CLASSIFICATION OF DOMAINS OF OPERATOR ALGEBRAS

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The structure of domains of Op*-algebras $\mathscr{A}(\mathscr{D})$ is investigated; particularly the case where the domain $\mathscr{D}$ is a Fréchet space with respect to a natural topology is considered. We obtain a complete classification and description of domains $\mathscr{D}$ of the form $\mathscr{D} = \bigcap_{n\geqslant 0} \mathscr{D}(A^n)$, $A = A^*$ a self-adjoint operator.

0. Introduction

In [17] there was given a complete classification of the domains of closed operators in a Hilbert space and a description of the structure of these domains. Now we present the first part of a classification of another type of domains, the domains of operator $*$-algebras (Op*-algebras). Thereby, we restrict ourselves in this paper to the special but very important case, where the domain is a Fréchet space with respect to a certain natural topology.

1. Preliminaries

We use the same notations and notions as in [17]. Let $\mathscr{H}$ be a separable Hilbert space, $\mathscr{D} \subset \mathscr{H}$ a dense linear manifold, $\langle\ .\ \rangle$, $\|\ \|$, resp., the scalar product, the norm resp. in $\mathscr{H}$. A *kernel* of $\mathscr{D}$ is an infinitely dimensional closed subspace $\mathscr{N} \subset \mathscr{D}$. A kernel $\mathscr{N}$ is said to be *maximal* if there is no kernel $\mathscr{N}' \subset \mathscr{D}, \mathscr{N} \subset \mathscr{N}'$ such that $\dim(\mathscr{N}' \ominus \mathscr{N}) = \infty$ [5]. By $\mathscr{L}^+(\mathscr{D})$ we denote the set of all linear operators A from $\mathscr{D}$ into $\mathscr{D}$, $A\mathscr{D} \subset \mathscr{D}$ such that $\mathscr{D} \subset \mathscr{D}(A^*)$ and $A^*\mathscr{D} \subset \mathscr{D}$. $\mathscr{L}^+(\mathscr{D})$ is a $*$-algebra with respect to the usual operations, and the involution is defined by $A \to A^+$, where A^+ is the restriction of A^* to $\mathscr{D}$. A $*$-subalgebra $\mathscr{A}(\mathscr{D}) = \mathscr{A}$ of $\mathscr{L}^+(\mathscr{D})$ containing the identity I will be called an Op*-algebra. An Op*-algebra $\mathscr{A}(\mathscr{D})$ is said to be *closed* if

$$\mathscr{D} = \bigcap_{A\in\mathscr{A}} \mathscr{D}(\overline{A}).$$

Equivalently, an Op*-algebra $\mathscr{A}(\mathscr{D})$ is closed if the domain $\mathscr{D}$ equipped with the topology t_A defined by the seminorms

$$\|\varphi\|_A = \|\varphi\| + \|A\varphi\|, \quad A \in \mathscr{A},\ \varphi \in \mathscr{D},$$

is complete (further considerations concerning Op*-algebras are contained, for example, in [14], [15], [16], [23]).

By a *diagonal operator* we mean an operator $S = S\{(a_n), (\varphi_n)\}$ with the domain

$$\mathscr{D} = \mathscr{D}(S) = \left\{\varphi = \sum x_n \varphi_n : \sum |x_n|^2 \, |a_n|^2 < \infty\right\}$$

where $\{\varphi_n\}$ is an orthonormal basis in $\mathscr{H}$ contained in $\mathscr{D}$ and $\{a_n\}$ a sequence of real numbers unbounded if $\mathscr{D} \neq \mathscr{H}$. Two linear manifolds $\mathscr{D}$, $\mathscr{D}'$ are called *linearly* (*unitarily* resp.) *equivalent* if $\mathscr{D}' = K\mathscr{D}$, where K is a bounded operator with bounded inverse (a unitary operator resp.).

In what follows we need some notions about sequences.

Let (a_n) be a sequence and (b_n) a subsequence of (a_n), $(b_n) \subset (a_n)$. Then by $(d_n) = (a_n) - (b_n)$ we denote the subsequence of (a_n) obtained by cancelling the elements (b_n). We write $(a_n) = (b_n) \cup (d_n)$. Let (a_n) be a sequence of naturals. By $(a_n)'$ we denote the set of accumulation points of (a_n) and by (a'_n) we denote the sequence $(a_n)' - \{\infty\}$ such that $a'_1 \leqslant a'_2 \leqslant \ldots$

An *iterated* sequence $(\hat{a}_n) = (a^\sigma_n)$ is a sequence obtained from (a_n) in the following way: let σ be a monotone map from N onto itself; then $(\hat{a}_n) = (a^\sigma_n) = (a_{\sigma(n)})$. For example, $(\hat{a}_n)$ could be $a_1, a_1, a_2, a_3, a_3, a_3, \ldots$ We call σ the "iteration" and, for brevity, we will often use the same sign "$\wedge$" for any iteration.

Let (a_n), (b_n) be two sequences of positive numbers. One says that (a_n) is *majorized* by (b_n), $\big((a_n) \prec (b_n)\big)$ if there is a constant $C > 0$ such that $a_n \leqslant C \cdot b_n$ for all n.

