TOPOLOGICAL ALGEBRAS OF OPERATORS

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Various methods of topologization of an algebra of unbounded operators are studied. For an algebra of unbounded operators with involution the uniform topology is defined and investigated. The notions of an $\tilde{O}^*$-algebra and $A\tilde{O}^*$-algebra are defined, which are generalizations of the notions of a C^*-algebra and B^*-algebra. Examples illustrate how the developed idea works.

1. Introduction

A growing interest in a general representation theory of topological algebras in the non-normed (unbounded) case may be observed. This is due to its applications in quantum theory where this problem arose at least ten years ago, when the quantum field theory was formulated in the language of the representation theory of algebras by Borchers [1] and Uhlmann [14]. From the mathematical point of view, it is astonishing that up to now the well-developed theory of normed algebras and their representations ([10], [12]) has such few generalization to the non-normed case.

The impressive results of the theory of Banach algebras and their representations are obtained by refined combinations of algebraical and topological methods. One would like in an algebra of unbounded operators to have a "natural" topology just like the uniform topology (operator-norm topology) in an algebra of bounded operators.

In this paper, we define for a $*$-algebra of unbounded operators (Op^*-algebras) a uniform topology $\tau_{\mathscr{D}}$ [4], which is expected to be a good generalization of the uniform topology in an algebra of bounded operators (Section 3). With this topology the notions of an $\tilde{O}^*$-algebra and $A\tilde{O}^*$-algebra are defined, which are generalizations of the notions of a C^*-algebra and B^*-algebra, and some theorems about representations of non-normed locally convex topological algebras (LK-algebras) are proved (Section 4). In Section 5 we investigate the question of a topology with respect to which the algebra $\mathscr{P}$ of all polynomials becomes an $A\tilde{O}^*$-algebra. The same question for the test function algebra $\mathscr{S}_{\otimes}$ of a quantum field is investigated in [5].

In a further paper [6] the topological method developed here is applied to LMC^*-algebras [8] and in [7] the continuity of normal states (trace-states) with respect to the uniform topology $\tau_{\mathscr{D}}$ is investigated.

We note that recently Vasilev ([15], [16] and other papers) has been strenuously searching for a general representation theory of non-normed algebras in view of its application to the quantum field theory.

2. Algebras of operators

An algebra $\mathscr{A}$ is a complex linear space with multiplication. We always assume $\mathscr{A}$ contains a unit element e. $\mathscr{A}$ is called a *$*$-algebra* if there is defined an involution in $\mathscr{A}$:

$$a \to a^*, \quad (\lambda a)^* = \bar{\lambda} a^*, \quad (a \cdot b)^* = b^* a^*, \quad a^{**} = a.$$

A *LK-algebra* $\mathscr{A}$ is an algebra in which a locally convex topology is defined so that the multiplication is separately continuous, i.e. $a \to ba$ and $a \to ab$ are linear continuous mappings of $\mathscr{A}$ for any $b \in \mathscr{A}$. A *LK*-algebra which is a $*$-algebra is called a *LK^*-algebra* if the involution $a \to a^*$ is continuous.

Let $\mathscr{D}$ be a unitary space with the scalar product $\langle \Phi, \Psi \rangle$, $\langle \bar{\lambda}\Phi, \Psi \rangle = \langle \Phi, \lambda \Psi \rangle = \lambda \langle \Phi, \Psi \rangle$, Φ, $\Psi \in \mathscr{D}$, $\lambda \in C$ (complex numbers) and let $\|\Phi\| = \langle \Phi, \Phi \rangle^{\frac{1}{2}}$ be the norm in $\mathscr{D}$. $\mathscr{H}$ denotes the Hilbert space which is the completion of $\mathscr{D}$.

By $\mathscr{L}_0(\mathscr{D})$ we denote the *set of all linear operators in* $\mathscr{D}$ which are closable in $\mathscr{H}$ and by $\mathscr{L}_+(\mathscr{D})$ we denote the *set of all operators* $A \in \mathscr{L}_0(\mathscr{D})$ for which there exists an operator $A^+ \in \mathscr{L}_0(\mathscr{D})$ such that $\langle \Phi, A\Psi \rangle = \langle A^+\Phi, \Psi \rangle$, $\forall \Phi, \Psi \in \mathscr{D}$.

In other words, $\mathscr{L}_+(\mathscr{D})$ contains all (unbounded) operators A of $\mathscr{H}$ with the domain $\mathscr{D}(A) = \mathscr{D}$ and satisfying the following conditions:

1. *$\mathscr{D}$ is invariant with respect to A, $A\mathscr{D} \subset \mathscr{D}$.*

2. *The adjoint operator A^* exists.*

3. *The domain $\mathscr{D}(A^*)$ of the adjoint operator A^* contains $\mathscr{D}$ and $\mathscr{D}$ is invariant with respect to A^*, $A^*\mathscr{D} \subset \mathscr{D}$.*

A^+ is the restriction of A^* to the domain $\mathscr{D}$. In general, $\mathscr{L}_0(\mathscr{D})$ is not an algebra, because for $A, B \in \mathscr{L}_0(\mathscr{D})$ $A \cdot B$ is not closable, in general. But it is easy to prove the following lemma:

LEMMA 2.1. *$\mathscr{L}_+(\mathscr{D})$ is an algebra. Equipped with the involution $A \to A^+$ $\mathscr{L}_+(\mathscr{D})$ becomes a $*$-algebra.*

We have also

LEMMA 2.2. (i) *For $\mathscr{D} = \mathscr{H}$, we have $\mathscr{L}_+(\mathscr{D}) = \mathscr{L}_0(\mathscr{D}) = \mathscr{B}(\mathscr{H})$ – the $*$-algebra of all bounded operators in $\mathscr{H}$.*

(ii) *If only one operator $A \in \mathscr{L}_+(\mathscr{D})$ is closed, then $\mathscr{D} = \mathscr{H}$ and consequently $\mathscr{L}_+(\mathscr{D}) = \mathscr{B}(\mathscr{H})$.*

Proof: (i) follows immediately from the closed graph theorem. We prove (ii): Let $\mathscr{D}$ be equipped with the scalar product $(\Phi, \Psi) = \langle \Phi, \Psi \rangle + \langle A\Phi, A\Psi \rangle$. Since A is assumed to be closed, we obtain a complete Hilbert space $\mathscr{H}_1 = \mathscr{D}$ ([3], chap. XII, 4). A becomes a continuous mapping from $\mathscr{H}_1$ into $\mathscr{H}$ and consequently $\langle \Phi, A\Psi \rangle$, $\Phi \in \mathscr{H}$ arbitrary, depends continuously on $\Psi \in \mathscr{H}_1$. Therefore, by the Riesz theorem there exists an element $\chi \in \mathscr{H}_1$ for which $\langle \Phi, A\Psi \rangle = (\chi, \Psi) = \langle \chi, \Psi \rangle + \langle A\chi, A\Psi \rangle = \langle \chi + A^+ A\chi, \Psi \rangle$ holds. Thus $\Phi \in \mathscr{D}(A^*)$, and because Φ was arbitrary in $\mathscr{H}$ we obtain $\mathscr{D}(A^*) = \mathscr{H}$. This implies

$$\mathscr{D}(A) = \mathscr{D} = \mathscr{H}.$$

The following interesting theorem will be important for the proof of Theorem 3.3:

