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COMPLETELY POSITIVE MAPPINGS AND UNBOUNDED OBSERVABLES*

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Completely positive mappings on algebras of unbounded observables are investigated. The main theorem of Stinespring on the structure of completely positive mappings is generalized to algebras of unbounded observables. It is shown that the consideration of unbounded observables leads to a natural physical topology on the state space, with respect to which the entropy is continuous.

1. Introduction

The so-called completely positive mappings [12] have gained importance in the theory of measurement as well as in the theory of time development of quantum systems in recent years. In general, the dynamics of an irreversible process is described by one-parametric semigroups of positive transformations [2], [5], which should be provided as completely positive.

Kraus [6] used to describe the change of states of quantum mechanical systems under measurement processes the complete positivity of the adjoint mapping of states. Lindblad [10] investigated the structure of generators of semigroups of completely positive mappings, but only for bounded generators. Gorini, Kossakowski and Sudarshan [4] obtained results analogous to those of Lindblad. All those investigations were carried out in the case of C^*-algebras.

Our aim is to go over to algebras of unbounded observables. The first step is to generalize the theorem of Stinespring [12] for algebras of unbounded operators, which is done in first half of this paper. In the last section we point out that inclusion of observable algebras with unbounded observables may lead in a natural way to physical topologies on the state space, with respect to which the entropy is continuous. We discuss this problem for a quantum mechanical system with finite degrees of freedom.

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2. Op*-algebras

We start with some definitions. Let $R[\tau]$ be a topological $*$-algebra. Then we define the convex hull of all elements a^*a

$$\mathscr{P}(R) = \operatorname{conv}\{a^*a,\ a \in R\}$$

and the τ-closure of this set

$$\bar{\mathscr{P}}(R) = \overline{\mathscr{P}(R)}^{\tau}.$$

We assume R to be semisimple in the sense that $\bar{\mathscr{P}}(R)$ is a proper cone.

Important for our considerations are $*$-algebras of unbounded operators over a pre-Hilbert space $\mathscr{D}$ with the scalar product $\langle\cdot\,,\,\cdot\rangle$. By $\mathscr{L}^+(\mathscr{D})$ we denote the set of all linear operators A of $\mathscr{D}$ into itself for which there exists an operator A^+ also mapping $\mathscr{D}$ into itself and satisfying $\langle\phi, A\psi\rangle = \langle A^+\phi, \psi\rangle$ for all $\phi, \psi \in \mathscr{D}$. $\mathscr{L}^+(\mathscr{D})$ is a $*$-algebra of operators with involution $A \to A^+$ and a $*$-subalgebra $\mathscr{A}$ of $\mathscr{L}^+(\mathscr{D})$ containing the identity is called an Op*-*algebra on the pre-Hilbert space* $\mathscr{D}$ [7].

Let $\mathscr{A}$ be an Op*-algebra on $\mathscr{D}$ and τ a locally convex topology on $\mathscr{A}$ such that $\mathscr{A}[\tau]$ becomes a topological $*$-algebra and such that τ is stronger than the weak topology σ defined by all seminorms

$$\sigma\colon\ p_{\phi,\psi}(A) = |\langle\phi, A\psi\rangle|, \qquad \phi, \psi \in \mathscr{D}.$$

This assumption about τ has the consequence that every $A \in \bar{\mathscr{P}}(\mathscr{A})$ is a positive operator, i.e.

$$\langle\phi, A\phi\rangle \geqslant 0 \qquad \text{for all } \phi \in \mathscr{D}.$$

Let us define besides

$$\mathscr{K} = \{A \in \mathscr{A},\ \langle\phi, A\phi\rangle \geqslant 0 \text{ for all } \phi \in \mathscr{D}\}.$$

If an Op*-algebra $\mathscr{A}[\tau]$ is a C^*-algebra then the three cones $\mathscr{P} \subset \bar{\mathscr{P}} \subset \mathscr{K}$ coincide, $\mathscr{P} = \bar{\mathscr{P}} = \mathscr{K}$. In the general case it is well known that the cones $\mathscr{P}$, $\bar{\mathscr{P}}$ and $\mathscr{K}$ may be different and so they define also different semiorders in $\mathscr{A}$.

