

Results for Thesis

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Preliminaries and Definitions

Definition. Let \mathcal{D} be a pre-Hilbert space. Denote by $\mathcal{L}^\dagger(\mathcal{D})$ the space of linear operators on \mathcal{D} such that for all $A \in \mathcal{L}^\dagger(\mathcal{D})$

1. $A(\mathcal{D}) \subseteq \mathcal{D}$,
2. $\mathcal{D} \subseteq \mathcal{D}(A^*)$ and $A^*(\mathcal{D}) \subseteq \mathcal{D}$.

Under the involution $A^\dagger = A^*|_{\mathcal{D}}$, $\mathcal{L}^\dagger(\mathcal{D})$ is a $*$ -algebra. An Op^* -algebra is a unital $*$ -subalgebra of $\mathcal{L}^\dagger(\mathcal{D})$.

Definition. Let \mathcal{A} be an Op^* -algebra. We define the convex hull of all elements $A^\dagger A$ as the set $\mathcal{P}(\mathcal{A}) = \text{co}(\{A^\dagger A ; A \in \mathcal{A}\})$. Similarly, we write $\mathcal{K}(\mathcal{A}) = \{A \in \mathcal{A} ; \langle A\psi, \psi \rangle \geq 0 \text{ for all } \psi \in \mathcal{D}\}$.

We have the relations $\mathcal{P} \subseteq \overline{\mathcal{P}} \subseteq \mathcal{K}$, where the middle closure may be taken over a topology finer than the ultraweak topology on \mathcal{A} . If \mathcal{A} is a C^* -algebra, then these all coincide. This is not the case for general Op^* -algebras. As a result there are a variety of positivity one can take in the unbounded setting. For the time being we take the weakest.

Definition. (Weak Positivity) Let $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{B}$ be a linear map between Op^* -algebras. We say \mathcal{E} is $(\mathcal{P}(\mathcal{A}), \mathcal{K}(\mathcal{B}))$ positive (usually just stated as "positive") if for any $A \in \mathcal{A}$ one has $\langle \mathcal{E}(A^\dagger A)\psi, \psi \rangle \geq 0$ for any $\psi \in \mathcal{D}(\mathcal{B})$. In a similar manner we say \mathcal{E} is completely positive if every corresponding matrix amplification $\mathcal{E}_n : \mathcal{A} \otimes M_n \rightarrow \mathcal{B} \otimes M_n$ is positive.

This of course carries the insinuation that $\mathcal{A} \otimes M_n$ is an Op^* -algebra over $\mathcal{D}(\mathcal{A})^n$, which is indeed the case. We may also consider the tensor product $\mathcal{A} \otimes M_\infty$, where M_∞ denotes the set of finitely-supported, but infinite

matrices. We will consider stability of this tensor product and its relationship to complete positivity later.

There is a generalization of the Stinespring Theorem for this class, and hence any more specific class of positive maps between Op^* -algebras.

Theorem. (*Stinespring's Theorem*) Let $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{B}$ be a completely positive map. This map is of the form

$$\mathcal{E}(A) = V^* \pi(A) V.$$

Where $\pi : \mathcal{A} \rightarrow \pi(\mathcal{A})$ is a $*$ -representation onto an Op^* -algebra $\pi(\mathcal{A})$ over a dilated pre-Hilbert space \mathcal{D}_π containing $\mathcal{D}(\mathcal{A})$ as a subspace, and $V : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}_\pi$ is a linear map continuous with respect to the graph topologies on $\mathcal{D}(\mathcal{A})$ and \mathcal{D}_π , V^* the corresponding TVS adjoint. Conversely, any such map is completely positive. If \mathcal{E} is ultraweakly continuous, then so is π .

