Here we discuss correctability and privacy for quantum channels, now taken to generality of Op^*-algebras as defined in the preliminary section. We are interested in two kinds of each notion; the exact case and the approximate case. The former is merely an algebraic notion, and is little different than in the bounded case, save for the generality of the objects. As discussed in the previous section however, we have a variety of generalizations of the operator norm which may or may not agree in the unbounded case, which leaves how best to think of correctability potentially more ambiguous, certainly compared to the setting which we are trying to generalize.

It's worth noting still however that even in the bounded case there remains some ambiguity with what the best notion of correctability and privacy are. First, let's state the definitions we are aiming to extend to unbounded algebras.

Definition. *Let $\mathcal{E} : \mathscr{A} \to \mathscr{B}$ be a quantum channel of C^*-algebras. A C^*-subalgebra $\mathscr{N} \subseteq \mathscr{B}$ is said to be **private** for $\mathcal{E}$ if*

$$\mathcal{E}(\mathscr{A}) \subseteq \mathscr{N}'.$$

Similarly, we saw that $\mathscr{N}$ is ε-private for $\mathcal{E}$ if there exists a quantum channel $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, private for $\mathscr{N}$, such that

$$\|\mathcal{E} - \mathcal{F}\|_{cb} < \varepsilon.$$

Definition. *Let $\mathcal{E} : \mathscr{A} \to \mathscr{B}$ be a quantum channels of C^*-algebras. a C^*-subalgebra $\mathscr{N} \subseteq \mathscr{B}$ is **correctable** for $\mathcal{E}$ if there exists a quantum channel $\mathcal{R} : \mathscr{N} \to \mathscr{A}$ such that*

$$\mathcal{E}\mathcal{R} = \mathcal{I}_{\mathscr{N}}.$$

$\mathscr{N}$ is ε-correctable for $\mathcal{E}$ if instead we have

$$\|\mathcal{E}\mathcal{R} - \mathcal{I}_{\mathscr{N}}\|_{cb} < \varepsilon.$$

Though we cannot write down the completely bounded norm for an unbounded channel, we can topologize the channel in a manner which is both more general and captures the approximation up to "arbitrary amplification" in a manner similar to the completely bounded norm.

However, completely bounded convergence or pertubation is arguably too strong of a requirement to describe certain physical pertubations. It is then perhaps desirable to develop a correspondence between privacy and correctability in a weaker setting.

We also note that in the above definitions are often restricted to the case where $\mathscr{A}$ is a von Neumann algebra, but the definition makes sense at this level of generality and is immediately analogous to our Op^*-algebra setting.

0.1 $\mathscr{A}$-amplifications

We aim to work with correctability and privacy in the sense of one of a few possible locally convex topologies. It makes sense then to describe these in terms of the seminorms of whatever topology we choose. What isn't immediately clear is how to capture the 'amplified' nature of the completely bounded topology, and so first a few things are in order.

Definition. *Let $\mathscr{A}$ be an Op^*-algebra. We denote by $\mathscr{A}_n = \mathscr{A} \otimes M_n$ the space of n by n $\mathscr{A}$-matrices, which we may verbally refer to as the $\mathscr{A}$-amplifications of degree n.*

$\mathscr{A}_n$ is an Op^*-algebra when acting on the direct sum of n copies of $\mathscr{D}$, which we write as $\mathscr{D}_n$. In this setting we have stability of the completeness of the domain.

Proposition. *Let $\mathscr{A}$ be an Op^*-algebra on a pre Hilbert space $\mathscr{D}$ such that $\mathscr{D}$ is complete with respect to $\mathcal{T}_{\mathscr{A}}$, then $\mathscr{D}_n$ is complete with respect to $\mathcal{T}_{\mathscr{A}_n}$.*

Proof. Let $\{f_\alpha\}_\alpha$ be Cauchy with respect to $\mathcal{T}_{\mathscr{A}_n}$. Take the $\mathscr{A}$-amplification consisting of an operator $A \in \mathscr{A}$ in the kth diagonal entry. We have, by our assumption, that $\|A(f_\alpha(k) - f_\beta(k))\| = \|f_\alpha(k) - f_\beta(k)\|_A$ converges to 0. Whence we extract a limit $f(k) = \lim_\alpha f_\alpha(k)$ by the completeness of $\mathscr{D}$. The seminorm $\|f_\alpha - f\|_X$, where X is any $\mathscr{A}$-amplification, is simply a finite sum of seminorms on $\mathscr{D}$ applied to $f_\alpha(k) - f(k)$, which converges to 0. $\square$

There is of course a natural embedding of $\mathscr{A}$ into $\mathscr{A}_n$ via the map $A \to A \otimes 1_n$. It would be convenient if any of the locally convex topologies one can put on an Op^*-algebra would be stable under taking the subspace topology with respect to this embedding. Thus we will assume that this embedding is a homeomorphism with respect to the topology taken for our algebras.

0.2 Topologizing the Channels

In the previous section we discussed a variety of generalizations of the operator norm on the operator algebras themselves. We must now discuss how to generalize the completely bounded norm given a locally convex topology on the algebras.

Let $(\mathscr{A}_n, \{q_\alpha\})$ and $(\mathscr{B}_n, \{q_\beta\})$ be locally convex amplification algebras, let $\mathcal{E} : \mathscr{A} \to \mathscr{B}$ be a quantum channel continuous relative to both topologies. Let $\|\cdot\|_{S,\beta}$ be the family of seminorms defined via

$$\|\mathcal{E}\|_{S_n,\beta} = \sup_{X \in S_n} (q_\beta(\mathcal{E}_n(X))),$$

where S_n is a bounded subset of $(\mathscr{A}_n, \{q_\alpha\})$. This gives a topology on the channels relative to whatever locally convex topology for $\mathscr{A}$ we choose.