Chapter 3

Frequency and Scale

University of Konstanz

Lecture “Image Analysis and Computer Vision”
Winter semester 2014/15

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Overview

1. **Thinking in Frequency: the Fourier transform**

Overview

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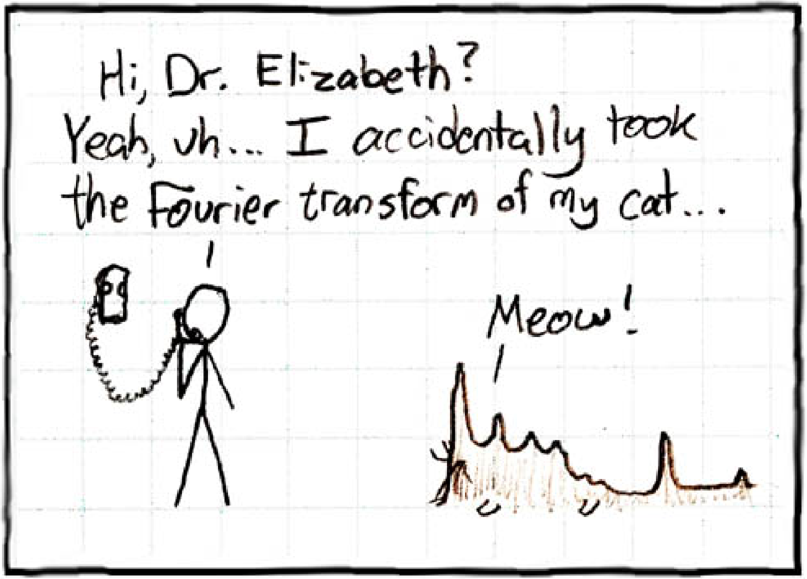
Thinking in frequency



XKCD #26: Fourier

Thinking in frequency

- We have already often mentioned the term “frequency” in an informal way - for example, when thinking about noise as “high-frequency” image content which is amplified by taking derivatives.

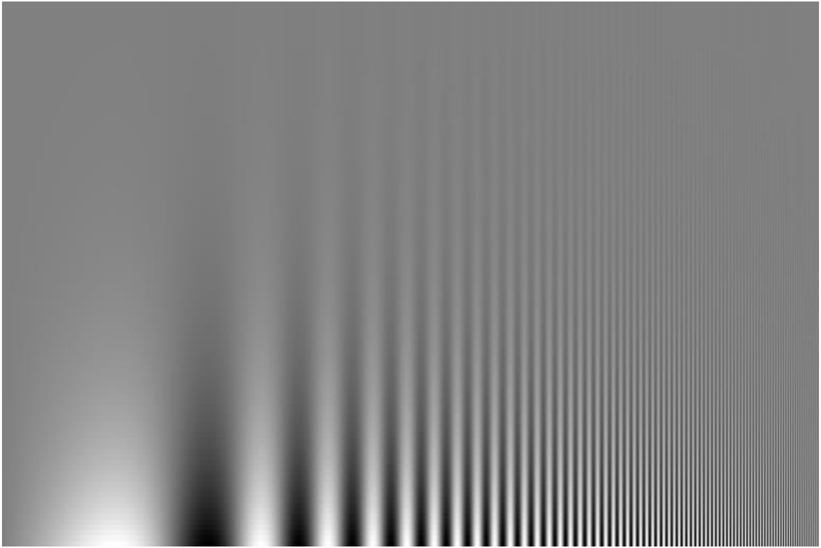
- In this chapter, we will formalize the idea of frequency in image content - however, not too much. While it is very important for filter design, it will not play a major role in the remainder of the lecture.

- However, everyone working with images should have an idea about the **Fourier transform, frequency space, aliasing and the relationships to filters and scale**.

- We will skip most of the mathematical detail to get an idea of the big picture - if it gets you interested, check out the Signal Processing class by Prof. Dietmar Saupe !

Let’s start with exploring how frequency
influences human vision ...

