Chapter 3

Frequency and Scale

University of Konstanz

Lecture "Image Analysis and Computer Vision"
Winter semester 2014/15

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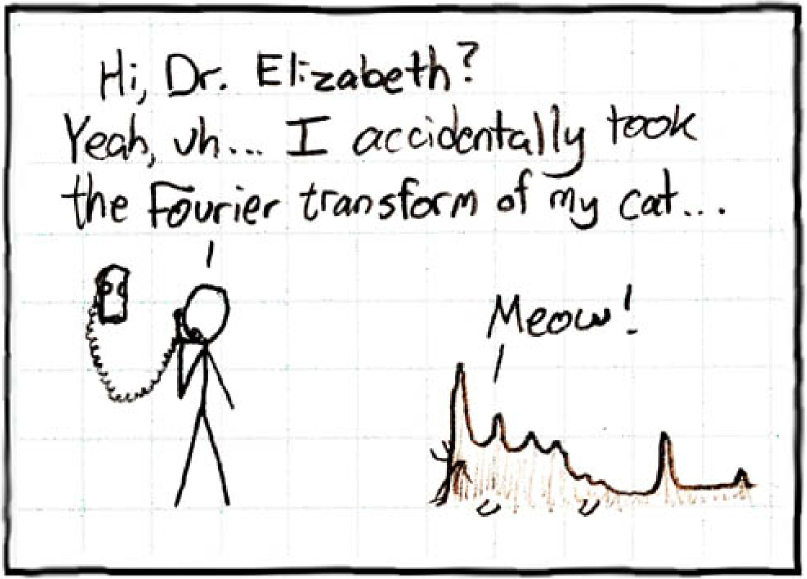
Overview

Overview

Thinking in frequency

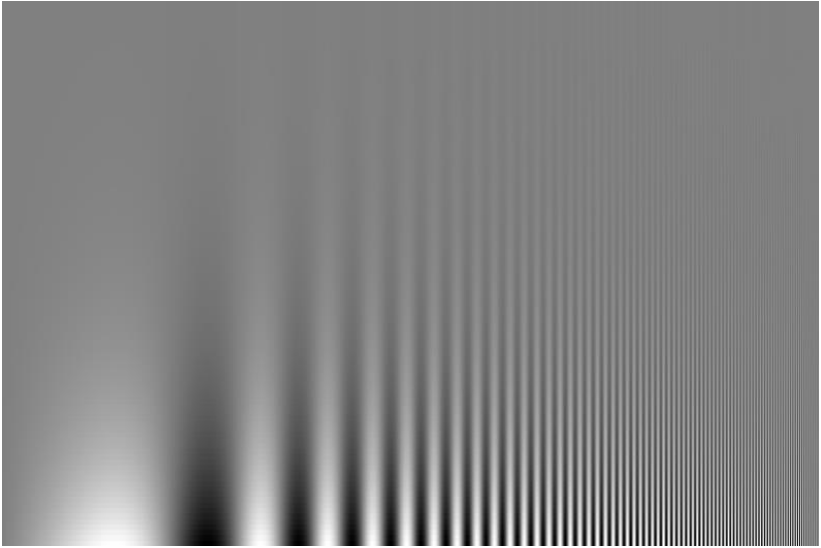


Thinking in frequency

- We have already often mentioned the term “frequency” in an informal way - for example, when thinking about noise as “high-frequency” image content which is amplified by taking derivatives.
- In this chapter, we will formalize the idea of frequency in image content - however, not too much. While it is very important for filter design, it will not play a major role in the remainder of the lecture.
- However, everyone working with images should have an idea about the **Fourier transform, frequency space, aliasing and the relationships to filters and scale**.
- We will skip most of the mathematical detail to get an idea of the big picture - if it gets you interested, check out the Signal Processing class by Prof. Dietmar Saupe !

Let’s start with exploring how frequency
influences human vision ...

The Campbell-Robson contrast sensitivity curve



For humans, there is a signal frequency for which constrast sensitivity attains a peak
(for now, think about frequency as the rate of change of the signal).

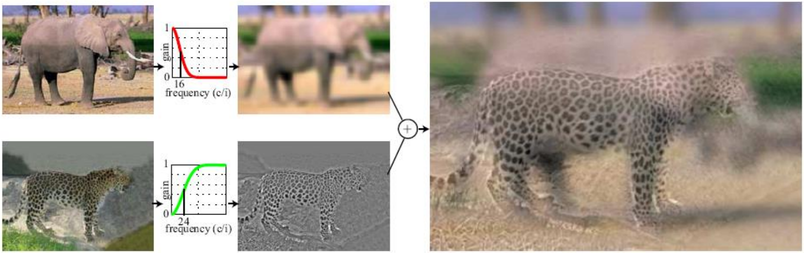
Hybrid images: exploiting the curve

Which animal do you see?

Hybrid images: exploiting the curve

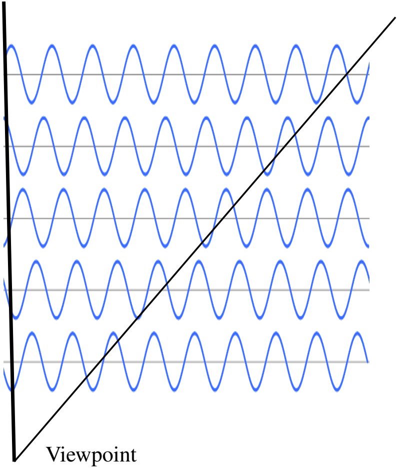
Man or cat?

How does it work?



Low frequency content from first image,
high frequency content from second image

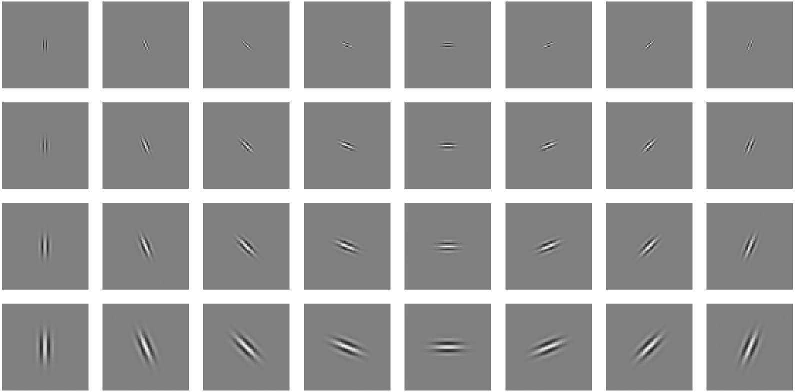
How does it work?



- If an object moves closer to a camera (or the eye), signal frequencies on the sensor (retina) decrease and vice versa.
- Thus, when changing observation distance, different frequencies of the hybrid image enter the range of highest contrast sensitivity.

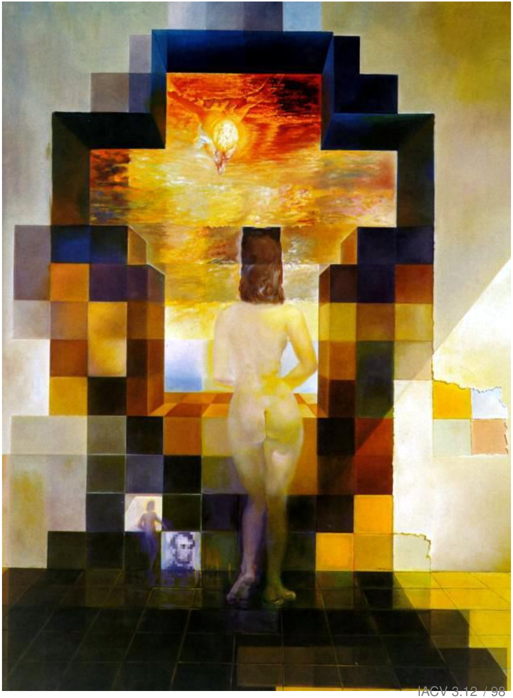
How does it work?

The human visual system actually measures frequency at the low level processing stage



Certain neurons perform local filtering with kernels of different frequency and orientation (“Gabor filters”) - **note:** a filtering kernel usually looks like what it is intended to detect (why? see cross correlation).

The inventor of hybrid images?



Salvador Dali
“Gala Contemplating the Mediterranean Sea, which at 30 meters becomes the portrait of Abraham Lincoln”, 1976.

Downsampled - only low frequencies remain.

Salvador Dali
“Gala Contemplating the Mediterranean Sea, which at 30 meters becomes the portrait of Abraham Lincoln”, 1976.

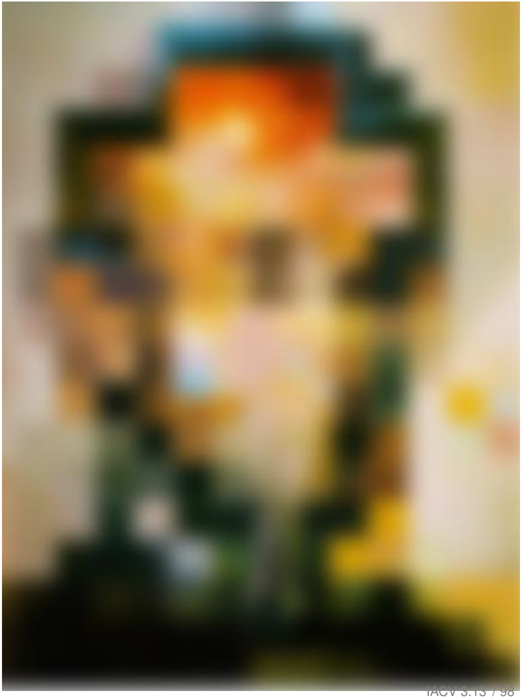
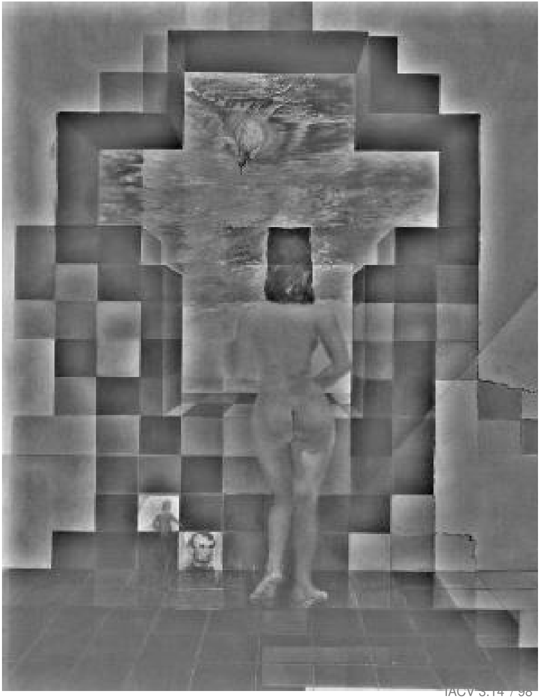


Image: J. Hays

This is the high-frequency content



Salvador Dali
"Gala Contemplating the Mediterranean Sea, which at 30 meters becomes the portrait of Abraham Lincoln", 1976.

Image: J. Hays

Wait ... what is the relationship between the abstract frequencies in the Campbell-Robson curve and actual images?

Fourier's idea

- Crazy idea (1807):

Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.

Jean Baptiste Joseph Fourier (1768-1830)

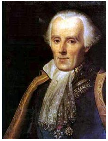
Fourier's idea

- Crazy idea (1807):

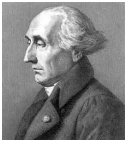
Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.