3. Completely positive mappings

Let R be a $*$-algebra and M_n the algebra of all $n\times n$-matrices of complex numbers, $R^{(n)} = R\otimes M_n$, $n = 1, 2, \ldots$, the $*$-algebra of all $n\times n$-matrices $a^{(n)} = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \cdot & \cdot\,\cdot & \cdot \\ a_{n1} & \ldots & a_{nn}\end{bmatrix}$ with $a_{ij} \in R$ with the usual algebraic operations. If $\mathscr{A}$ is an Op*-algebra over $\mathscr{D}$ then $\mathscr{A}^{(n)}$ is an Op*-algebra over $\mathscr{D}^{(n)} = \mathscr{D}\oplus\mathscr{D}\oplus \ldots \oplus\mathscr{D}$ and therefore the cone $\mathscr{K}(\mathscr{A}^{(n)})$ is also well defined.

For a linear mapping μ of a $*$-algebra R_1 into a $*$-algebra R_2 by $\mu^{(n)}$, $n = 1, 2, \ldots$, we denote the mapping of $R_1^{(n)}$ into $R_2^{(n)}$ defined by

$$\mu^{(n)}(a^{(n)}) = \begin{bmatrix} \mu(a_{11}) & \ldots & \mu(a_{1n}) \\ \cdot & \cdot\,\cdot\,\cdot & \cdot \\ \mu(a_{n1}) & \ldots & \mu(a_{nn})\end{bmatrix}.$$

DEFINITION 3.1. A linear mapping μ of a $*$-algebra R_1 into a $*$-algebra R_2 is called *$(\mathscr{P}, \mathscr{P})$-completely positive* if for any $n = 1, 2, \ldots$ $\mu^{(n)}$ maps $\mathscr{P}(R_1^{(n)})$ into $\mathscr{P}(R_2^{(n)})$. The set of all $(\mathscr{P}, \mathscr{P})$-completely positive mappings we denote by $[\mathscr{P}, \mathscr{P}]$.

Quite analogously we define for two topological $*$-algebras the $(\mathscr{P}, \bar{\mathscr{P}})$ and $(\bar{\mathscr{P}}, \bar{\mathscr{P}})$-completely positive mappings and the corresponding sets $[\mathscr{P}, \bar{\mathscr{P}}]$ and $[\bar{\mathscr{P}}, \bar{\mathscr{P}}]$ of these mappings.

DEFINITION 3.2. If R is a $*$-algebra and $\mathscr{A}$ an Op*-algebra, then we define $(\mathscr{P}, \mathscr{K})$ and *$(\bar{\mathscr{P}}, \mathscr{K})$-completely positive mappings* quite analogously as in Definition 3.1 and the corresponding sets $[\mathscr{P}, \mathscr{K}]$ and $[\bar{\mathscr{P}}, \mathscr{K}]$ of these mappings.

Between the different sets of completely positive mappings of a topological $*$-algebra R into a topological Op*-algebra we have the following relations:

$$[\mathscr{P}, \mathscr{K}] \supset [\bar{\mathscr{P}}, \mathscr{K}] \supset [\bar{\mathscr{P}}, \bar{\mathscr{P}}]$$

and

$$[\mathscr{P}, \mathscr{K}] \supset [\mathscr{P}, \bar{\mathscr{P}}] \supset [\mathscr{P}, \mathscr{P}],$$

where we have the different concepts of complete positivity defined only for the cases that the second cone is "larger" than the first.

