m 的物体通过一根质量可以忽略不计的绳子绕水平棒$1\frac{1}{4}$周后于另一端加一水平力 F，如图所示．若绳子和棒之间的摩擦因素为 μ，要使物体保持静止状态，应施加多大的水平拉力？



$$T+\mathrm{d}T+\mu\mathrm{d}N=T$$

$$\mathrm{d}N=2T\sin\frac{\mathrm{d}\varphi}{2}=T\mathrm{d}\varphi \quad \mathrm{d}T=-\mu T\mathrm{d}\varphi$$

when $\varphi=0,\ T=F_{\max},\ \varphi=\frac{5}{2}\pi,\ T=mg$



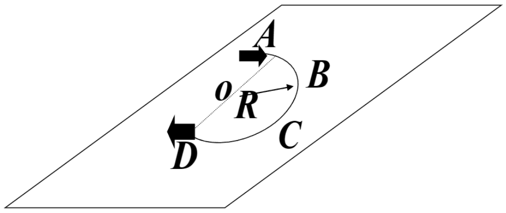
$$\int_{F_{\max}}^{mg}\frac{\mathrm{d}T}{T}=-\int_0^{\frac{5}{2}\pi}\mu\mathrm{d}\varphi$$

$$F_{\max}=mge^{\frac{5}{2}\mu\pi}$$

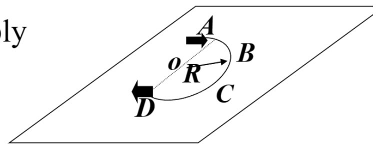
$$\boxed{mge^{-\frac{5}{2}\mu\pi}<F_{\max}<mge^{\frac{5}{2}\mu\pi}}$$

例题二：

Example: As shown in the figure, a block of mass ***m*** with a velocity v_0 moves into a semicircular wall ABCD from point A. The coefficient of friction between the block and the wall is μ. The wall is fixed on a frictionless horizontal table, Find the final speed of the block at point D.



Solution: normal force N supply centripetal force, and produce tangential friction force f

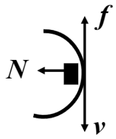


$$\begin{cases} N = ma_n = m\dfrac{v^2}{R} \\ -f = ma_\tau = m\dfrac{dv}{dt} \\ f = \mu N \end{cases}$$

$$-\mu\frac{v^2}{R} = \frac{dv}{dt} = \frac{dv}{dl}\cdot\frac{dl}{dt}$$

$$\int_{v_0}^{v}\frac{dv}{v} = -\int_0^{\pi R}\frac{\mu}{R}dl$$

$$v = v_0 e^{-\mu\pi}$$



3.3. 功与动能

$$W = \int_{r_i}^{r_f} F \cdot dr$$

3.4. 机械能守恒

$$E = K + U = \text{Constant}$$

稳定平衡(stable equilibrium)是系统能量的最小值点，也即

$$\frac{dU}{dx} = 0 \text{ 并且 } \frac{d^2U}{dx^2} > 0$$

同样，不稳定平衡(unstable equilibrium)是系统能量的最大点，也即

$$\frac{dU}{dx} = 0 \text{ 并且 } \frac{d^2U}{dx^2} < 0$$

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3.5. 势能

势能要求力是保守力(conservative)，即力做的功与路径无关

$$U = -\int F \cdot \mathrm{d}r$$

万有引力:

$$F = -\frac{G(m_1 m_2)}{r^2} = -\frac{\mathrm{d}U}{\mathrm{d}r}$$
$$U = -\frac{G(m_1 m_2)}{r}$$

开普勒定律：Orbital period：

$$T = 2\pi\sqrt{\frac{a^3}{G(m_1 + m_2)}}$$

,i.e.

$$T^2 \propto a^3$$

在距离地球中心为 r 的轨道稳定运行所具有的能量:

$$E = -\frac{G(m_1 m_2)}{2r}$$

Escaped speed:

$$v = \sqrt{\frac{2Gm_1}{r}}$$

from:

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = 0$$

火箭发射:

$$v_f = v_i + u\ln\left(\frac{M_i}{M_f}\right)$$

其中，u为燃料速度，M_i为初始质量，M_f为最终质量
火箭的推力(thrust)：

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$$F=\frac{\mathrm{d}m}{\mathrm{d}t}u$$