- Don't believe it?
  - Neither did Lagrange, Laplace, Legendre and other big wigs
  - Not translated into English until 1878!

Mixed initial reviews



Lagrange



Laplace

Legendre

"...the manner in which the author arrives at these equations is not exempt of difficulties and...his analysis to integrate them still leaves something to be desired on the score of generality and even rigour."

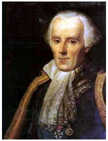
Fourier's idea

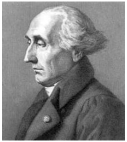
- Crazy idea (1807):

Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.

- Don't believe it?
  - Neither did Lagrange, Laplace, Legendre and other big wigs
  - Not translated into English until 1878!
- But it's mostly true - with some subtle restrictions.

Mixed initial reviews


Lagrange


Laplace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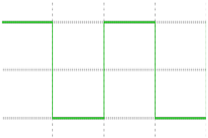
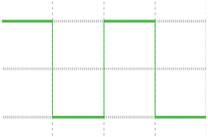
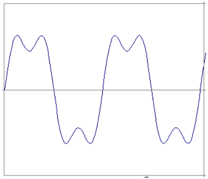
Fourier spectrum: building a signal from waves



Fourier spectrum: building a signal from waves



$\approx$



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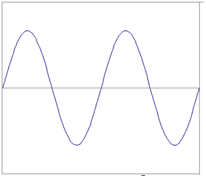
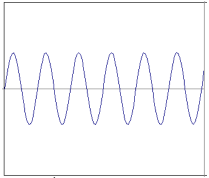


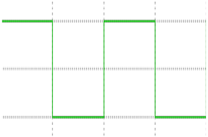
$\sin(2\pi t)$

$+$

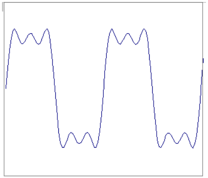


$\frac{1}{2}\sin(4\pi t)$

Fourier spectrum: building a signal from waves



$\approx$



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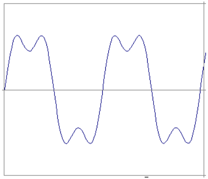

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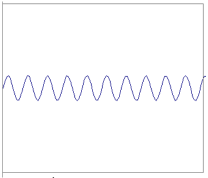
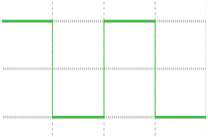
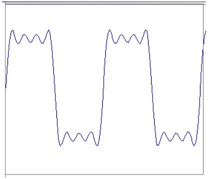


$\frac{1}{3}\sin(6\pi t)$

Fourier spectrum: building a signal from waves



$\approx$



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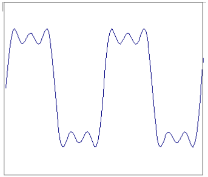

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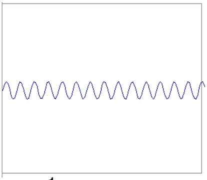
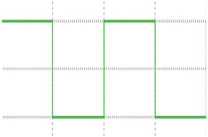
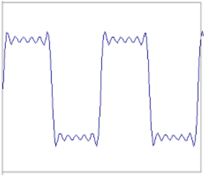


$\frac{1}{4}\sin(8\pi t)$

Fourier spectrum: building a signal from waves



$\approx$



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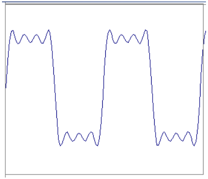

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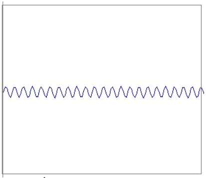
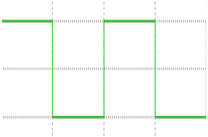
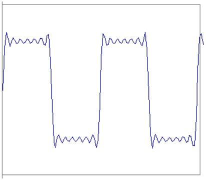


$\frac{1}{5}\sin(10\pi t)$

Fourier spectrum: building a signal from waves



$\approx$



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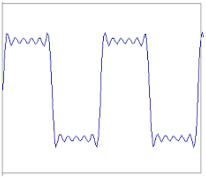

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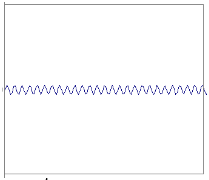
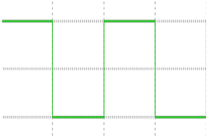
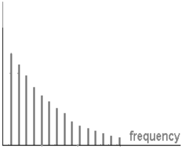


$\frac{1}{6}\sin(12\pi t)$

Fourier spectrum: building a signal from waves



$$= \quad \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi kt)$$



Frequency spectrum

- All well and good, but how about images, which are 2D?

- And how do we find out which frequencies we need with what amplitude?

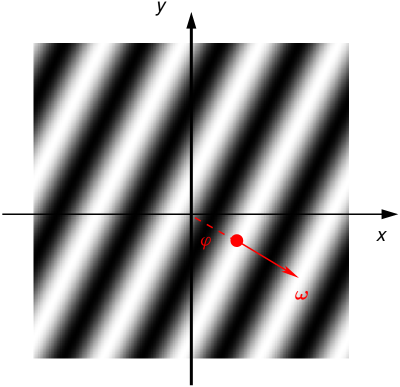
- And what **are** frequency and amplitude anyway?

Waves in 2D

- Elementary building block:

  $$f(x, y) = r \cos\big(2\pi(\omega_x x + \omega_y y) + \varphi\big).$$

- $\boldsymbol{\omega} = [\omega_x \ \omega_y]$ is called the **wave number**.
- $f = |\boldsymbol{\omega}|_2$ gives the number of peaks per unit length, and thus the frequency.
- $\boldsymbol{\omega}/f$ is the direction of the wave.
- The **phase** $\varphi \in \mathbb{R}$ gives the distance of the first peak to the origin.
- The **amplitude** $r \geq 0$ gives the maximum peak height.



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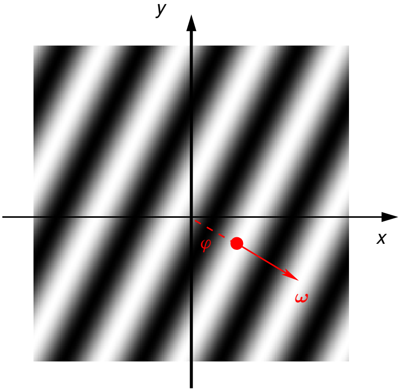
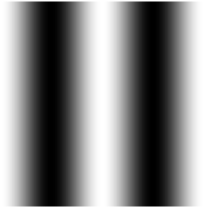


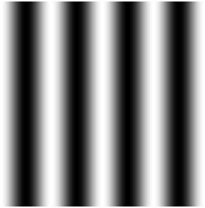
Let's see some examples ...

Varying the wave number ...

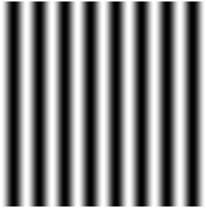
All with phase $\varphi = 0$, origin in the center of the image.
Window size $[-1, 1] \times [1, 1]$, positive x is right, positive y is up



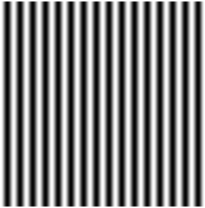
$\boldsymbol{\omega} = [1\ 0]$



$\boldsymbol{\omega} = [2\ 0]$



$\boldsymbol{\omega} = [4\ 0]$



$\boldsymbol{\omega} = [8\ 0]$

$\boldsymbol{\omega} = [0\ 1]$

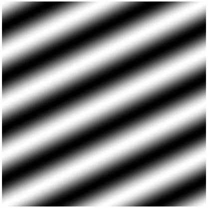
$\boldsymbol{\omega} = [0\ 2]$

$\boldsymbol{\omega} = [0\ 4]$

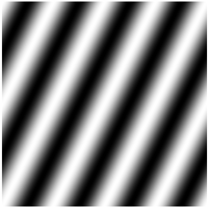
$\boldsymbol{\omega} = [0\ 8]$

Varying the wave number ...

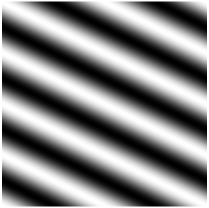
All with phase $\varphi = 0$, origin in the center of the image.
Window size $[-1, 1] \times [1, 1]$, positive x is right, positive y is up



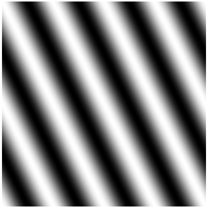
$\boldsymbol{\omega} = [1\ 2]$



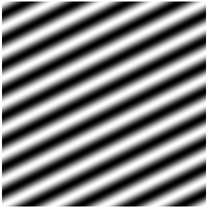
$\boldsymbol{\omega} = [2\ 1]$



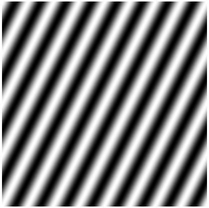
$\boldsymbol{\omega} = [-1\ 2]$



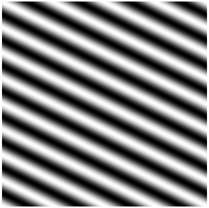
$\boldsymbol{\omega} = [-2\ 1]$



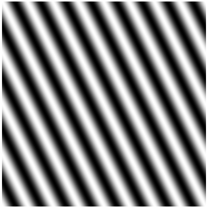
$\boldsymbol{\omega} = [2\ 4]$



$\boldsymbol{\omega} = [4\ 2]$



$\boldsymbol{\omega} = [-2\ 4]$

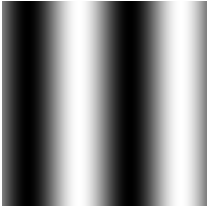


$\boldsymbol{\omega} = [-4\ 2]$

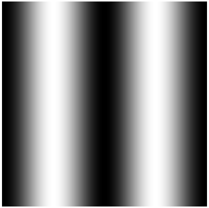
Varying the phase ...

Top row: $\boldsymbol{\omega} = [1\ 0]$, Bottom row: $\boldsymbol{\omega} = [1\ 2]$.
Window size $[-1, 1] \times [1, 1]$, positive x is right, positive y is up

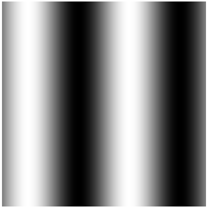
$\varphi = 0$



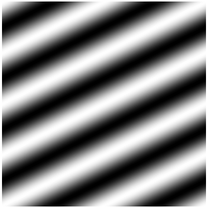
$\varphi = \pi/2$



$\varphi = \pi$

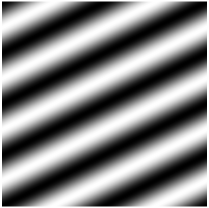


$\varphi = 3\pi/2$

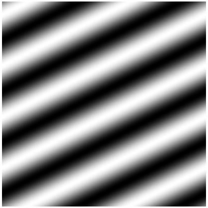


$\varphi = 0$

$\varphi = \pi/2$



$\varphi = \pi$



$\varphi = 3\pi/2$

Complex numbers and waves ....

- In science, waves are nearly always written with complex numbers
- The reason is that many formulas become more simple and intuitive.
- In order to understand the output of the Fourier transform (as e.g. from Matlab), one needs to understand a bit about complex numbers.
- I'll try to keep it to the necessary minimum to not detour too much.