LEMMA 3.3. *For continuous mappings μ we have only the two types $[\bar{\mathscr{P}}, \bar{\mathscr{P}}]$ and $[\bar{\mathscr{P}}, \mathscr{K}]$ of completely positive mappings. If R_1 and R_2 are C*-algebras, then all the above regarded types of completely positive mappings coincide.*

The class $[\bar{\mathscr{P}}, \mathscr{K}]$ is in general larger than $[\bar{\mathscr{P}}, \bar{\mathscr{P}}]$.

The first part of the lemma is a consequence of well-known facts. To prove the last statement we give an example, using the well-known Hilbert polynomial $v(x, y)$ of degree six with two variables, which cannot be approximated by linear combinations of polynomials of the form $\bar{a}(x, y) \cdot a(x, y)$.

Let $\mathscr{D} = C_0(R^2) \subset L_2(R^2)$ be the pre-Hilbert space of all continuous functions in two real variables with compact support and $\mathscr{A}$ the algebra of all polynomials $a(x, y)$ in two variables. $\mathscr{A}$ is an Op*-algebra over $\mathscr{D}$, where the action of $a(x, y)$ on a vector $\phi = \phi(x, y) \in \mathscr{D}$ is simply the multiplication

$$a\phi = a(x, y) \cdot \phi(x, y).$$

For the topology τ in $\mathscr{A}$ we take the strongest locally convex topology. Then $\mathscr{K}$ is the cone of all positive polynomials and $\bar{\mathscr{P}}$ is the cone of all polynomials which can be approximated by linear combinations of $\bar{a}(x, y)a(x, y)$, $a \in \mathscr{A}$. The Hilbert polynomial $v(x, y)$ is an element of $\mathscr{K}$ but not of $\bar{\mathscr{P}}$. Now we can regard the linear mapping

$$\mu(a) = v \cdot a.$$

It is easy to see that $\mu \in [\bar{\mathscr{P}}, \mathscr{K}]$. But $\mu \notin [\bar{\mathscr{P}}, \bar{\mathscr{P}}]$, then $\mu(1) = v \notin \bar{\mathscr{P}}$.

4. The General Stinespring Theorem

Fundamental for the applications of completely positive mappings to the description of physical operations was a structure theorem for completely positive mappings between C^*-algebras proved by Stinespring [12]. We shall generalize this theorem to the case of arbitrary Op*-algebras. For this we need the following facts.

An Op*-algebra $\mathscr{A}$ defines a locally convex topology $t_{\mathscr{A}}$ on $\mathscr{D}$ by all seminorms

$$\|\phi\|_A = \|A\phi\|, \qquad A \in \mathscr{A}.$$

Any $A \in \mathscr{A}$ is then a continuous mapping of $\mathscr{D}[t_{\mathscr{A}}]$ into itself ([7]). In this way we have a rigged Hilbert space

$$\mathscr{D}[t_{\mathscr{A}}] \subset \mathscr{H} \subset \mathscr{D}[t_{\mathscr{A}}]^*,$$

where $\mathscr{H}$ is imbedded in $\mathscr{D}[t_{\mathscr{A}}]^*$ in the canonical way.

THEOREM 4.1. *Let R be a $*$-algebra with identity e and $\mathscr{A}$ an* Op*-*algebra over* $\mathscr{D}[t_{\mathscr{A}}]$. *A $(\mathscr{P}, \mathscr{K})$-completely positive mapping μ of R into $\mathscr{A}$ is of the form*

$$\mu(a) = V^* \cdot \varrho(a) \cdot V$$

where $a \to \varrho(a)$ is a $*$-representation of R onto an Op*-*algebra* $\varrho(R) = \mathscr{B}$ *over* $\mathscr{D}_1[t_{\mathscr{B}}]$, $V \in \mathscr{L}(\mathscr{D}[t_{\mathscr{A}}], \mathscr{D}_1[t_{\mathscr{B}}])$ *and V^* is the adjoint mapping of V,* $V^* \in \mathscr{L}(\mathscr{D}_1[t_{\mathscr{B}}]^*, \mathscr{D}[t_{\mathscr{A}}]^*)$.