例题

A rocket moving in free space has a speed of 3.0×10^3 m/s relative to the Earth. Its engines are turned on, and fuel is ejected in a direction opposite the rocket's motion at a speed of 5.0×10^3 m/s relative to the rocket. (a) What is the speed of the rocket relative to the Earth once the rocket's mass is reduced to one-half its mass before ignition?

$$v_f = v_i + v_e \ln\left(\frac{M_i}{M_f}\right) = \boxed{6.5\times10^3\ \mathrm{m/s}}$$

(b) What is the thrust on the rocket if it burns fuel at the rate of 50 kg/s?

$$\text{Thrust} = \left|v_e\frac{dM}{dt}\right| = \boxed{2.5\times10^5\ \mathrm{N}}$$

Figure 11: PPT 例题

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3.6. 动量

Impulse:

$$I=\int_{t_i}^{t_f}\vec{F}\cdot\mathrm{d}t$$

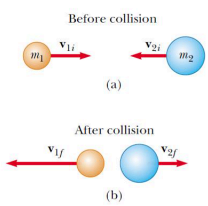
$$\vec{p}=m\vec{v}$$

注意弹性碰撞和非弹性碰撞中动量与动能情况

在两种碰撞中，动量均守恒，完全非弹性碰撞损失能量最多。对于完全弹性碰撞，如图：

$$m_1v_{1i}+m_2v_{2i} = m_1v_{1f}+m_2v_{2f}$$

$$\frac{1}{2}m_1v_{1i}^2+\frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2+\frac{1}{2}m_2v_{2f}^2$$



Show that

$$v_{1f}=\left(\frac{m_1-m_2}{m_1+m_2}\right)v_{1i}+\left(\frac{2m_2}{m_1+m_2}\right)v_{2i}$$

$$v_{2f}=\left(\frac{2m_1}{m_1+m_2}\right)v_{1i}+\left(\frac{m_2-m_1}{m_1+m_2}\right)v_{2i}$$

Figure 12: 完全弹性碰撞

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注意：每当两个相同质量的物体发生弹性碰撞且其中一个最初处于静止状态时，它们的最终速度总是彼此成直角。

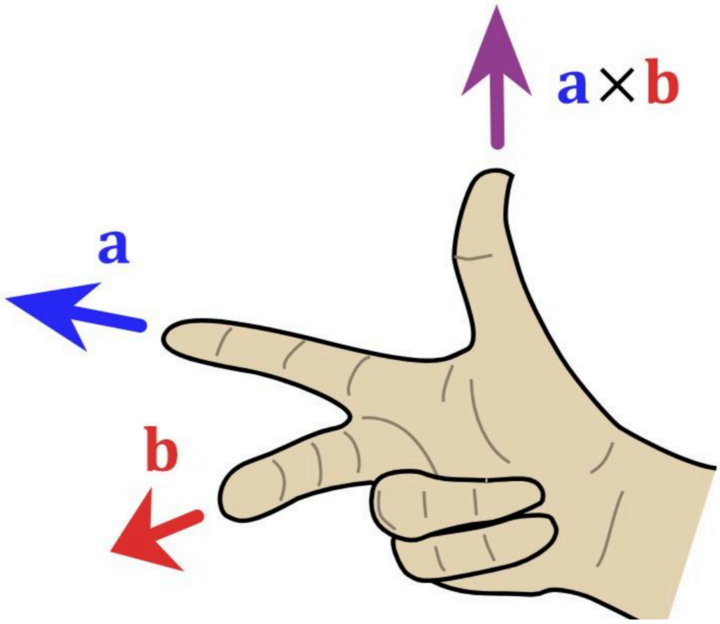
3.7. 转动

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

叉乘的右手定则

方向：



$$\vec{v} = \vec{r} \times \vec{\omega}$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$i.e.$$

$$\vec{r} = I\alpha\hat{\omega}$$

$$\tau = I\alpha$$

$$W = \int_{\theta_i}^{\theta_f} \tau \,\mathrm{d}\theta = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \tau\omega$$