A quick reminder: The complex number plane $\mathbb{C}$

- **Problem:** No solution to

$$x^2 = -1.$$

- **Solution:** Let's invent a new number!
- Let i have the property that

$$i^2 = -1.$$

- Obviously, $i \notin \mathbb{R}$.
- However, i shall obey all the algebraic rules valid for numbers in $\mathbb{R}$.

Carl Friedrich Gauss (1777-1855)

A quick reminder: The complex number plane $\mathbb{C}$

- When you add i to your set of numbers, you automatically get a bunch of other new numbers as well:

  $$2i,\ -3i,\ \pi i,\ i+1,\ 5i+\sqrt{2},\ldots$$

  All the numbers you now have are called the **complex numbers**.

- These are usually visualized in the complex number plane $\mathbb{C}$, which is a two-dimensional vector space over $\mathbb{R}$.

- The first axis contains the real numbers, the other axis has the new number i as a unit.

- That means every complex number $z \in \mathbb{C}$ can be written in the form $x + iy$, where $x = Re(z)$ is called the **real part** and $y = Im(z)$ the **imaginary part**.

Complex numbers are nothing mysterious!

- Just treat them like you would any other number
- Handle the complex unit i formally like you would a variable
- Remember $i^2 = -1$ to simplify stuff

Side note: complex numbers are beautiful!

They might be called complex, but they are actually much simpler than real numbers.

- **Algebraically complete:** Every polynomial with complex coefficients and degree $n \geq 1$ has exactly n roots counting multiples, i.e. it can be factored into n linear terms.
- **Complex analysis** (*Funktionentheorie*), the theory of complex valued functions on complex spaces, just might be one of the most amazing mathematical disciplines.

Side note: complex numbers are beautiful!

Fun with complex numbers ... a mystery Matlab script.

```
% example from mathworks.de, slightly modified
% Parameters
maxIterations = 500;
gridSize = 1000;
xlim = [-0.748766713922161, -0.748766707771757];
ylim = [ 0.123640844894862,  0.123640851045266];
% Setup
x = linspace( xlim(1), xlim(2), gridSize );
y = linspace( ylim(1), ylim(2), gridSize );
[xGrid,yGrid] = meshgrid( x, y );
z0 = xGrid + 1i*yGrid;
count = ones( size(z0) );

% Calculate
z = z0;
for n = 0:maxIterations
    z = z.*z + z0;
    inside = abs( z )<=2;
    count = count + inside;
end
count = log( count );

% Show
imagesc( x, y, count );
axis image
colormap( [bone();flipud( bone() );0 0 0] );
```

It just iterates $z_0 := c, z_{n+1} \mapsto z_n^2 + c$ for complex numbers c in a rectangle of the complex plane, and counts the iterations until $|z| > 2$.

Euler's formula

One of the most important functions is exp - it's defined for complex numbers as well. For purely imaginary numbers it, $t \in \mathbb{R}$, it is given by **Euler's formula**

$$\boxed{\exp(it) = e^{it} = \cos(t) + i\sin(t).}$$

In particular:

- e^{it} lies on the unit circle
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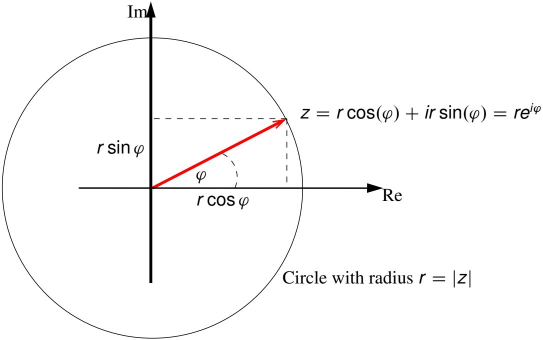
Note: this already shows the connection of exp to waves.

Polar representation of a complex number



Any complex number can be written in the form $z = re^{i\varphi}$, where $r = |z|$ is called the **norm** of z and φ the **argument**.

The complex elementary wave

For wave number $\boldsymbol{\omega} = [\omega_x \; \omega_y] \in \mathbb{R}^2$:

$$W_{\boldsymbol{\omega}}(\mathbf{p}) = e^{2\pi i(\boldsymbol{\omega}\cdot\mathbf{p})}$$

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For wave number $\boldsymbol{\omega} = [\omega_x \; \omega_y] \in \mathbb{R}^2$:

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\begin{aligned}
W_{\boldsymbol{\omega}}(\mathbf{p}) &= e^{2\pi i (\boldsymbol{\omega} \cdot \mathbf{p})} \\
&= \cos\big(2\pi(\boldsymbol{\omega} \cdot \mathbf{p})\big) + i \sin\big(2\pi(\boldsymbol{\omega} \cdot \mathbf{p})\big)
\end{aligned}
$$

Phase zero, amplitude one.

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For wave number $\boldsymbol{\omega} = [\omega_x \ \omega_y] \in \mathbb{R}^2$:

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Phase zero, amplitude one.

But ... you lied! this looks **more** complicated, not less.
Wasn't it supposed to be simple?

What becomes simple now is changing phase and amplitude

Multiply an elementary wave with a complex number $z = re^{i\varphi}$, and we get

$$\begin{aligned} zW_{\boldsymbol{\omega}}(\mathbf{p}) &= re^{i\varphi} e^{2\pi i(\boldsymbol{\omega}\cdot\mathbf{p})} \\ &= re^{i(2\pi(\boldsymbol{\omega}\cdot\mathbf{p})+\varphi)} \end{aligned}$$

... the new wave has amplitude r and phase φ.

Central idea of the Fourier transform

Write a function $f : \mathbb{R}^2 \to \mathbb{C}$ as an "infinite linear combination" of elementary waves with complex coefficients $\hat{f}(\omega)$:

$$f(\mathbf{p}) \;=\; \int_{\mathbb{R}^2} \hat{f}(\omega)\, W_\omega(\mathbf{p})\, \mathrm{d}\omega$$

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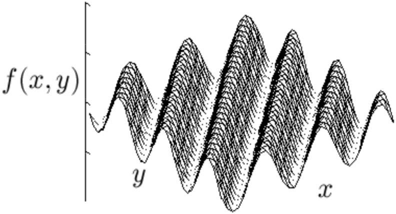
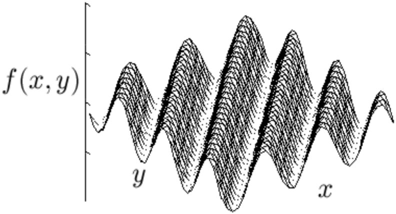
- For each point ω in **frequency space**, the value $\hat{f}(\omega) = re^{i\varphi}$ gives the amplitude and phase of the elementary wave W_ω contained in the signal f.

Examples

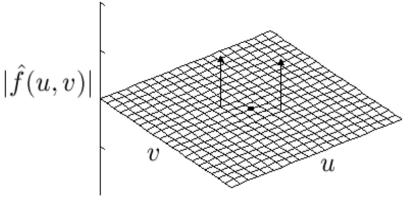


Real-valued elementary wave

Examples



Real-valued elementary wave



Two complex-valued components

The real-valued wave consists of two complex elementary waves with exactly the same amplitude ... why?

Getting rid of the imaginary part in a complex wave

- **Note:** if you add two complex waves with the same amplitude, but opposite wave number and phase ...

$$
\begin{aligned}
& e^{i(2\pi\boldsymbol{\omega}\cdot\mathbf{p}+\varphi)} + e^{i(2\pi(-\boldsymbol{\omega})\cdot\mathbf{p}-\varphi)} \\
= & \cos(2\pi\boldsymbol{\omega}\cdot\mathbf{p}+\varphi) + i\sin(2\pi\boldsymbol{\omega}\cdot\mathbf{p}+\varphi) \\
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- ... you get a real-valued elementary wave with twice the amplitude.

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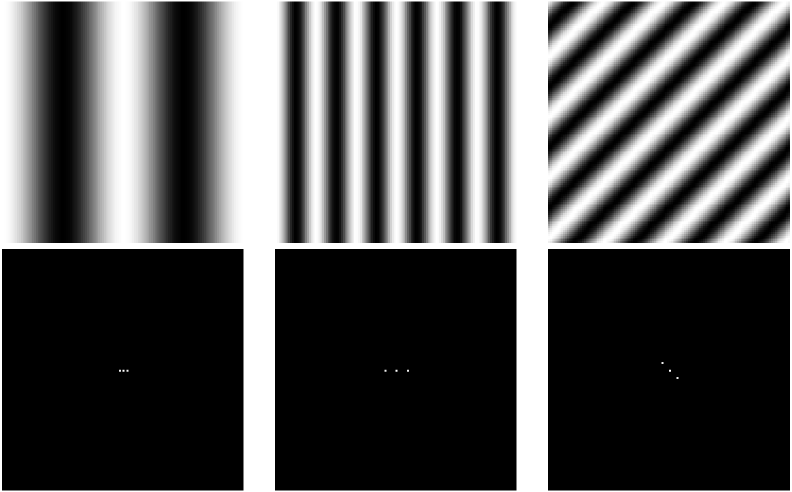
- ... you get a real-valued elementary wave with twice the amplitude.

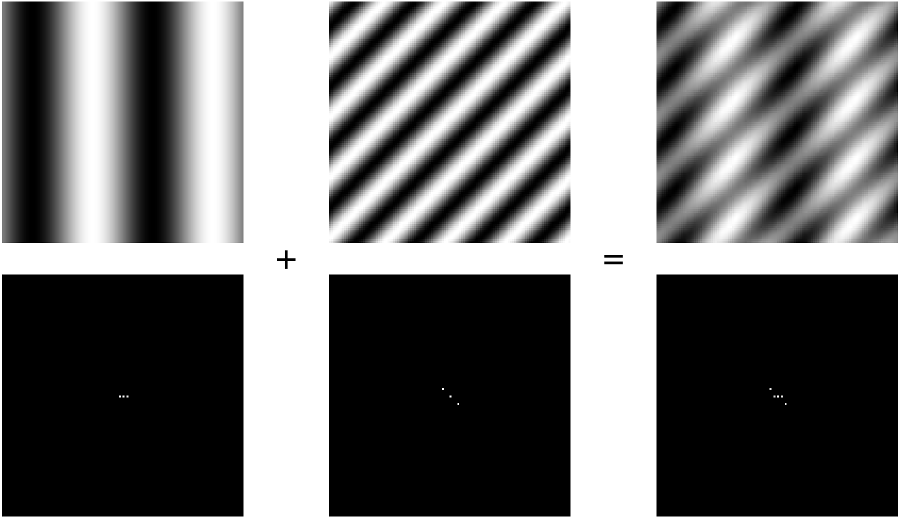
- This is the reason why the amplitude spectrum of images is always symmetric to the origin, while the phase spectrum is anti-symmetric (i.e. you get the negative number on the opposite side).

Elementary waves: changing frequency



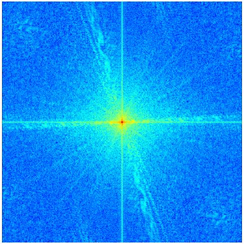
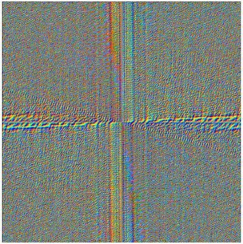
Top: input image, **Bottom:** Amplitude spectrum.

Elementary waves: adding signals



The Fourier transform of a sum is the sum of the transforms.

