NORMAL STATES ON ALGEBRAS OF UNBOUNDED OPERATORS

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The problem of the normality of positive functionals on algebras of unbounded operators is investigated. We formulate conditions under which the normal states are uniformly continuous and we prove the normality of any uniformly continuous state on the maximal Op^*-algebras on certain unitary spaces.

Introduction

This paper is a contribution to the problem of description of the normal states on algebras of unbounded operators. In [6], [8] it was shown that for certain classes of algebras of unbounded operators all strongly positive functionals, $f(A)\geqslant 0$ for $A\geqslant 0$, are normal. In this paper we study the connection between the normality and uniform continuity [2] of states (Theorems 1, 2, 3).

Let $\mathscr{D}$ be a unitary space and $\mathscr{H}$ its completion. By $\mathscr{L}_+(\mathscr{D})$ we denote the set of all operators $A\in \operatorname{End}(\mathscr{D})$ for which there exists an operator $A^+\in \operatorname{End}(\mathscr{D})$ satisfying $\langle\Phi, A\Psi\rangle=\langle A^+\Phi, \Psi\rangle$ for all $\Phi, \Psi\in\mathscr{D}$. $\mathscr{L}_+(\mathscr{D})$ becomes a $*$-algebra of operators with the involution $A\to A^+$. A $*$-subalgebra $\mathscr{A}=\mathscr{A}(\mathscr{D})$ of $\mathscr{L}_+(\mathscr{D})$ containing the identity I will be called an Op^*-algebra [2]. An Op^*-algebra $\mathscr{A}$ is said to be *closed* if $\mathscr{D}=\bigcap_{A\in\mathscr{A}}\mathscr{D}(\bar{A})$ and *self-adjoint* if

$$\mathscr{D}=\mathscr{D}_*\underset{\text{def}}{=}\bigcap_{A\in\mathscr{A}}\mathscr{D}(A^*)\qquad([5],[7]).$$

For an Op^*-algebra $\mathscr{A}(\mathscr{D})$ we denote by $\tau_{\mathscr{D}}$ the *uniform topology* of $\mathscr{A}$ defined by the seminorms

$$\|A\|_{\mathscr{M}}=\sup_{\Phi,\Psi\in\mathscr{M}}|\langle\Phi, A\Psi\rangle|$$

taken for all sets $\mathscr{M}\subset\mathscr{D}$ for which $\|A\|_{\mathscr{M}}<\infty$ for any $A\in\mathscr{A}$, [2]. Such a set is called $\mathscr{A}$-*bounded.*

A linear functional f on an Op^*-algebra $\mathscr{A}(\mathscr{D})$ is said to be *strongly positive* if $f(A)\geqslant 0$ for any $A\geqslant 0$ of $\mathscr{A}$.

We call the functional f on $\mathscr{A}$ *normal,* if $f(A)=\operatorname{tr} AT$ for all $A\in\mathscr{A}$. Any normal functional is strongly positive. Woronowicz [8] proves that any strongly positive functional on $\mathscr{L}_+(\mathscr{S})$, $\mathscr{S}$ being the Schwartz space, is normal. From this follows the normality of the strongly positive functionals on the Op^*-algebra $\mathscr{A}=\mathscr{A}(\mathscr{S})$ generated by the position and momentum operators $q_k=x_k$, $p_k=\frac{1}{i}\cdot\frac{\partial}{\partial x_k}$, $k=1, 2, \ldots$ (finite), on the Schwartz space $\mathscr{S}$ in n variables. The nuclearity of the space is important for the proof of these results. Using quite different methods Sherman [6] proves the following theorem. Let $\mathscr{A}(\mathscr{D})$ be a closed Op^*-algebra of countable dimension containing an operator N which is the restriction to $\mathscr{D}$ of the inverse of a completely continuous operator on $\mathscr{H}$; then any strongly positive functional on $\mathscr{A}$ is normal. Sherman obtains his results by refined combinations of this algebraic problem with measure-theoretical considerations.

We prove in this paper that the uniformly continuous positive functionals on $\mathscr{L}_+(\mathscr{D})$ are normal if $\mathscr{L}_+(\mathscr{D})$ contains the restriction to $\mathscr{D}$ of the inverse of a nuclear operator (Theorem 3). It is shown further that the normal states on an Op^*-algebra $\mathscr{A}(\mathscr{D})$ are uniformly continuous, if $\mathscr{D}$ is contained in the range of a nuclear operator or if $\mathscr{A}$ has a countable dimension. Sections 1 and 2 contain results concerning Op^*-algebras needed for the proofs of theorems, but these results are of interest concluding regardless of their applications in Section 3.

1. In this section we give some lemmas about closable operators which we use in the Sections 2 and 3. We make use of the Kato–Heinz inequalities [1].