THEOREM 2.1 *If there exists a norm $\|\cdot\|_1$ in $\mathscr{D}$ stronger than the Hilbert norm $\|\cdot\|$, with respect to which a symmetric operator $A = A^+ \in \mathscr{L}_+(\mathscr{D})$ is continuous, $\|A\Phi\|_1 \leqslant K\|\Phi\|_1$, then A is a bounded operator.*

Proof: For every $\Phi \in \mathscr{D}$,

$$\|A^\nu \Phi\| \leqslant c\,\|A^\nu \Phi\|_1 \leqslant cK^\nu \|\Phi\|_1, \qquad \nu = 0, 1, 2, \ldots,$$

i.e. every $\Phi \in \mathscr{D}$ is an analytic vector for A and consequently, A with the domain $\mathscr{D}$ is an essentially self-adjoint operator ([9], [11]). Thus, there exists a spectral decomposition

$$A\Phi = \int_{-\infty}^{+\infty} \lambda \, dE_\lambda \Phi, \qquad \Phi \in \mathscr{D}.$$

Now, we suppose A to be unbounded, say $F_\lambda = I - E_\lambda \neq O$ for all $\lambda < +\infty$. Then there are vectors $\Phi_\lambda \in \mathscr{D}$ such that $F_\lambda \Phi_\lambda \neq O$, and for $\nu = 0, 1, 2, \ldots$

$$\lambda^\nu \|F_\lambda \Phi_\lambda\| \leqslant \|A^\nu F_\lambda \Phi_\lambda\| \leqslant \|A^\nu \Phi_\lambda\| \leqslant cK^\nu \|\Phi_\lambda\|_1.$$

This is a contradiction to $\lambda > K$. Hence, A cannot be unbounded.

Remark. Without the assumption $A = A^+$ the last theorem does not hold (see also Example 4.2, Statemant 2).

Now we give the

DEFINITION 2.1. An *Op*-algebra is a subset $\mathscr{A} = \mathscr{A}(\mathscr{D})$ of $\mathscr{L}_0(\mathscr{D})$, which is an algebra with respect to the usual operations. A $*$-subalgebra $\mathscr{A}$ of $\mathscr{L}_+(\mathscr{D})$ is called an *Op*-algebra.*

$\mathscr{L}_+(\mathscr{D})$ is the maximal *Op**-algebra on $\mathscr{D}$, but in general there does not exist a maximal *Op*-algebra. Note that any *Op**-algebra is an *Op*-algebra.

3. Topologization of algebras of operators

Let $\mathscr{A} = \mathscr{A}(\mathscr{D})$ be an *Op*-algebra; then by $t_{\mathscr{A}}$ we denote the weakest locally convex topology in $\mathscr{D}$ with respect to which every operator $A \in \mathscr{A}$ is a continuous linear mapping of the locally convex space $\mathscr{D}[t_{\mathscr{A}}]$ into $\mathscr{D} \subset \mathscr{H}$ (with the Hilbert space topology).

Straightforwardly one proves the following lemma:

LEMMA 3.1. (i) *The topology $t_{\mathscr{A}}$ is defined by the seminorms*

$$\|\Phi\|_A = \|A\Phi\|, \qquad A \in \mathscr{A}, \ \Phi \in \mathscr{D},$$

and is stronger than the Hilbert space topology, because we assume $\mathscr{A}$ to contain the identity operator.

(ii) *Every operator $A \in \mathscr{A}$ is not only a continuous operator of $\mathscr{D}[t_{\mathscr{A}}]$ into $\mathscr{H}$, but also a continuous linear transformation of the LK-space $\mathscr{D}[t_{\mathscr{A}}]$ into itself.*

(iii) *If every operator $A \in \mathscr{A}$ is bounded, then $t_{\mathscr{A}}$ coincides with the Hilbert space topology defined by $\| \ \|$.*

(iv) *Let Σ be an algebraic linear basis in the Op-algebra $\mathscr{A}(\mathscr{D})$. Then the topology $t_{\mathscr{A}}$ is already defined by the seminorms $\| \ \|_A$ with $A \in \Sigma$.*

The locally convex space $\mathscr{D}[t_{\mathscr{A}}]$ is, in general, not complete. We denote its completion ([2], I, 5) by $\tilde{\mathscr{D}}[\tilde{t}_{\mathscr{A}}]$.

LEMMA 3.2 *Let $\mathscr{A}(\mathscr{D})$ be an Op-algebra, where $\mathscr{D}$ is dense in $\mathscr{H}$. The naturally given injection of $\mathscr{D}[t_{\mathscr{A}}]$ into $\mathscr{H}$ can be uniquely extended to a continuous injection of $\tilde{\mathscr{D}}[\tilde{t}_{\mathscr{A}}]$ into $\mathscr{H}$, and then $\tilde{\mathscr{D}}[\tilde{t}_{\mathscr{A}}] = \bigcap_{A \in \mathscr{A}} \mathscr{D}(\bar{A})$ where $\mathscr{D}(\bar{A})$ is the domain of the closure $\bar{A}$ of the operator A.*

Proof: $\mathscr{D}(\bar{A})$ is complete with respect to the norm $\|\Phi\|_A^1 = [\langle\Phi, \Phi\rangle + \langle\bar{A}\Phi, \bar{A}\Phi\rangle]^{\frac{1}{2}}$ ([3], chap. XII, 4) and $\mathscr{D}$ is dense in $\mathscr{D}(\bar{A})$ and therefore also in $\bigcap_{A \in \mathscr{A}} \mathscr{D}(\bar{A})$ equipped with the topology $\tilde{t}_{\mathscr{A}}$ defined by all seminorms $\|\Phi\|'_A$, $A \in \mathscr{A}$. Since $\bigcap_{A \in \mathscr{A}} \mathscr{D}(\bar{A})$ is complete with respect to $\tilde{t}_{\mathscr{A}}$ and the topology $\tilde{t}_{\mathscr{A}}$ induces $t_{\mathscr{A}}$ on $\mathscr{D}$, the statement indeed follows.

Now let us have a look at two examples: Let $\mathscr{H} = L_2(\boldsymbol{R}^1)$ be the Hilbert space of all square summable functions on $\boldsymbol{R}^1$, $\mathscr{D} = C_0^\infty(\boldsymbol{R}^1)$ the space of the infinitely differentiable funtions with compact supports and $\mathscr{A}_1$ the Op^*-algebra of all differential operators

$$A = \sum_{n, m \geqslant 0}^{\text{Finite}} a_{nm} x^n \left(\frac{d}{dx}\right)^m ;$$

then the completion of $\mathscr{D}[t_{\mathscr{A}_1}]$ is the Shwartz space $\mathscr{S} = \tilde{\mathscr{D}}[\tilde{t}_{\mathscr{A}_1}]$. If $\mathscr{A}_2$ is the Op^*-algebra of all differential operators

$$A = \sum_{n \geqslant 0} f_n(x) \left(\frac{d}{dx}\right)^n, \qquad \text{finite for any } x,$$

$f_n(x)$ an arbitrary infinitely differentiable function, then $\mathscr{D}[t_{\mathscr{A}_2}]$ is the Schwartz space $\mathscr{D}$.