Real-world image: noise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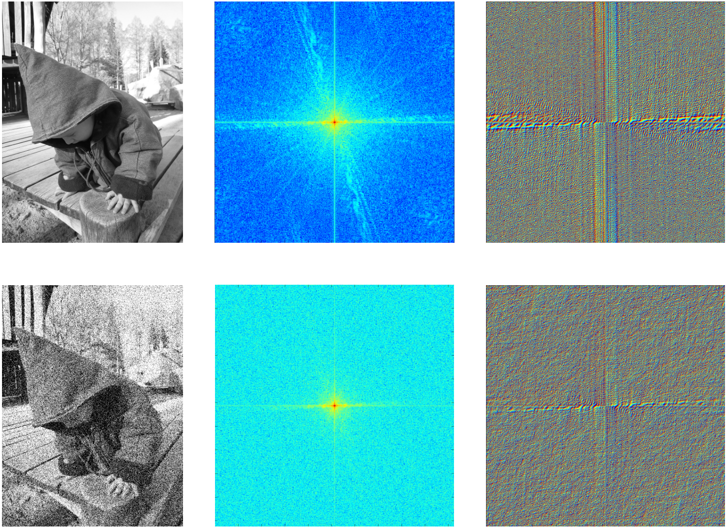
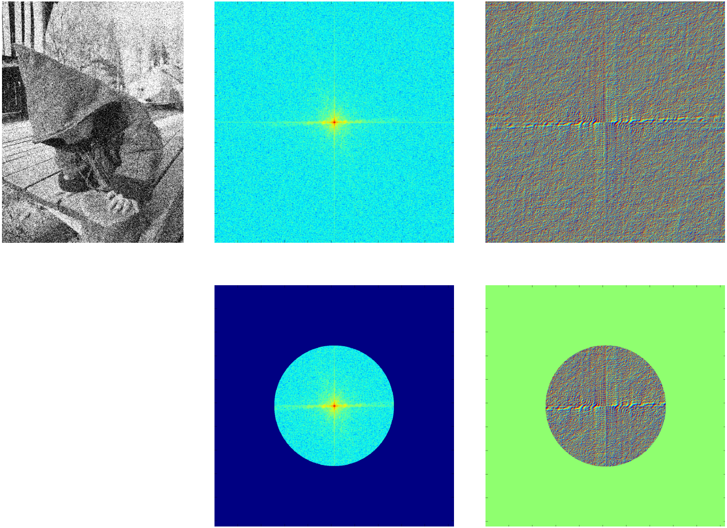


Input image f $\quad$ $\log|\hat{f}|$ $\quad$ $\arg \hat{f}$

... noise amplifies large frequencies.

What if we "cut away" large frequencies?



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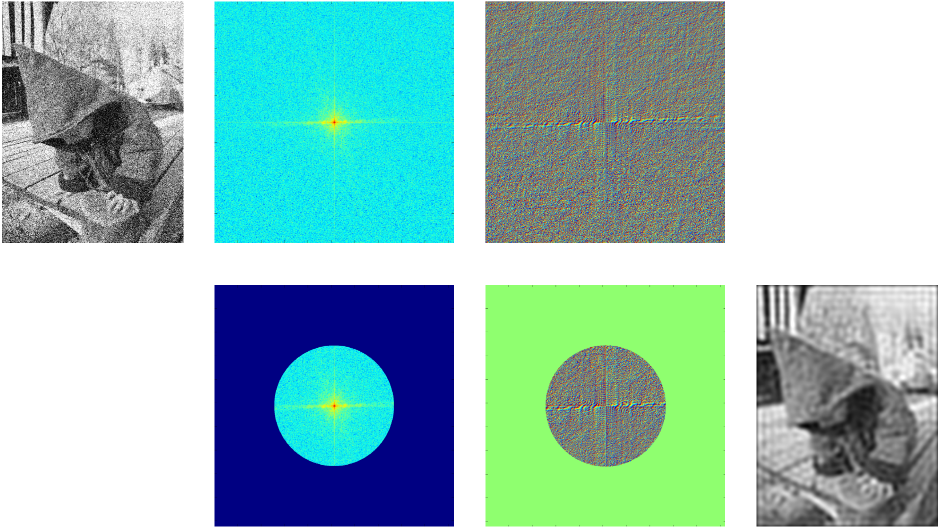
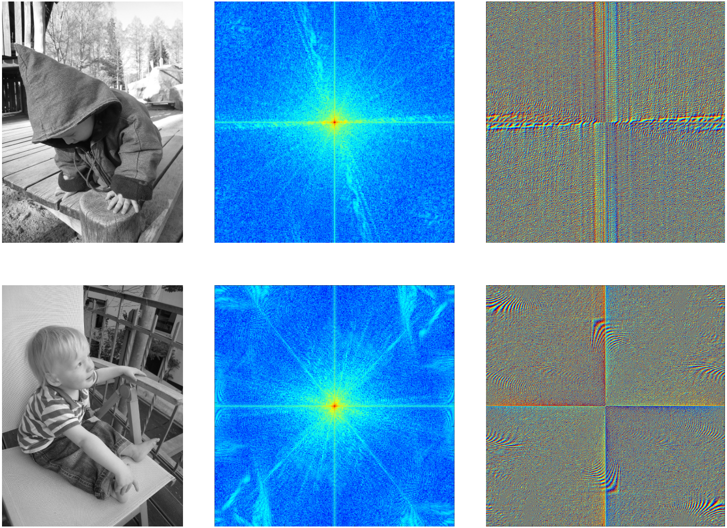


Input image f $\log|\hat{f}|$ $\arg \hat{f}$

... similar to blurring, but not very good quality - let’s revisit this later.

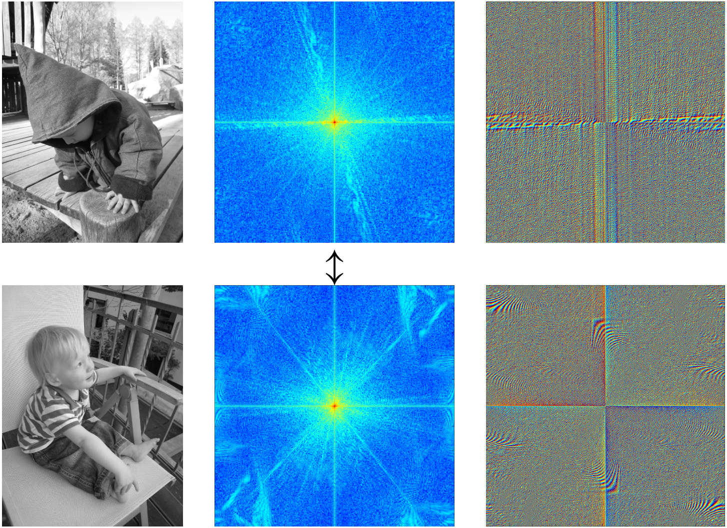
Comparing two real-world images



Input image f | $\log|\hat{f}|$ | $\arg \hat{f}$

... hard to tell much from the Fourier transform.

Weird question: what happens when we exchange these?



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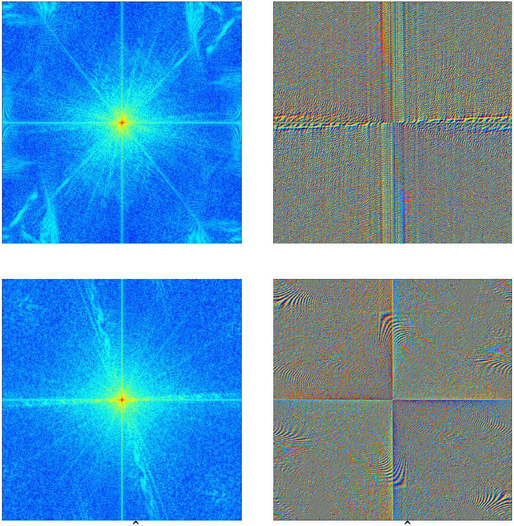

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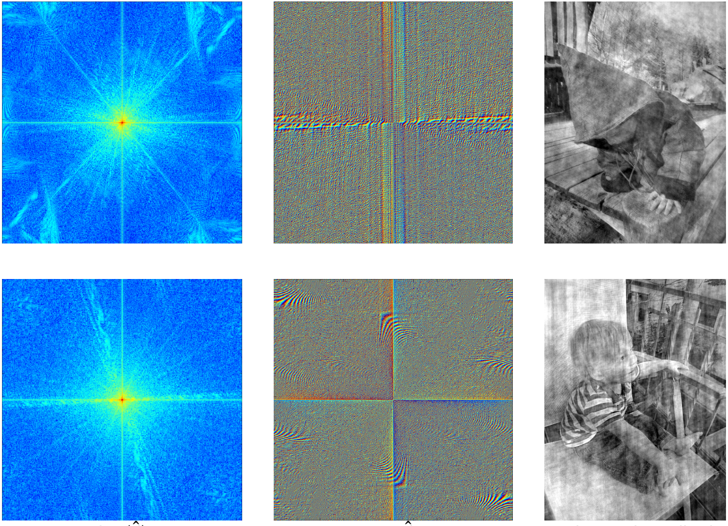

$\log|\hat{f}|$ $\arg \hat{f}$ exchanged amplitude

Phase seems to be more important for how an image looks like.

How do we compute the coefficients of the Fourier transform?
Let's review some linear algebra on $\mathbb{R}^n$ first.

Reminder: computing basis coefficients in $\mathbb{R}^n$

If $(b_1, \ldots, b_n)$ is an orthonormal basis of $\mathbb{R}^n$, then for any vector $a \in \mathbb{R}^n$,

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For the Fourier transform, we formally do something similar, **but or basis consists of the elementary waves W_ω, i.e. complex-valued functions.**

Reminder: the inner product for complex numbers $z, w \in \mathbb{C}$

- Given two complex numbers $z, w \in \mathbb{C}$, how to define an inner product $(z, w)_{\mathbb{C}}$?

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- The product is also called a **"Hermitian" inner product** - it has most algebraic rules in common with the "normal" inner product on real vector spaces, with some subtle differences.

The inner product for functions $f, g : \mathbb{R}^2 \to \mathbb{C}$

- When defining an inner product $(f, g)_{\mathbb{C}}$ for two complex valued functions e.g. on $\mathbb{R}^2$, one thinks of a function as a “vector with infinitely many components”.

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- In consequence, we define

$$
\begin{aligned}
(f, g)_{\mathbb{C}} :&= \int_{\mathbb{R}^2} f(\mathbf{p})\, \bar{g}(\mathbf{p})\, \mathrm{d}\mathbf{p} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y)\, \bar{g}(x, y)\, \mathrm{d}x\, \mathrm{d}y.
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&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) e^{-2\pi i(\omega_x x + \omega_y y)} \, dx \, dy
\end{aligned}$$

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- The first one you don't often see, but I believe it is the most intuitive by far.

Theorem: the Fourier transform

Let $f : \mathbb{R}^2 \to \mathbb{C}$ be "sufficiently nice" (for experts: $f \in L^1(\mathbb{R}^2, \mathbb{C})$). Then

$$\boxed{\begin{aligned} f(\mathbf{p}) &= \int_{\mathbb{R}^2} \hat{f}(\omega)\, W_\omega(\mathbf{p})\, \mathrm{d}\omega \\ \text{with } \hat{f}(\omega) &:= (f, W_\omega)_{\mathbb{C}}. \end{aligned}}$$

The function $\hat{f} : \mathbb{R}^2 \to \mathbb{C}$ is called the **(continuous) Fourier transform** of f.