LEMMA 1.1. *Let A be a closable operator and T a bounded operator. TA is bounded if and only if $T^*\mathscr{H}\subset\mathscr{D}(A^*)$.*

Proof: If TA is bounded, $\langle\Phi, TA\Psi\rangle=f(\Psi)$ is a continuous functional on $\mathscr{D}(A)$. Then $\langle T^*\Phi, A\Psi\rangle$ is continuous in Ψ and consequently $T^*\Phi\in\mathscr{D}(A^*)$.

If $T^*\mathscr{H}\subset\mathscr{D}(A^*)$, then for $\Phi\in\mathscr{H}$, $\Psi\in\mathscr{D}(A)$: $B(\Phi, \Psi)=\langle T^*\Phi, A\Psi\rangle=\langle\Phi, TA\Psi\rangle$ $=\langle A^*T^*\Phi, \Psi\rangle$ and consequently $B(\Phi, \Psi)$ is continuous in Φ and Ψ. As a consequence of the Theorem of Mazur, Orlicz [4] we get $|B(\Phi, \Psi)|\leqslant C||\Phi||\cdot||\Psi||$ and therefore $||TA\Psi||$ $\leqslant C_1||\Psi||$, q.e.d.

LEMMA 1.2. *Let A be a closed operator and T a bounded operator for which AT is densely defined. AT is bounded if and only if $T\mathscr{H}\subset\mathscr{D}(A)$.*

Proof: Let $\mathscr{D}_0=\{\Phi\in\mathscr{H}\colon T\Phi\in\mathscr{D}(A)\}$. $\mathscr{D}_0$ is dense in $\mathscr{H}$. $\mathscr{D}(A)$ is a Banach space with the norm $||\Phi||_A=||\Phi||+||A\Phi||$. Since

$$||T\Phi||_A=||AT\Phi||+||T\Phi||\leqslant C||\Phi||,$$

T is a continuous operator on $\mathscr{D}_0$ in $\mathscr{D}(A)$ and consequently

$$T\tilde{\mathscr{D}}_0=T\mathscr{H}\subset\mathscr{D}(A).$$

Let $T\mathscr{H}\subset\mathscr{D}(A)$. Then we have $AT\mathscr{H}\subset\mathscr{H}$. Hence, AT is a closed operator on $\mathscr{H}$ and therefore continuous, q.e.d.

Now we prove some lemmas about the ranges of operators.

LEMMA 1.3. *Let A, B be two self-adjoint operators and let* $||A\Phi||\leqslant||B\Phi||$ *for all* $\Phi\in\mathscr{D}(B)\subset\mathscr{D}(A)$ *and suppose that B^{-1} exists. Then*

$$\mathscr{R}(B)\supset\mathscr{R}(A).$$

Proof: Let $\Psi\in\mathscr{R}(A)$, $\Phi\in\mathscr{R}(B)=\mathscr{D}(B^{-1})$. Then we have

$$\begin{aligned}|\langle B^{-1}\Phi,\Psi\rangle|&=|\langle B^{-1}\Phi,A\chi\rangle|=|\langle AB^{-1}\Phi,\chi\rangle|\\&\leqslant||AB^{-1}\Phi||\,||\chi||\leqslant C\,||\Phi||\,||\chi||.\end{aligned}$$

Therefore $\psi\in\mathscr{D}(B^{-1})=\mathscr{R}(B)$, q.e.d.

LEMMA 1.4. *Let A, B be two self-adjoint bounded operators and suppose that B^{-1} exists. If $\mathscr{R}(A)\subset\mathscr{R}(B)$ then*

$$||A\Phi||\leqslant C\,||B\Phi||.$$

Proof: The mapping $\Phi\to B^{-1}A\Phi$ is continuous, say $||B^{-1}A\Phi||\leqslant C||\Phi||$. Then it follows for Φ, $\Psi\in\mathscr{H}$:

$$\begin{aligned}|\langle\Psi,A\Phi\rangle|&=|\langle\Psi,AB^{-1}B\Phi\rangle|=|\langle B^{-1}A\Psi,B\Phi\rangle|\\&\leqslant||B^{-1}A\Psi||\cdot||B\Phi||\leqslant C\,||\Psi||\cdot||B\Phi||.\end{aligned}$$

Hence, for $\Psi=A\Phi$, $||A\Phi||\leqslant C||B\Phi||$, q.e.d.

