Preliminaries and Definitions

Definition. *Let $\mathscr{D}$ be a pre-Hilbert space. Denote by $\mathscr{L}^\dagger(\mathscr{D})$ the space of linear operators on $\mathscr{D}$ such that for all $A \in \mathscr{L}^\dagger(\mathscr{D})$*

1. $A(\mathscr{D}) \subseteq \mathscr{D}$,
2. $\mathscr{D} \subseteq \mathcal{D}(A^*)$ *and* $A^*(\mathscr{D}) \subseteq \mathscr{D}$.

Under the involution $A^\dagger = A^*|_{\mathscr{D}}$, $\mathscr{L}^\dagger(\mathscr{D})$ is a $*$-algebra. An Op*-algebra is a unital $*$-subalgebra of $\mathscr{L}^\dagger(\mathscr{D})$.

Definition. *Let $\mathscr{A}$ be an Op*-algebra. We define the convex hull of all elements $A^\dagger A$ as the set $\mathcal{P}(\mathscr{A}) = co(\{A^\dagger A \ ; \ A \in \mathscr{A}\})$. Similarly, we write $\mathcal{K}(\mathscr{A}) = \{A \in \mathscr{A} \ ; \ \langle A\psi, \psi\rangle \geq 0$ for all $\psi \in \mathscr{D}\}$.*

We have the relations $\mathcal{P} \subseteq \overline{\mathcal{P}} \subseteq \mathcal{K}$, where the middle closure may be taken over a topology finer that the ultraweak topology on $\mathscr{A}$. If $\mathscr{A}$ is a C*-algebra, then these all coincide. This is not the case for general Op*-algebras. As a result there are a variety of positivity one can take in the unbounded setting. For the time being we take the weakest.

Definition. *(Weak Positivity) Let $\mathcal{E} : \mathscr{A} \to \mathscr{B}$ be a linear map between Op*-algebras. We say $\mathcal{E}$ is $(\mathcal{P}(\mathscr{A}), \mathcal{K}(\mathscr{B}))$ positive (usually just stated as "postive") if for any $A \in \mathscr{A}$ one has $\langle \mathcal{E}(A^\dagger A)\psi, \psi\rangle \geq 0$ for any $\psi \in \mathcal{D}(\mathscr{B})$. In a similar manner we say $\mathcal{E}$ is completely positive if every corresponding matrix amplification $\mathcal{E}_n : \mathscr{A} \otimes M_n \to \mathscr{B} \otimes M_n$ is positive.*

This of course carries the insinuation that $\mathscr{A} \otimes M_n$ is an Op*-algebra over $\mathcal{D}(\mathscr{A})^n$, which is indeed the case. We may also consider the tensor product $\mathscr{A} \otimes M_\infty$, where M_∞ denotes the set of finitely-supported, but infinite matrices. We will consider stability of this tensor product and its relationship to complete positivity later.

There is a generalization of the Stinespring Theorem for this class, and hence any more specific class of positive maps between Op*-algebras.

Theorem. *(Stinespring's Theorem) Let $\mathcal{E} : \mathscr{A} \to \mathscr{B}$ be a completely positive map. This map is of the form*

$$\mathcal{E}(A) = V^* \pi(A) V.$$

Where $\pi : \mathscr{A} \to \pi(\mathscr{A})$ is a $*$-representation onto an Op^*-algebra $\pi(\mathscr{A})$ over a dilated pre-Hilbert space $\mathcal{D}_\pi$ containing $\mathcal{D}(\mathcal{A})$ as a subspace, and $V : \mathcal{D}(\mathscr{A}) \to \mathcal{D}_\pi$ is a linear map continuous with respect to the graph topologies on $\mathcal{D}(\mathscr{A})$ and $\mathcal{D}_\pi$, V^* the corresponding TVS adjoint. Conversely, any such map is completely positive. If $\mathcal{E}$ is ultraweakly continuous, then so is π.