In [17] we defined different notions of equivalence of such sequences to obtain a description of the structure of domains of closed operators. Now we deal with systems of sequences and give the following

DEFINITION 1. Let $\mathscr{F}_1 = \{(a_n)\}$, $\mathscr{F}_2 = \{(b_n)\}$ be two systems of sequences of positive numbers. $\mathscr{F}_1$ and $\mathscr{F}_2$ are said to be *equivalent* ($\sim$) if any sequence $(a_n) \in \mathscr{F}_1$ can be majorized by a suitable sequence $(b_n) \in \mathscr{F}_2$ and conversely, any sequence of $\mathscr{F}_2$ can be majorized by a suitable sequence of $\mathscr{F}_1$; *essentially equivalent* ($\underset{e}{\sim}$) if there are $s, t \in N$ such that the systems $\{(a_{s+n})\}$ and $\{(b_{t+n})\}$ are equivalent; *weakly equivalent* ($\underset{w}{\sim}$) if there are suitable monotone maps σ, τ of N onto itself such that the systems $(\mathscr{F}_1)_\sigma$, $(\mathscr{F}_2)_\tau$ of iterated sequences $\{(a^\sigma_n)\}$, $\{(b^\tau_n)\}$ obtained from σ and τ are equivalent.

If the systems $\mathscr{F}_1$ and $\mathscr{F}_2$ contain only one sequence, we obtain the same notions as in [17], Definition 1.

2. Classification of the domains of Op*-algebras

In this section we give a classification of the domains of Op*-algebras and note some properties of the structure of these domains.

First we remark that these considerations are proper extensions of those of [17]. This is due to the following fact ([14], [2], [23]):

Let $\mathscr{D}$ be the domain of an Op*-algebra containing at least one unbounded operator. Then $\mathscr{D}$ cannot be the domain of a closed operator, i.e. these two types of domains are essentially different.

With the following classification we continue the enumeration begun in [17].

C_4: *Classification of the domains of closed* Op*-*algebras*

Let $\mathscr{A}(\mathscr{D})$ be a closed Op*-algebra. $\mathscr{D}$ is said to be of

Class I iff $\mathscr{D}$ does not contain a kernel,
Class II iff $\mathscr{D}$ contains a maximal kernel,
Class III iff $\mathscr{D}$ contains kernels but not maximal one.

We write $\mathscr{D} \in I$, $\in II$, and $\in III$.

The same definition of domain of closed operators was given in [17], and this geometrical picture was translated into the language of equivalence of sequences.

The classification C_4 is complete and disjoint. No class is empty, as can easily be seen (see also Section 3).

The following remark shows, roughly speaking, that the class to which a domain $\mathscr{D}$ belongs is not determined by the class to which $\mathscr{D}(\bar{A})$ belongs, for all $A \in \mathscr{L}^+(\mathscr{D})$, in an invariant manner.

Remark 1. Let $\mathscr{D}$ be the domain of a closed Op*-algebra. Then there can arise the following cases:

(i) $\mathscr{D} \in I$, then there are $A, B \in \mathscr{L}^+(\mathscr{D})$ such that $\mathscr{D}(\bar{A}) \in II$, $\mathscr{D}(\bar{B}) \in III$.

(ii) $\mathscr{D} \in II$, then there is an $A \in \mathscr{L}^+(\mathscr{D})$ such that $\mathscr{D}(\bar{A}) \in III$, but there is no $B \in \mathscr{L}^+(\mathscr{D})$ such that $\mathscr{D}(\bar{B}) \in I$.

(iii) $\mathscr{D} \in III$, then there is an $A \in \mathscr{L}^+(\mathscr{D})$ such that $\mathscr{D}(\bar{A}) \in II$, but there is no $B \in \mathscr{L}^+(\mathscr{D})$ such that $\mathscr{D}(\bar{B}) \in I$.

Proof: All statements can be proved in a similar way by constructing examples from diagonal operators. We consider for example (i) and construct the operator B.

Let $\mathscr{H} = l^2$, $\mathscr{D} = s = \{(a_n) : \sum n^L |a_n|^2 < \infty, \ \forall\, L \in N\}$. Let $\varphi_n = (0, \ldots, 1, 0, \ldots)$, 1 being the nth element. Now consider an arbitrary decomposition of the set of naturals: $N = \bigcup\limits_n M_n$, where the M_n are infinite sets, $M_l \cap M_k = \emptyset$ for $k \neq l$. Put $m_n = \min M_n$ and let C be the following diagonal operator:

$$C = C\{(b_n), (\varphi_n)\} \qquad \text{with } b_j = m_n \text{ for all } j \in M_n.$$

It can easily be seen that $\mathscr{D} \subset \mathscr{D}(C)$ and $B = C_{|\mathscr{D}} \in \mathscr{L}^+(\mathscr{D})$ and $\mathscr{D}(\bar{B}) = \mathscr{D}(C) \in III$, but $\mathscr{D}$ obviously $\in I$. Q.E.D.

In order to perform a detailed classification and description of the structure of the domains it is important to investigate the following

PROBLEM. Under what conditions can the domain $\mathscr{D}$ of a closed Op*-algebra $\mathscr{A}(\mathscr{D})$ be represented in the form

$$\mathscr{D} = \left\{\varphi = \sum x_n \varphi_n : \sum |x_n|^2 (a_n^{(\alpha)})^2 < \infty, \ \forall \alpha \in \mathfrak{A}\right\}?$$

$(\varphi_n) \subset \mathscr{D}$ is an orthonormal basis, and $\{(a_n^{(\alpha)}), \alpha \in \mathfrak{A}\}$ is a suitable system of positive sequences.