LEMMA 1.5. *Let A be a positive self-adjoint operator for which A^{-1} exists and is a bounded operator. Let T be a bounded positive operator with $\mathscr{R}(T)\subset\mathscr{D}(A)$. Then*

$$\mathscr{R}(T^{\nu})\subset\mathscr{D}(A^{\nu})\quad \textit{for}\quad 0\leqslant\nu\leqslant1.$$

Proof: We have $\mathscr{R}(T)\subset\mathscr{D}(A)=\mathscr{R}(A^{-1})$. Therefore, by Lemma 1.4, we obtain $||T\Phi||\leqslant C||A^{-1}\Phi||$. Since $\mathscr{D}(A)=\mathscr{D}(CA)$, we can take $C=1$ without any loss of generality. Then it follows by the Kato–Heinz inequality [1]

$$||T^{\nu}\Phi||\leqslant||A^{-\nu}\Phi||.$$

This gives by Lemma 1.3,

$$\mathscr{R}(T^{\nu})\subset\mathscr{R}(A^{-\nu})=\mathscr{D}(A^{\nu}),\qquad \text{q.e.d.}$$

2. In this section we introduce special algebras of bounded operators with respect to given Op^*-algebras. The elements of these algebras are used for the definition of linear functionals on the Op^*-algebras by the trace-representation. Some results about self-adjoint Op^*-algebras are obtained.

We begin with the following definition:

DEFINITION 2.1. Let $\mathscr{A}$ be a closed Op-algebra. Then we define

$$\mathfrak{S}_1(\mathscr{A})=\{T:\ AT \text{ and } AT^* \text{ nuclear for all } A\in\mathscr{A}\},$$

$$_1\mathfrak{S}(\mathscr{A})=\{T:\ TA \text{ and } T^*A \text{ nuclear for all } A\in\mathscr{A}\}.$$

Remark. Since I (identity) $\in \mathscr{A}$, it is clear that any operator of $\mathfrak{S}_1(\mathscr{A})$, ${}_1\mathfrak{S}(\mathscr{A})$ is a nuclear one.

Now we turn to the proof of

LEMMA 2.1. *Let $\mathscr{A}$ be a closed Op^*-algebra; then it holds*

(i) $T \in \mathfrak{S}_1(\mathscr{A}) \rightarrow T\mathscr{H} \subset \mathscr{D}$.

(ii) $T \in {}_1\mathfrak{S}(\mathscr{A}) \rightarrow T\mathscr{H} \subset \mathscr{D}_*$,

(iii) $\mathfrak{S}_1(\mathscr{A})$ *and* ${}_1\mathfrak{S}(\mathscr{A})$ *are $*$-algebras,*

(iv) $\mathscr{A}\mathfrak{S}_1(\mathscr{A}) = \mathfrak{S}_1(\mathscr{A})$ *and* ${}_1\mathfrak{S}(\mathscr{A})\mathscr{A} = {}_1\mathfrak{S}(\mathscr{A})$.

Proof: (i), (ii) Since $\mathscr{D} = \bigcap_{A \in \mathscr{A}} \mathscr{D}(\bar{A})$ and $\mathscr{D}_* = \bigcap_{A \in \mathscr{A}} \mathscr{D}(A^*)$, these statements follow by Lemmas 1.1 and 1.2.

(iii) The $*$-property and the linearity are clear by definition. The statement follows from the fact that AB is nuclear if A is nuclear and B bounded or A bounded and B nuclear.

(iv) These statements follow immediately from the algebra-property of $\mathscr{A}$, q.e.d.

Now we define linear functionals on Op^*-algebras.

DEFINITION 2.2. For every $T \in \mathfrak{S}_1(\mathscr{A})$ we define the linear functional

$$f_T(A) = \operatorname{tr} AT$$

and for every $T \in {}_1\mathfrak{S}(\mathscr{A})$ we define the linear functional

$${}_Tf(A) = \operatorname{tr} TA .$$

Just as in the case of bounded operators one proves the following lemma.

LEMMA 2.2 (i) $\mathfrak{S}_1(\mathscr{A}) \subset {}_1\mathfrak{S}(\mathscr{A})$ *and* $f_T(A) = \operatorname{tr} AT = \operatorname{tr} TA$ *for* $T \in \mathfrak{S}_1(\mathscr{A})$,

(ii) f_T *is a positive functional if* $T \geqslant 0$.

We prove now a lemma about self-adjoint Op^*-algebras.

LEMMA 2.3. *For a closed Op^*-algebra $\mathscr{A}$, $\mathfrak{S}_1(\mathscr{A}) = {}_1\mathfrak{S}(\mathscr{A})$ if and only if $\mathscr{A}$ is self-adjoint.*

Proof: If the Op^*-algebra $\mathscr{A}$ is self-adjoint, then we have $\mathscr{D} = \mathscr{D}_*$ and for $T \in {}_1\mathfrak{S}(\mathscr{A})$ we obtain $T\mathscr{H} \subset \mathscr{D}$ by Lemma 2.1. Therefore, for $A \in \mathscr{A}$: $(T^*A)^* \supset A^*T \supset A^+T$. But the operator on the right-hand side is defined on the whole space $\mathscr{H}$ and therefore nuclear, too. This implies $T \in \mathfrak{S}_1(\mathscr{A})$. The other inclusion follows by Lemma 2.2.