Now we shall define different topologies in Op-algebras, in particular, in Op^*-algebras.

A system $\mathfrak{S}$ of bounded sets $\mathscr{M}$ of the locally convex space $\mathscr{D}[t_{\mathscr{A}}]$ is said to be *admissible* if (1) for $\mathscr{M} \in \mathfrak{S}$ and $A \in \mathscr{A}$, $A\mathscr{M} \in \mathfrak{S}$, (2) $\bigcup_{\mathscr{M} \in \mathfrak{S}} \mathscr{M}$ is dense in $\mathscr{H}$ and (3) for $\mathscr{M}, \mathscr{N} \in \mathfrak{S}$, $\mathscr{M} \cup \mathscr{N} \in \mathfrak{S}$.

DEFINITION 3.1. Let $\mathscr{A} = \mathscr{A}(\mathscr{D})$ be an Op-algebra and $\mathfrak{S}$ an admissible system of bounded sets of $\mathscr{D}[t_{\mathscr{A}}]$. Then we define in $\mathscr{A}$ the *topology* $\tau^{\mathfrak{S}}$ by the seminorms

$$\tau^{\mathfrak{S}}: \quad p_{B, \mathscr{M}}(A) = \sup_{\Phi \in \mathscr{M}} \|A\Phi\|_B, \quad \mathscr{M} \in \mathfrak{S}, \ B \in \mathscr{A},$$

and the *topology* $\tau_{\mathfrak{S}}$ by the seminorms

$$\tau_{\mathfrak{S}}: \quad \|A\|_{\mathscr{M}} = \sup_{\Phi, \Psi \in \mathscr{M}} |\langle\Phi, A\Psi\rangle|, \quad \mathscr{M} \in \mathfrak{S}.$$

If $\mathfrak{S}_{\max}$ is the system of all bounded sets of $\mathscr{D}[t_{\mathscr{A}}]$ we write

$$\tau^{\mathscr{D}} \equiv \tau^{\mathfrak{S}_{\max}} \quad \text{and} \quad \tau_{\mathscr{D}} \equiv \tau_{\mathfrak{S}_{\max}}.$$

$\mathscr{A}[\tau^{\mathfrak{S}}]$ and $\mathscr{A}[\tau_{\mathfrak{S}}]$ are locally convex spaces and the topologies satisfy the axiom of separation.

THEOREM 3.1. (i) *If $\mathscr{A}$ is an Op-algebra, then $\mathscr{A}[\tau^{\mathfrak{S}}]$ is a LK-algebra.*

(ii) *If $\mathscr{A}$ is an Op*-algebra, then $\mathscr{A}[\tau_{\mathfrak{S}}]$ is a LK*-algebra.*

Proof: The property (i) follows immediately from the both relations

$$p_{C,\mathscr{M}}(BA) = \sup_{\Phi \in \mathscr{M}} \|CBA\Phi\| = p_{CB,\mathscr{M}}(A),$$

$$p_{C,\mathscr{M}}(AB) = \sup_{\Phi \in \mathscr{M}} \|CAB\Phi\| = p_{C,B\mathscr{M}}(A).$$

(ii) is a consequence of the relations

$$\|BA\|_{\mathscr{M}} = \sup_{\Phi,\Psi \in \mathscr{M}} |\langle B^{+}\Phi, A\Psi\rangle| \leqslant \|A\|_{\mathscr{M} \cup B^{+}\mathscr{M}},$$

$$\|AB\|_{\mathscr{M}} = \sup_{\Phi,\Psi \in \mathscr{M}} |\langle \Phi, AB\Psi\rangle| \leqslant \|A\|_{\mathscr{M} \cup B\mathscr{M}},$$

$$\|A\|_{\mathscr{M}} = \|A^{+}\|_{\mathscr{M}}.$$

Remark. If $\mathscr{A}$ is an *Op*-algebra, then in general $\mathscr{A}[\tau_{\mathfrak{S}}]$ is not a *LK*-algebra, even not for the strongest $\tau_{\mathfrak{S}}$-topology $\tau_{\mathscr{D}}$ and for an *Op*-algebra the involution $A \to A^{+}$ is in general not continuous with respect to $\tau^{\mathfrak{S}}$ (see Examples 4.1 and 4.2).

THEOREM 3.2. (i) *If every operator A of an Op-algebra $\mathscr{A}$ is bounded, then $\tau^{\mathscr{D}} = \tau_{\mathscr{D}} = \tau_{\|\,\|}$, where $\tau_{\|\,\|}$ is the operator-norm topology.*

(ii) *$\tau^{\mathscr{D}}$ is stronger than $\tau_{\mathscr{D}}$. For an Op*-algebra $\tau^{\mathscr{D}} = \tau_{\mathscr{D}}$ if and only if the multiplication is continuous with respect to the topology $\tau_{\mathscr{D}}$.*

Proof: (i) is a consequence of Lemma 3.1 (iii). We prove (ii): Let $\mathscr{M}$ be an arbitrary bounded set of $\mathscr{D}[t_{\mathscr{A}}]$; then

$$\|A\|_{\mathscr{M}} = \sup_{\Phi,\Psi \in \mathscr{M}} |\langle \Phi, A\Psi\rangle| \leqslant \sup_{\Phi \in \mathscr{M}} \|\Phi\| \cdot p_{I,\mathscr{M}}(A).$$

Consequently, $\tau^{\mathscr{D}}$ is stronger than $\tau_{\mathscr{D}}$.

Let $\mathscr{A}$ be an *Op**-algebra with $\tau^{\mathscr{D}} = \tau_{\mathscr{D}}$. Then for every bounded set $\mathscr{M}$ there is a bounded set $\mathscr{N}$ with $p_{I,\mathscr{M}}(A) \leqslant \|A\|_{\mathscr{N}}$. Therefore we obtain

$$\|AB\|_{\mathscr{M}} = \sup_{\Phi,\Psi \in \mathscr{M}} |\langle \Phi, AB\Psi\rangle| = \sup_{\Phi,\Psi \in \mathscr{M}} |\langle A^{+}\Phi, B\Psi\rangle|$$

$$\leqslant p_{I,\mathscr{M}}(A^{+})\, p_{I,\mathscr{M}}(B) \leqslant \|A\|_{\mathscr{N}} \|B\|_{\mathscr{N}}$$

and the multiplication is continuous with respect to $\tau_{\mathscr{D}}$. *Vice versa,* if the multiplication is continuous with respect to $\tau_{\mathscr{D}}$, then for every bounded set $\mathscr{M} \subset \mathscr{D}[t_{\mathscr{A}}]$ there is a bounded set $\mathscr{N}$ such that

$$\|A^+B^+BA\|_{\mathscr{M}} \leqslant \|A^+\|_{\mathscr{N}}\|B^+\|_{\mathscr{N}}\|A\|_{\mathscr{N}}\|B\|_{\mathscr{N}} = \|A\|_{\mathscr{N}}^2\|B\|_{\mathscr{N}}^2$$

holds for $A, B \in \mathscr{A}$. Consequently,

$$\begin{aligned} p_{B,\mathscr{M}}(A)^2 &= \sup_{\Phi \in \mathscr{M}} |\langle BA\Phi, BA\Phi\rangle| \\ &\leqslant \sup_{\Phi, \Psi \in \mathscr{M}} |\langle \Psi, A^+B^+BA\Phi\rangle| = \|A^+B^+BA\|_{\mathscr{M}} \\ &\leqslant \|A\|_{\mathscr{N}}^2\|B\|_{\mathscr{N}}^2. \end{aligned}$$

Thus $p_{B,\mathscr{M}}(A) \leqslant \|B\|_{\mathscr{N}}\|A\|_{\mathscr{N}}$ and this implies $\tau^{\mathscr{D}} = \tau_{\mathscr{D}}$.