Conversely, any linear mapping μ of such a form is $(\mathscr{P}, \mathscr{K})$-completely positive.

Proof: The second part of the theorem can be shown by straightforward calculations. So we have only to prove the first part.

Suppose that μ is completely positive. As in the proof by Stinespring [12] and Arveson [1] we consider the vector space $R\otimes D$, the algebraic tensor product of R and $\mathscr{D}$, and define for $\xi = \sum_i a_i\otimes\phi_i$ and $\eta = \sum_i b_i\otimes\psi_j$ of $R\otimes\mathscr{D}$ the bilinear form

$$\langle \xi, \eta\rangle = \sum_{i,j} \langle \phi_i, \mu(a_i^* b_j)\psi_j\rangle, \qquad \langle \xi, \xi\rangle \geqslant 0;$$

$\langle \xi, \eta\rangle$ is then a semi-definite scalar product in $R\otimes\mathscr{D}$. Now we define for $a \in R$

$$\pi(a)\left(\sum b_i\otimes\psi_i\right) = \sum ab_i\otimes\psi_i.$$

Then $\pi(a)$ is a representation of R on $R\otimes\mathscr{D}$ and

$$\langle \xi, \pi(a)\eta\rangle = \langle \pi(a^*)\xi, \eta\rangle.$$

Now let $\mathscr{N} = \{\xi \in R\otimes\mathscr{D}, \|\xi\| = \langle\xi,\xi\rangle^{1/2} = 0\}$ be the kernel of the scalar product. $\mathscr{N}$ is invariant under $\pi(a)$, since for $\xi \in \mathscr{N}$ and $a \in R$

$$\|\pi(a)\xi\|^2 = \langle \xi, \pi(a^*a)\xi\rangle \leqslant \|\xi\|\,\|\pi(a^*a)\xi\| = 0.$$

Hence $\pi(a)$ defines a linear operator $\varrho(a)$ on the pre-Hilbert space $\mathscr{D}_1 = R\otimes\mathscr{D}/\mathscr{N}$ with the positive definite scalar product $\langle\cdot, \cdot\rangle$. $\mathscr{B} = \varrho(R)$ is then an Op*-algebra on $\mathscr{D}_1$ and $t_{\mathscr{B}}$ the corresponding topology on $\mathscr{D}_1$.

Now we define a mapping V of $\mathscr{D}$ into $\mathscr{D}_1$ by

$$V\phi = \widetilde{1\otimes\phi} = 1\otimes\phi+\mathscr{N}$$

and show that $V \in \mathscr{L}(\mathscr{D}[t_{\mathscr{A}}], \mathscr{D}_1[t_{\mathscr{B}}])$.

In fact, for $B = \varrho(a) \in \mathscr{B}$, $a \in R$, we have

$$\begin{aligned}||V\phi||^2_{\mathscr{B}} &= \langle BV\phi, BV\phi\rangle = \langle\varrho(a)\,(\widetilde{1\otimes\phi}), \varrho(a)\,(\widetilde{1\otimes\phi})\rangle\\ &= \langle\widetilde{a\otimes\phi}, \widetilde{a\otimes\phi}\rangle\\ &= \langle\mu(a^*a)\phi, \phi\rangle \leqslant ||\mu(a^*a)\phi||\,||\phi||\\ &\leqslant ||A\phi||^2 = ||\phi||^2_{\mathscr{A}},\end{aligned}$$

where $A = I+\mu(a^*a) \in \mathscr{A}$.