The Campbell-Robson contrast sensitivity curve



For humans, there is a signal frequency for which constrast sensitivity attains a peak
(for now, think about frequency as the rate of change of the signal).

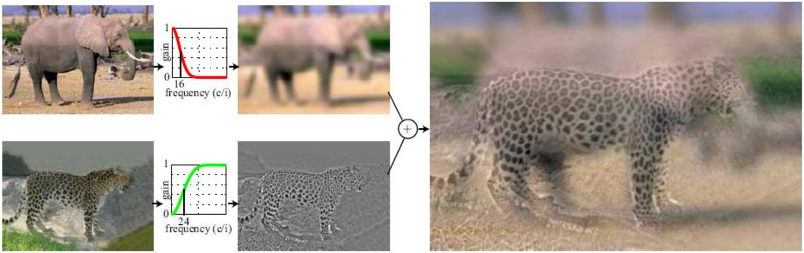
Hybrid images: exploiting the curve

Which animal do you see?

Hybrid images: exploiting the curve

Man or cat?

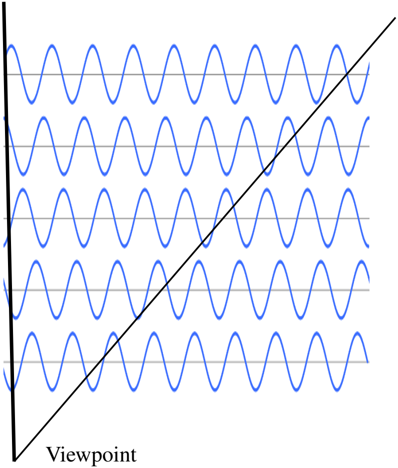
How does it work?



Low frequency content from first image,

high frequency content from second image

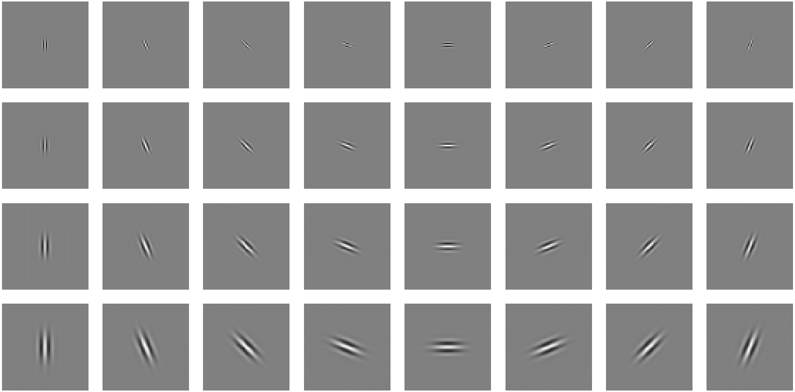
How does it work?



- If an object moves closer to a camera (or the eye), signal frequencies on the sensor (retina) decrease and vice versa.
- Thus, when changing observation distance, different frequencies of the hybrid image enter the range of highest contrast sensitivity.

How does it work?

The human visual system actually measures frequency at the low level processing stage



Certain neurons perform local filtering with kernels of different frequency and orientation ("Gabor filters") - **note:** a filtering kernel usually looks like what it is intended to detect (why? see cross correlation).

Wait ... what is the relationship between the abstract frequencies in the Campbell-Robson curve and actual images?

Fourier's idea

- Crazy idea (1807):

Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.

Jean Baptiste Joseph Fourier (1768-1830)

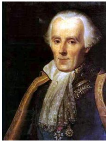
Fourier's idea

- Crazy idea (1807):

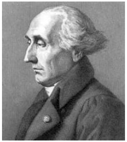
Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.

- Don't believe it?
  - Neither did Lagrange, Laplace, Legendre and other big wigs
  - Not translated into English until 1878!

Mixed initial reviews



Lagrange



Laplace

Legendre

"...the manner in which the author arrives at these equations is not exempt of difficulties and...his analysis to integrate them still leaves something to be desired on the score of generality and even rigour."