It is remarkable that for an Op^*-algebra and the topology $\tau_{\mathscr{D}}$ a certain conversion of Theorem 3.2 (i) can be proved.

THEOREM 3.3. *If there exists a norm* $\|\ \|_0$ *in an* Op^**-algebra* $\mathscr{A} = \mathscr{A}(\mathscr{D})$ *defining a stronger topology than* $\tau_{\mathscr{D}}$, *then every operator* $A \in \mathscr{A}$ *is bounded.*

We remark that $\|\ \|_0$ is not assumed to be a norm which makes A a normed*-algebra, i.e. we have not assumed the multiplication or the involution to be continuous with respect to $\|\ \|_0$.

Proof: Let $\mathfrak{N}$ be the system of all bounded sets of $\mathscr{D}[t_{\mathscr{A}}]$ with the property $\|A\|_{\mathscr{M}} \leqslant \|A\|_0$ for all $A \in \mathscr{A}$. Below we shall prove that

1. *there exists a maximal bounded set* $\mathscr{M}_0$ *in* $\mathfrak{N}$ and
2. $\mathscr{M}_0$ *is a closed, absolutely convex set and it absorbs every bounded set* $\mathscr{K}$ *of* $\mathscr{D}[t_{\mathscr{A}}]$.

Now we define in $\mathscr{D}$ the Minkowski functional for $\mathscr{M}_0$,

$$\|\Phi\|_1 = \inf\left\{r > 0 : \frac{1}{r}\,\Phi \in \mathscr{M}_0\right\}.$$

Since $\mathscr{M}_0$ is absorbing and bounded, $\|\ \|_1$ is a norm in $\mathscr{D}$ stronger than the Hilbert space norm $\|\ \|$. Let $A^+ = A$ be a symmetric operator of $\mathscr{A}$. Since $A\mathscr{M}_0$ is also a bounded set, there exists a $K > 0$ such that $A\mathscr{M}_0 \subset K\mathscr{M}_0$. This implies that $\|A\Phi\|_1 \leqslant K\|\Phi\|_1$ and hence A is bounded in consequence of Theorem 2.1. Since the symmetric operators are bounded, every operator A of $\mathscr{A}$ is bounded.

It remains to prove conditions 1 and 2. Let $\{\mathscr{N}_\gamma\}$ be a system of elements of $\mathfrak{N}$, ordered by the inclusion $\subset$. One proves straightforwardly that also $\mathscr{N} = \bigcup_\gamma \mathscr{N}_\gamma$ is an element of $\mathfrak{N}$. Then the existence of $\mathscr{M}_0$ follows from the Zorn's Lemma. Since $\mathscr{M}_0$ is maximal, it is closed and absolutely convex.

Now we shall show that $\mathcal{M}_0$ absorbs any bounded set $\mathcal{K}$ of $\mathcal{D}[t_{\mathcal{A}}]$. To prove that, we consider the set $\mathcal{M} \cup \mathcal{K}$ which is also bounded. Since the norm $\|\ \|_0$ is stronger than the topology $\tau_{\mathcal{D}}$, there is a positive $0<\mu\leqslant 1$ with $\|A\|_{\mathcal{M}_0 \cup \mathcal{K}} \leqslant \frac{1}{\mu} \|A\|_0$ for all $A \in \mathcal{A}$. We put $\mathcal{M}' = \mathcal{M}_0 \cup (\mu\mathcal{K})$ and take $\Phi, \Psi \in \mathcal{M}'$. Then there are the following three possibilities:

(a) $\Phi, \Psi \in \mathcal{M}'$,

(b) $\Phi \in \mathcal{M}_0$, $\Psi \in \mu\mathcal{K}$ or $\Psi \in \mathcal{M}_0$, $\Phi \in \mu\mathcal{K}$ and

(c) $\Phi, \Psi \in \mu\mathcal{K}$.

For each case one can prove $|\langle \Psi, A\Phi\rangle| \leqslant \|A\|_0$. For (a),

$$|\langle \Psi, A\Phi\rangle| \leqslant \|A\|_{\mathcal{M}_0} \leqslant \|A\|_0 .$$

For (b) one obtains

$$|\langle \Psi, A\Phi\rangle| = \mu \left|\left\langle \frac{1}{\mu} \Psi, A\Phi\right\rangle\right| \leqslant \mu \|A\|_{\mathcal{M}_0 \cup \mathcal{K}} \leqslant \|A\|_0,$$

and similiarly for (c). This implies $\|A\|_{\mathcal{M}_0 \cup \mu\mathcal{K}} \leqslant \|A\|_0$ for all $A \in \mathcal{A}$ and therefore $\mathcal{M}_0 \cup \mu\mathcal{K} \in \mathfrak{N}$. Since $\mathcal{M}_0$ is maximal in $\mathfrak{N}$, it follows that $\mu\mathcal{K} \subset \mathcal{M}_0$ and the proof of property 2 is completed.

There is none such theorem for *Op*-algebras and the topology $\tau^{\mathcal{D}}$ (see Example 4.2).

We conclude this section with a lemma characterizing the topology $\tau_{\mathcal{D}}$ without an explicit relation to the topology $t_{\mathcal{A}}$ of $\mathcal{D}$.

LEMMA 3.4. *Let $\mathcal{A}$ be an Op^*-algebra over $\mathcal{D}$. The topology $\tau_{\mathcal{D}}$ in $\mathcal{A}$ is defined by all seminorms*

$$\|A\|_{\mathcal{M}} = \sup_{\Phi, \Psi \in \mathcal{M}} |\langle \Phi, A\Psi\rangle|,$$

where $\mathcal{M}$ is an arbitrary subset of $\mathcal{D}$ such that $\|A\|_{\mathcal{M}}$ is finite for all $A \in \mathcal{A}$.

Proof: Let $\mathcal{M}$ be a set with this property. Then

$$\sup_{\Phi \in \mathcal{M}} \|A\Phi\| = \sup_{\Phi \in \mathcal{M}} |\langle \Phi, A^+A\Phi\rangle| \leqslant \|A^+A\|_{\mathcal{M}} < \infty$$

and consequently $\mathcal{M}$ is bounded in $\mathcal{D}[t_{\mathcal{A}}]$. Thus $\|A\|_{\mathcal{M}}$ is a seminorm for the topology $\tau_{\mathcal{D}}$.

4. $\tilde{O}^*$-algebras

In that what follows we study Op^*-algebras only. As we see from Theorems 3.2 and 3.3, the topology $\tau_{\mathcal{D}}$ may be a good generalization of the uniform topology (norm-topology) in an algebra of bounded operators. In this section we shall take into account some general properties of such topological algebras of unbounded operators more precisely. Let us begin with the following definition.