In other words, $\mathscr{D}$ is isomorphic to the space of sequences $\mathscr{D}^s \subset l^2$:

$$\mathscr{D}^s = \left\{(x_n) : \sum |x_n|^2 (a_n^{(\alpha)})^2 < \infty, \ \forall \alpha \in \mathfrak{A}\right\}.$$

Then with the diagonal operators $T_\alpha = T_\alpha\{(a_n^{(\alpha)}), (\varphi_n)\}$, $\alpha \in \mathfrak{A}$

$$D = \bigcap_{\alpha \in \mathfrak{A}} D(T_\alpha).$$

The crucial point of this problem is the fact that the basis (φ_n) must be the same for all operators T_α.

This problem leads to the following

STATEMENT. *If the domain $\mathscr{D}$ of a closed* Op*-*algebra $\mathscr{A}(\mathscr{D})$ can be realized as a space of sequences, then the maximal* Op*-*algebra $\mathscr{L}^+(\mathscr{D})$ is self-adjoint, that is,*

$$\mathscr{D} = \bigcap_{A \in \mathscr{L}^+(\mathscr{D})} \mathscr{D}(\bar{A}) = \bigcap_{A \in \mathscr{L}^+(\mathscr{D})} \mathscr{D}(A^*)$$

(cf. also [14], [16], [1], [23]).

In general it is an open question whether $\mathscr{L}^+(\mathscr{D})$ is self-adjoint in any case when $\mathscr{D}$ is the domain of an arbitrary closed Op*-algebra.

In what follows we consider algebras $\mathscr{A}(\mathscr{D})$ with metrizable topology $t_{\mathscr{A}}$, i.e. $\mathscr{D}[t_{\mathscr{A}}]$ is an F-space because we regard only closed algebras. Therefore one has

$$\mathscr{D} = \bigcap_n \mathscr{D}(A_n), \quad A_{n|\mathscr{D}} \in \mathscr{L}^+(\mathscr{D}), \tag{1}$$

The following class of domains is a special case of (1):

$$\mathscr{D} = \mathscr{D}^\infty(T) = \bigcap_{n \geq 0} \mathscr{D}(T^n), \quad T = T^*, \quad T_{|\mathscr{D}} \in \mathscr{L}^+(\mathscr{D}). \tag{2}$$

Without loss of generality one always may assume that $T = T^* \geq I$, $I \leq A_n \subset A_n^*$, and $\mathscr{D}(A_{n+1}) \subset \mathscr{D}(A_n)$ for all n. Domains of the form (2) were investigated by many authors from other points of view than we do it here. They were studied, for example, in connection with differential operators or in the interpolation theory as centres of Hilbert or Banach-scales (cf. [13], [18], [19], [20], [21], [24] and references there quoted). Here we only remark that an important example of a space of the form (2) is the Schwartz space $\mathscr{S}$ of rapidly increasing functions.

Remark 2. Obviously, the class of domains of the form (2) is contained in the class of domains of the form (1), but the two classes do not coincide, i.e. there are domains $\mathscr{D} = \bigcap_{n \geqslant 0} \mathscr{D}(A_n)$ such that there is no closed operator T with $\mathscr{D} = \bigcap_{n \geqslant 0} \mathscr{D}(T^n)$.

We give an example: Let $\{\varphi_n\}$ be an orthonormal basis in a Hilbert space $\mathscr{H}$, (t_n) a sequence of naturals with $\lim t_n = \infty$. Further, let $\{M_n\}$ be a sequence of infinite sets with the following properties:

1. $M_n \subset N$, i.e. M_n is an infinite set of naturals.
2. $M_{n+1} \subset M_n$.
3. The sets $N_1 = N \setminus M_1$, $N_j = M_{j-1} \setminus M_j$ are infinite for all $j > 1$.

Finally, we use the notion (t_{j+n}) for the sequence $(t_j, t_{j+1}, \ldots)$. Now we define a system of diagonal operators $A_n = A_n\{(a_l^{(n)}), (\varphi_l)\}$ as follows:

$$a_L^{(1)} = \begin{cases} t_L & \text{if } L \in N_1, \\ 1 & \text{if } L \in M_1, \end{cases}$$

and for $n > 1$:

$$a_L^{(n)} = \begin{cases} t_L & \text{if } L \in N_1, \\ t_{j+L} & \text{if } L \in N_j \quad \text{for } 2 \leqslant j \leqslant n, \\ 1 & \text{if } L \in M_n. \end{cases}$$

For the operators A_n constructed above we have:

1. $\mathscr{D}(A_n) = \mathscr{H}_n \oplus \mathscr{D}_n$, $\mathscr{D}_n \in I$, $\mathscr{H}_n$ closed, infinitely dimensional,
2. $\mathscr{D}(A_{n+1}) \subset \mathscr{D}(A_n)$, more precisely:

$$\mathscr{H}_{n+1} \subset \mathscr{H}_n, \qquad \mathscr{D}_n \subset \mathscr{D}_{n+1},$$

$$\mathscr{H}_n = \mathscr{H}_{n+1} \oplus \hat{\mathscr{H}}_{n+1}, \qquad \mathscr{D}_{n+1} = \mathscr{D}_n \oplus \hat{\mathscr{D}}_{n+1}, \qquad \hat{\mathscr{D}}_{n+1} \subset \hat{\mathscr{H}}_{n+1}.$$

Roughly speaking, the operators A_n are formed in such a way that, if A_n is bounded on $\mathscr{H}_n$, A_{n+1} is bounded only on an infinitely dimensional subspace of $\mathscr{H}_n$ and unbounded on the infinitely dimensional orthogonal complement.

