Results

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1 Preliminaries

Definition. *Let $\mathscr{D}$ be a pre-Hilbert space. Denote by $\mathscr{L}^\dagger(\mathscr{D})$ the space of linear operators on $\mathscr{D}$ such that for all $A \in \mathscr{L}^\dagger(\mathscr{D})$*

1. *$A(\mathscr{D}) \subseteq \mathscr{D}$,*
2. *$\mathscr{D} \subseteq \mathcal{D}(A^*)$ and $A^*(\mathscr{D}) \subseteq \mathscr{D}$.*

Under the involution $A^\dagger = A^*|_{\mathscr{D}}$, $\mathscr{L}^\dagger(\mathscr{D})$ is a $*$-algebra. An Op*-algebra is a unital $*$-subalgebra of $\mathscr{L}^\dagger(\mathscr{D})$.

Definition. *Let $\mathscr{A}$ be an Op*-algebra. We define the convex hull of all elements $A^\dagger A$ as the set $\mathcal{P}(\mathscr{A}) = co(\{A^\dagger A\ ;\ A \in \mathscr{A}\})$. Similarly, we write $\mathcal{K}(\mathscr{A}) = \{A \in \mathscr{A}\ ;\ \langle A\psi, \psi\rangle \geq 0$ for all $\psi \in \mathscr{D}\}$.*

We have the relations $\mathcal{P} \subseteq \overline{\mathcal{P}} \subseteq \mathcal{K}$, where the middle closure may be taken over a topology finer that the ultraweak topology on $\mathscr{A}$. If $\mathscr{A}$ is a C*-algebra, then these all coincide. This is not the case for general Op*-algebras. As a result there are a variety of positivity one can take in the unbounded setting. For the time being we take the weakest.

Definition. *(Weak Positivity) Let $\mathcal{E} : \mathscr{A} \to \mathscr{B}$ be a linear map between Op*-algebras. We say $\mathcal{E}$ is $(\mathcal{P}(\mathscr{A}), \mathcal{K}(\mathscr{B}))$ positive (usually just stated as "postive") if for any $A \in \mathscr{A}$ one has $\langle \mathcal{E}(A^\dagger A)\psi, \psi\rangle \geq 0$ for any $\psi \in \mathcal{D}(\mathscr{B})$. In a similar manner we say $\mathcal{E}$ is completely positive if every corresponding matrix amplification $\mathcal{E}_n : \mathscr{A} \otimes M_n \to \mathscr{B} \otimes M_n$ is positive.*

This of course carries the insinuation that $\mathscr{A} \otimes M_n$ is an Op*-algebra over $\mathcal{D}(\mathscr{A})^n$, which is indeed the case. We may also consider the tensor product $\mathscr{A} \otimes M_\infty$, where M_∞ denotes the set of finitely-supported, but infinite matrices. We will consider stability of this tensor product and its relationship to complete positivity later.

There is a generalization of the Stinespring Theorem for this class, and hence any more specific class of positive maps between Op*-algebras.

Theorem. *(Stinespring's Theorem) Let $\mathcal{E} : \mathscr{A} \to \mathscr{B}$ be a completely positive map. This map is of the form*

$$\mathcal{E}(A) = V^* \pi(A) V.$$

Where $\pi : \mathscr{A} \to \pi(\mathscr{A})$ is a $*$-representation onto an Op*-algebra $\pi(\mathscr{A})$ over a dilated pre-Hilbert space $\mathcal{D}_\pi$ containing $\mathcal{D}(\mathcal{A})$ as a subspace, and $V : \mathcal{D}(\mathscr{A}) \to \mathcal{D}_\pi$ is a linear map continuous with respect to the graph topologies on $\mathcal{D}(\mathscr{A})$ and $\mathcal{D}_\pi$, V^* the corresponding TVS adjoint. Conversely, any such map is completely positive. If $\mathcal{E}$ is ultraweakly continuous, then so is π.