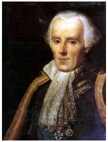
Fourier's idea

- Crazy idea (1807):

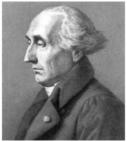
Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.

- Don't believe it?
  - Neither did Lagrange, Laplace, Legendre and other big wigs
  - Not translated into English until 1878!
- But it's mostly true - with some subtle restrictions.

Mixed initial reviews



Lagrange



Laplace

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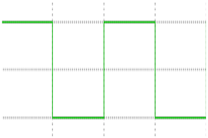
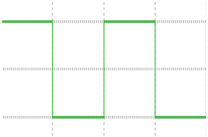
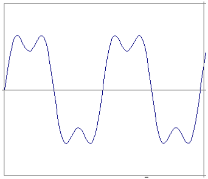
Fourier spectrum: building a signal from waves



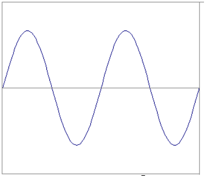
Fourier spectrum: building a signal from waves



$\approx$

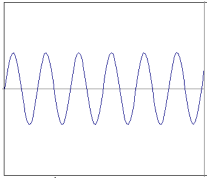


$=$



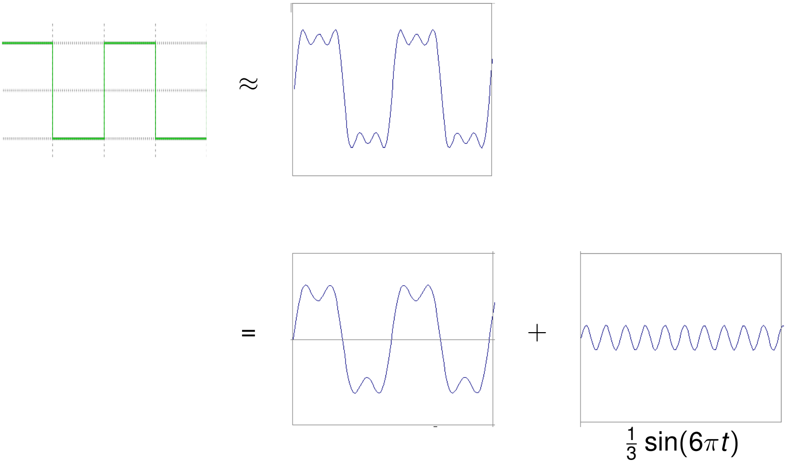
$\sin(2\pi t)$

$+$

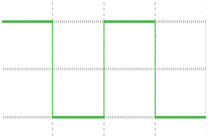


$\frac{1}{2}\sin(4\pi t)$

Fourier spectrum: building a signal from waves



Fourier spectrum: building a signal from waves



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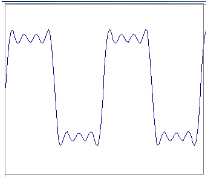

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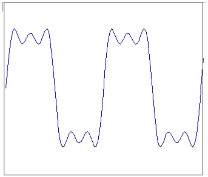

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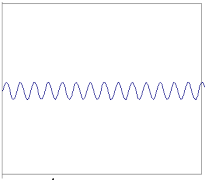
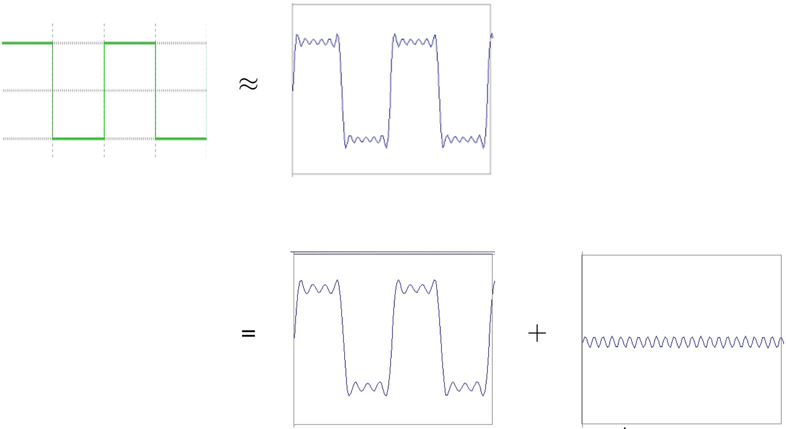