DEFINITION 4.1. An Op^*-algebra $\mathcal{A} = \mathcal{A}(\mathcal{D})$ equipped with the topology $\tau_{\mathcal{D}}$ is called an *$\tilde{O}^*$-algebra* and denoted by $\mathcal{A}[\tau_{\mathcal{D}}]$ The topology $\tau_{\mathcal{D}}$ is called the *uniform topology* of $\mathcal{A}$.

A complete $\tilde{O}^*$-algebra is an $\tilde{O}^*$-algebra which is complete as a locally convex space. The concept of a complete $\tilde{O}^*$-algebra is a generalization of the concept of a C^*-algebra. But in the case of unbounded operators the difference between uncomplete and complete $\tilde{O}^*$-algebras is much more essential than the difference between an uncomplete normed algebra of bounded operators and a complete normed algebra of bounded operators, i.e. a C^*-algebra. Whereas every normed $*$-algebra of bounded operators is a dense subalgebra of a C^*-algebra this does not hold in general for an $\tilde{O}^*$-algebra, as it will be shown in Example 4.1.

DEFINITION 4.2. A $*$-representation $a \to A(a)$ of a $*$-algebra $\mathscr{R}$ is a $*$-homomorphism ([12], IV) of $\mathscr{R}$ onto an Op^*-algebra $\mathscr{A} = \mathscr{A}(\mathscr{D})$.

A $*$-representation of a LK^*-algebra (locally convex $*$-algebra) is said to be *weakly continuous*, if $\langle \Phi, A(a)\Psi \rangle$ depends continuously on a for all $\Phi, \Psi \in \mathscr{D}$. The $*$-representation is said to be *uniformly continuous*, if $a \to A(a)$ is a continuous mapping of $\mathscr{R}$ onto the $\tilde{O}^*$-algebra $\mathscr{A}[\tau_{\mathscr{D}}]$.

It is well known that any $*$-representation of a Banach $*$-algebra with an identity element is uniformly continuous. We can now prove certain generalizations of this fact.

THEOREM 4.1. *Let $a \to A(a)$ be a $*$-representation of a LK^*-algebra $\mathscr{R}$. If for any $\Phi \in \mathscr{D}$, $F_\Phi(a) = \langle \Phi, A(a)\Phi \rangle$ is continuous in a, then $a \to A(a)$ is weakly continuous. If furthermore $\mathscr{R}$ is a barreled space* (tonnelé [2]), *then $a \to A(a)$ is also uniformly continuous.*

Proof: For symmetric elements $a = a^* \in \mathscr{R}$

$$4\langle \Phi, A(a)\Psi \rangle = F_{\Phi+\Psi}(a) - F_{\Phi-\Psi}(a) - iF_{\Phi+i\Psi}(a) + iF_{\Phi-i\Psi}(a).$$

From this decomposition and continuity of $a \to a^*$ we see that $\langle \Phi, A(a)\Psi \rangle$ continuously depends on all $a \in \mathscr{R}$. Let $\mathscr{M}$ be a bounded set of $\mathscr{D}[t_{\mathscr{A}}]$ and $\varepsilon > 0$. We define

$$\mathscr{U}_{\mathscr{M},\varepsilon} = \{a \in \mathscr{R} : \|A(a)\|_{\mathscr{M}} \leq \varepsilon\} = \bigcap_{\Phi, \Psi \in \mathscr{M}} \{a \in \mathscr{R} : |\langle \Phi, A(a)\Psi \rangle| \leq \varepsilon\}.$$

Since $\langle \Phi, A(a)\Psi \rangle$ is continuous in a, every set $\{a \in \mathscr{R} : |\langle \Phi, A(a)\Psi \rangle| \leq \varepsilon\}$ is absolutely convex and closed. Thus $\mathscr{U}_{\mathscr{M},\varepsilon}$ is absolutely convex and closed also. Furthermore, $\mathscr{U}_{\mathscr{M},\varepsilon}$ is an absorbing set. Thus for any $a \in \mathscr{R}$, $\varepsilon \|A(a)\|_{\mathscr{M}}^{-1} a \in \mathscr{U}_{\mathscr{M},\varepsilon}$, if not $\|A(a)\|_{\mathscr{M}} = 0$. If $\mathscr{R}$ is barreled, then $\mathscr{U}_{\mathscr{M},\varepsilon}$ is a neighbourhood and consequently $a \to A(a)$ is uniformly continuous.

THEOREM 4.2. *Every $*$-representation of a F^*-algebra $\mathscr{R}$ is uniformly continuous.*

A *F^*-algebra* is a complete LK^*-algebra the topology of which is defined by a denumerable system of seminorms $p_n(a)$ $(n = 1, 2, \ldots)$ satisfying $p_n(a, b) \leq p_n(a)p_n(b)$.

Proof: A F^*-algebra is a barreled space. For $\Phi \in \mathscr{D}$ $\langle \Phi, A(a)\Phi \rangle$ is a positive functional on $\mathscr{R}$ and in consequence of [13], Theorem 1, it is continuous. Thus we can apply the foregoing theorem.

The following definition generalizes the notion of a B^*-algebra.

DEFINITION 4.3. We say that a LK^*-algebra is an *$A\tilde{O}^*$-algebra* (*abstract $\tilde{O}^*$-algebra*) if it is algebraically and topologically $*$-isomorphic to an $\tilde{O}^*$-algebra.

Undoubtedly, it would be interesting to give an abstract characterization of an (complete) $A\tilde{O}^*$-algebra, similar to the property $\|xx^*\|=\|x\|^2$ of a B^*-algebra. This problem is unsolved. We must restrict ourselves to some non-trivial examples. In the next section we discuss the question with respect to what topology does the algebra of polynomials become an $A\tilde{O}^*$-algebra. In an othe paper ([6]) we show that a barreled LMC^*-algebra (locally multiplicative-convex algebra [8]) is an $A\tilde{O}^*$-algebra.

As a corollary from Theorem 3.3 we note here the following property of an $A\tilde{O}^*$-algebra.

THEOREM 4.3. *If the topology τ of an $A\tilde{O}^*$-algebra $\mathscr{R}$ is not a norm-topology, then there does not exist any norm which defines in $\mathscr{R}$ a topology stronger than τ.*

This result is rather queer, because a stronger norm can exist in a normed $A\tilde{O}^*$-algebra. For example, the algebra $C^1[0, 1]$ equipped with the norm $\|f\|=\sup_x|f(x)|$ is an $A\tilde{O}^*$-algebra (namely a subalgebra of the B^*-algebra $C[0, 1]$) and in $C^1[0, 1]$ there exists a stronger norm, e.g. $\|f\|_1=\|f\|+\left\|\frac{d}{dx}f\right\|$.

We conclude this section with two examples, already referred to in the context.