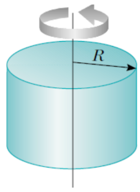
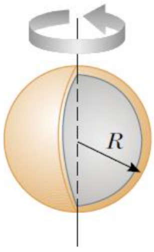
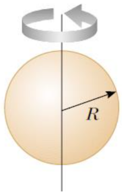
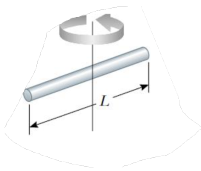
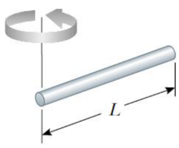
普物复习

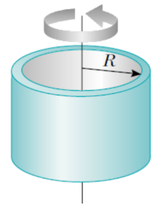
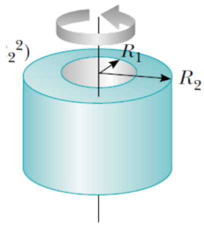
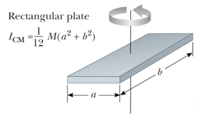
转动惯量：

$$I = \sum_i m_i r_i^2$$

$$K_R = \frac{1}{2} I \omega^2$$

常用转动惯量：

几何体	转动惯量	图片
实心圆柱体	$I = \frac{1}{2} MR^2$	
圆环	$I = MR^2$	none
薄球壳	$I = \frac{2}{3} MR^2$	
实心球体	$I = \frac{2}{5} MR^2$	
棒绕中心点转动	$I = \frac{1}{12} ML^2$	
棒绕一端转动	$I = \frac{1}{3} ML^2$	

圆柱壳	$I = MR^2$	
中空圆柱	$I = \frac{1}{2}M(R_1^2 + R_2^2)$	
矩形盘	$I = \frac{1}{12}M(a^2 + b^2)$	

3.8. 角动量

$$\tau = \frac{\mathrm{d}L}{\mathrm{d}t}$$

$$\begin{aligned} \vec{L} &= \vec{r} \times \vec{p} \\ &= I\vec{\omega} \end{aligned}$$

P,W 分别为功率与功

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \tau\omega$$

$$W = \tau\theta$$

如果外力矩为零，则角动量守恒
注意：角动量的计算依赖于原点的选取

3.9. 例题

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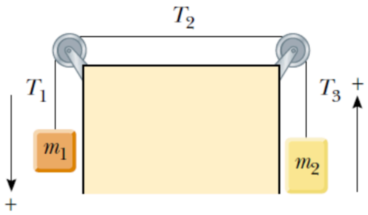
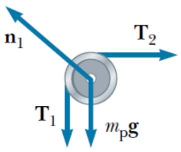
•**Two blocks having masses m_1 and m_2 are connected to each other by a light cord that passes over two identical, frictionless pulleys, each having a moment of inertia I and radius R. Find the acceleration of each block and the tensions T_1, T_2, and T_3 in the cord.**

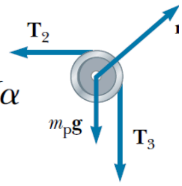
(Assume no slipping between cord and pulleys.)





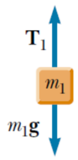
$$(T_1 - T_2)R = I\alpha$$

$$(T_2 - T_3)R = I\alpha$$



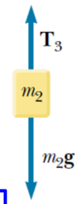
$$\Rightarrow \boxed{(T_1 - T_3)R = 2I\alpha}$$

$$\boxed{\alpha = a/R}$$



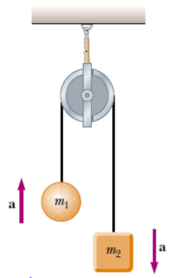
$$m_1 g - T_1 = m_1 a$$

$$T_3 - m_2 g = m_2 a$$



$$\Rightarrow \boxed{T_1 - T_3 = (m_1 - m_2)g - (m_1 + m_2)a}$$

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2 + 2\dfrac{I}{R^2}}$$



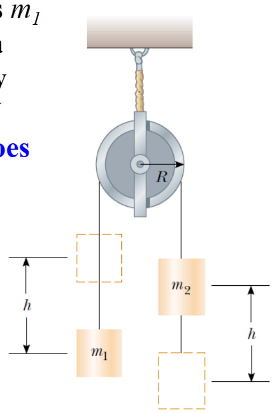
•**Discussion:**

- **Equal mass: At equilibrium.**
- **Unequal mass: Normally we assume a direction for the acceleration. If the result is negative, the real acceleration is in the opposite direction.**
- **$I = 0$: Goes back to the same old Newton's laws.**