Proof. We show the first part of the theorem, which closely follows the bounded case. Consider the algebraic tensor product $\mathscr{A} \otimes \mathcal{D}$, where $\mathcal{D} = \mathcal{D}(\mathscr{A})$. For $\xi = \sum_i A_i \otimes \phi_i$ and $\eta = \sum_j B_j \otimes \psi_j$ the bilinear form $\langle \xi, \eta \rangle = \sum_{i,j} \langle \phi_i, \mathcal{E}(A_i^\dagger B_j) \psi_j \rangle$. That this is positive semi-definite follows from positivity of $\mathcal{E}$. Define the action of π by

$$\pi(X)\xi = \sum_i X A_i \otimes \phi_i.$$

This defines a representation of $\mathscr{A}$ on $\mathscr{A} \otimes \mathcal{D}$ satisfying

$$\langle \xi, \pi(A)\eta \rangle = \langle \pi(A^\dagger)\xi, \eta \rangle.$$

Now let $\mathcal{N}$ be the kernel of the previously defined inner product on $\mathscr{A} \otimes \mathcal{D}$. Given that $\xi \in \mathcal{N}$, one has

$$\|\pi(A)\xi\|^2 = \langle \xi, \pi(A^\dagger A)\xi \rangle \leq \|\xi\| \cdot \|\pi(A^\dagger A)\xi\| = 0.$$

Thus the action of π on the pre Hilbert space $\mathcal{D}_\pi = (\mathscr{A} \otimes \mathcal{D})/\mathcal{N}$ admits a well-defined linear operator. $\pi(\mathscr{A})$ is then an Op*-algebra.

Now define $V : \mathcal{D} \to \mathcal{D}_\pi$ via $V\psi = 1 \otimes \psi + \mathcal{N}$. Let ρ_X denote a seminorm corresponding to $X \in \mathscr{A}$ or $X \in \pi(\mathscr{A})$ in either graph topology. We have

$$\begin{aligned}
\rho_{\pi(A)}(V\psi)^2 &= \langle \pi(A)V\psi, \pi(A)V\psi\rangle \\
&= \langle \pi(A)(1\otimes\psi), \pi(A)(1\otimes\psi)\rangle \\
&= \langle A\otimes\psi, \pi(A)\otimes\psi\rangle \\
&= \langle \mathcal{E}(A^\dagger A)\psi, \psi\rangle \\
&\leq \|\mathcal{E}(A^\dagger A)\psi\| \cdot \|\psi\| \\
&\leq \|B\psi\|^2 \\
&= \rho_B(\psi)^2.
\end{aligned}$$

Where $B = I + \mathcal{E}(A^\dagger A) \in \mathscr{B}$ and $1 \otimes \psi$ is understood here as an equivalence class. Moreover, we have

$$\langle V\phi, \pi(A)V\psi\rangle = \langle 1\otimes\phi, A\otimes\psi\rangle = \langle \phi, \mathcal{E}(A)\psi\rangle$$

and so $\mathcal{E}(A) = V^*\pi(A)V$.

Now suppose that $\mathcal{E}$ is ultraweakly continuous and that $\{A_\alpha\}_\alpha$ converges to 0 ultraweakly in $\mathscr{A}$. Let $\{\sum_k X_{n,k} \otimes \xi_{n,k}\}_n$ and $\{\sum_k Y_{n,k} \otimes \eta_{n,k}\}_n$ be two square summable sequences in $\mathcal{D}_\pi$. One has

$$\begin{aligned}
\sum_n \langle \pi(A_\alpha)\xi_n, \eta_n\rangle &= \sum_{n,i,j} \langle \pi(A_\alpha)X_{n,i}\otimes\xi_{n,i}, Y_{n,j}\otimes\eta_{n,j}\rangle \\
&= \sum_{n,i,j} \langle A_\alpha X_{n,i}\otimes\xi_{n,i}, Y_{n,j}\otimes\eta_{n,j}\rangle \\
&= \sum_{n,i,j} \langle \psi_{n,i}, \mathcal{E}((A_\alpha X_{n,i})^\dagger Y_{n,j})\eta_{n,j}\rangle.
\end{aligned}$$

Where ξ_n and η_n are the nth terms of the previous defined sequences. It follows by our assumption on the continuity of $\mathcal{E}$, the seperate continuity of multiplication, and continuity of the involution that the last line converges to zero, as required. □

The previous definitions and Stinespring's Theorem motivates the following definition.