EXAMPLE 4.1. Let $\mathscr{H}$ be a separable Hilbert space and $\Phi_1, \Phi_2, \ldots$ an orthonormal basis in $\mathscr{H}$. By $\mathscr{D}$ we denote the set of all finite linear combinations of the basic vectors. Every $A\in\mathscr{L}_+(\mathscr{D})$ is uniquely determined by a matrix $A=(a_{\mu\nu})$ defined by

$$A\Phi_\mu=\sum_\nu a_{\mu\nu}\Phi_\nu.$$

For $A^+=(a^+_{\mu\nu})$,

$$a^+_{\mu\nu}=\bar{a}_{\nu\mu}.$$

Furthermore, $a_{\mu\nu}=0$ for $\mu\geqslant\mu_0(\nu)$ or $\nu\geqslant\nu_0(\mu)$. Hence $\mathscr{L}_+(\mathscr{D})$ is the set of all matrices containing in every row or column only a finite number of elements different from zero. Let γ_n, $n=1, 2, \ldots$, be an arbitrary sequence of positive numbers and $A_{(\gamma_n)}\in\mathscr{L}_+(\mathscr{D})$ the operator $a_{\mu\nu}=\delta_{\mu\nu}\gamma_\nu$; then for $\Phi=\sum_{n\geqslant 1}x_n\Phi_n\in\mathscr{D}$,

$$\|\Phi\|_{A(\gamma_n)}\equiv\|\Phi\|_{(\gamma_n)}=\Big(\sum_{n\geqslant 1}|x_n|^2\gamma_n^2\Big)^{\frac{1}{2}}.$$

This system of seminorms defines in $\mathscr{D}$ the strongest possible locally convex topology t and consequently $t_{\mathscr{L}_+}=t$.

Since every bounded set $\mathscr{M}$ of $\mathscr{D}[t]$ is contained in a finite-dimensional subspace of $\mathscr{D}$, we have

STATEMENT 1. *In the Op^*-algebra $\mathscr{L}_+(\mathscr{D})$ the uniform topology $\tau_{\mathscr{D}}$ coincides with the weak topology τ_s, defined by the seminorms $\langle\Phi, A\Psi\rangle$, $\Phi, \Psi\in\mathscr{D}$. $\mathscr{L}_+(\mathscr{D})\,[\tau_{\mathscr{D}}]$ is not com-*

plete and since $\mathscr{L}_+(\mathscr{D})$ is the maximal Op-algebra on $\mathscr{D}$, the $\tilde{O}$*-algebra $\mathscr{L}_+(\mathscr{D})\,[\tau_{\mathscr{D}}]$ cannot be extended to a complete $\tilde{O}$*-algebra.*

Now we prove the following

STATEMENT 2. *The involution $\mathscr{L}_+(\mathscr{D})$ is not continuous with respect to the topology $\tau^{\mathscr{D}}$.*

To prove this, we consider the sequence A_n, $n=1,2,\ldots$, of operators defined by $A_n\Phi_\nu=\Phi_{\nu-n}$, where $\Phi_{\nu-n}=0$ for $\nu-n\leqslant 0$. Then $A_n^+\Phi_\nu=\Phi_{\nu+n}$. Since every bounded $\mathscr{M}\subset\mathscr{D}[t]$ is contained in a $L[\Phi_1,\ldots,\Phi_N]$, we have $p_{B,\mathscr{M}}(A_n)=0$ provided $n>N$ (see Definition 3.1). Thus A_n converges to zero with respect to $\tau^{\mathscr{D}}$. But

$$p_{I,\mathscr{M}}(A_n^+)=\sup_{\Phi\in\mathscr{M}}\|A_n^+\Phi\|=\sup_{\Phi\in\mathscr{M}}\|\Phi\|\neq 0$$

and consequently A_n^+ does not converge to zero.

In the next example we study an *Op*-algebra which is not an *Op**-algebra.

EXAMPLE 4.2. Let $\mathscr{H}$ be a Hilbert space with the orthonormal basis $\{\Phi_1,\Phi_2,\ldots,\Psi_1,\Psi_2,\ldots\}$ and $\mathscr{D}$ the algebraic linear hull of the basis vectors as in the foregoing example. For $\mathscr{A}$ we take the *Op*-algebra generated by the operators A_n, $n=1,2,\ldots$, and B (and the identity operator I), defined by

$$A_n\Phi_m=\delta_{nm}\Phi_m,\qquad A_n\Psi_m=O,$$

$$B\Phi_m=m\Psi_m,\qquad B\Psi_m=O.$$

Since A_n and B are operators of $\mathscr{L}_+(\mathscr{D})$, the *Op*-algebra $\mathscr{A}$ is a subalgebra of $\mathscr{L}_+(\mathscr{D})$, but not a sub-$*$-algebra.

STATEMENT 1. *The linear mapping $A\to BA$ of $\mathscr{A}$ is not continuous with respect to the topology $\tau_{\mathscr{D}}$, i.e. $\mathscr{A}[\tau_{\mathscr{D}}]$ is not a LK-algebra.*

It is easy to see that the topology $t_{\mathscr{A}}$ is defined by the single norm

$$\|\Phi\|_1=(\|\Phi\|^2+\|B\Phi\|^2)^{\frac{1}{2}},\qquad \Phi\in\mathscr{D}.$$

Consequently, the topology $\tau_{\mathscr{D}}$ of $\mathscr{A}$ can be defined by the single norm

$$\|A\|_0=\sup_{\Phi,\Psi\in\mathscr{K}}|\langle\Phi,A\Psi\rangle|,\qquad A\in\mathscr{A},$$

where $\mathscr{K}=\{\Phi\in\mathscr{D}\colon\ \|\Phi\|_1\leqslant 1\}$ is the unit sphere with respect to the norm $\|\ \|_1$. $\mathscr{K}$ is exactly the set of all

$$\Phi=\sum_{n\geqslant 1}(x_n\Phi_n+y_n\Psi_n)\in\mathscr{D}\qquad\text{with}\qquad\sum_{n\geqslant 1}(|x_n|^2(n^2+1)+|y_n|^2)\leqslant 1.$$

Let now

$$\Phi=\sum_n(x_n\Phi_n+y_n\Psi_n)\qquad\text{and}\qquad\Psi=\sum_n(x_n'\Phi_n+y_n'\Psi_n)$$

be two elements of $\mathscr{K}$. Then

$$|\langle\Phi, A_n\Psi\rangle|=|x_n x_n'|\leq(n^2+1)^{-1},$$

i.e. $\|A_n\|_0\leq(n^2+1)^{-1}$. Consequently, A_n converges to zero for $n\to\infty$. On the other hand, for $\chi_n=(n^2+1)^{-\frac{1}{2}}\Phi_n\in\mathscr{K}$ and Ψ_n we get

$$\|BA_n\|_0\geq|\langle\Psi_n, BA_n\chi_n\rangle|=\frac{n}{\sqrt{n^2+1}}\geq\frac{1}{2}$$

and thus BA_n does not converge to zero for $n\to\infty$.

From the foregoing considerations we get the following

STATEMENT 2. *The unbounded operator $B\in\mathscr{L}_+(\mathscr{D})$ is continuous with respect to the norm $\|\ \|_1$. The topology $\tau_{\mathscr{D}}$ is given by a norm, although the algebra contains unbounded operators.*

This statement shows that the assumption of Theorem 2.1 of A being a symmetric operator and the assumption of Theorem 3.3 of $\mathscr{A}(\mathscr{D})$ being an Op^*-algebra are essential.