The Fourier transform (verbose version)

Let $f : \mathbb{R}^2 \to \mathbb{C}$ be "sufficiently nice" (for experts: $f \in L^1(\mathbb{R}^2, \mathbb{C})$). Then the **Fourier transform** of f is the function $\hat{f} : \mathbb{R}^2 \to \mathbb{C}$ defined by

$$\hat{f}(\omega_x, \omega_y) \ := \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) e^{-2\pi i(\omega_x x + \omega_y y)} \, \mathrm{d}x \, \mathrm{d}y.$$

The **inverse Fourier transform** of $\hat{f} : \mathbb{R}^2 \to \mathbb{C}$ is

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The Fourier transform (verbose version)

Let $f : \mathbb{R}^2 \to \mathbb{C}$ be "sufficiently nice" (for experts: $f \in L^1(\mathbb{R}^2, \mathbb{C})$). Then the **Fourier transform** of f is the function $\hat{f} : \mathbb{R}^2 \to \mathbb{C}$ defined by

$$\hat{f}(\omega_x, \omega_y) \ := \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) e^{-2\pi i(\omega_x x + \omega_y y)} \, \mathrm{d}x \, \mathrm{d}y.$$

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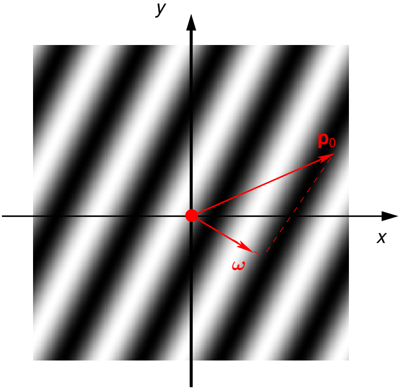
Note: *The formulas are deceptively similar, but the above is an inner product, while the one below is a linear combination (superposition) of waves.*

Overview

How do convolution, derivatives and the Fourier transform interact?

Shifting an elementary wave

An elementary wave $W_\omega(\mathbf{p}) = e^{2\pi i\omega\cdot\mathbf{p}}$ is shifted by a vector $\mathbf{p}_0$. How does the equation of the wave change?



$$e^{2\pi i\omega\cdot(\mathbf{p}-\mathbf{p}_0)} = e^{2\pi i\omega\cdot\mathbf{p}}\ e^{-2\pi i\omega\cdot\mathbf{p}_o}$$

This is a wave-number dependent phase shift by $-2\pi\boldsymbol{\omega}\cdot\mathbf{p}_0$.

→ makes sense, since amplitude/frequency are not changed by shifts, and only shifts in the direction of $\boldsymbol{\omega}$ influence phase.

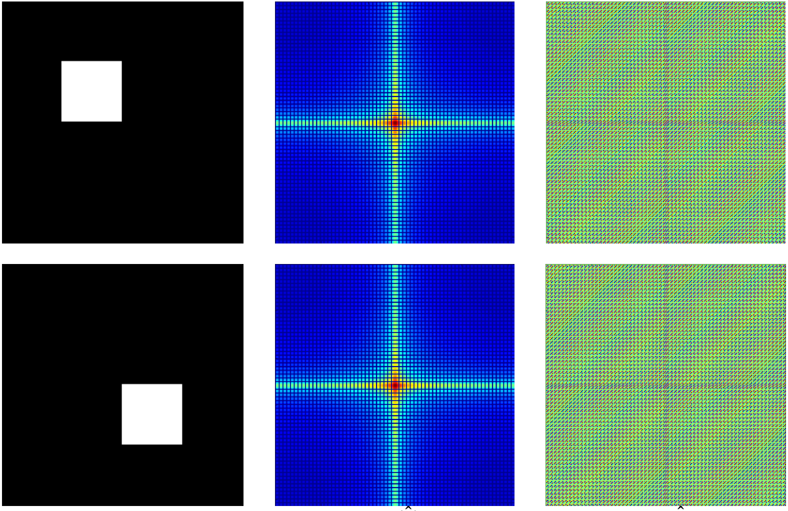
The shift theorem

Generalized to the Fourier transform of a function, this means:

- When a function is shifted, all elementary waves in the Fourier decomposition are shifted as well
- In consequence, the phase of all Fourier coefficients changes according to the formula above.

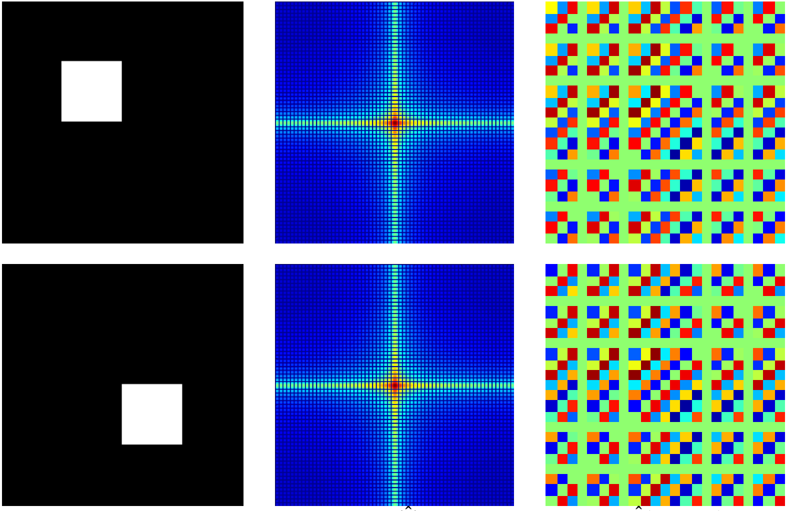
$$\boxed{\widehat{f(\mathbf{p}-\mathbf{p}_0)}(\omega) \;=\; e^{-2\pi i \omega\cdot\mathbf{p}_o}\,\hat{f}(\omega).}$$

Example: the shift theorem



Input image f | $\log |\hat{f}|$ | $\arg \hat{f}$

Example: the shift theorem



Input image f | $\log|\hat{f}|$ | $\arg \hat{f}(zoomed)$

What does convolution do to an elementary wave?

Convolution of an elementary wave

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a filter kernel. Then the convolution of g with an elementary wave W_ω can be interpreted as a superposition (linear combination) of shifted waves, whose amplitudes are given by the kernel entries.

$$(g * W_\omega)(\mathbf{p}) \;=\; \int_{\mathbf{p}_0 \in \mathbb{R}^2} g(\mathbf{p}_0)\, W_\omega(\mathbf{p} - \mathbf{p}_0)\, \mathrm{d}\mathbf{p}_0$$

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&= W_\omega(\mathbf{p}) \int_{\mathbf{p}_0 \in \mathbb{R}^2} g(\mathbf{p}_0)\; e^{-2\pi i \omega \cdot \mathbf{p}_o}\, d\mathbf{p}_0 \\
&= W_\omega(\mathbf{p})\, \hat{g}(\omega).
\end{aligned}
$$

When convolved with f, an elementary wave W_ω undergoes a phase and amplitude change given by the Fourier transform $\hat{g}(\omega)$ of the filter.

The convolution theorem of the Fourier transform

- Let's again generalize the insight to a function f, which is a superposition of elementary waves.

The convolution theorem of the Fourier transform

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- The above means that to arrive at $g * f$, for each wave number $\boldsymbol{\omega}$, the coefficient $\hat{f}(\omega)$ must be multiplied by $\hat{g}(\boldsymbol{\omega})$.

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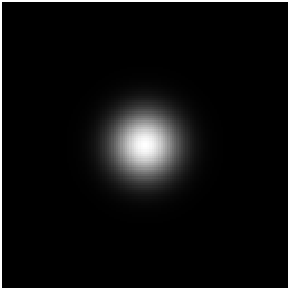
In particular, a convolution of f and g can be computed by multiplying the Fourier transforms point-wise, then computing the inverse transform. For large kernel sizes, this is much more efficient than straight-forward convolution.

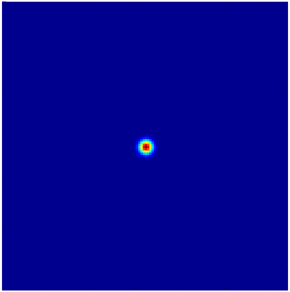
Example: the convolution theorem

The Fourier transform of a Gaussian is a Gaussian-like function:

$$f(\mathbf{p}) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{\mathbf{p}^2}{2\sigma^2}\right) \quad \Rightarrow \quad \hat{f}(\omega) = \exp\left(-\frac{4\pi^2\omega^2}{2\sigma^{-2}}\right)$$



Gaussian kernel g



$|\hat{g}|$

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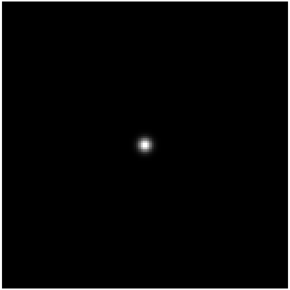

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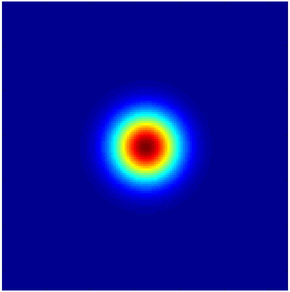
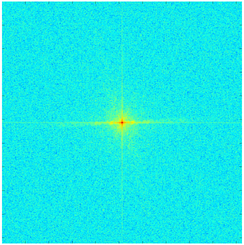


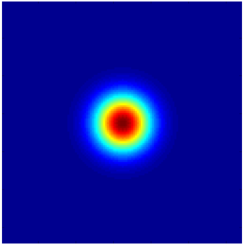
$|\hat{g}|$

Example: the convolution theorem

Input image f



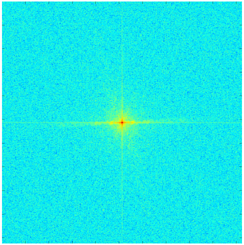
$\log|\hat{f}|$



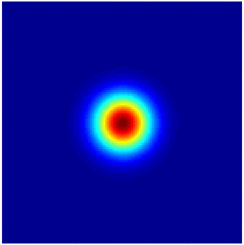
$\log|\hat{g}|$

Example: the convolution theorem

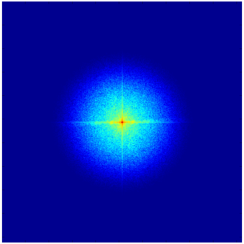
Input image f



$\log|\hat{f}|$



$\log|\hat{g}|$

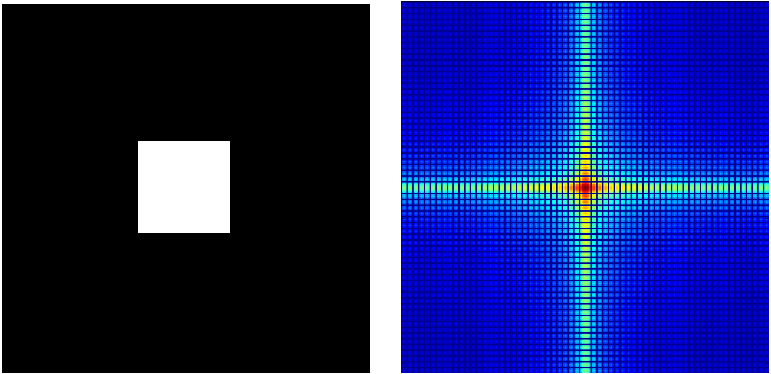


$\log|\hat{f} \cdot \hat{g}|$

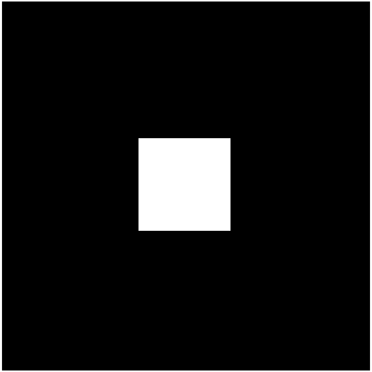
Result (inverse FT)

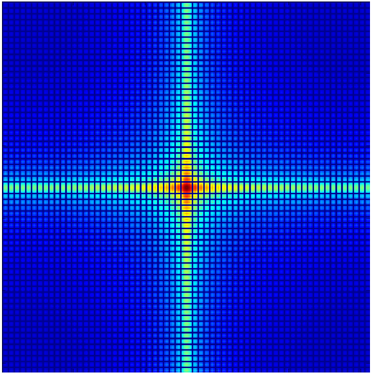
Exactly the same as computing Gaussian convolution $g * f$.