Definition. *A quantum channel of Op*-algebras is a completely positive, ultraweakly continuous linear map $\mathcal{E} : \mathscr{A} \to \mathscr{B}$, where $\mathscr{A}$ and $\mathscr{B}$ are Op*-algebras.*

2 Topologies

Here we will record results pertaining to the topologies one can put on an Op*-algebra. Our notions of correctability and privacy for quantum channels of unbounded operators will be stated in terms of the seminorms generating a locally convex topology. However, it is not immediately clear which topology is best for the task of proving the desired results, which are most satisfying mathematically or physically, and how each topology relates to one another.

Definition. *An O*-algebra equipped with a locally convex topology for which multiplication is seperately continuous is called a locally convex algebra. If the involution is also continuous, we say that it is a locally convex $*$-algebra.*

2.1 The Uniform Topology

Here we will outline the results of Lassner concerning their so-called uniform topology. Before we can define what that is, we require a topology on the domain of an Op*-algebra.

Let $\mathscr{A}$ be an Op*-algebra. We denote by $\mathcal{T}_{\mathscr{A}}$ the coarsest locally convex topology on $\mathscr{D} = \mathcal{D}(\mathscr{A})$ for which every operator $A \in \mathscr{A}$ is a continuous linear mapping of $(\mathscr{D}, \mathcal{T}_{\mathscr{A}}) \to (\mathscr{D}, \langle \cdot, \cdot \rangle)$. This topology is given by the seminorms $\|\psi\|_A = \|A\psi\|$, where $A \in \mathscr{A}$ on $\mathscr{D}$. This is stronger than the Hilbert space topology since $\mathscr{A}$ is unital. Though we assumed the codmain to have the Hilbert space topology, every $A \in \mathscr{A}$ is continuous if it also has $\mathcal{T}_{\mathscr{A}}$.

Theorem. *Given an Op*-algebra $\mathscr{A}$ on a pre-Hilbert space $\mathscr{D}$ we have the following.*

1. *Every linear operator $A \in \mathscr{A}$ is a continuous operator of $(\mathscr{D}, \mathcal{T}_{\mathscr{A}}) \to (\mathscr{D}, \mathcal{T}_{\mathscr{A}})$.*

2. *If every operator in $\mathscr{A}$ is bounded, then $\mathcal{T}_{\mathscr{A}}$ coincides with the Hilbert space topology on $\mathscr{D}$.*

3. *Given an algebraic linear basis of $\mathscr{A}$, $\mathcal{T}_{\mathscr{A}}$ is generated by those seminorms q_A with A in the basis.*

The locally convex space $(\mathscr{D}, \mathcal{T}_{\mathscr{A}})$ is not in general complete. We may denote the completion of $\mathscr{D}$ by $\overline{\mathscr{D}}$.

Lemma. *Let $\mathscr{A}$ be an Op*-algebra on $\mathscr{D} \subseteq \mathscr{H}$. The injection $\mathscr{D} \to \mathscr{H}$ can be extended to a continuous injection of $\overline{\mathscr{D}} \to \mathscr{H}$. Moreover, $\overline{\mathscr{D}} = \bigcap_{\mathscr{A}} \mathcal{D}\left(\overline{A}\right)$.*

We will now begin our overview of locally convex topologies on an Op*-algebra $\mathscr{A}$.

Definition. *Let $\mathscr{A}$ be an Op*-algebra on $\mathscr{D}$, $\mathscr{B}$ a collection of bounded sets of a locally convex space $(\mathscr{D}, \mathcal{T})$. We say that that $\mathscr{B}$ is admissible if*

1. *for $S \in \mathscr{B}$ and $A \in \mathscr{A}$, $A(S) \in \mathscr{B}$,*
2. *$\bigcup_{S \in \mathscr{B}} S$ is dense in $\mathcal{H}(\mathscr{D})$,*
3. *for $S_1, S_2 \in \mathscr{B}$, $S_1 \cup S_2 \in \mathscr{B}$.*

These so-called admissable systems give class of locally convex topologies through the topology $\mathcal{T}_{\mathscr{A}}$ on $\mathscr{D}$.