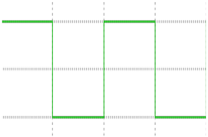
$\frac{1}{4}\sin(8\pi t)$

Fourier spectrum: building a signal from waves

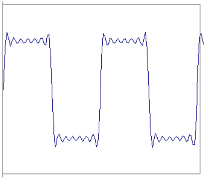


$$\frac{1}{5}\sin(10\pi t)$$

Fourier spectrum: building a signal from waves



$\approx$



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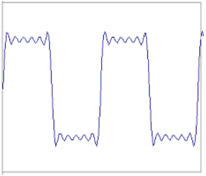

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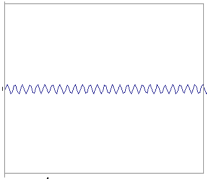
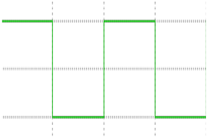
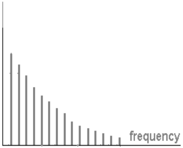


$\frac{1}{6}\sin(12\pi t)$

Fourier spectrum: building a signal from waves



$$\approx \qquad \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi k t)$$



Frequency spectrum

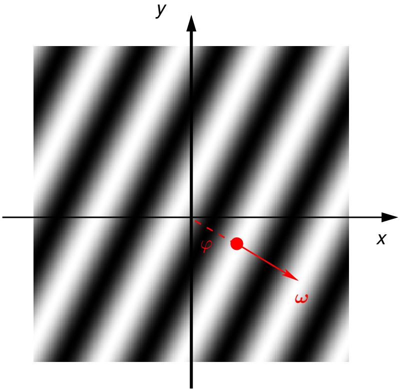
- All well and good, but how about images, which are 2D?
- And how do we find out which frequencies we need with what amplitude?

Waves in 2D

- Elementary building block:

  $$f(x,y) = r\,\cos\big(\,2\pi(\omega_x x + \omega_y y) + \varphi\,\big).$$

- $\boldsymbol{\omega} = [\omega_x\ \omega_y]$ is called the **wave number**.
- $f = |\boldsymbol{\omega}|_2$ gives the number of peaks per unit length, and thus the frequency.
- $\boldsymbol{\omega}/f$ is the direction of the wave.
- The **phase** $\phi \in \mathbb{R}$ gives the distance of the first peak to the origin.
- The **amplitude** $r \geq 0$ gives the maximum peak height.



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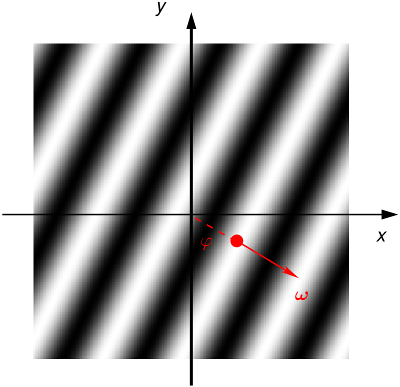
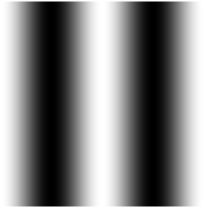


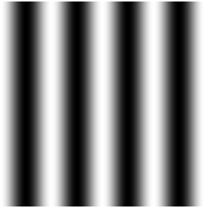
Let's see some examples ...

Varying the wave number ...