3. The representation

$$\mathscr{D} = \bigcap_{n \geqslant 0} \mathscr{D}(T^n), \qquad T \text{ closed},$$

where $\mathscr{D}$ is defined to be the domain of the algebra generated by the operators A_n, cannot hold, i.e.

$$\mathscr{D} = \bigcap_{m, n \geqslant 0} \mathscr{D}(A_n^m).$$

In fact, from the closed graph theorems it would otherwise follow that

$$||T\varphi|| \leqslant C(||A_l^k \varphi|| + ||\varphi||)$$

for suitable $k, l \in N$, $C > 0$ and all $\varphi \in \mathscr{D}$. But then the boundedness of A_l^k on the dense subspace $\mathscr{H}_l \cap \mathscr{D}$ of $\mathscr{H}_l$ leads to

$$||T\varphi|| \leqslant C(||A_l^k \varphi|| + ||\varphi||) \leqslant D||\varphi||, \quad \text{for all } \varphi \in \mathscr{H}_l \cap \mathscr{D}.$$

This means that T is bounded on $\mathscr{H}_l \cap \mathscr{D}$, and because T is closed, we have $\mathscr{H}_l \subset \mathscr{D}(T)$. Therefore $\mathscr{H}_l \subset \mathscr{D}$, which contradicts 2. This concludes the example. Because the topology $t_{\mathscr{A}}$ on the domains of form (1) can be given by a system of scalar products $\{\langle\ ,\ \rangle_n\}$, these domains are countable Hilbert spaces ([10]). With respect to this and in connection with the construction of the space $\mathscr{D}^s$ of sequences isomorphic to $\mathscr{D}$ we recall the following

Definition ([4], [9]). A system (x_n) of elements of a linear topological space E is called a *basis* of E if there is a system (f_n) of linear functionals $f_n \in E'$ such that for any $x \in E$ there exists a unique representation

$$x = \sum_{n=1}^{\infty} f_n(x) x_n. \tag{3}$$

The basis (x_n) is called *unconditional* if $(x_{\pi(n)})$ is a basis for any permutation π of N, or equivalently, if the series (3) is unconditionally convergent for any $x \in E$.

Now we make use of the following proposition [20]:

Proposition. *Let $\mathscr{D}$ be a countable Hilbert space $\mathscr{D} = \bigcap \mathscr{H}_n$ with the topology τ defined by the scalar products $\{\langle\ .\ \rangle_n\}$. If (x_j) is an unconditional basis in $\mathscr{D}$, then there is a system $\{\langle\ ,\ \rangle_n'\}$ of scalar products defining the same topology τ such that (x_j) is an orthogonal system with respect to any $\langle\ ,\ \rangle_n'$.*

Hence we arrive at the following

Corollary. *If the domain $\mathscr{D} = \bigcap \mathscr{D}(A_n)$ contains an unconditional basis (ψ_n), then there are diagonal operators $T_l = T_l\{(t_n^{(l)}), (\psi_n)\}$ such that $\mathscr{D} = \bigcap \mathscr{D}(T_l)$, namely, we choose $t_n^{(l)} = \|\psi_n\|_l'$, where $\|\ \|_l'$ denotes the norm corresponding to the new scalar product $\langle\ ,\ \rangle_l'$.*

In the special case $\mathscr{D} = \mathscr{D}^\infty(T) = \bigcap \mathscr{D}(T^n)$, $T = T^*$, we easily deduce from the spectral theorem that there is an unconditional basis $(\psi_n) \subset \mathscr{D}$ such that $\mathscr{D} = \mathscr{D}^\infty(A)$, where $A = A\{(a_n), (\psi_n)\}$ is a suitable diagonal operator.

Thus, the existence of an (unconditional) basis allows to treat $\mathscr{D}$ as a sequence space. But to speak about *the* sequence space associated with $\mathscr{D}$ it must be shown that this space is independent of the choice of the (unconditional) basis. This is the "problem of quasi-equivalence of unconditional bases" which can be formulated as follows:

Let $\mathscr{D}[\tau]$ be a locally convex space, (φ_n), (ψ_n) two unconditional bases of $\mathscr{D}[\tau]$. Is there a permutation π of N and a sequence of positive numbers (r_n) such that the operator T defined by $T\varphi_n = r_n \psi_{\pi(n)}$ is a homeomorphism of $\mathscr{D}[\tau]$ onto itself?

For the domains in which we are interested (namely (F)-spaces) a positive answer to this question can be given, essentially, only in two cases. The first one deals with so-called "regular" bases introduced by Dragilev [8]. The results along this line concern nuclear (F)-spaces and countable Hilbert spaces and were obtained by Crone, Robinson [3], Kondakov [11] and Djakov [7]. The second case deals with centres of Hilbert scales and has been investigated by Mityagin [20]. In the sequel we make essential use of his result which can be summarized as follows:

THEOREM ([20]). *Let* $\mathscr{D} = \bigcap_{n\geqslant 0} \mathscr{D}(T^n) = \bigcap_{n\geqslant 0} \mathscr{D}(S^n)$ *with* $S = S\{(s_n), (\varphi_n)\}$ *and* $T = T\{(t_n), (\psi_n)\}$, $S, T \geqslant I$. *Then there exist constants* $R > 1$, $C > 0$ *such that with a suitable permutation* π *of* N

$$\frac{1}{C}(s_n)^{1/R} \leqslant t_{\pi(n)} \leqslant C(s_n)^R \quad \text{holds for all } n \in N. \tag{M}$$

This theorem can be regarded as a generalization of the theorem of Köthe [12] essentially used in [17].