Further we have

$$\langle V\phi, \varrho(a)V\psi\rangle = \langle 1\otimes\phi, a\otimes\psi\rangle = \langle\phi, \mu(a)\psi\rangle$$

and therefore

$$\mu(a) = V^*\varrho(a)V.$$

Remarks. We may assume that in Theorem 4.1 $\mathscr{A}$ and $\mathscr{B}$ are closed Op*-algebras, i.e. $\mathscr{D}[t_{\mathscr{A}}]$ and $\mathscr{D}_1[t_{\mathscr{B}}]$ are complete. If then R and $\mathscr{A}$ are C^*-algebras then also $\mathscr{B}$ is a C^*-algebra. Therefore $t_{\mathscr{A}}$ and $t_{\mathscr{B}}$ are the usual norm-topologies, i.e. $\mathscr{D}$ and $\mathscr{D}_1$ are complete Hilbert spaces. V maps $\mathscr{D}$ into $\mathscr{D}_1$ and the adjoint operator V^* maps $\mathscr{D}_1$ into $\mathscr{D}$. Therefore in this case our theorem coincides with the original Stinespring theorem.

Powers [11] has given another generalization of Stinespring's theorem. But he works purely algebraically, with linear spaces and bilinear forms, so that also in the C^*-case his theorem is more general.

In an earlier paper we introduced the concept of *c-evolution* as a special class of irreversible processes [9]. A c-evolution (concave evolution) is characterized by the fact that in the Schrödinger picture a state ϱ_t (density matrix) on an Op*-algebra $\mathscr{A}$ at the time $t > 0$ is "more mixed" than ϱ_0 at the time $t = 0$ in the sense of Uhlmann [13], i.e.

$$\varrho_t \in \overline{\mathrm{conv}}^{\tau}\{U\varrho_0 U^*,\ U \text{ all unitary operators}\}.$$

An important class of c-evolutions are the so-called *quantum stochastic processes*, investigated by Kossakowski [5],

$$\varrho \to \varrho_t = \int\limits_G U(g)\varrho U(g^{-1})\mu_t(dg) = U_t\varrho U_t^*,$$

where G is a locally compact separable group, $g \in G$, $g \to U(g)$ a unitary representation of G in Hilbert space $\mathscr{H}$ and μ_t a one-parameter convolution semi-group of probability measures over G.

These c-evolutions are completely positive mappings in the set of bounded density matrices.

By going over to the Heisenberg picture we are led to concern completely positive mappings in non-normable algebras in general. A more detailed investigation of this connection will be given in another paper.

In the last section of this paper we want to outline an argument showing the advantage of working with observable algebras which contain also unbounded observables.

5. On the continuity of the entropy

Let $\mathscr{D}$ be a pre-Hilbert space, $\mathscr{H}$ its completion, and $\mathscr{L}^+(\mathscr{D})$ the maximal Op*-algebra over $\mathscr{D}$ (see Sec. 2).

DEFINITION 5.1. By $\mathfrak{D}_1(\mathscr{D})$ we denote the set of all nuclear operators $\varrho \in \mathscr{L}^+(\mathscr{D})$ for which

(i) $A\varrho B$, $A\varrho^* B$ are nuclear for all $A, B \in \mathscr{L}^+(\mathscr{D})$ and

(ii) $\operatorname{tr} AB\varrho C = \operatorname{tr} B\varrho CA$ for all $A, B, C \in \mathscr{L}^+(\mathscr{D})$.

Then $(\mathfrak{D}_1(\mathscr{D}), \mathscr{L}^+(\mathscr{D}))$ is a dual pair with respect to the bilinear form $(\varrho, A) = \operatorname{tr}\varrho A$. The positive ϱ of $\mathfrak{D}_1(\mathscr{D})$ with $\operatorname{tr}\varrho = 1$ form the set $\mathscr{Z}_1(\mathscr{D})$ of states on $\mathscr{L}^+(\mathscr{D})$ given by density matrices.