Now we assume $\mathscr{A}$ not to be self-adjoint. Let be $\Psi \in \mathscr{D}_*$, $\Psi \notin \mathscr{D}$. By P we denote the projection onto Ψ. In consequence of Lemma 2.1 (i), PA is bounded and therefore nuclear, because it is one-dimensional, i.e. $P \in {}_1\mathfrak{S}(\mathscr{A})$. On the other hand, P does not belong to $\mathfrak{S}_1(\mathscr{A})$, since $\mathscr{A}$ contains at least one operator A with $\Psi \notin \mathscr{D}(\bar{A})$, q.e.d.

LEMMA 2.4. *Let $\mathscr{A}(\mathscr{D})$ be a self-adjoint Op^*-algebra and T a positive bounded self-adjoint operator with $T\mathscr{H} \subset \mathscr{D}$. Then $T^\nu\mathscr{H} \subset \mathscr{D}$ for all $\nu > 0$.*

Proof: First we consider the case $0 \leqslant \nu \leqslant 1$.

We have $\mathscr{D}=\bigcap_{A\in\mathscr{A}}\mathscr{D}(\bar{A})$, and one can prove that $\mathscr{D}=\bigcap_{S\in\mathscr{F}}\mathscr{D}(S)$, where $\mathscr{F}$ is the set of all operators $S=\overline{(A^+A+I)}$, $A\in\mathscr{A}$. This follows because $\mathscr{D}\overline{(A^+A+I)}\subset\mathscr{D}(\bar{A})$. Moreover, from the self-adjointness of $\mathscr{A}$ we obtain the equation $\mathscr{D}=\bigcap_{S_1\in\mathscr{F}_1}\mathscr{D}(S_1)$, where $\mathscr{F}_1$ is the set of self-adjoint Friedrichs extensions of the operators S of $\mathscr{F}$, $S\subset S_1=S_1^*\subset S^*$. Since $\mathscr{D}$ is invariant, $\mathscr{D}\subset\mathscr{D}(S_1^{1/\nu})$ for all $S_1\in\mathscr{F}_1$. Consequently, $T\mathscr{H}\subset\mathscr{D}(S_1^{1/\nu})$. We note that S_1^{-1} is bounded because S^{-1} is bounded and from Lemma 1.5 $T^\nu\mathscr{H}\subset\mathscr{D}(S_1)$, but this implies $T\mathscr{H}\subset\mathscr{D}$.

Now let $\nu>1$ be arbitrary. We choose a positive integer $n>\nu$. Then $T^n\mathscr{H}\subset\mathscr{D}$ trivially and therefore $T^\nu\mathscr{H}=(T^n)^{\nu/n}\mathscr{H}\subset\mathscr{D}$ in consequence of the foregoing proof, q.e.d.

Now we make a few remarks about diagonal operators which we shall need later. Let $\mathscr{H}$ be a Hilbert space and $\{\Phi_n\}_{n=1,2,\ldots}$ an orthonormal basis in $\mathscr{H}$. A self-adjoint diagonal operator $A=A(\{a_n\},\{\Phi_n\})$ in $\mathscr{H}$ is an operator A defined by $A\Phi_n=a_n\Phi_n$ on the domain

$$\mathscr{D}(A)=\{\Phi=\sum_n x_n\Phi_n:\ \sum_n |a_n x_n|^2<\infty\}.$$

Now we need the following lemma (cf. [3]) which we state here without proof.

Lemma 2.5. *Let $\{\Phi_n\}_{n=1,2,\ldots}$ be an orthonormal basis and $\{\Psi_K\}_{K=1,2,\ldots}$ an orthonormal system in $\mathscr{H}$, $\mathscr{H}_1$ the Hilbert space generated by $\{\Psi_K\}$. Let A and B be the following diagonal operators*:

$$A\Phi_n=a_n\Phi_n,\quad B\Psi_n=b_n\Psi_n,\quad B\Psi=0\quad \textit{for}\quad \Psi\in\mathscr{H}_0=\mathscr{H}\ominus\mathscr{H}_1,$$

where $\{a_n\}$ and $\{b_n\}$ are positive and increasing sequences with $\lim b_n=\infty$. *For the domain of the diagonal operator B we have the decomposition $\mathscr{D}(B)=\mathscr{H}_0\oplus\mathscr{D}_\Psi(B)$. If $\mathscr{D}_\Psi(B)\subset\mathscr{D}(A)$ then there exists a constant γ such that $a_n\leqslant\gamma b_n$ for all n.*

By the help of this lemma we are able to prove another one, which is the basis for the proofs of Theorems 1 and 3.