Definition. *Let $\mathscr{A}$ be an Op*-algebra on $\mathscr{D}$ and $\mathscr{B}$ an admissable system of bounded sets of $(\mathscr{D}, \mathcal{T}_{\mathscr{A}})$. We define the topology $\mathcal{T}^{\mathscr{B}}$ by the seminorms*

$$q_{B,S}(A) = \sup_{\psi \in S} \|A\psi\|_B\,,\ S \in \mathscr{B}, B \in \mathscr{A}.$$

Similarly, we define the topology $\mathcal{T}_{\mathscr{B}}$ by the seminorms

$$\|A\|_S = \sup_{\phi,\psi \in S} |\langle \phi, A\psi \rangle|,\ S \in \mathscr{B}.$$

If $\mathscr{B}$ is the system of all bounded sets on $(\mathscr{D}, \mathcal{T}_{\mathscr{A}})$, we write these topologies as $\mathcal{T}^{\mathscr{D}}$ and $\mathcal{T}_{\mathscr{D}}$, respectively.

Under any admissable system $\mathscr{B}$ with respect to an Op*-algebra $\mathscr{A}$ on $\mathscr{D}$, $(\mathscr{A}, \mathcal{T}^{\mathscr{B}})$ and $(\mathscr{A}, \mathcal{T}_{\mathscr{B}})$ are locally convex spaces. Moreover, we have the following;

Theorem. *If $\mathscr{A}$ is an Op*-algebra, then $(\mathscr{A}, \mathcal{T}^{\mathscr{B}})$ is a locally convex algebra, and $(\mathscr{A}, \mathcal{T}_{\mathscr{B}})$ is a locally convex $*$-algebra.*

Lassner calls this the uniform topology, as a generalization of the operator norm in the bounded case due to the next theorem.

Theorem. *If every operator A of an Op*-algebra $\mathscr{A}$ on $\mathscr{D}$ is bounded, then $\mathcal{T}^{\mathscr{D}} = \mathcal{T}_{\mathscr{D}} = \mathcal{T}_{\|\cdot\|}$ where $\mathcal{T}_{\|\cdot\|}$ is the operator norm topology. $\mathcal{T}^{\mathscr{D}}$ is finer than $\mathcal{T}_{\mathscr{D}}$ and they are equivalent if and only if multiplication is $\mathcal{T}_{\mathscr{D}}$-continous.*

There is something of a converse to the first part of the previous theorem.

Theorem. *If there exists a norm on an Op*-algebra $\mathscr{A}$ on $\mathscr{D}$ defining a stronger topology than $\mathcal{T}_{\mathscr{D}}$, then every operator $A \in \mathscr{A}$ is continuous.*

Finally, Lassner gives a lemma characterizing the topology $\mathcal{T}_{\mathscr{D}}$ without reference to $\mathcal{T}_{\mathscr{A}}$.

Lemma. *Let $\mathscr{A}$ be an Op*-algebra over $\mathscr{D}$. The topology $\mathcal{T}_{\mathscr{D}}$ is defined by all seminorms*

$$\|A\|_S = \sup_{\phi,\psi \in S} |\langle \phi, A\psi \rangle|,$$

where S is an arbitrary subset of $\mathscr{D}$ such that the written supremum exists for any $A \in \mathscr{A}$.

2.2 The β Topology

Here we will discuss another topology given by Lassner which emerges from a dual pairing familiar to bounded operator algebras.

Definition. *By $\mathscr{T}_1(\mathscr{D})$ we refer to those nuclear operator $\rho \in \mathscr{L}^\dagger(\mathscr{D})$ for which*

1. *$A\rho B$ and $A\rho^\dagger B$ are nuclear for all $A, B \in \mathscr{L}^\dagger(\mathscr{D})$.*
2. *$\langle AB, \rho C \rangle = \langle B\rho, CA \rangle$ for all $A, B, C \in \mathscr{L}^\dagger(\mathscr{D})$.*

Where $\langle \rho, A \rangle = Tr(\rho A)$ and $\langle A, \rho \rangle = Tr(A\rho)$.

Then $\langle \rho, A \rangle$ gives a dual pairing for $(\mathscr{T}_1(\mathscr{D}), \mathscr{L}^\dagger(\mathscr{D}))$. We denote by β the topology induced by this dual pairing; the locally convex topology given by the seminorms

$$\beta_S(A) = \sup_{\rho \in S} |\langle \rho, A \rangle|.$$

Where S is a weakly bounded subset of $\mathscr{T}_1(\mathscr{D})$. This topology is "physically motivated" in the sense of the duality between states ρ and observables A. The β topology also generalizes the uniform topology on $\mathscr{B}(\mathscr{H})$ with, in the bounded case, $\mathscr{T}_1(\mathscr{D})$ simply being all nuclear operators and the dual pairing being the familiar one.