All the results known up to now about bases and quasi-equivalence of bases clearly show that it is very hard to give a detailed description of the structure of domains of Op*-algebras by means of spaces of sequences in the quite general case $\mathscr{D} = \bigcap_{\alpha} \mathscr{D}(A_\alpha)$, $\alpha \in \mathfrak{A}$. Therefore we restrict ourselves to the case $\mathscr{D} = \mathscr{D}^\infty(T) = \bigcap_{n\geqslant 0} \mathscr{D}(T^n)$, $T = T^* \geqslant I$ which will be investigated in the next section.

3. Classification of domains $\mathscr{D} = \mathscr{D}^\infty(T)$

For the domains of this form we have: $\mathscr{D} \in I$ ($\in II$, $\in III$, resp.) if and only if $T \in I$ ($\in II$, $\in III$, resp.) with respect to the classification of operators given in [17]. Recalling the facts from the corollary above, we find a suitable diagonal operator $A = A\{(a_n), (\psi_n)\}$ on $\mathscr{D}(T)$ such that

$$\mathscr{D} = \left\{\psi = \sum x_n \psi_n : \sum |x_n|^2 (a_n)^{2j} < \infty,\ \forall j = 1, 2, \ldots\right\}$$

and the system of sequences $\mathscr{F} = \{(a_n^j), j = 1, 2, \ldots\}$, $a_n \in N$, $\forall n$, describes the associated sequence space. That is, we have an isomorphism between $\mathscr{D}$ and the space of sequences

$$\mathscr{D}^s = \left\{(x_n) : \sum |x_n|^2 a_n^{2j} < \infty,\ \forall j \in N\right\}.$$

It is clear that we have to prove that the system $\mathscr{F}$ does in fact characterize the domain $\mathscr{D}$ up to unitary equivalence. By using the theorem of Mityagin we obtain the desired proof and a complete description of the regarded domains. We remark that it is sufficient to restrict oneself in the following theorems to the case $\mathscr{D} = \mathscr{D}'$ and to give the proof only in one direction: that from $\mathscr{D} = \mathscr{D}'$ something follows about the systems $\mathscr{F}$, $\mathscr{F}'$ of sequences. The remaining parts of the theorems are obvious or follow by simple considerations.

THEOREM 1. *Let* $\mathscr{D}, \mathscr{D}' \in I$. $\mathscr{D}$ *and* $\mathscr{D}'$ *are unitarily equivalent if and only if there are diagonal operators* $S = S\{(s_n), (\varphi_n)\}$ *and* $T = T\{(t_n), (\psi_n)\}$ *with*

$$\mathscr{D} = \mathscr{D}^\infty(S), \qquad \mathscr{D}' = \mathscr{D}^\infty(T)$$

such that the systems of sequences

$$\mathcal{F} = \{(s_n^l),\ l = 1, 2, \ldots\}, \quad \mathcal{F}' = \{(t_n^l),\ l = 1, 2, \ldots\}$$

are equivalent.

Proof: In consideration of the remark before Theorem 1 we use the inequalities (M) and can prove, as it is done in [5], Lemma 5.4, that (M) holds with the identical permutation. But this simply means the equivalence of $\mathcal{F}$ and $\mathcal{F}'$.

We note that the assertion of this theorem can also be obtained from the closed-graph theorem and some considerations like in [17] (cf. [23] for such a proof). Now we pass to the domains of class *II*.

THEOREM 2. *Let $\mathcal{D}$, $\mathcal{D}' \in II$. $\mathcal{D}$ and $\mathcal{D}'$ are unitarily equivalent if and only if there are diagonal operators $S = S\{(s_n), (\varphi_n)\}$, $T = T\{(t_n), (\psi_n)\}$ with $(s_n) = (s_n^b) \cup (s_n^\infty)$, $(t_n) = (t_n^b) \cup (t_n^\infty)$ and $\mathcal{D} = \mathcal{D}^\infty(S)$, $\mathcal{D}' = \mathcal{D}^\infty(T)$ such that the systems of sequences*

$$\mathcal{F} = \{(s_n^{\infty j}),\ j = 1, 2, \ldots\}, \quad \mathcal{F}' = \{(t_n^{\infty j}),\ j = 1, 2, \ldots\}$$

are essentially equivalent.

Before proving the theorem we recall that from $\mathcal{D}$, $\mathcal{D}' \in II$ it follows that the diagonal operators S, T are also of the class II (cf. [17] and the remark at the beginning of this section). From the considerations in [17] it follows that the sequences (s_n), (t_n) resp., can be decomposed in the above-mentioned way, where (s_n^b) is a bounded sequence and for $(s_n^\infty): \lim s_n^\infty = \infty$ (analogously for (t_n)). Recall that the decomposition is unique up to a finite number of elements in any subsequence.

Proof of the theorem: From (M) we obtain in particular

$$\frac{1}{C}(s_n^b)^{1/R} \leqslant t_{k_n} \leqslant C(s_n^b)^R, \tag{1}$$

$$\frac{1}{C}(s_n^\infty)^{1/R} \leqslant t_{j_n} \leqslant C(s_n^\infty)^R \tag{2}$$

for suitable subsequences (t_{k_n}) and (t_{j_n}) of (t_n). (Clearly, (1) means that if $s_n^b = s_{l_n}$, then $t_{k_n} = s_{\pi(l_n)}$ and an analogous interpretation for (2).)