another one

普物复习

•Consider two cylinders having masses m_1 and m_2, where $m_1 \neq m_2$, connected by a string passing over a pulley. The pulley has a radius R and moment of inertia I about its axis of rotation. The string does not slip on the pulley, and the system is released from rest. Find the linear speeds of the cylinders after cylinder 2 descends through a distance h, and the angular speed of the pulley at this time.



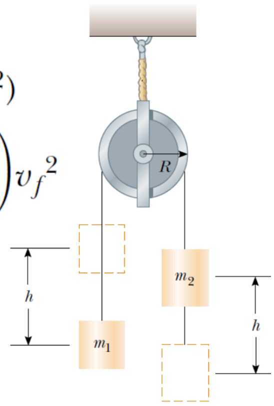
$$\Delta K + \Delta U_1 + \Delta U_2 = 0$$

$$\Delta K = (\tfrac{1}{2}m_1 v_f^{\,2} + \tfrac{1}{2}m_2 v_f^{\,2} + \tfrac{1}{2}I\omega_f^{\,2})$$

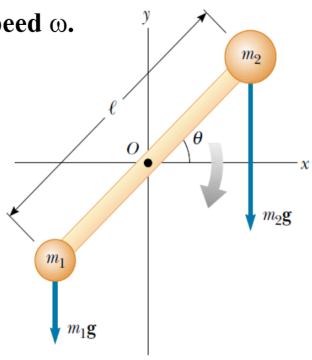
$$\xrightarrow{v_f = R\omega_f} \Delta K = \tfrac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)v_f^{\,2}$$

$$\Delta U_1 = m_1 g h \qquad \Delta U_2 = -m_2 g h$$

$$v_f = \left[\frac{2(m_2 - m_1)gh}{\left(m_1 + m_2 + \dfrac{I}{R^2}\right)}\right]^{1/2}$$



A rigid rod of mass M and length l is pivoted without friction at its center. Two particles of masses m_1 and m_2 are connected to its ends. The combination rotates in a vertical plane with an angular speed ω.
(a) Find an expression for the magnitude of the angular momentum of the system.
(b) Find an expression for the magnitude of the angular acceleration of the system when the rod makes an angle θ with the horizontal.



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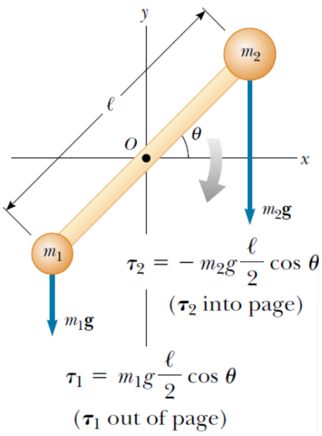
$$
\begin{aligned}
I &= \frac{1}{12}M\ell^2 + m_1\left(\frac{\ell}{2}\right)^2 + m_2\left(\frac{\ell}{2}\right)^2 \\
&= \frac{\ell^2}{4}\left(\frac{M}{3} + m_1 + m_2\right)
\end{aligned}
$$

$$
L = I\omega = \boxed{\frac{\ell^2}{4}\left(\frac{M}{3} + m_1 + m_2\right)\omega}
$$

$$
\sum \tau_{\text{ext}} = \tau_1 + \tau_2 = \tfrac{1}{2}(m_1 - m_2)g\ell\cos\theta
$$

$$
\alpha = \frac{\sum \tau_{\text{ext}}}{I} = \boxed{\frac{2(m_1 - m_2)g\cos\theta}{\ell(M/3 + m_1 + m_2)}}
$$



3.10. 质心

$$
\overrightarrow{r_{\text{CM}}} = \frac{1}{M}\sum_i m_i\vec{r}_i = \frac{1}{M}\int r\,\mathrm{d}m
$$

