QUANTUM ERROR CORRECTION IN ALGEBRAS OF UNBOUNDED OPERATORS

1. Background & Motivation

Operator algebra quantum error correction (OAQEC) [1, 2] aims to correct errors induced by quantum channels (i.e., unital completely positive maps) on algebras of observables of quantum systems. OAQEC provides a unifying framework for subspace/subsystem error correction, as well as hybrid classical-quantum error correction (e.g., [6, 12]) and holographic quantum error correction in the anti-de Sitter space/conformal field theory (AdS/CFT) correspondence (e.g., [4, 7, 10]). To date, OAQEC has primarily focused on the correction of errors induced on algebras of bounded observables. However, arguably the most important physically relevant observables are unbounded, e.g., position, momentum, energy, and quantum field observables. In some special cases, e.g., in the presence of essentially self-adjoint operators, spectral theory provides a suitable passage between algebras of unbounded observables and the associated algebras of bounded operators to which they are affiliated. However, as is well-known in quantum field theory, these assumptions are not always valid, and a generalized theory of quantum error correction in such settings is warranted.

In a related direction, energy-limited channels have been studied from multiple perspectives (e.g.,[5, 11, 14, 17, 18, 19]). Physically, they correspond to interactions which input a finite amount of energy into the system, measured relative to a reference Hamiltonian. Mathematically, they correspond to channels which admit well-defined extensions to unbounded operators which are relatively bounded with respect to the reference Hamiltonian (self-adjoint operator). This growing body of work provides additional motivation for a generalized theory of OAQEC.

2. Objectives

The first objective is to generalize the basic definitions and correctability conditions of OAQEC [1, 2] to the unbounded setting. In particular, we aim to generalize the main result of [3, Theorem 4.7] on the complementarity of privacy/correctability (which is just another way of interpreting the generalized Knill-Laflamme conditions from [1]). We will start with channels whose domain is the Op*-algebra $\mathcal{M} = \mathcal{L}^+(D)$, where D is a sense subspace of a Hilbert space H, and then investigate generalized von Neumann algebras (and potentially more).

The second objective is to generalize the complementary recovery result [4, Theorem 1.1] in terms of Connes' cocycle derivatives to algebras of unbounded operators. The methodology will incorporate the modular theory of Op*-algebras (e.g., [9]) together with the existing notion of sufficiency for *-algebras [8]. Our generalization will allow one to phrase the recent holographic OAQEC conditions in AdS/CFT directly at the quantum field level, which is highly desirable from a physical perspective.

3. Suggested Reading Order

Objective 1:

(1) Basics of unital completely positive maps between C^*-algebras, Stinespring's dilation theorem (Paulsen's book).
(2) Stinespring dilation for unbounded operator algebras [13].
(3) Basics of operator algebra quantum error correction [1, 2]. Just the main ideas, no need for detailed understanding.

(4) Operator algebraic privacy and complementarity theorem [3]. In particular, understand the proof of [3, Theorem 4.7].
(5) Apply unbounded Stinespring dilation theory [13] to establish a generalized complementarity theorem [3, Theorem 4.7], starting with the exact ($\varepsilon = 0$) case. For the approximate case, we would likely need to use the topologies of [13, §5] (or similar). See [17, §4] for energy-limited continuity of Stinespring dilations, which may admit a generalization via the topologies of [13, §5].

Objective 2:

(1) Basics of modular theory, relative modular operators and spatial derivatives. Relative entropy in von Neumann algebras e.g., [15, 16]. No need to understand the technical proofs at this point. Just the basics. Can give lectures if desired.
(2) Basics of Cocycle derivatives. Discussion in [15] is sufficient at this point.
(3) Understand proof of complementary recovery for von Neumann algebras [4, Theorem 1.1].
(4) Relative modular operators and cocycle derivates for "standard" generalized von Neumann algebras [9, §3]. Warning: technical. I can give lectures on this material when the time comes.
(5) Comparison with relative entropy/sufficiency framework of [8].
(6) Formulation of complementary recovery result [4, Theorem 1.1] in op*-algebras, starting with generalized von Neumann algebras.

Completing objective 1 is sufficient for a Master's thesis (and could lead to a publication by itself). Having 2 closely related problems to think about simultaneously can be advantageous (in terms of productivity). Objective 2 would be a nice application of objective 1. If we successfully achieve both objectives, their combination would make for a quality publication.

References

[1] C. Bény C, A. Kempf and D. W. Kribs, *Generalization of quantum error correction via the Heisenberg picture*, Phys. Rev. Lett., 98 (2007), 100502.

[2] C. Bény C, A. Kempf and D. W. Kribs, *Quantum error correction on infinite-dimensional Hilbert spaces*, J. Math. Phys. 50 (2009), 062108.

[3] J. Crann, D. W. Kribs, R. H. Levene and I. G. Todorov, *Private algebras in quantum information and infinite-dimensional complementarity,* J. Math. Phys. 57 (2016), no. 1, 015208.

[4] J. Crann and M. J. Kang, *Algebraic approach to spacetime bulk reconstruction*, arXiv:2412.00298.

[5] N. Davis, M. E. Shirokov, and M. M. Wilde, *Energy-constrained two-way assisted private and quantum capacities of quantum channels*, Physical Review A 97.6 (June 2018).

[6] G. Dauphinais, D. W. Kribs and M. Vasmer, *Stabilizer formalism for operator algebra quantum error correction*, Quantum 8, 1261 (2024).

[7] D. Harlow, *The Ryu-Takayanagi formula from quantum error correction* Commun. Math. Phys. 354 (2017), 865-912.

[8] F. Hiai, M. Ohya and M. Tsukada, *Sufficiency and relative entropy in *-algebras with applications in quantum systems*, Pacific Jour. Math. 107, no. 1 (1983), 117-140.

[9] A. Inoue, *Standard generalized vectors for algebras of unbounded operators*, J. Math. Soc. Japan 47, no. 2 (1995), 329-347.

[10] M. J. Kang and D. K. Kolchmeyer, *Holographic relative entropy in infinite-dimensional Hilbert spaces*, Commun. Math. Phys. 400, no.3, 1665-1695 (2023).

[11] R. König and C. Rouzé, *Limitations of local update recovery in stabilizer-GKP codes: a quantum optimal transport approach*, arXiv:2309.16241.

[12] G. Kuperberg, *The capacity of hybrid quantum memory*, IEEE Trans. Inform. Theory 49 (2003), 1465-1473.

[13] G. Lassner and G. A. Lassner, *Completely positive mappings and unbounded observables*, Rep. Math. Phys. 11, no. 1 (1977), 133-140.

[14] L. van Luijk, *Energy-limited quantum dynamics*, arXiv:2405.10259.

[15] M. Ohya and D. Petz, *Quantum entropy and its use*, Texts and Monographs in Physics, Springer-Verlag, Berlin-Heidelberg 1993.

[16] B. M. Reible, *Monotonicity of the relative entropy and the two-sided Bogoliubov inequality in von Neumann algebras*, arXiv:2501.04564.

[17] M. E. Shirokov, *Operator E-norms and their use*, Sb. Math., 211:9 (2020), 1323-1353.

[18] M. E. Shirokov, *On Extension of Quantum Channels and Operations to the Space of Relatively Bounded Operators*, Lobachevskii Journal of Mathematics 41.4 (Apr. 2020), pp. 714–727.

[19] A. Winter, *Energy-Constrained Diamond Norm with Applications to the Uniform Continuity of Continuous Variable Channel Capacities*, arXiv:1712.10267.