From (1) and (2) it follows that (t_{k_n}) is a bounded sequence and for $(t_{j_n}): \lim t_{j_n} = \infty$. But this means that (t_{k_n}) coincides, up to a finite number of elements, with (t_n^b) and (t_{j_n}) coincides, up to a finite number of elements, with (t_n^∞). Consequently, (2) gives the equivalence of $\{(s_n^{\infty j})\}$ and $\{(t_{j_n}^j)\}$, but this means the essential equivalence of $\{(s_n^{\infty j})\}$ and $\{(t_n^{\infty j})\}$. Q.E.D.

Now we pass to the class *III*. Like in [17], we shall distinguish three subclasses described below.

Let $\mathcal{D} \in III$, $\mathcal{D} = \mathcal{D}^\infty(S)$, $S = S\{(s_n), (\varphi_n)\}$ with the associated sequences (s_n') (cf. Preliminaries) and (s_n^0), the sequence of all eigenvalues of S with finite multiplicity, $s_1^0 \leqslant s_2^0 \leqslant \ldots$

Put $\mathscr{F} = \{(s_n'^j),\ j = 1, 2, \ldots\}$. It can easily be seen that

$$\mathscr{D} = \sum\nolimits_{\mathscr{F}} \oplus \mathscr{H}_n \oplus \mathscr{D}_0,$$

where $\mathscr{H}_n$ is the infinitely dimensional eigenspace corresponding to the eigenvalue s_n' and $\sum_{\mathscr{F}} \oplus \mathscr{H}_n$, $\mathscr{D}_0$ resp. mean

$$\sum\nolimits_{\mathscr{F}} \oplus \mathscr{H}_n = \Big\{\chi = \sum \chi_n \colon \chi_n \in \mathscr{H}_n,\ \sum \|\chi_n\|^2 (s_n')^{2j} < \infty,\ \forall j \in N\Big\},$$

$$\mathscr{D}_0 = \Big\{\psi = \sum x_n \psi_n \colon \sum |x_n|^2 (s_n^0)^{2j} < \infty,\ \forall j \in N\Big\},$$

where ψ_n is the eigenvector of S corresponding to the eigenvalue s_n^0. We also use the notation $\mathscr{D}_0 \triangleq \{(s_n^{0j}), j = 1, 2, \ldots\}$. Like in [17], we use the notion of "*reduction*" of $\mathscr{D}$ (or $\mathscr{D}_0$). Let $\mathscr{D} = \sum_{\mathscr{F}} \oplus \mathscr{H}_n \oplus \mathscr{D}_0$. We say that $\mathscr{D}$ (or $\mathscr{D}_0$) can be reduced if $\mathscr{D}$ also has the representation

$$\mathscr{D} = \sum\nolimits_{\mathscr{F}} \oplus \hat{\mathscr{H}}_n \oplus \hat{\mathscr{D}}_0$$

with $\mathscr{H}_n \subset \hat{\mathscr{H}}_n$, $\dim(\hat{\mathscr{H}}_n \ominus \mathscr{H}_n) < \infty$, $\dim(\bar{\mathscr{D}} \ominus \bar{\hat{\mathscr{D}}}) = \infty$. (Because finite-dimensional reductions, i.e., $\dim(\bar{\mathscr{D}} \ominus \bar{\hat{\mathscr{D}}}) < \infty$, are trivial and always possible, we regard only infinite dimensional reductions such as just defined).

More informally but roughly speaking, the possibility of reduction of $\mathscr{D} = \sum_{\mathscr{F}} \oplus \mathscr{H}_n \oplus \mathscr{D}_0$ means that we can find an infinitely dimensional submanifold of $\mathscr{D}_0$ and can "add" it to $\sum_{\mathscr{F}} \oplus \mathscr{H}_n$. Now we give the following definition of three cases which can arise in class *III*.

Class III_A If $\mathscr{D} = \sum_{\mathscr{F}} \oplus \hat{\mathscr{H}}_n$, we say that $\mathscr{D}_0$ can be *completely reduced.*

Class III_B If $\mathscr{D} = \sum_{\mathscr{F}} \oplus \hat{\mathscr{H}}_n \oplus \mathscr{D}_1$, $\mathscr{D}_1 \triangleq \{(a_n^l),\ l = 1, 2, \ldots\}$ and $\mathscr{D}_1$ cannot be reduced further, we say that D_0 can be *reduced to the highest degree.*

Class III_C If for any reduction of $\mathscr{D}_0$ which leads to $\mathscr{D} = \sum_{\mathscr{F}} \oplus \hat{\mathscr{H}}_n \oplus \mathscr{D}_2$, $\mathscr{D}_2$ can be reduced further, we say that $\mathscr{D}_0$ can be reduced but not to the highest degree.

Like in [17] one can easily obtain the following statement which we give here without proof.

STATEMENT. (i) *$\mathscr{D}_0$ can be completely reduced iff there is a subsequence $(s_{n_l}') \subset (s_n')$ such that with a suitable iteration $\wedge$* (cf. Preliminaries) *$\{(\hat{s}_{n_l}'^j)\}$ and $\{(s_n^{0j})\}$ are equivalent.*

(ii) *$\mathscr{D}_0$ can be reduced to the highest degree iff there is a decomposition $(s_n^0) = (a_n) \cup (b_n)$ and a subsequence $(s_{j_n}') \subset (s_n')$ such that with a suitable iteration $\wedge$:*

$$\{(a_n^l),\ l = 1, 2, \ldots\} \quad \text{and} \quad \{(\hat{s}_{j_n}'^l),\ l = 1, 2, \ldots\}$$

are equivalent, and $\mathscr{D}_2 \triangleq \{(b_n^l)\}$ cannot be reduced.