5. Algebra of polynomials

In this section we search for a topology τ with respect to which the $*$-algebra $\mathscr{P}$ of all polynomials $p=p(x)=\sum_{n\geq 0}\alpha_n x^n$, $p^*(x)=\overline{p(x)}$, becomes an $A\tilde{O}^*$-algebra $\mathscr{P}[\tau]$. The problem can be reformulated in the following way. Let $\mathscr{D}$ (dense in $\mathscr{H}$) be a unitary space and $T=T^+$ a symmetric operator of $\mathscr{L}_+(\mathscr{D})$. T is called *infinite*, if the $*$-homomorphism

$$\mathscr{P}\ni p\to p(T)=\sum_{n\geq 0}\alpha_n T^n\in\mathscr{L}_+(\mathscr{D})$$

is a $*$-isomorphism. Let $\mathscr{P}(T)$ be the $*$-algebra of all operators $p(T)$. If T is infinite, then the uniform topology $\tau_{\mathscr{D}}$ of $\mathscr{P}(T)$ induces in $\mathscr{P}$ a locally convex topology which we denote by τ_T. The problem consists in determining all topologies τ_T. If T is a bounded operator, then it is infinite if and only if the spectrum $\sigma=\sigma(\bar{T})$, $\bar{T}$ being the unique extension of T to $\mathscr{H}$, contains an infinite number of points. The topology τ_T in $\mathscr{P}$ is then the norm-topology

$$\tau_T:\ \|p\|_T=\sup_{x\in\sigma}|p(x)|. \tag{1}$$

Hence the LK^*-algebras $\mathscr{P}[\tau_T]$ with the topologies (1) are $A\tilde{O}^*$-algebras. They are not complete. Now the question arises what topologies τ_T does one obtain for unbounded T. We conjecture that for any unbounded T the topology τ_T is equal to the topology τ_∞, defined by all seminorms

$$\tau_\infty:\ \|p\|_{(\gamma_n)}=\sum_{n\geq 0}\gamma_n|\alpha_n|,\quad (\gamma_n)\in\Gamma_\infty, \tag{2}$$

where Γ_∞ is the system of all positive sequences and $p=\sum_{n\geq 0}\alpha_n x^n$. We prove $\tau_T=\tau_\infty$ for a large class of operators.

THEOREM 5.1. *Let $T=T^+ \in \mathscr{L}_+(\mathscr{D})$ be an operator for which there exists a sequence of $\Phi_n \in \mathscr{D}$, $n=1,2,\dots$, a monotonic sequence $\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \dots$ of positive numbers and a sequence r_n, $n=1,2,\dots$, of positive numbers with $3 \leqslant r_n + 2 \leqslant r_{n+1}$ such that*

(a) $\langle T^k \Phi_n, T^e \Phi_m \rangle = O$ *for* $n \neq m$, $k, l = 0, 1, \dots,$

(b) $\lambda_m (r_n - 1)^m \leqslant \|T^m \Phi_n\| \leqslant M \lambda_m (r_n)^m$, $m=0,1,2,\dots;\ n=1,2,\dots;\ M \geqslant 1.$

Then $\tau_T = \tau_\infty$.

Before we prove this theorem we show that T satisfies the assumptions of the theorem in two important cases.

EXAMPLE 5.1. *If the operator $T=T^+ \in \mathscr{L}_+(\mathscr{D})$ has a spectral decomposition*

$$T\Phi = \int_{-\infty}^{+\infty} \lambda \, dE(\lambda) \Phi, \qquad \Phi \in \mathscr{D},$$

such that $\mathscr{D}$ is invariant for $E(x_1, x_2) = E(x_2) - E(x_1)$, $\infty < x_1 < x_2 < +\infty$, then T satisfies assumptions (a) *and* (b) *of the theorem.*

Proof: Since T is unbounded, $E(\lambda) \neq I$ for all $\lambda < \infty$ and consequently there exists a sequence $r_1, r_2, \dots, 3 \leqslant r_n + 2 \leqslant r_{n+1}$ such that $E_n = E(r_n) - E(r_n - 1) \neq 0$ and hence $E_n \mathscr{D} \neq 0$. Let $\Phi_1, \Phi_2, \dots$ be a sequence of vectors with $\Phi_n \in E_n \mathscr{D}$, $\|\Phi_n\| = 1$, then assumptions (a) and (b) are satisfied with $\lambda_0 = \lambda_1 = \dots = 1$.

EXAMPLE 5.2. *Let Ω be a subset of the Euclidean space R^s, $\mathscr{H} = L_2(\Omega)$ and $\mathscr{D} = C_0^\infty(\Omega)$ the space of all functions with compact support in Ω, which have derivations of any order. If*

$$T = \sum_{i_1, \dots, i_s} a_{i_1, \dots, i_s} \frac{\partial^{i_1 + \dots + i_s}}{\partial x_1^{i_1} \dots \partial x_s^{i_s}}$$

is a symmetric differential operator with constant coefficients, then assumptions (a) *and* (b) *of the theorem are satisfied.*

Proof: T can be written in the form $T = T_p + S$, where T_p is a homogeneous differential operator of the degree $p \geqslant 1$ and S is a differential operator of order less than p. Let $f(x)$ be a function of $C^\infty(R^s)$ with the support in the unit ball around the origin of R^s. We put

$$\nu_m = \|T_p^m f\| \quad \text{and} \quad \mu_m = \sup_{k=0,\dots,m-1} m \binom{m}{k} \|T_p^k S^{m-k} f\|.$$

The function f can be chosen in such a way that $\nu_m \geqslant 2\mu_m$, $\nu_{m+1} \geqslant \nu_m$. Now, let $K_1, K_2, \dots$ be a sequence of mutually disjoint balls contained in Ω with centres in t_n and the radii $\rho_n \leqslant 1$, $n=1,2,\dots$ We define

$$\Phi_n(x) = \rho_n^{-s/2} f\left(\frac{x - t_n}{\rho_n}\right). \tag{3}$$

$\Phi_n = \Phi_n(x)$ are functions with supports in K_n. Since the balls are mutually disjoint, asumption (a) is satisfied for the Φ_n, $n=1, 2, \ldots$ Furthermore

$$\|T_p^m \Phi_n\| - \|\sum_{k=0}^{m-1} \binom{m}{k} T^k S^{m-k} \Phi_n\| \leqslant \|T^m \Phi_n\| \leqslant \|T_p^m \Phi_n\| + \|\sum_{k=0}^{m-1} \binom{m}{k} T^k S^{m-k} \Phi_n\| . \tag{4}$$

Since

$$\|T_p^m \Phi_n\| = (\rho_n^{-p})^m \|T_p^m f\| = r_n^m v_m , \qquad r_n = \rho_n^{-p} ,$$

and

$$\|\sum_{k=0}^{m-1} \binom{m}{k} T^k S^{m-k} \Phi_n\| \leqslant r_n^m \mu_m \leqslant r_n^m \frac{v_m}{2} ,$$

we obtain from (4) the estimation

$$\tfrac{1}{2} v_m (r_n)^m \leqslant \|T^m \Phi_n\| \leqslant \tfrac{3}{2} v_m (r_n)^m . \tag{5}$$

Consequently, also assumption (b) is satisfied with $\lambda_m = \frac{1}{2} v_m$, $M=3$, since the ρ_n can be chosen so that the r_n have the required properties.