Proof. We show the first part of the theorem, which closely follows the bounded case. Consider the algebraic tensor product $\mathcal{A} \otimes \mathcal{D}$, where $\mathcal{D} = \mathcal{D}(\mathcal{A})$. For $\xi = \sum_i A_i \otimes \phi_i$ and $\eta = \sum_j B_j \otimes \psi_j$ the bilinear form $\langle \xi, \eta \rangle = \sum_{i,j} \langle \phi_i, \mathcal{E}(A_i^\dagger B_j) \psi_j \rangle$. That this is positive semi-definite follows from positivity of \mathcal{E} . Define the action of π by

$$\pi(X)\xi = \sum_i X A_i \otimes \phi_i.$$

This defines a representation of \mathcal{A} on $\mathcal{A} \otimes \mathcal{D}$ satisfying

$$\langle \xi, \pi(A)\eta \rangle = \langle \pi(A^\dagger)\xi, \eta \rangle.$$

Now let \mathcal{N} be the kernel of the previously defined inner product on $\mathcal{A} \otimes \mathcal{D}$. Given that $\xi \in \mathcal{N}$, one has

$$\|\pi(A)\xi\|^2 = \langle \xi, \pi(A^\dagger A)\xi \rangle \leq \|\xi\| \cdot \|\pi(A^\dagger A)\xi\| = 0.$$

Thus the action of π on the pre Hilbert space $\mathcal{D}_\pi = (\mathcal{A} \otimes \mathcal{D})/\mathcal{N}$ admits a well-defined linear operator. $\pi(\mathcal{A})$ is then an Op^* -algebra.

Now define $V : \mathcal{D} \rightarrow \mathcal{D}_\pi$ via $V\psi = 1 \otimes \psi + \mathcal{N}$. Let ρ_X denote a seminorm

corresponding to $X \in \mathcal{A}$ or $X \in \pi(\mathcal{A})$ in either graph topology. We have

$$\begin{aligned}
\rho_{\pi(A)}(V\psi)^2 &= \langle \pi(A)V\psi, \pi(A)V\psi \rangle \\
&= \langle \pi(A)(1 \otimes \psi), \pi(A)(1 \otimes \psi) \rangle \\
&= \langle A \otimes \psi, \pi(A) \otimes \psi \rangle \\
&= \langle \mathcal{E}(A^\dagger A)\psi, \psi \rangle \\
&\leq \|\mathcal{E}(A^\dagger A)\psi\| \cdot \|\psi\| \\
&\leq \|B\psi\|^2 \\
&= \rho_B(\psi)^2.
\end{aligned}$$

Where $B = I + \mathcal{E}(A^\dagger A) \in \mathcal{B}$ and $1 \otimes \psi$ is understood here as an equivalence class. Moreover, we have

$$\langle V\phi, \pi(A)V\psi \rangle = \langle 1 \otimes \phi, A \otimes \psi \rangle = \langle \phi, \mathcal{E}(A)\psi \rangle$$

and so $\mathcal{E}(A) = V^*\pi(A)V$.

Now suppose that \mathcal{E} is ultraweakly continuous and that $\{A_\alpha\}_\alpha$ converges to 0 ultraweakly in \mathcal{A} . Let $\{\sum_k X_{n,k} \otimes \xi_{n,k}\}_n$ and $\{\sum_k Y_{n,k} \otimes \eta_{n,k}\}_n$ be two square summable sequences in \mathcal{D}_π . One has

$$\begin{aligned}
\sum_n \langle \pi(A_\alpha)\xi_n, \eta_n \rangle &= \sum_{n,i,j} \langle \pi(A_\alpha)X_{n,i} \otimes \xi_{n,i}, Y_{n,j} \otimes \eta_{n,j} \rangle \\
&= \sum_{n,i,j} \langle A_\alpha X_{n,i} \otimes \xi_{n,i}, Y_{n,j} \otimes \eta_{n,j} \rangle \\
&= \sum_{n,i,j} \langle \psi_{n,i}, \mathcal{E}((A_\alpha X_{n,i})^\dagger Y_{n,j})\eta_{n,j} \rangle.
\end{aligned}$$

Where ξ_n and η_n are the n th terms of the previous defined sequences. It follows by our assumption on the continuity of \mathcal{E} , the separate continuity of multiplication, and continuity of the involution that the last line converges to zero, as required. \square

The previous definitions and Stinespring's Theorem motivates the following definition.

Definition. A quantum channel of Op^* -algebras is a completely positive, ultraweakly continuous linear map $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{B}$, where \mathcal{A} and \mathcal{B} are Op^* -algebras.

Lemmas

Propositions

Theorems