Lemma 2.6. *Let $\mathscr{A}(\mathscr{D})$ be a closed Op*-algebra containing the restriction N of the inverse of a nuclear self-adjoint operator on $\mathscr{H}$. If T is a positive operator with $T\mathscr{H}\subset\mathscr{D}$ then T^ν, $0<\nu\leqslant 1$, is a nuclear operator.*

Proof: The operator N^{-k} is nuclear and the operator N^k is defined on $\mathscr{D}$ for $k=1,2,\ldots$ Let $\{\Phi_n\}$ be the orthonormal basis of eigenvectors of N^{-1}, $N^{-1}\Phi_l=1/n_l\Phi_l$. Then

$$\sum_{l=1}^{\infty}1/n_l<\infty\,;$$

let be $n_i\leqslant n_{i+1}$ for all i. Let $\mathscr{K}$ be the unit ball in $\mathscr{H}$. Then $T\mathscr{K}=\mathscr{M}$ is compact, because of $\mathscr{M}=T\mathscr{K}=N^{-1}NT\mathscr{K}$ and NT is bounded. Therefore T is compact and $T\Psi_i=t_i\Psi_i$, where $\{\Psi_i\}$ is an orthonormal basis in $\mathscr{H}_1$, $T\Psi=0$ for $\Psi\in\mathscr{H}_0$, $\mathscr{H}=\mathscr{H}_0\oplus\mathscr{H}_1$. We consider the operator T^{-1} on $\mathscr{D}_\Psi=\mathscr{R}(T)\subset\mathscr{D}$. By Lemma 2.5, we obtain

$$n_l^k\leqslant C_k/t_l\quad \text{for all } l=1,2,\ldots \text{ and } k=1,2,\ldots$$

Hence

$$t_l \leqslant C_k/n_l^k \quad \text{and} \quad t_l^\nu \leqslant C_k^\nu/n_l^{k\nu} \leqslant C'/n_l$$

for suitable $k=k(\nu)$, $0<\nu\leqslant 1$. Therefore we obtain

$$\sum_{l=1}^{\infty} t_l^\nu \leqslant C' \sum_{l=1}^{\infty} 1/n_l < \infty$$

and that means T^ν is nuclear for $0<\nu\leqslant 1$, q.e.d.

3. Now we investigate linear positive functionals on certain classes of Op^*-algebras. We begin with the following definitions.

DEFINITION 3.1. A state ω on a closed Op^*-algebra $\mathscr{A}(\mathscr{D})$ is called *normal* if it can be represented in the form

$$\omega(A)=\omega_T(A)=\operatorname{tr} AT,$$

where $T\in\mathfrak{S}_1(\mathscr{A})$ is a positive operator.

DEFINITION 3.2. An operator A on $\mathscr{D}$ is called *finite dimensional* if $\dim(\mathscr{R}(A))<\infty$.

We prove a lemma about the density of the set of finite dimensional operators with respect to the topology $\tau_{\mathscr{D}}$ (cf. Introduction).

LEMMA 3.1. *Let $\mathscr{D}$ be the domain of an Op^*-algebra and assume there exists in $\mathscr{L}_+(\mathscr{D})$ an operator N, $N\Phi_l=n_l\Phi_l$ which is the restriction to $\mathscr{D}$ of the inverse of a completely continuous operator on $\mathscr{H}$, $\{\Phi_l\}$ being an orthonormal basis. Then the finite dimensional operators of $\mathscr{L}_+(\mathscr{D})$ are dense in $\mathscr{L}_+(\mathscr{D})$ with respect to the uniform topology $\tau_{\mathscr{D}}$.*

More precisely: For any $A\in\mathscr{L}_+(\mathscr{D})$ one can find a sequence $\{A_k\}\in\mathscr{L}_+(\mathscr{D})$ of finite dimensional operators such that $A_k\to A$ with respect to $\tau_{\mathscr{D}}$.

Proof: Let $A\in\mathscr{L}_+(\mathscr{D})$ arbitrary. We put

$$A_k=A\sum_{j=1}^{k-1}|\Phi_j\rangle\langle\Phi_j|,$$

where $\{\Phi_l\}$ is the orthonormal basis of eigenvectors of N. Let

$$R_k=\sum_{j=k}^{\infty} 1/n_j\,|\Phi_j\rangle\langle\Phi_j|.$$

Then

$$A-A_k=AR_kN.$$

Let $\mathscr{M}$ be a $\mathscr{L}_+(\mathscr{D})$-bounded set. Then

$$\begin{aligned}\|A-A_k\|_{\mathscr{M}}&=\sup_{\Phi,\Psi\in\mathscr{M}}|\langle\Phi,(A-A_k)\Psi\rangle|=\sup|\langle\Phi,AR_kN\Psi\rangle|\\&=\sup|\langle A^+\Phi,R_kN\Psi\rangle|\leqslant\sup_{\Phi\in\mathscr{M}}\|A^+\Phi\|\cdot\sup_{\Psi\in\mathscr{M}}\|R_kN\Psi\|.\end{aligned}$$

Thus, according to the next lemma

$$\lim_{k\to\infty} \sup_{\Psi\in\mathcal{M}} \|R_k N\Psi\| = 0, \quad \text{q.e.d.}$$

