Here we discuss correctability and privacy for quantum channels, now taken to generality of Op^*-algebras as defined in the preliminary section. We are interested in two kinds of each notion; the exact case and the approximate case. The former is merely an algebraic notion, and is little different than in the bounded case, save for the generality of the objects. As discussed in the previous section however, we have a variety of generalizations of the operator norm which may or may not agree in the unbounded case, which leaves how best to think of correctability potentially more ambiguous, certainly compared to the setting which we are trying to generalize.

It's worth noting still however that even in the bounded case there remains some ambiguity with what the best notion of correctability and privacy are. First, let's state the definitions we are aiming to extend to unbounded algebras.

Definition. *Let $\mathcal{E} : \mathscr{A} \to \mathscr{B}$ be a quantum channel of C^*-algebras. A C^*-subalgebra $\mathscr{N} \subseteq \mathscr{B}$ is said to be* ***private*** *for $\mathcal{E}$ if*

$$\mathcal{E}(\mathscr{A}) \subseteq \mathscr{N}'.$$

Similarly, we saw that $\mathscr{N}$ is ε-private for $\mathcal{E}$ if there exists a quantum channel $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, private for $\mathscr{N}$, such that

$$\|\mathcal{E} - \mathcal{F}\|_{cb} < \varepsilon.$$

Definition. *Let $\mathcal{E} : \mathscr{A} \to \mathscr{B}$ be a quantum channels of C^*-algebras. a C^*-subalgebra $\mathscr{N} \subseteq \mathscr{B}$ is* ***correctable*** *for $\mathcal{E}$ if there exists a quantum channel $\mathcal{R} : \mathscr{N} \to \mathscr{A}$ such that*

$$\mathcal{E}\mathcal{R} = \mathcal{I}_{\mathscr{N}}.$$

$\mathscr{N}$ is ε-correctable for $\mathcal{E}$ if instead we have

$$\|\mathcal{E}\mathcal{R} - \mathcal{I}_{\mathscr{N}}\|_{cb} < \varepsilon.$$

Though we cannot write down the completely bounded norm for an unbounded channel, we can topologize the channel in a manner which is both more general and captures the approximation up to "arbitrary amplification" in a manner similar to the completely bounded norm.

However, completely bounded convergence or pertubation is arguably too strong of a requirement to describe certain physical pertubations. It is then perhaps desirable to develop a correspondence between privacy and correctability in a weaker setting.

We also note that in the above definitions are often restricted to the case where $\mathscr{A}$ is a von Neumann algebra, but the definition makes sense at this level of generality and is immediately analogous to our Op^*-algebra setting.

0.1 $\mathscr{A}$-amplifications

We aim to work with correctability and privacy in the sense of one of a few possible locally convex topologies. It makes sense then to describe these in terms of the seminorms of whatever topology we choose. What isn't immediately clear is how to capture the 'amplified' nature of the completely bounded topology, and so first a few things are in order.

Definition. *Let $\mathscr{A}$ be an Op^*-algebra. We denote by $\mathscr{A}_\infty = \mathscr{A} \otimes M_\infty$ the space of infinite $\mathscr{A}$-matrices, called the $\mathscr{A}$-amplifications.*

It is an algebraic matter how one makes a $*$-algebra out of this space, but what space should this act on? Let $\mathscr{D}_\infty$ be the space of finitely supported, $\mathscr{D}$-valued sequences. This is a pre-Hilbert space under the usual 'direct sum' inner product for Hilbert spaces. $\mathscr{A}_\infty$ acts on these sequences in the usual manner. It is clear that $\mathscr{A}_\infty$ is an Op^*-algebra on $\mathscr{D}_\infty$.

What is advantageous with this choice of domain is the stability of completeness of the domain with respect to the topologies induced by $\mathscr{A}$ and $\mathscr{A}_\infty$.

Proposition. *Let $\mathscr{A}$ be an Op^*-algebra on a pre Hilbert space $\mathscr{D}$ such that $\mathscr{D}$ is complete with respect to $\mathcal{T}_{\mathscr{A}}$, then $\mathscr{D}_\infty$ is complete with respect to $\mathcal{T}_{\mathscr{A}_\infty}$.*

Proof. Let $\{f_\alpha\}_\alpha$ be Cauchy with respect to $\mathcal{T}_{\mathscr{A}_\infty}$. Take the $\mathscr{A}$-amplification consisting of an operator $A \in \mathscr{A}$ in the nth diagonal entry. We have, by our assumption, that $\|A(f_\alpha(n) - f_\beta(n))\| = \|f_\alpha(n) - f_\beta(n)\|_A$ converges to 0. Whence we extract a limit $f(n) = \lim_\alpha f_\alpha(n)$ by the completeness of $\mathscr{D}$. The seminorm $\|f_\alpha - f\|_X$, where X is any $\mathscr{A}$-amplification, is simply a finite sum of seminorms on $\mathscr{D}$ applied to $f_\alpha(j) - f(j)$, $j \leq n$ for some $n \in \mathbb{N}$, which converges to zero. □

This property is helpful for when it is important that our Op^*-algebra is complete, as it allows us to still get a handle on the operators and vectors on this amplified space. It also meshes well with how amplifications are defined in the bounded setting.

We wish to apply channels of the original algebras to this amplified space. In order to use topologies on $\mathscr{A}_\infty$ in this setting it's convenient to have a natural embedding of $\mathscr{A}$ into $\mathscr{A}_\infty$, motivating why we take infinite matrices and finitely supported vectors.

0.2 Topologizing the Channels

In the previous section we discussed a variety of generalizations of the operator norm on the operator algebras themselves. We must now discuss how to generalize the completely bounded norm given a locally convex topology on the algebras.

Let $(\mathscr{A}_\infty, \{q_\alpha\})$ and $(\mathscr{B}_\infty, \{q_\beta\})$ be locally convex amplification algebras, let $\mathcal{E} : \mathscr{A} \to \mathscr{B}$ be a quantum channel continuous relative to both topologies. Let $\|\cdot\|_{S,\beta}$ be the family of seminorms defined via

$$\|\mathcal{E}\|_{S,\beta} = \sup_{X \in S}(q_\beta(\mathcal{E}_\infty(X))),$$

where S is a bounded subset of $(\mathscr{A}, \{q_\alpha\})$. This gives a topology on the channels relative to whatever locally convex topology for $\mathscr{A}$ we choose.

There is one more step to make a 'completely bounded' topology out of this.