We denote by $s = \beta(\mathfrak{D}_1(\mathscr{D}), \mathscr{L}^+(\mathscr{D}))$ the strong topology in $\mathfrak{D}_1(\mathscr{D})$ and by $\beta = \beta(\mathscr{L}^+(\mathscr{D}), \mathfrak{D}_1(\mathscr{D}))$ the strong topology in $\mathscr{L}^+(\mathscr{D})$ (Edwards [3], Chap. 8). The topologies are given by the following system of seminorms:

s: $\quad P_{\mathfrak{A}}(\varrho) = \sup\limits_{A\in\mathfrak{A}} |\operatorname{tr}\varrho A|,$

where $\mathfrak{A}$ runs over all weakly bounded sets in $\mathscr{L}^+(\mathscr{D})$

β: $\quad \|A\|_{\mathfrak{M}} = \sup\limits_{\varrho\in\mathfrak{M}} |\operatorname{tr}\varrho A|,$

where $\mathfrak{M}$ runs over all weakly bounded sets in $\mathfrak{D}_1(\mathscr{D})$.

By the duality of observables A and states ϱ the topologies s, β are "physically motivated".

If $\mathscr{D} = \mathscr{H}$ is a complete Hilbert space, then $\mathscr{L}^+(\mathscr{D}) = \mathscr{B}(\mathscr{H})$ is the C^*-algebra of all bounded operators on $\mathscr{H}$ and $\mathfrak{D}_1(\mathscr{H}) = \mathfrak{D}_1$ is the set of all nuclear operators. The topology β is then the usual uniform topology on $\mathscr{B}(\mathscr{H})$ given by the operator norm $\|\cdot\|$ and s is the topology given by the trace norm $\|\ \|_1$ on the linear space of nuclear operators.

Now it is well known that the entropy

$$S(\varrho) = -\operatorname{tr}\varrho\ln\varrho, \qquad \varrho \geqslant 0,$$

is not continuous with respect to the trace norm, i.e. the topology s generated by the trace norm $\|\ \|_1$ is too weak from a physical point of view. This is connected with the fact that the observable algebra is "too small"; for instance, it does not contain the position, momentum and energy operators.

Let us now regard a quantum mechanical system of finite degrees of freedom, where the observable algebra is generated by the position and momentum observables Q_i, P_i, $i = 1, 2, \ldots, N = 3f$. Let us realize these observables by the operators $Q_i = x_i$, P_i

$= \frac{1}{i} \cdot \frac{\partial}{\partial x_i}$ on the Schwartz space $\mathscr{S} = \mathscr{S}^N$ of rapidly decreasing functions in N variables. Then $Q_i, P_i \in \mathscr{L}^+(\mathscr{S})$ and we take $\mathscr{L}^+(\mathscr{S})$ as the observable algebra, but the following theorem remains valid if we take a smaller Op*-algebra $\mathscr{A} \subset \mathscr{L}^+(\mathscr{S})$ containing the Q_i and P_i.

THEOREM 5.1. *The entropy* $S(\varrho) = -\operatorname{tr}\varrho\ln\varrho$ *is continuous on* $\mathfrak{D}_1^p(\mathscr{S}) = \{\varrho; \varrho \in \mathfrak{D}_1(\mathscr{S}); \varrho \geqslant 0\}$ *with respect to the strong topology s defined on* $\mathfrak{D}_1^p(\mathscr{S})$ *by the dual pair* $(\mathscr{L}^+(\mathscr{S}), \mathfrak{D}_1(\mathscr{S}))$.

This theorem is a special case of a more general theorem proved in [9]. Here we prove only a special variant of this theorem, namely the following lemma.

LEMMA 5.2. *Let* $H = \frac{1}{2}m\sum P_i^2 + k\sum Q_i^2$ *be the Hamiltonian of the harmonic oscillator with complete set of eigenvectors* $\Omega_1, \Omega_2, \ldots \in \mathscr{S}$ *and the eigenvalues* $h_1, h_2, \ldots$ *Further let* $\mathscr{X}$ *be the set of all density matrices* $\varrho \in \mathfrak{D}_1(\mathscr{S})$ *which have the same eigenvectors* $\Omega_1, \Omega_2, \ldots, \varrho\Omega_i = \varrho_i\Omega_i$, *and the eigenvalues* $\varrho_1, \varrho_2, \ldots \geqslant 0$.