2.3 The ρ topology

Let $\mathscr{A}$ be an Op*-algebra over $\mathscr{D}$ and $\{A_n\}_n \subseteq \mathscr{A}$ a positive, increasing sequence. Suppose that $\mathscr{A} = \bigcup_n \mathscr{A}_n$ where

$$\mathscr{A}_n := \{A \in \mathscr{A};\ \text{for any } \psi \in \mathscr{D},\ |\langle A\psi, \psi\rangle| \leq \lambda \cdot |\langle A_n\psi, \psi\rangle| \text{ for some } \lambda > 0\}.$$

Then we have the norms $\|\cdot\|_n : \mathscr{A}_n \to \mathbb{R}^+$ defined by

$$\|A\|_n = \inf\{\lambda > 0\ ;\ |\langle A\psi, \psi\rangle| \leq \lambda \cdot |\langle A_n\psi, \psi\rangle| \text{ for all } \psi \in \mathscr{D}\}.$$

Suppose further that $\mathscr{A}$ is the inductive limit of the locally convex spaces $\{(\mathscr{A}_n, \|\cdot\|_n)\}_n$. Since each $\mathscr{A}_n$ is a normed space, a neighbourhood basis of 0 is given by unions of open balls with respect to each norm. Moreover, this topology is generated by the following seminorms indexed by $\alpha \in (\mathbb{R}^+)^{\mathbb{N}}$

$$\rho_\alpha(A) := \inf\left\{\sum_{k=1}^{N} \alpha(k)\, \|A_k\|_k\ ;\ A = \sum_{k=1}^{N} A_k,\ A_k \in \mathscr{A}_k\right\}.$$

Certain examples of Op*-algebras satsify the conditions necessary to define this topology, and in many cases this topology coincides with the β topology, such as the vital position/momentum algebra on Schwarz space. It is unkown whether this this true for all Op*-algebras. It is easy to see that once again this topology generalizes the operator norm in the bounded setting.

3 Error Correction

Here we discuss correctability and privacy for quantum channels, now taken to generality of *Op**-algebras as defined in the preliminary section. We are interested in two kinds of each notion; the exact case and the approximate case. The former is merely an algebraic notion, and is little different than in the bounded case, save for the generality of the objects. As discussed in the previous section however, we have a variety of generalizations of the operator norm which may or may not agree in the unbounded case, which leaves how best to think of correctability potentially more ambiguous, certainly compared to the setting which we are trying to generalize.

It's worth noting still however that even in the bounded case there remains some ambiguity with what the best notion of correctability and privacy are. First, let's state the definitions we are aiming to extend to unbounded algebras.

Definition. *Let $\mathcal{E} : \mathscr{A} \to \mathscr{B}$ be a quantum channel of C^*-algebras. A C^*-subalgebra $\mathscr{N} \subseteq \mathscr{B}$ is said to be **private** for $\mathcal{E}$ if*

$$\mathcal{E}(\mathscr{A}) \subseteq \mathscr{N}'.$$

Similarly, we saw that $\mathscr{N}$ is ε-private for $\mathcal{E}$ if there exists a quantum channel $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, private for $\mathscr{N}$, such that

$$\|\mathcal{E} - \mathcal{F}\|_{cb} < \varepsilon.$$

Definition. *Let $\mathcal{E} : \mathscr{A} \to \mathscr{B}$ be a quantum channels of C^*-algebras. a C^*-subalgebra $\mathscr{N} \subseteq \mathscr{B}$ is **correctable** for $\mathcal{E}$ if there exists a quantum channel $\mathcal{R} : \mathscr{N} \to \mathscr{A}$ such that*

$$\mathcal{E}\mathcal{R} = \mathcal{I}_{\mathscr{N}}.$$

$\mathscr{N}$ is ε-correctable for $\mathcal{E}$ if instead we have

$$\|\mathcal{E}\mathcal{R} - \mathcal{I}_{\mathscr{N}}\|_{cb} < \varepsilon.$$

Though we cannot write down the completely bounded norm for an unbounded channel, we can topologize the channel in a manner which is both more general and captures the approximation up to "arbitrary amplification" in a manner similar to the completely bounded norm.