Proof. We show the first part of the theorem, which closely follows the bounded case. Consider the algebraic tensor product $\mathscr{A} \otimes \mathcal{D}$, where $\mathcal{D} = \mathcal{D}(\mathscr{A})$. For $\xi = \sum_i A_i \otimes \phi_i$ and $\eta = \sum_j B_j \otimes \psi_j$ the bilinear form $\langle \xi, \eta \rangle = \sum_{i,j} \langle \phi_i, \mathcal{E}(A_i^\dagger B_j)\psi_j \rangle$. That this is positive semi-definite follows from positivity of $\mathcal{E}$. Define the action of π by

$$\pi(X)\xi = \sum_i X A_i \otimes \phi_i.$$

This defines a representation of $\mathscr{A}$ on $\mathscr{A} \otimes \mathcal{D}$ satisfying

$$\langle \xi, \pi(A)\eta \rangle = \langle \pi(A^\dagger)\xi, \eta \rangle.$$

Now let $\mathcal{N}$ be the kernel of the previously defined inner product on $\mathscr{A} \otimes \mathcal{D}$. Given that $\xi \in \mathcal{N}$, one has

$$\|\pi(A)\xi\|^2 = \langle \xi, \pi(A^\dagger A)\xi \rangle \leq \|\xi\| \cdot \|\pi(A^\dagger A)\xi\| = 0.$$

Thus the action of π on the pre Hilbert space $\mathcal{D}_\pi = (\mathscr{A} \otimes \mathcal{D})/\mathcal{N}$ admits a well-defined linear operator. $\pi(\mathscr{A})$ is then an Op*-algebra.

Now define $V : \mathcal{D} \to \mathcal{D}_\pi$ via $V\psi = 1 \otimes \psi + \mathcal{N}$. Let ρ_X denote a seminorm corresponding to $X \in \mathscr{A}$ or $X \in \pi(\mathscr{A})$ in either graph topology. We have

$$\begin{aligned}
\rho_{\pi(A)}(V\psi)^2 &= \langle \pi(A)V\psi, \pi(A)V\psi \rangle \\
&= \langle \pi(A)(1 \otimes \psi), \pi(A)(1 \otimes \psi) \rangle \\
&= \langle A \otimes \psi, \pi(A) \otimes \psi \rangle \\
&= \langle \mathcal{E}(A^\dagger A)\psi, \psi \rangle \\
&\leq \|\mathcal{E}(A^\dagger A)\psi\| \cdot \|\psi\| \\
&\leq \|B\psi\|^2 \\
&= \rho_B(\psi)^2.
\end{aligned}$$

Where $B = I + \mathcal{E}(A^\dagger A) \in \mathscr{B}$ and $1 \otimes \psi$ is understood here as an equivalence class. Moreover, we have

$$\langle V\phi, \pi(A)V\psi \rangle = \langle 1 \otimes \phi, A \otimes \psi \rangle = \langle \phi, \mathcal{E}(A)\psi \rangle$$

and so $\mathcal{E}(A) = V^*\pi(A)V$.

Now suppose that $\mathcal{E}$ is ultraweakly continuous and that $\{A_\alpha\}_\alpha$ converges to 0 ultraweakly in $\mathscr{A}$. Let $\{\sum_k X_{n,k} \otimes \xi_{n,k}\}_n$ and $\{\sum_k Y_{n,k} \otimes \eta_{n,k}\}_n$ be two square summable sequences in $\mathcal{D}_\pi$. One has

$$\begin{aligned}
\sum_n \langle \pi(A_\alpha)\xi_n, \eta_n\rangle &= \sum_{n,i,j} \langle \pi(A_\alpha)X_{n,i} \otimes \xi_{n,i}, Y_{n,j} \otimes \eta_{n,j}\rangle \\
&= \sum_{n,i,j} \langle A_\alpha X_{n,i} \otimes \xi_{n,i}, Y_{n,j} \otimes \eta_{n,j}\rangle \\
&= \sum_{n,i,j} \langle \psi_{n,i}, \mathcal{E}((A_\alpha X_{n,i})^\dagger Y_{n,j})\eta_{n,j}\rangle.
\end{aligned}$$

Where ξ_n and η_n are the nth terms of the previous defined sequences. It follows by our assumption on the continuity of $\mathcal{E}$, the seperate continuity of multiplication, and continuity of the involution that the last line converges to zero, as required. □

The previous definitions and Stinespring's Theorem motivates the following definition.

Definition. *A quantum channel of Op*-algebras is a completely positive, ultraweakly continuous linear map $\mathcal{E} : \mathscr{A} \to \mathscr{B}$, where $\mathscr{A}$ and $\mathscr{B}$ are Op*-algebras.*