LEMMA 3.2. *Let $\mathcal{D}$ be as in Lemma 3.1. If $\mathcal{M}$ is an arbitrary $\mathcal{L}_+(\mathcal{D})$-bounded set, then*

$$\lim_{k\to\infty} \sup_{\Psi\in\mathcal{M}} \|R_k N\Psi\| = 0.$$

Proof: We assume that $\sup_{\Psi\in\mathcal{M}} \|R_k N\Psi\| = \alpha_k$ and $\inf_k \alpha_k = \alpha > 0$. Let us take ε such that $\alpha/2 > \varepsilon > 0$. Then there exists a sequence $\{\Psi_k\} \in \mathcal{M}$ such that $\|R_k N\Psi_k\| > \alpha_k - \varepsilon$.

Hence, $\inf_k \|R_k N\Psi_k\| > \alpha - \varepsilon > \alpha/2$ and therefore $\sup_{\Psi\in\mathcal{M}} \|N\Psi\| \geqslant \sup_k \|N\Psi_k\| = \infty$, but this contradicts the assumption that $\mathcal{M}$ is $\mathcal{L}_+(\mathcal{D})$-bounded, q.e.d.

Now we shall state and prove the main results of this paper. First we give conditions which imply the uniform continuity of the normal states.

THEOREM 1. *Let $\mathcal{A}(\mathcal{D})$ be a self-adjoint Op*-algebra such that $\mathcal{L}_+(\mathcal{D})$ contains the restriction of the inverse of a nuclear operator. Then any normal state ω on $\mathcal{A}$ is uniformly continuous.*

Proof: Let be $\omega(A) = \operatorname{tr} AT$, $T\Phi_i = t_i\Phi_i$. In consequence of Lemma 2.6, and Lemma 2.4,

$$\sum_i t_i^\nu < \infty \quad \text{for} \quad 0 < \nu \leqslant 1 \text{ and } T^\nu \mathcal{H} \subset \mathcal{D}.$$

Therefore AT^ν, $0 < \nu \leqslant 1$, is bounded. This implies that $T^\nu \mathcal{K}$ is an $\mathcal{A}$-bounded set, where $\mathcal{K}$ is the unit ball in $\mathcal{H}$.

Now we obtain by applying Lemma 2.2,

$$|\omega(A)| = \Big|\sum_i \langle \Phi_i, At_i\Phi_i\rangle\Big| \leqslant \sum t_i^{1/3} |\langle T^{1/3}\Phi_i, AT^{1/3}\Phi_i\rangle|$$
$$\leqslant \sum t_i^{1/3} \|A\|_{T^{1/3}\mathcal{K}}, \quad \text{q.e.d.}$$

Now we modify the assumptions of Theorem 1 and obtain the same result for algebras with countable algebraic dimension.

THEOREM 2. *Let $\mathcal{A}(\mathcal{D})$ be a closed Op*-algebra of countable algebraic dimension. Then any normal state ω on $\mathcal{A}$ is uniformly continuous.*

Proof: Without any loss of generality we may assume $\mathcal{A}$ to have a countable linear generating system $\mathfrak{A} = \{A_1, A_2, \ldots\}$ of positive operators $A_i \geqslant I$. Further we assume that together with A_n also $A_n^2 = A_m$ is an element of $\mathfrak{A}$. Let $\omega(A) = \operatorname{tr} AT$ be a normal state on $\mathcal{A}$, $T\Phi_i = t_i\Phi_i$, $\{\Phi_i\}$ an orthonormal basis. Then $\omega(A_n) = \sum_i t_i \langle \Phi_i, A_n\Phi_i\rangle < \infty$. We shall prove below the existence of a positive sequence $\alpha_i \geqslant 1$ such that

$$\sum t_i\alpha_i < \infty, \quad \sup_i 1/\alpha_i \langle \Phi_i, A_n\Phi_i\rangle < \infty \quad \text{for all } n. \tag{1}$$

Let

$$\mathscr{M}=\left\{\frac{1}{\sqrt{\alpha_i}}\,\Phi_i\,,\ i=1,2,\ldots\right\}.$$

$\mathscr{M}$ is $\mathscr{A}$-bounded for any $A\in\mathscr{A}$ because it is A-bounded for any A_n. In fact, since with A_n also A_n^2 is an element of $\mathfrak{A}$, we have

$$\|A_n\|_{\mathscr{M}}\leqslant\sup_{i,j}\frac{1}{\sqrt{\alpha_i}\sqrt{\alpha_j}}\,|\langle\Phi_i,A_n\Psi_j\rangle|\leqslant\sup_j\frac{1}{\sqrt{\alpha_j}}\,\|A_n\Psi_j\|$$
$$=\left(\sup_j\frac{1}{\alpha_j}\,\langle\Psi_j,A_n^2\Psi_j\rangle\right)^{\frac{1}{2}}<\infty\,.$$

Therefore

$$|\omega(A)|\leqslant\sum_i t_i\alpha_i\left|\frac{1}{\alpha_i}\,\langle\Phi_i,A\Phi_i\rangle\right|\leqslant\left(\sum t_i\alpha_i\right)\|A\|_{\mathscr{M}}$$

and this is just the uniform continuity of ω.