(iii) *$\mathscr{D}_0$ can be reduced but not to the highest degree iff for any decomposition $(s_n^0) = (a_n) \cup \cup (b_n)$ such that $\{(\hat{s}_{l_n}'^j)\}$ and $\{(a_n^j)\}$ are equivalent for a suitable subsequence of (s_n') and* *a suitable iteration there are a further decomposition* $(b_n) = (c_n) \cup (d_n)$ *and subsequence* $(s'_{h_n}) \subset (s'_n)$ *and an iteration such that* $\{(\hat{s}'^{j}_{h_n})\}$ *and* $\{(c^j_n)\}$ *are equivalent.*

The next proposition deals with the proof that the membership of a domain $\mathscr{D}$ to one of the classes III_A, III_B or III_C does not depend on the choice of the diagonal operator S in the representation $\mathscr{D} = \mathscr{D}^\infty(S)$.

PROPOSITION 1. *Let* $\mathscr{D} = \mathscr{D}^\infty(S) = \bigcap \mathscr{D}(S^n)$. *If* $\mathscr{D} \in III_A$ $(\in III_B, \in III_C)$ *with respect to the representation* $\mathscr{D} = \mathscr{D}^\infty(S)$, *then* $\mathscr{D} \in III_A$ $(\in III_B, \in III_C)$ *with respect to any representation* $\mathscr{D} = \mathscr{D}^\infty(T)$, *where* $S = S\{(s_n), (\varphi_n)\}$, $T = T\{(t_n), (\psi_n)\}$.

Proof: 1. Suppose that $\mathscr{D}^\infty(S) \in III_A$. Consider again the sequences (s_n), (s'_n) and (t_n), (t'_n), (t^0_n). Note that by the assumption $\mathscr{D}^\infty(S) \in III_A$, the set (s^0_n) is void.

From the estimations

$$\frac{1}{C}(s_n)^{1/R} \leqslant t_{\pi(n)} \leqslant C(s_n)^R \tag{M}$$

it follows in particular for $(t^0_n) = (t_{l_n})$ that

$$\frac{1}{C}(s_{\pi^{-1}(l_n)})^{1/R} \leqslant t_{l_n} \leqslant C(s_{\pi^{-1}(l_n)})^R. \tag{3}$$

Let $(s_{\pi^{-1}(l_n)}) = (s_{h_n})$. Since any s'_n has infinite multiplicity, we can find a further subsequence (s_{u_n}) with:

$$s_{u_n} = s_{h_n} \text{ for all } n; \quad (s_{u_n}) \subset \{(s_n) \setminus (s_{h_n})\}.$$

The latter and (3) lead to

$$\frac{1}{C} s^{1/R}_{u_n} \leqslant t_{\pi(u_n)} \leqslant C s^R_{u_n}, \quad (t_{\pi(u_n)}) \subset \{(t_n) \setminus (t^\infty_n)\}, \tag{4}$$

that is, $(t_{\pi(u_n)}) = (\hat{t}'_{j_n})$ for a suitable subsequence $(t'_{j_n}) \subset (t'_n)$ and a suitable iteration. From (3) and (4) we obtain

$$t^0_n \leqslant C(s_{h_n})^R \leqslant C \cdot C^{R^2} t^{R2}_{\pi(u_n)} = E(\hat{t}'_{j_n})^{R^2}, \tag{5}$$

$$t^0_n \geqslant \frac{1}{C} s^{1/R}_{\pi^{-1}(l_n)} \geqslant \left(\frac{1}{C}\right)^{1+\frac{1}{R2}} (t^{1/R2}_{\pi(u_n)}) = D(\hat{t}'_{j_n})^{1/R^2}. \tag{6}$$

Consequently,

$$D(\hat{t}'_{j_n})^{1/R2} \leqslant t^0_n \leqslant E(\hat{t}'_{j_n})^{R2},$$

i.e., the systems

$$\{(t^{0l}_n)\} \text{ and } \{(\hat{t}'^{l}_{j_n})\} \text{ are equivalent.}$$

Hence $\mathscr{D}^\infty(T) \in III_A$.

2. Suppose $\mathscr{D}^\infty(S) \in III_B$, $\mathscr{D}^\infty(T) \in III_C$ and consider

$$(s_n),\ (s'_n),\ (s^0_n) \quad \text{and} \quad (t_n),\ (t'_n),\ (t^0_n).$$

From (M) it follows that

$$\frac{1}{C}(s_n^0)^{1/R} \leqslant t_{k_n} \leqslant C(s_n^0)^R \qquad \text{for all } n, \tag{7}$$

$$\frac{1}{C}(s_{l_n})^{1/R} \leqslant (t_n^0) \leqslant C(s_{l_n})^R \qquad \text{for all } n. \tag{8}$$

This leads to $(t_{k_n}) \subset (t_n^0)$ up to a finite number of elements, because, if this were not the case one could obtain a contradiction with $\mathscr{D}^\infty(S) \in III_B$ by considerations similar to those in 1. (Because finite dimensional reductions are always possible, we may without loss of generality assume that

$$(t_{k_n}) \subset (t_n^0) \qquad \text{for all } n.)$$

From (7) and (M) it follows that for $\{(t_n^0)\setminus(t_{k_n})\} = (u_n)$, $\{(u_n^k)\}$ is equivalent to a suitable $\{(\hat{s}'^k_{r_n})\}$ and, again, by (M) and the considerations of 1: $\{(u_n^k)\}$ is equivalent to a suitable $\{(\hat{t}'^k_{h_n})\}$.