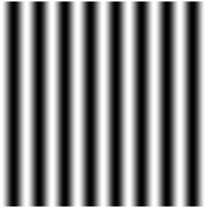
All with phase $\phi = 0$, origin in the center of the image.
Window size $[-1, 1] \times [1, 1]$, positive x is right, positive y is up



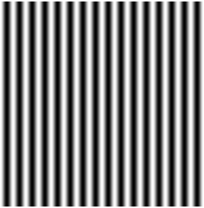
$\boldsymbol{\omega} = [1\ 0]$



$\boldsymbol{\omega} = [2\ 0]$



$\boldsymbol{\omega} = [4\ 0]$



$\boldsymbol{\omega} = [8\ 0]$

$\boldsymbol{\omega} = [0\ 1]$

$\boldsymbol{\omega} = [0\ 2]$

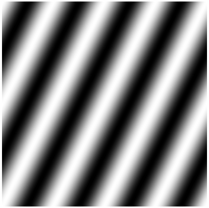
$\boldsymbol{\omega} = [0\ 4]$

$\boldsymbol{\omega} = [0\ 8]$

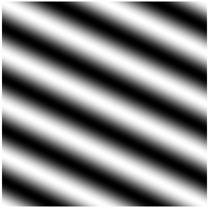
Varying the wave number ...

All with phase $\phi = 0$, origin in the center of the image.
Window size $[-1, 1] \times [1, 1]$, positive x is right, positive y is up

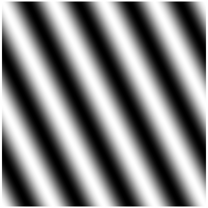
$\boldsymbol{\omega} = [1\ 2]$



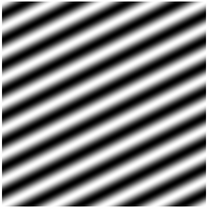
$\boldsymbol{\omega} = [2\ 1]$



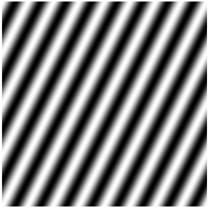
$\boldsymbol{\omega} = [-1\ 2]$



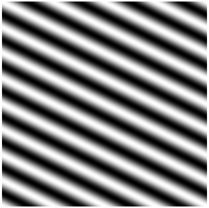
$\boldsymbol{\omega} = [-2\ 1]$



$\boldsymbol{\omega} = [2\ 4]$



$\boldsymbol{\omega} = [4\ 2]$

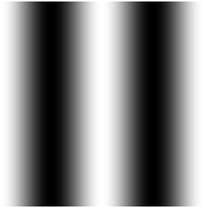


$\boldsymbol{\omega} = [-2\ 4]$

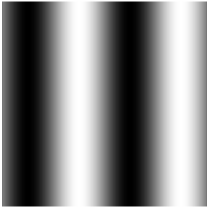
$\boldsymbol{\omega} = [-4\ 2]$

Varying the phase ...

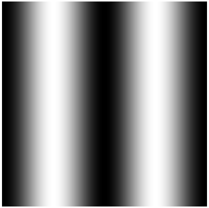
Top row: $\boldsymbol{\omega} = [1\ 0]$, Bottom row: $\boldsymbol{\omega} = [1\ 2]$.
Window size $[-1, 1] \times [1, 1]$, positive x is right, positive y is up



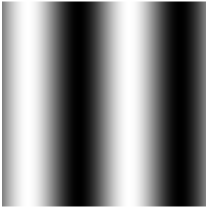
$\phi = 0$



$\phi = \pi/2$



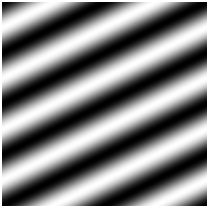
$\phi = \pi$



$\phi = 3\pi/2$

$\phi = 0$

$\phi = \pi/2$



$\phi = \pi$

$\phi = 3\pi/2$

Complex numbers and waves ....

- In science, waves are nearly always written with complex numbers
- The reason is that many formulas become more simple and intuitive.
- In order to understand the output of the Fourier transform (as e.g. from Matlab), one needs to understand a bit about complex numbers.
- I'll try to keep it to the necessary minimum to not detour too much.

A quick reminder: The complex number plane $\mathbb{C}$

to be continued ...

if you have time, refresh your knowledge
about complex numbers a bit before next lecture !