It remains to prove the existence of the sequence (α_i) with the properties (1). First we define for $n=1,2,\ldots$ the sequences

$$a_i^{(n)}=\langle\Phi_i,(A_1+\ldots+A_n)\Phi_i\rangle\,,\qquad i=1,2,\ldots$$

Then

$$\sum t_i a_i^{(n)}<\infty\qquad\text{for all }n\,.$$

We can now obtain the sequence (α_i) in the following way:

$$(\alpha_1,\alpha_2,\ldots,\alpha_{n_2},\alpha_{n_2+1},\ldots,\alpha_{n_3},\alpha_{n_3+1},\ldots)$$
$$=(a_1^{(1)},a_2^{(1)},\ldots,a_{n_2}^{(2)},a_{n_2+1}^{(2)},\ldots,a_{n_3}^{(3)},a_{n_3+1}^{(3)},\ldots).$$

We choose n_2 to satisfy

$$\sum_{i\geqslant n_2}t_i a_i^{(2)}\leqslant\tfrac{1}{2}\,.$$

We choose n_3 to satisfy

$$\sum_{i\geqslant n_3}t_i a_i^{(3)}\leqslant\tfrac{1}{4}\qquad\text{and so on.}$$

(α_i) has the desired properties. This completes the proof, q.e.d.

THEOREM 3. *Let $\mathscr{D}$ be the domain of a self-adjoint Op*-algebra $\mathscr{L}_+(\mathscr{D})$ and suppose there exists in $\mathscr{L}_+(\mathscr{D})$ an operator N which is the restriction of the inverse of a nuclear operator on $\mathscr{H}$. Then any uniformly continuous state ω on $\mathscr{L}_+(\mathscr{D})$ is a normal one.*

Proof: For the proof of this theorem we use an idea from [8]. Let ω be uniformly continuous, i.e. $|\omega(A)|\leqslant C\|A\|_{\mathscr{M}}$. For $A=|\Phi\rangle\langle\Psi|$ we obtain

$$|\omega(|\Phi\rangle\langle\Psi|)|\leqslant C\,\||\Phi\rangle\langle\Psi|\|_{\mathscr{M}}\leqslant C\sup_{\chi,\rho\in\mathscr{M}}|\langle\Psi,\rho\rangle|\,|\langle\chi,\Phi\rangle|$$
$$\leqslant C\|\Psi\|\,\|\Phi\|\sup_{\chi\in\mathscr{M}}\|\chi\|\sup_{\rho\in\mathscr{M}}\|\rho\|\leqslant C_1\|\Psi\|\,\|\Phi\|\,.$$

Thus there exists a bounded operator T with the domain $\tilde{\mathscr{D}}=\mathscr{H}$ such that

$$\omega(|\Phi\rangle\langle\Psi|)=\langle\Psi, T\Phi\rangle. \tag{2}$$

Since $|\Phi\rangle\langle\Phi|\geqslant 0$, we obtain $\omega(|\Phi\rangle\langle\Phi|)\geqslant 0$ for any $\Phi\in\mathscr{D}$, i.e. T is a positive operator.

Now we prove that $T\mathscr{H}\subset\mathscr{D}$. We note that for $A\in\mathscr{L}_+(\mathscr{D})$

$$\begin{aligned}|\omega(|\Phi\rangle\langle A\Psi|)|&\leqslant C\sup_{\chi,\rho\in\mathscr{M}}|\langle\chi,\Phi\rangle|\,|\langle A\Psi,\rho\rangle|\\ &\leqslant C\sup_{\chi,\rho\in\mathscr{M}}\|\Phi\|\,\|\Psi\|\,\|\chi\|\,\|A^+\rho\|\leqslant C_A\|\Phi\|\,\|\Psi\|.\end{aligned}$$

That means

$$|\omega(|\Phi\rangle\langle A\Psi|)|=|\langle A\Psi, T\Phi\rangle|\leqslant C_A\|\Phi\|\,\|\Psi\|.$$