From this and $\mathscr{D}^\infty(T) \in III_C$ it follows that there is a further decomposition $(t_{k_n}) = (a_n) \cup (b_n)$ such that the system $\{(a_n^k)\}$ is again equivalent to a suitable $\{(\hat{t}'^k_{j_n})\}$. But this means, after all, that the subsequence $(s_{k_n}^0)$ of (s_n^0), the elements of which stand in (7), together with the elements of (a_n), "could be reduced", which is a contradiction with the assumption $\mathscr{D}^\infty(S) \in III_B$. Q.E.D.

The next theorem gives us information about unitary equivalence of domains of class III.

THEOREM 3. *Let $\mathscr{D}$, $\mathscr{D}' \in III$. $\mathscr{D}$ and $\mathscr{D}'$ are unitarily equivalent if and only if the following requirements hold*:

1. *There are operators $S = S\{(s_n), (\varphi_n)\}$ and $T = T\{(t_n), (\psi_n)\}$ such that $\mathscr{D} = \mathscr{D}^\infty(S)$ and $\mathscr{D}' = \mathscr{D}^\infty_{\approx}(T)$.*

2. *Both $\mathscr{D}$ and $\mathscr{D}'$ are of the same class III_A, III_B, III_C, resp.*

3. *For the sequences (s_n'), (t_n'), (s_n^0), (t_n^0) one has:*

(i) *(s_n') and (t_n') are weakly equivalent, that is, there are suitable iterations σ and τ such that*

$$(s'_{\sigma(n)}) \sim (t'_{\tau(n)}).$$

(ii) *For the classes III_B and III_C in addition to condition* (i) *one has*:

III_B: the systems $\{(s_n^{0l})\}$, $\{(t_n^{0l})\}$ are essentially equivalent.

III_C: there are decompositions $(s_n^0) = (a_n) \cup (b_n)$, $(t_n^0) = (c_n) \cup (d_n)$, subsequences $(u_n) \subset (s_n')$, $(v_n) \subset (t_n')$ and suitable iterations such that the three pairs of systems of sequences

$$\{(a_n^l)\},\ \{(c_n^l)\},$$
$$\{(b_n^l)\},\ \{(\hat{u}_n^l)\},$$
$$\{(d_n^l)\},\ \{(\hat{v}_n^l)\}$$

are equivalent.

Remark 3. Let us note that the conditions for classes III_B and III_C must be formulated in the language of systems of sequences (as it is done) while the condition for class III_A is a requirement only for the sequences themselves.

Proof of the theorem: Let

$$M_k = \{s_j \in (s_n) : s_j = k\},$$
$$N_k = \{t_j \in (t_n) : t_j = k\}.$$

From the estimations

$$\frac{1}{C} s_n^{1/R} \leqslant t_{\pi(n)} \leqslant C s_n^R \tag{M}$$

it follows that:

$$\left.\begin{array}{l} \text{if } s_i \in M_k, \text{ then } t_{\pi(i)} \in \bigcup_{j=1}^{n_k} N_j, \\ \text{if } t_i \in N_k, \text{ then } s_{\pi^{-1}(i)} \in \bigcup_{j=1}^{m_k} M_j, \end{array}\right. \quad m_k, n_k < \infty, \ \forall k.$$

But this is the same situation as in the proof of Theorem 3 of [17] from which it follows that (s_n') and (t_n') are weakly equivalent.

III_B: Like in the proof of Proposition 1 we deduce from (M) the following properties of the sequences (s_n^0) and (t_n^0):

(i) If the elements s_n^0 stand on the left-hand side and on the right-hand side of (M), then the corresponding $t_{\pi(n)}$ must belong to (t_n^0) up to a finite number elements.

(ii) If the elements of (t_n^0) stand for the $t_{\pi(n)}$, then on the right-hand side and on the left-hand side of (M) elements of (s_n^0) must stand up to a finite number of elements (if this is not the case, we obtain in both cases a contradition with $\mathscr{D}^\infty(S) \in III_B$, i.e. $\mathscr{D}$ reduced to the highest degree). (i) and (ii) mean that in (M) the elements of (s_n^0) and (t_n^0) "stand together" up to a finite number of elements. But this shows the essential equivalence of $\{(s_n^{0l})\}$ and $\{(t_n^{0l})\}$.

III_C: Consider the following decomposition of (t_n^0):

$$(t_n^0) = (c_n) \cup (d_n),$$

where c_n are those elements of (t_n^0) which stand in (M) together with elements of (s_n^0). Denote these elements of (s_n^0) by (a_n). Let $(d_n) = \{(t_n^0) \setminus (c_n)\}$; $(b_n) = \{(s_n^0) \setminus (a_n)\}$. The elements of (s_n) which stand in (M) together with the elements d_n form a sequence $(\hat{s}'_{j_n})$. From our "standard argumentations" it follows that $\{(\hat{s}'^{l}_{j_n})\}$ is equivalent to a suitable system

$$\{(\hat{t}'^{l}_{h_n})\} = \{(\hat{v}_n^l)\},$$

i.e., $\{(d_n^l)\}$ is equivalent to $\{(\hat{v}_n^l)\}$.

Analogously, one finds that $\{(b_n^l)\}$ is equivalent to $\{(\hat{s}'^{l}_{p_n})\} = \{(\hat{u}_n^l)\}$ for a suitable subsequence $(s'_{p_n}) = (u_n) \subset (s_n')$ and a suitable iteration. This concludes the proof of the theorem.

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