Proof of Theorem 5.1: Let (s_n) be an arbitrary strongly monotonic increasing sequence of naturals, $s_0 < s_1 < s_2 < \ldots$ We put

$$\begin{aligned} y_i &= \lambda_{2i}^{-1} (r_{s_i})^{-i-\alpha_i} , \\ x_i &= \lambda_i^{-1} (r_{s_i})^{-\alpha_i} , \qquad \alpha_i = (2i-1)/4 , \; i = 0, 1, \ldots , \end{aligned} \tag{6}$$

and construct the elements

$$\begin{aligned} \Psi_j &= \sum_{i=0}^{j} y_i T^i \Phi_{s_i} , \\ \Phi_j(\varepsilon) &= \sum_{i=0}^{j} \varepsilon_i x_i \Phi_{s_i} , \qquad j = 0, 1, 2, \ldots , \end{aligned} \tag{7}$$

where $(\varepsilon) = (\varepsilon_0, \varepsilon_1, \ldots)$ is an arbitrary sequence of complex numbers with $|\varepsilon_i| = 1$.

STATEMENT 1. *The set $\mathcal{M}_{(s_n)}$ of all Ψ_j, $\Phi_j(\varepsilon)$ is a bounded set of $\mathcal{D}[t_{\mathcal{P}(T)}]$.*

To prove Statement 1, we have to show that for each n $\|T^n \Phi\|$ is bounded for all $\Phi \in \mathcal{M}_{(s_n)}$. For $n = 0, 1, 2, \ldots$

$$\|T^n \Psi_j\| \leqslant \sum_{i=0}^{j} y_i \|T^{n+i} \Phi_{s_i}\| \leqslant M \sum_{i=0}^{\infty} \frac{\lambda_{n+i}}{\lambda_{2i}} (r_{s_i})^{n-\alpha_i} = \sigma_1(n) < \infty .$$

These series converge, because $\lambda_{n+i} \leqslant \lambda_{2i}$, $r_{s_i} \geqslant s_i \geqslant i$ for $i \geqslant n$ and $\alpha_i \to \infty$. Consequently, $\sup_j \|T^n \Psi_j\| \leqslant \sigma_1(n)$. For the $\Phi_j(\varepsilon)$ we obtain

$$\|T^n \Phi_j(\varepsilon)\| \leqslant M \sum_{i=0}^{\infty} \frac{\lambda_n}{\lambda_i} (r_{s_i})^{n-\alpha_i}$$

and these series also converge.

Next we prove

STATEMENT 2. *The numbers* (6) x_n, y_n *(depending on* s_n*) are chosen in such a way that*

$$\lim_{s_n\to\infty} x_n y_n \|T^n\Phi_{s_n}\|^2 = +\infty, \tag{8}$$
$$\lim_{s_n\to\infty} x_n y_n \|T^n\Phi_{s_n}\| \, \|T^m\Phi_{s_n}\| = 0 \quad \textit{for} \quad m \leqslant n-1.$$

Applying (b) we obtain

$$x_n y_n \|T^n\Phi_{s_n}\|^2 \geqslant \frac{\lambda_n}{\lambda_{2n}} \frac{(r_{s_n}-1)^{2n}}{(r_{s_n})^{n+2\alpha_n}} \geqslant \frac{2^{-2n}\lambda_n}{\lambda_{2n}} (r_{s_n})^{n-2\alpha_n} \geqslant \frac{2^{-2n}\lambda_n}{\lambda_{2n}} (r_{s_n})^{\frac{1}{2}}$$

and for $i \leqslant n-1$

$$x_n y_n \|T^n\Phi_{s_n}\| \, \|T^i\Phi_{s_n}\| \leqslant M^2 (r_{s_n})^{n-2\alpha_n-1} = M^2 (r_{s_n})^{-\frac{1}{2}}.$$

The statement follows from both these estimations since $r_{s_n} \to \infty$ for $s_n \to \infty$.

After this introduction we can now prove the

STATEMENT 3. *For every sequence of positive numbers* (γ_ν), $\nu = 0, 1, 2, \ldots,$ *there exists a sequence* (s_n), $n = 0, 1, 2, \ldots,$ *of natural numbers such that*

$$\|p\|_{(\gamma_n)} = \sum_{\nu\geqslant 0} |\alpha_\nu| \gamma_\nu \leqslant \|p(T)\|_{\mathcal{M}(s_n)}, \tag{9}$$

where $p(T) = \sum_{\nu\geqslant 0} \alpha_\nu T^\nu$. *Consequently, the topology* τ_T *is stronger than* τ_∞.

We construct the sequence (s_n) in the following way. Suppose we have already chosen s_i for $i \leqslant n-1$; we take s_n big enough so that firstly

$$x_n y_n \|T^n\Phi_{s_n}\|^2 \geqslant 1 + \gamma_n + \sum_{i\leqslant n-1} x_i y_i \|T^i\Phi_{s_i}\| \, \|T^n\Phi_{s_i}\| \tag{10}$$

and secondly

$$x_n y_n \|T^n\Phi_{s_n}\| \, \|T^m\Phi_{s_n}\| \leqslant \frac{1}{2^n} \tag{11}$$

for $m \leqslant n-1$. This is possible in consequence of (8).

Let $\mathcal{M}_{(s_i)}$ be the bounded set of vectors (7) constructed above. Then

$$\|p(T)\|_{\mathcal{M}(s_n)} \geqslant \left|\langle \Psi_j, \sum_{n\geqslant 0} \alpha_n T^n \Phi_j(\varepsilon)\rangle\right| \geqslant \left|\sum_{n\geqslant 0} \alpha_n \sum_{i=0}^{j} \varepsilon_i x_i y_i \langle T^i\Phi_{s_i}, T^n\Phi_{s_i}\rangle\right|, \tag{12}$$

where we have applied the property (a). Now, for $p(T) = \sum_{n=0}^{N} \alpha_n T^n$, N being the degree of p, we take $j = N$, $\varepsilon_i = \frac{\overline{\alpha_i}}{|\alpha_i|}$ $(=1$ for $\alpha_i = 0)$. Then we get from (12)

$$\begin{aligned} \|p(T)\|_{\mathcal{M}(s_n)} \geqslant \sum_{n\geqslant 0} |\alpha_n| x_n y_n \|T^n\Phi_{s_n}\|^2 - \\ - \sum_{n\geqslant 0} |\alpha_n| \sum_{i\leqslant n-1} x_i y_i \|T^i\Phi_{s_i}\| \, \|T^n\Phi_{s_i}\| - \\ - \sum_{n\geqslant 0} |\alpha_n| \sum_{i\geqslant n+1} x_i y_i \|T^i\Phi_{s_i}\| \, \|T^n\Phi_{s_i}\|. \end{aligned} \tag{13}$$

From (10) and (11) we obtain finally

$$\begin{aligned}||p(T)||_{\mathscr{M}_{(s_n)}} &\geqslant \sum_{n\geqslant 0}|\alpha_n|\left(\gamma_n+1-\sum_{i\geqslant n+1}\frac{1}{2^i}\right)\\ &\geqslant \sum_{n\geqslant 0}|\alpha_n|\,\gamma_n=||p||_{(\gamma_n)}. \end{aligned}\tag{14}$$

The last inequality proves Statement 3. Since τ_∞ is the strongest possible locally convex topology in $\mathscr{P}$, Statement 3 implies $\tau_T=\tau_\infty$ and the theorem is completely proved.

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