However, completely bounded convergence or pertubation is arguably too strong of a requirement to describe certain physical pertubations. It is then perhaps desirable to develop a correspondence between privacy and correctability in a weaker setting.

We also note that in the above definitions are often restricted to the case where $\mathscr{A}$ is a von Neumann algebra, but the definition makes sense at this level of generality and is immediately analogous to our Op^*-algebra setting.

3.1 $\mathscr{A}$-amplifications

We aim to work with correctability and privacy in the sense of one of a few possible locally convex topologies. It makes sense then to describe these in terms of the seminorms of whatever topology we choose. What isn't immediately clear is how to capture the 'amplified' nature of the completely bounded topology, and so first a few things are in order.

Definition. *Let $\mathscr{A}$ be an Op^*-algebra. We denote by $\mathscr{A}_\infty = \mathscr{A} \otimes M_\infty$ the space of infinite $\mathscr{A}$-matrices, called the $\mathscr{A}$-amplifications.*

It is an algebraic matter how one makes a $*$-algebra out of this space, but what space should this act on? Let $\mathscr{D}_\infty$ be the space of finitely supported, $\mathscr{D}$-valued sequences. This is a pre-Hilbert space under the usual 'direct sum' inner product for Hilbert spaces. $\mathscr{A}_\infty$ acts on these sequences in the usual manner. It is clear that $\mathscr{A}_\infty$ is an Op^*-algebra on $\mathscr{D}_\infty$.

What is advantageous with this choice of domain is the stability of completeness of the domain with respect to the topologies induced by $\mathscr{A}$ and $\mathscr{A}_\infty$.

Proposition. *Let $\mathscr{A}$ be an Op^*-algebra on a pre Hilbert space $\mathscr{D}$ such that $\mathscr{D}$ is complete with respect to $\mathcal{T}_{\mathscr{A}}$, then $\mathscr{D}_\infty$ is complete with respect to $\mathcal{T}_{\mathscr{A}_\infty}$.*

Proof. Let $\{f_\alpha\}_\alpha$ be Cauchy with respect to $\mathcal{T}_{\mathscr{A}_\infty}$. Take the $\mathscr{A}$-amplification consisting of an operator $A \in \mathscr{A}$ in the nth diagonal entry. We have, by our assumption, that $\|A(f_\alpha(n) - f_\beta(n))\| = \|f_\alpha(n) - f_\beta(n)\|_A$ converges to 0. Whence we extract a limit $f(n) = \lim_\alpha f_\alpha(n)$ by the completeness of $\mathscr{D}$. The seminorm $\|f_\alpha - f\|_X$, where X is any $\mathscr{A}$-amplification, is simply a finite sum of seminorms on $\mathscr{D}$ applied to $f_\alpha(j) - f(j)$, $j \leq n$ for some $n \in \mathbb{N}$, which converges to zero. □

This property is helpful for when it is important that our Op^*-algebra is complete, as it allows us to still get a handle on the operators and vectors on this amplified space. It also meshes well with how amplifications are defined in the bounded setting.

We wish to apply channels of the original algebras to this amplified space. In order to use topologies on $\mathscr{A}_\infty$ in this setting it's convenient to have a natural embedding of $\mathscr{A}$ into $\mathscr{A}_\infty$, motivating why we take infinite matrices and finitely supported vectors.

3.2 Topologizing the Channels

In the previous section we discussed a variety of generalizations of the operator norm on the operator algebras themselves. We must now discuss how to generalize the completely bounded norm given a locally convex topology on the algebras.

Let $(\mathscr{A}_\infty, \{q_\alpha\})$ and $(\mathscr{B}_\infty, \{q_\beta\})$ be locally convex amplification algebras, let $\mathcal{E} : \mathscr{A} \to \mathscr{B}$ be a quantum channel continuous relative to both topologies. Let $\|\cdot\|_{S,\beta}$ be the family of seminorms defined via

$$\|\mathcal{E}\|_{S,\beta} = \sup_{X \in S}(q_\beta(\mathcal{E}_\infty(X))),$$

where S is a bounded subset of $(\mathscr{A}, \{q_\alpha\})$. This gives a topology on the channels relative to whatever locally convex topology for $\mathscr{A}$ we choose.

There is one more step to make a 'completely bounded' topology out of this.