Therefore $T\Phi\in\mathscr{D}(A^*)$ and consequently $T\mathscr{H}\subset\mathscr{D}_*=\mathscr{D}$. By Lemma 2.6, T is a nuclear operator and since $AT\mathscr{H}\subset\mathscr{D}$ for any $A\in\mathscr{L}_+(\mathscr{D})$, AT is a nuclear operator, too. This means $T\in\mathfrak{S}_1(\mathscr{L}_+(\mathscr{D}))$. By Theorem 1 the positive functional $\omega_T(A)=\operatorname{tr} AT$ is uniformly continuous. Furthermore

$$\operatorname{tr}(|\Phi\rangle\langle\Psi|\,T)=\langle\Psi, T\Phi\rangle=\omega(|\Phi\rangle\langle\Psi|)$$

and therefore $\omega(A)=\omega_T(A)$ for all finite dimensional operators of $\mathscr{L}_+(\mathscr{D})$. Because of the uniform continuity of both functionals ω, ω_T it follows from Lemma 3.1 that $\omega(A)=\operatorname{tr} AT$ for all $A\in\mathscr{L}_+(\mathscr{D})$, q.e.d.

Now we shall discuss the applications of the proved results to Op^*-algebras generated by the *CCR*.

Let

$$p_k=\frac{1}{i}\frac{\partial}{\partial x_k},\qquad q_k=x_k,\qquad k=1,2,\dots,\nu \text{ (finite)},$$

be the position-momentum operators defined on the Schwartz space $\mathscr{S}^\nu$ in ν variables. From the theorems proved above we find that the normal states on $\mathscr{L}_+(\mathscr{S}^\nu)$ are exactly the uniformly continuous ones, because N^2 is the inverse of a nuclear operator. $N=\sum a_k^+a_k$ is the number operator.

If we denote by $\mathscr{P}_\nu$ the algebra of all polynomials in p_k, q_k, then $\mathscr{P}_\nu''=\mathscr{L}_+(\mathscr{S}^\nu)$. The commutant $\mathscr{A}'$ of an Op^*-algebra $\mathscr{A}\subset\mathscr{L}_+(\mathscr{D})$ is defined by

$$\mathscr{A}'=\{B\in\mathscr{L}_+(\mathscr{D}), [A, B]=0, \text{ for all } A\in\mathscr{A}\}.$$

We say that a closed Op^*-algebra $\mathscr{A}$ is a *generalized von Neumann algebra* if $\mathscr{A}''=\mathscr{A}$. In this sense $\mathscr{L}_+(\mathscr{S}^\nu)$ is the generalized von Neumann algebra of a mechanical system of finite degree and all of its uniformly continuous states are normal states.

In the infinitely dimensional case $\nu=\infty$ the normality of all states can be derived only from the existence of certain energy operators $H=\sum\varepsilon_k a_k^+a_k$.

Let $|n\rangle=|n_1, n_2, \ldots\rangle$, $n_i=0, 1, 2, \ldots$, be the orthonormal basis in the Fock space and

$$a_i|n\rangle=\sqrt{n_i}\,|\ldots, n_i-1, \ldots\rangle,$$

$$a_i^+|n\rangle=\sqrt{n_i+1}\,|\ldots, n_i+1, \ldots\rangle,$$

$$q_i=\frac{1}{\sqrt{2}}(a_i+a_i^+), \qquad p_i=\frac{1}{\sqrt{2}\cdot i}(a_i-a_i^+).$$

Further let $\mathscr{D}$ be a unitary space containing the $|n\rangle$ and let $\mathscr{A}=\mathscr{A}(\mathscr{D})$ be a self-adjoint Op^*-algebra, the observable algebra, containing the operators p_i, q_i. We make the following assumptions:

1. *$\mathscr{A}(\mathscr{D})$ contains the energy-operator*

$$H=\sum \varepsilon_i a_i^+ a_i, \qquad \lim_i \varepsilon_i=+\infty .$$

2. *$\mathscr{A}(\mathscr{D})$ contains also certain functions of the energy H, for example e^H.*

If ε_i is a sufficiently rapidly increasing sequence, then e^{-H} is a nuclear operator. In fact,

$$e^{-H}\,|\,n\rangle=e^{-\sum \varepsilon_i n_i}\,|\,n\rangle$$

and therefore

$$\operatorname{tr} e^{-H}=\sum_{|n\rangle} e^{-\sum \varepsilon_i n_i}=\sum_{|n\rangle}\prod_i e^{-\varepsilon_i n_i}=\prod_i \frac{1}{1-e^{-\varepsilon_i}}<\infty .$$

Under these assumptions it follows from the theorems proved above that any uniformly continuous state on $\mathscr{L}_+(\mathscr{D})$ is a normal one, i.e. this result holds if the domain $\mathscr{D}$ is small enough to have the operator e^H is also defined on it.

It follows already from the assumption 1 that $(I+H)^{-1}$ is a completely continuous operator. Therefore one obtains from Sherman's result [6] that any strongly positive state on $\mathscr{A}(\mathscr{D})$ is a normal one, if $\mathscr{A}(\mathscr{D})$ is generated by a countable set of operators, e.g. by q_i, p_i, H.

A more detailed discussion of the facts mentioned in these concluding remarks will be published in another paper.

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