对于非均匀的物体,如图

$$
\lambda = \alpha x
$$

$$
\begin{aligned}
x_{\text{CM}} &= \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\lambda\,dx = \frac{1}{M}\int_0^L x\alpha x\,dx \\
&= \frac{\alpha}{M}\int_0^L x^2\,dx = \frac{\alpha L^3}{3M}
\end{aligned}
$$

$$
x_{\text{CM}} = \frac{\alpha L^3}{3\alpha L^2/2} = \boxed{\frac{2}{3}L}
$$

$$
M = \int dm = \int_0^L \lambda\,dx = \int_0^L \alpha x\,dx = \frac{\alpha L^2}{2}
$$

$$
\overrightarrow{v_{\text{CM}}} = \frac{1}{M}\sum_i m_i\vec{v}_i
$$

$$
K = K_{\text{CM}} + K' = \frac{1}{2}Mv_{\text{CM}}^2 + \frac{1}{2}\sum_i m_i(\vec{v}_i - \overrightarrow{v_{\text{CM}}})^2
$$

平行轴定理:

$$
I = I_{\text{CM}} + Mh^2
$$

4. 简谐运动

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$$x - x_0 = A\cos(\omega t + \phi),$$

where omega equals to $\sqrt{\frac{k}{m}}$
形如

$$\ddot{x} + \omega^2 x = 0$$

的解为：

$$x = A\cos(\omega t + \phi)$$

5. Waves

right moving wave:

$$y(x,t) = A\cos(x - vt)$$

left moving wave:

$$y(x,t) = A\cos(x + vt)$$

principle of superposition:

$$y'(x,t) = y_1(x,t) + y_2(x,t)$$

对于线性波(linear wave)

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

$$v = \sqrt{\frac{T}{\mu}}$$

v 为波传播的速度，T 为张力，μ为线密度

6. sinusoidal wave

$$y(x,t) = A\cos(kx - \omega t + \phi)$$

$$v = \frac{\omega}{k} = \frac{\lambda}{T}$$

interference:

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$$
\begin{aligned}
y(x,t) &= A\cos(kx-\omega t+\varphi)+A\cos(kx-\omega t) \\
&= 2A\cos\left(\frac{\varphi}{2}\right)\sin\left(kx-\omega t+\frac{\varphi}{2}\right)
\end{aligned}
$$

相长：

$$\varphi_1-\varphi_2=\Delta\varphi=2\pi n$$

相消：

$$\varphi_1-\varphi_2=\Delta\varphi=(2n+1)\pi$$

Beat(频率不同的线性波的暂时干涉)

$$\lambda'\sim\frac{2\pi v}{\Delta\omega/2}$$

$$f_{\text{beat}}=\frac{\Delta\omega}{2\pi}$$

6.1. standing wave

$$
\begin{aligned}
y_1 &= A\sin(kx-\omega t) \\
y_2 &= A\sin(kx+\omega t) \\
y &= y_1+y_2=A\sin(kx)\cos(\omega t)
\end{aligned}
$$

nodes and antinodes

- node:波节，

  $$kx=n\pi$$

- antinode:波腹，

  $$kx=(2n+1)\frac{\pi}{2}$$

6.2. Sound Waves

Intensity:

$$I=\frac{P}{4\pi r^2}=\frac{1}{2}\rho v\omega^2 s_m^2$$

for a wave:

$$S(x,t)=S_m\cos(kx-\omega t)$$

Sound lever :

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$$\beta = 10\lg\left(\frac{I}{I_0}\right)$$

$$I_0 = 1 \times 10^{-12} W/m^2$$

Doppler effect:

$$f' = f\frac{v \pm v_s}{v \pm v_r}$$

其中，v_s为观察者速度，v_r为声源速度,v为声速

7. 相对论

7.1. 速度合成

在非相对论体系下，速度合成为：

$$w = v + u$$

其中，w 可以当作是物体相对于地面的速度，v 是物体相对于火车的速度，u 是火车相对于地面的速度。

但当速度接近光速时，速度合成公式为：

$$w = \frac{v + u}{1 + \frac{vu}{c^2}}$$