Can you now imagine why box filtering is sub-optimal?



Can you now imagine why box filtering is sub-optimal?





Not invariant to rotation, leaves some high frequencies intact.

High-pass, low-pass and band-pass filters

Some terminology:

- Filters which leave low frequencies mostly unchanged and suppress high frequencies are called **low-pass filters**.

  Example: Gaussian.

High-pass, low-pass and band-pass filters

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  Example: Impulse (identity filter) minus Gaussian.

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- Filters which leave frequencies in a certain range mostly unchanged and suppress high and low frequencies are called **band-pass filters**.

  Example: Difference of Gaussians.

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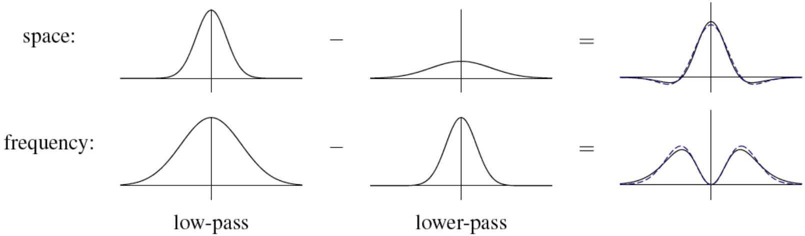
  Example: Impulse (identity filter) minus Gaussian.

- Filters which leave frequencies in a certain range mostly unchanged and suppress high and low frequencies are called **band-pass filters**.

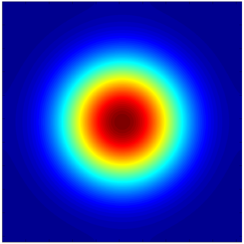
  Example: Difference of Gaussians.

- Side note: it might make more sense to you now why filters are called "filters" - they are often imagined as "filtering out" certain frequency ranges.

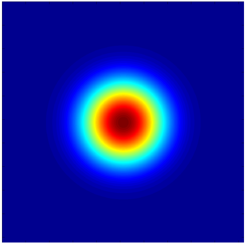
The Difference of Gaussians (DoG) filter revisited



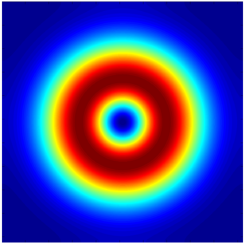
The Difference of Gaussians (DoG) filter revisited



Gaussian 1 FFT $\log|\hat{g}_1|$

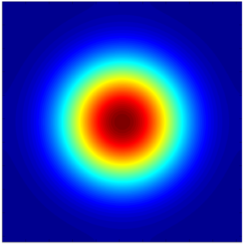


Gaussian 2 FFT $\log|\hat{g}_2|$



DoG $\log|\hat{g}_1 - \hat{g}_2|$

The Difference of Gaussians (DoG) filter revisited



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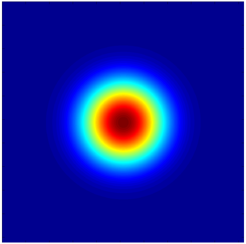
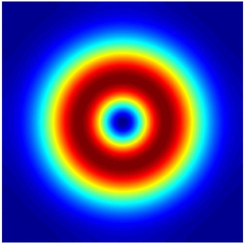
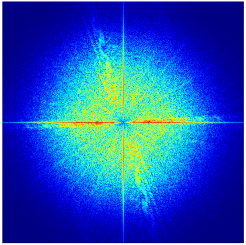


Gaussian 2 FFT $\log |\hat{g}_2|$



DoG $\log |\hat{g}_1 - \hat{g}_2|$

Input image f



FFT result $\log |\hat{f}(\hat{g}_1 - \hat{g}_2)|$

Result (inverse FT)

What about the Fourier transform and derivatives?

The derivative theorem for the Fourier transform

The Fourier coefficients of partial derivatives of a function can easily[1] be computed using the chain rule:

$$\begin{aligned}
\partial_x^n \partial_y^m f(\mathbf{p}) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\omega)\, \partial_x^n \partial_y^m \left[\exp\big(2\pi i(\omega_x x + \omega_y y)\big)\right] \mathrm{d}x\, \mathrm{d}y \\
&= \int_{\mathbb{R}^2} (2\pi i \omega_x)^n (2\pi i \omega_y)^m \hat{f}(\omega)\, W_\omega(\mathbf{p})\, \mathrm{d}\mathbf{p}.
\end{aligned}$$

[1] actual difficulty rating depends on experience

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From this we read off:

$$\boxed{\widehat{\partial_x^n \partial_y^m f}(\boldsymbol{\omega}) = (2\pi i \omega_x)^n (2\pi i \omega_y)^m \hat{f}(\omega).}$$

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Taking derivatives amplifies high frequencies - the higher the frequency and derivative order, the higher the amplification.

[1] actual difficulty rating depends on experience

Example: the derivative theorem

Examples for derivative filters: see exercise sheet !

The nitty-gritty details: discrete FT, FFT and Matlab (1)

- In a computer implementation of the Fourier transform, you can only work on grids, e.g. with discrete image and frequency spaces.

- In particular, you only compute Fourier coefficients for a discrete set of elementary waves, and all integrals turn to sums. This is the **discrete Fourier transform** (DFT).

- We skip all details regarding the set of frequencies chosen for a certain grid size.

- A very efficient and famous algorithm to compute the DFT is the Fast Fourier Transform (FFT), which is of complexity $\mathcal{O}(n \log(n))$ for n pixels - Gauss already used it in 1805 for astronomical computations.

The nitty-gritty details: discrete FT, FFT and Matlab (2)

- In Matlab, for the 2D FFT you use the function `fft2`, which takes as arguments a grayscale image and an optional size of the output grid (should be a power of two for maximum efficiency).
- The ordering of the coefficients is strange, in order to transform the result such that low frequencies are in the center (like in the examples in the lecture), you use `fftshift`.
- To go back from frequency domain to a 2D image, use `ifft2`. If you previously shifted the coefficients, remember to first shift them back using `ifftshift`.
- Examples for all of this accompany the exercise sheet number 4.

Summary: properties of the Fourier transform

For functions f, g and constants $\alpha, \beta \in \mathbb{C}$:

- **Linearity:**
$$\widehat{\alpha f + \beta g} = \alpha \hat{f} + \beta \hat{g}.$$

- **Rotation invariance:** when f is rotated, the Fourier transform $\hat{f}$ is rotated by the same angle.

- **Shift theorem:** shifting by $\mathbf{p}_0$ leads to a phase change according to
$$\widehat{f(\mathbf{p} - \mathbf{p}_0)}(\boldsymbol{\omega}) \; = \; e^{-2\pi i \boldsymbol{\omega} \cdot \mathbf{p}_o} \hat{f}(\boldsymbol{\omega}).$$

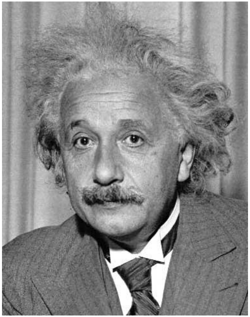
- **Convolution theorem:**
$$\widehat{f * g} = \hat{f} \, \hat{g}.$$

- **Derivatives:**
$$\widehat{\partial_x^n \partial_y^m} f(\boldsymbol{\omega}) = (2\pi i \omega_x)^n (2\pi i \omega_y)^m \hat{f}(\omega).$$

Overview

Reminder: detecting patches

Goal: Find  in the image.

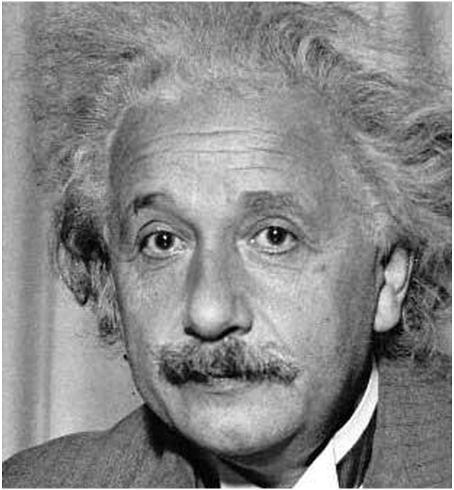


Method: Compare patch to be found to patch of same size at every image location, threshold result.

Not scale invariant!

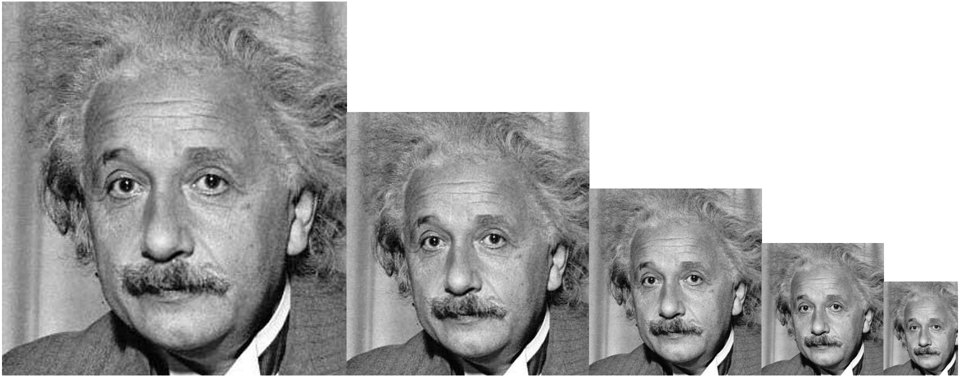
What if the image has too many pixels compared to the patch?

Goal: Find  in the image.