(i) *Then* $\sum h_i^{-2} < \infty$ *and the convex functionals* $p_k(\varrho) = \sum_{i=1}^{\infty} h_i^k \varrho_i$ *on* $\mathscr{X}$ *are finite for any* $k = 0, 1, 2, \ldots$ *and continuous with respect to the strong topology s on* $\mathscr{X}$.

(ii) *The entropy* $S(\varrho) = -\sum \varrho_i \ln\varrho_i$ *is continuous on* $\mathscr{X}$ *with respect to s.*

Proof: First we show that (ii) is a consequence of (i). Let $\varrho' \in \mathscr{X}$ and $\varepsilon > 0$. We have to prove the existence of a s-neighbourhood U of ϱ' in $\mathscr{X}$ such that $|S(\varrho)-S(\varrho')| < \varepsilon$ for all $\varrho \in U$. It follows from (i) that we can choose a neighbourhood U_1 of ϱ' such that

$$|p_3(\varrho)-p_3(\varrho')| < 1 \quad \text{for} \quad \varrho \in U_1.$$

Then we have $\varrho_i \leqslant ch_i^{-3}$ for all $\varrho \in U_1$ with a certain c. Now we have

$$|S(\varrho)-S(\varrho')| \leqslant \sum_{i=1}^{n} |\varrho_i\ln\varrho_i - \varrho_i'\ln\varrho_i'| + \sum_{i=n+1}^{\infty} (|\varrho_i\ln\varrho_i| + |\varrho_i'\ln\varrho_i'|).$$

Because of $\sum h_i^{-2} < \infty$ we can choose n so large that for $\varrho \in U_1$

$$\sum_{i=n+1}^{\infty} |\varrho_i\ln\varrho_i| \leqslant \sum_{i=n+1}^{\infty} (-ch_i^{-3}\ln ch_i^{-3}) \leqslant \sum_{i=n+1}^{\infty} ch_i^{-3}(3h_i - \ln c) < \tfrac{1}{4}\varepsilon.$$

Then

$$|S(\varrho)-S(\varrho')| \leqslant \sum_{i=1}^{n} |\varrho_i\ln\varrho_i - \varrho_i'\ln\varrho_i'| + \tfrac{1}{2}\varepsilon$$

for $\varrho, \varrho' \in U_1$ and we can take a smaller neighbourhood $U \subset U_1$ of ϱ' such that also the first summand on the right-hand side is smaller than $\frac{1}{2}\varepsilon$.

It remains to prove (i). Clearly, $\sum h_i^{-2} < \infty$. Since H maps $\mathscr{S}$ continuously into itself, for any $\varrho \in \mathscr{X}$ $H^k \cdot \varrho$ is a continuous operator of $\mathscr{H}$ into $\mathscr{S}$ and therefore nuclear. Consequently,

$$\operatorname{tr} H^k \varrho = \sum h_i^k \varrho_i < \infty .$$

We have yet to prove the continuity of $p_k(\varrho)$ with respect to s. But this is an immediate consequence of the equality

$$|p_k(\varrho') - p_k(\varrho)| = \left|\sum h_i^k(\varrho_i' - \varrho_i)\right| = |\operatorname{tr} H^k(\varrho' - \varrho)| .$$

This means that $p_k(\varrho)$ is already continuous in the weak topology on $\mathscr{X}$.

Remark. From the proof one can see that the special choice of H was not essential. In fact, one can take in the Lemma any essentially-self-adjoint positive operator H on $\mathscr{S}$, so that $\mathscr{S} = \bigcap_{k=1}^{\infty} \mathscr{D}(\bar{H}^k)$. Such an operator H has always a discrete spectrum h_i, $i = 1, 2, \ldots$, and there exists a natural number n such that $\sum h_i^{-n} < \infty$.

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