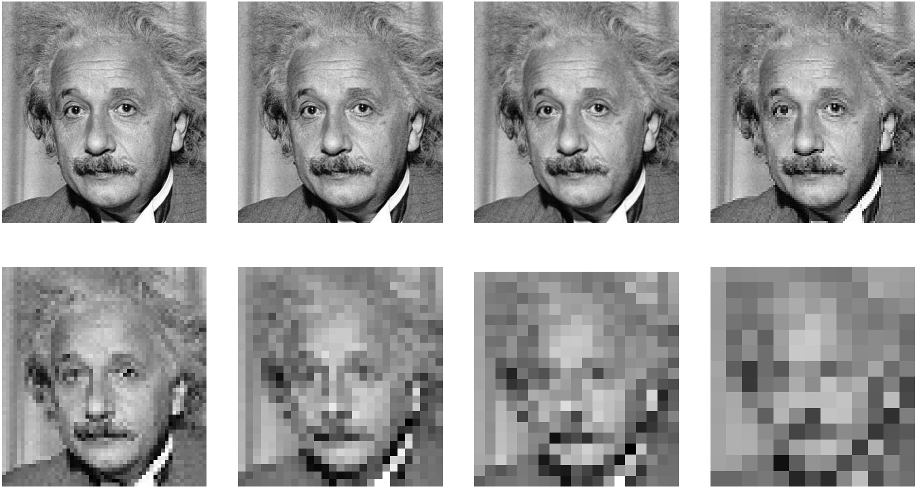
We reduce number of pixels and look at different “scales”!

Goal: Find  in the image.

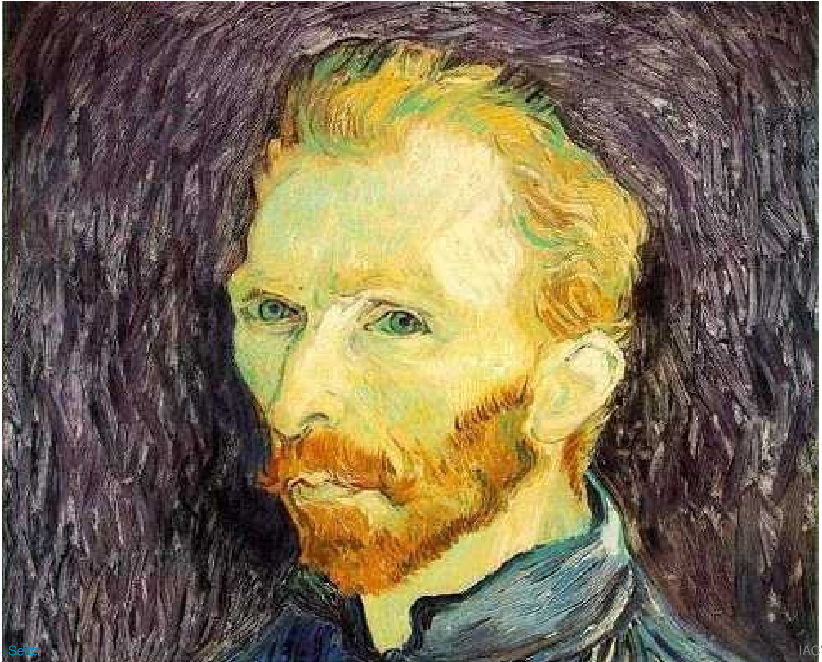


We reduce number of pixels and look at different “scales”!

Note: for patch comparison to work, both patch and image need to have the same resolution.

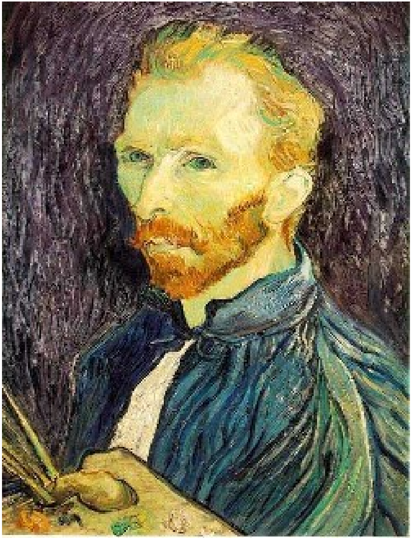


How do we lower the number of pixels in an image?

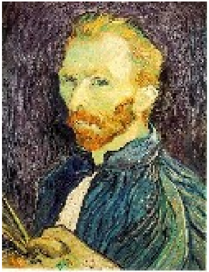


Source: S. Seitz

Brute force approach: throw away every second column/row



downsampled by factor 2

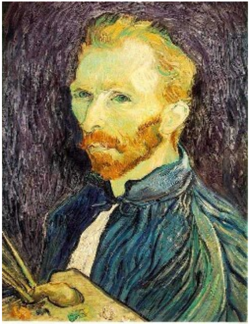


factor 4

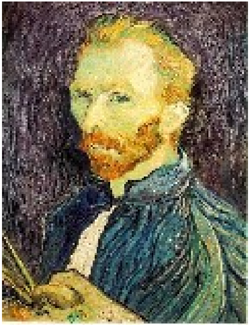
factor 8

Source: S. Seitz

Why does it look so bad?



downsampled by factor 2

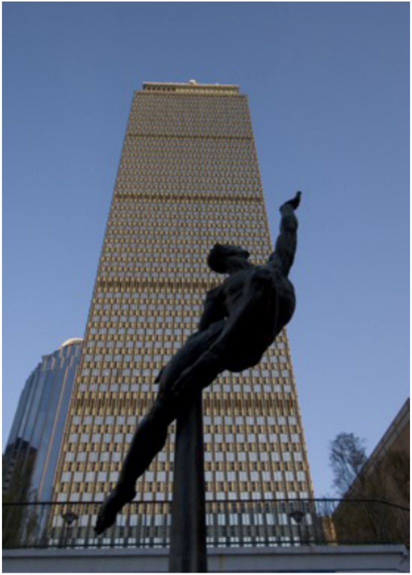


factor 4, zoom x2

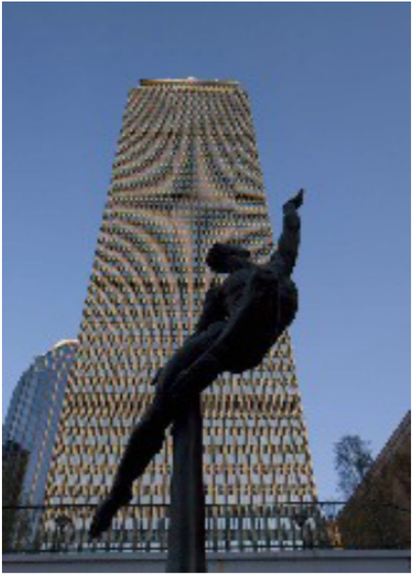
factor 8, zoom x4

Source: S. Seitz / R. Urtasun

What happens here?



original



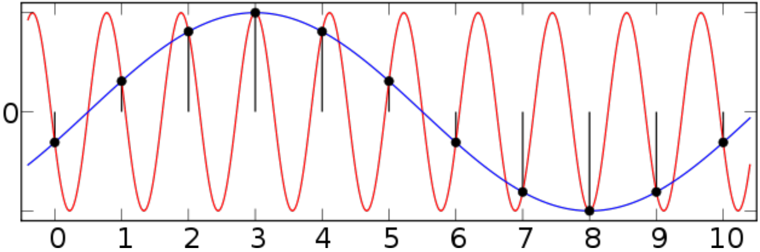
downsampled by factor 4, zoom x4

Source: F. Durand

And here?

Source: L. Zhang

Aliasing effect



Aliasing effect: if the sampling rate is too low, high frequent components are observed as low frequent artifacts.

Sampling Theorem (Abtasttheorem)

- Let a signal f be **band-limited**, i.e. there exists a highest frequency W such that

$$\hat{f}(\omega) = 0 \text{ if } |\omega|_2 > W.$$

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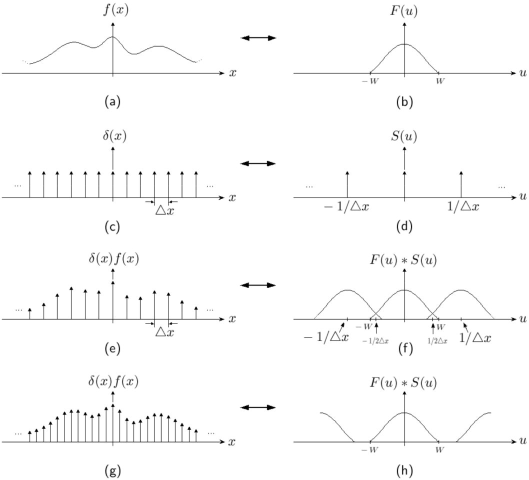
- The critical frequency $2W$ where aliasing starts is called the **Nyquist frequency**.

[Whittaker 1915, Nyquist 1928, Kotelnikov 1933, Shannon 1949]

Sampling theorem: illustration of proof idea

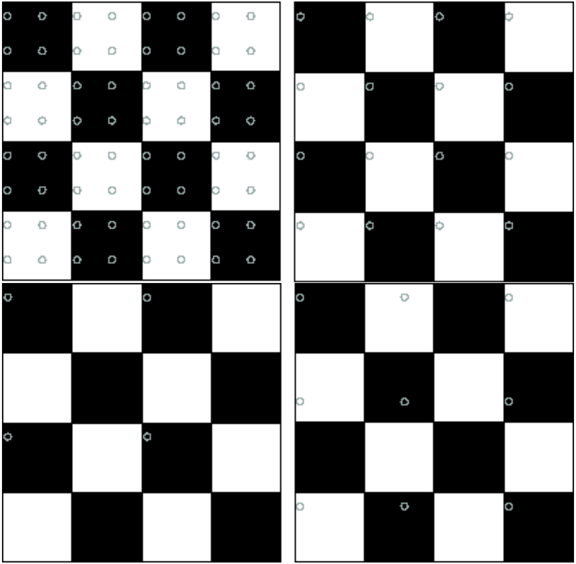


(a) Band-limited function. (b) Fourier spectrum. (c) Delta comb. (d) FT of the delta comb is a delta comb with reciprocal grid distance. (e) Sampling a band-limited function is multiplication with a delta comb in the spatial domain. (f) In the Fourier domain this gives convolution of the Fourier transforms of (b) and (d). Overlapping frequency bands from different periods create aliasing. (g) Reduction of the sampling distance. (h) In the Fourier domain the frequency bands do no longer overlap and no aliasing effects arise. Author: N. Khan (2005).

Nyquist limit 2D example

Good sampling

Bad sampling



Back to downsampling ...

Problem with downsampling:

- Assume the input image is band-limited close to the Nyquist frequency.
- Downsampling means increasing the sampling distance h, which reduces the Nyquist frequency.
- This means the input contains frequencies which cause aliasing at the coarser scale.

Solution:

Back to downsampling ...

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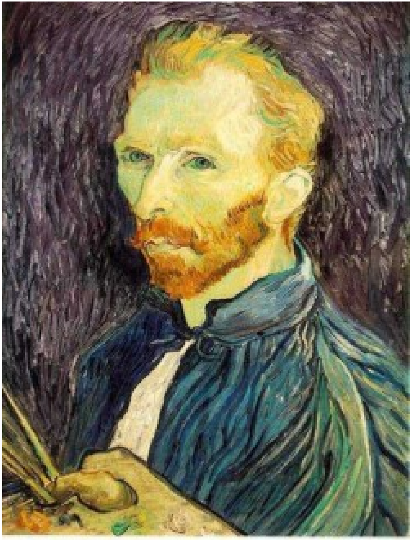
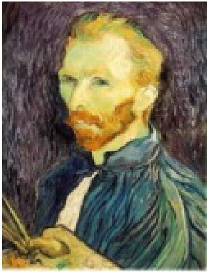
Solution:

- Further band-limit the signal before downsampling.
- Apply e.g. a Gaussian filter, then reduce resolution.

Gaussian filter, then throw away every second column/row



filtered, then downsampled by factor 2

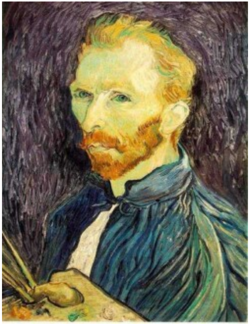


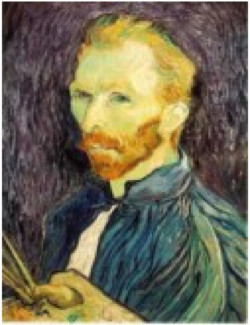
factor 4

factor 8

Source: S. Seitz / R. Urtasun

Looks much better now - no artifacts at low resolution

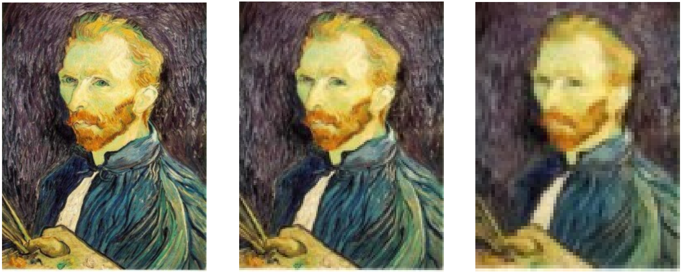

downsampled by factor 2


factor 4, zoom x2

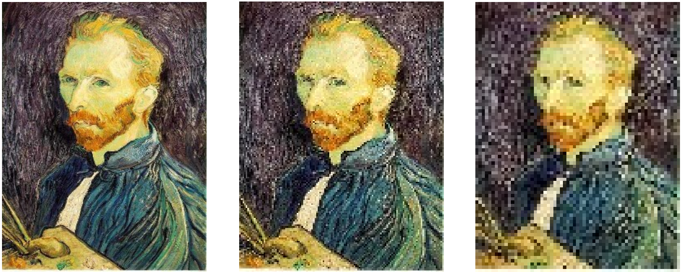

factor 8, zoom x4

Source: S. Seitz / R. Urtasun

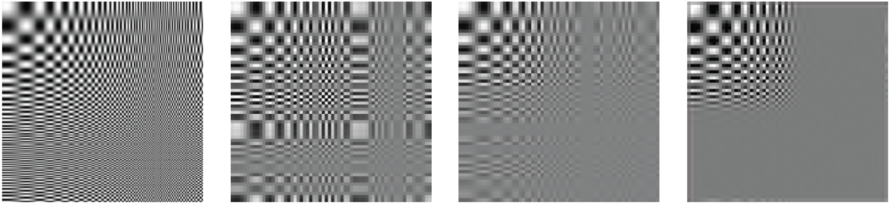
Compare to previous result ...



downsampled by factor 2 | factor 4, zoom x2 | factor 8, zoom x4



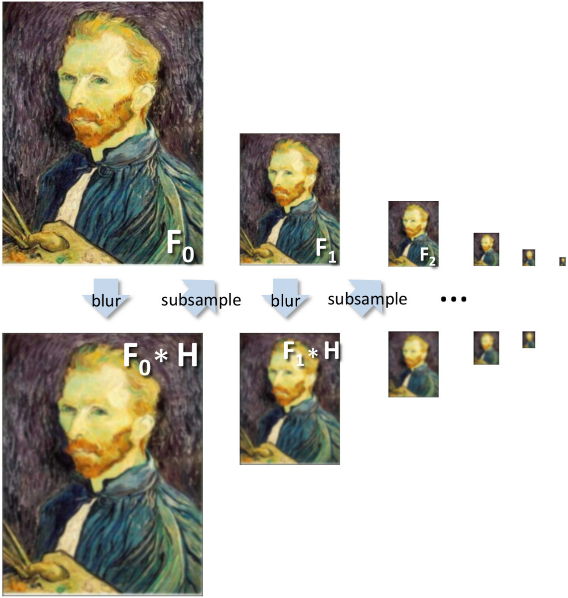
Compare to previous result ...



Left to right: Original signal, downsampled by 4 without filtering, pre-filtered with 4×4 box filter, pre-filtered with Gaussian.

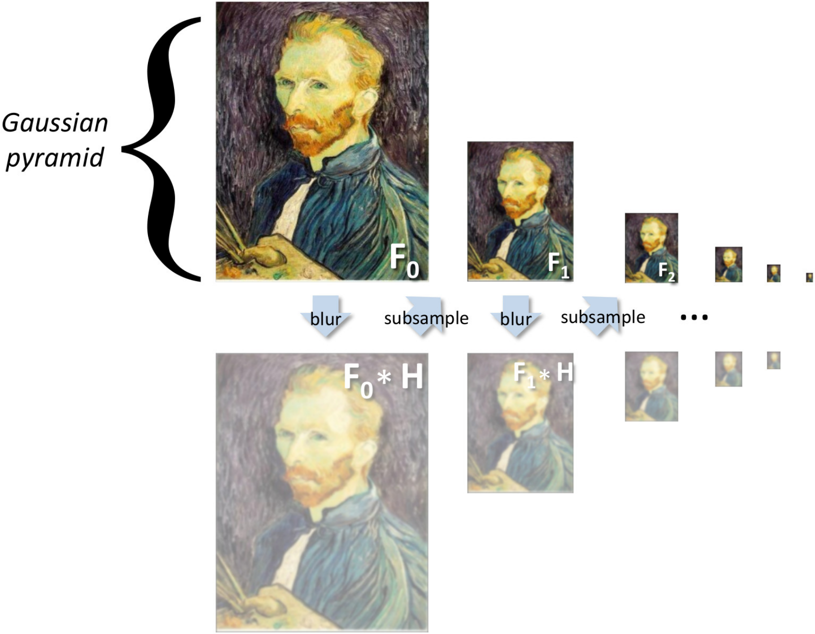
Source: R. Szeliski

Gaussian pre-filtering



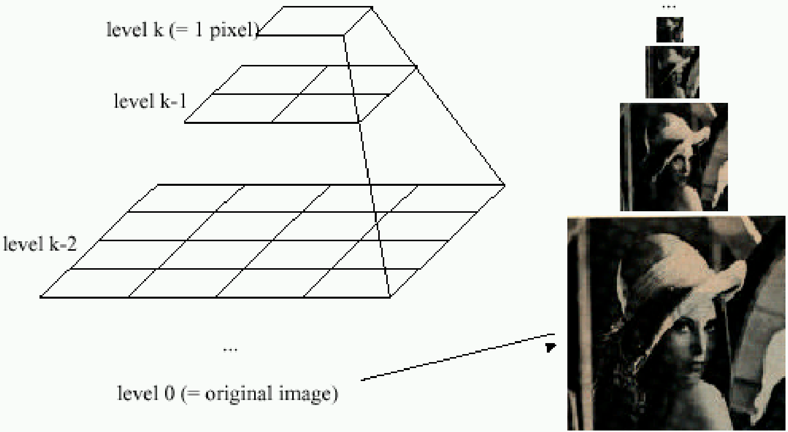
Source: N. Snavelny

Gaussian pre-filtering



Gaussian pyramids

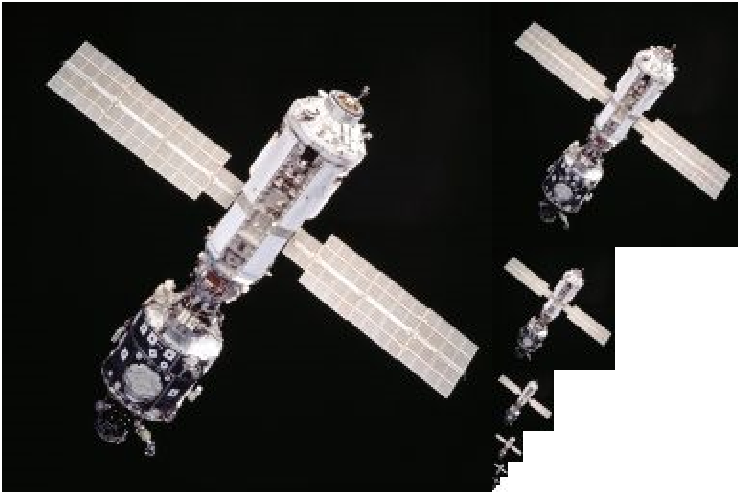
Idea: Represent NxN image as a “pyramid” of $1\text{x}1$, $2\text{x}2$, $4\text{x}4$,…, $2^k\text{x}2^k$ images (assuming $N=2^k$)



Gaussian pyramids: Burt and Adelson, 1983, *Mipmaps in CG:* Williams, 1983

Source: S. Seitz

Gaussian pyramids

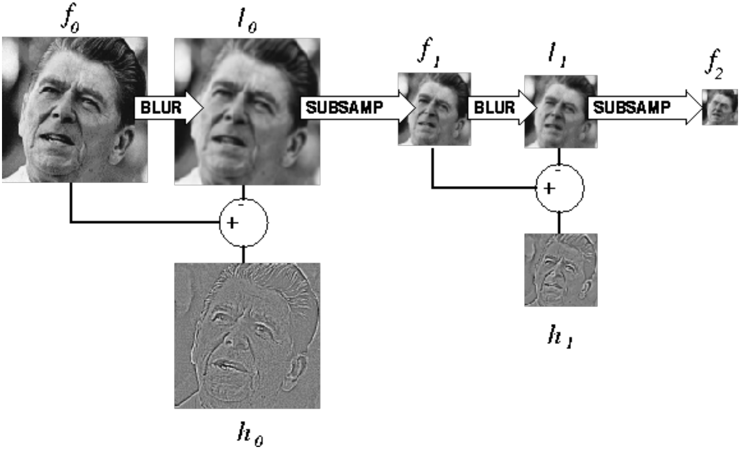


Required storage space compared to original image approximately:

$$1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{32} + \cdots = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}.$$

Image: N. Snavelny

Alternative: Laplacian pyramids



- Store the difference of Gaussians (i.e. detail in a certain frequency range) at subsequent levels of coarseness
- Perfect reconstruction of original possible from f_k and $h_0, \ldots, h_{k-1}$.
- We'll get back to this when we discuss SIFT, next chapter.

Overview

Summary: the Fourier transform

- The continuous Fourier transform analyses the frequency content of images.
- It decomposes the image into a superposition (linear combination) of elementary waves.
- The complex-valued Fourier coefficients represent amplitude and phase for the corresponding elementary wave.
- The Fourier transform is linear and invariant under rotations.
- Spatial shifts become phase shifts.
- Differentiation becomes multiplication with the frequency.
- Convolution in the image domain becomes point-wise multiplication in the Fourier domain, and vice versa.
- For this reason, the Fourier transform is highly useful for analysing the frequency behaviour of convolution filters.

Some bullet points from A. Bruhn (adapted)

Summary: aliasing, subsampling and image pyramids

- A frequency must be sampled more than twice per period in order to avoid aliasing.
- For this reason, images must be filtered before downsampling.
- Continued downsampling leads to image pyramids, which are representations of the image at different scales in the spatial domain.
- The Gaussian pyramid is a lowpass filter,
  the Laplacian pyramid a bandpass decomposition.
- Both are redundant: require more disk space than the original image.

Some bullet points from A. Bruhn (adapted)

References

- **Fourier transform:**
  - Szeliski 3.4
  - Forsyth and Ponce 7.3

- **Sampling and image pyramids:**
  - Szeliski 2.3, 3.5
  - Forsyth and Ponce 7.4, 7.7