Things to do:

- $\rho : \varphi(\mathcal{M})'_c \longrightarrow \pi(\mathcal{M})'_c$ is a $*$-rep and ultraweakly cts
- If $\varphi : \mathcal{M} \longrightarrow \mathcal{L}^+(\mathcal{D})$ is ultraweakly cts, is $\pi : \mathcal{M} \longrightarrow \mathcal{L}^+(\mathcal{D}_1)$ ultraweakly cts?

(Extra: comparing $\sigma(\mathcal{L}^+(\mathcal{D}), T(\mathcal{D}))$ and ultraweak topology).

First go through what Jason wrote:

Commutants:

Let $(\mathcal{M}, \mathcal{D})$ be a closed Op^*-algebra on $\mathcal{D} \leq \mathcal{H}$. The weak commutant $\mathcal{M}'_w$ is

$$\mathcal{M}_w' = \{ C \in \mathcal{B}(\mathcal{H}) : \langle CA(x), y \rangle = \langle C(x), A^*(y) \rangle \quad \forall A \in \mathcal{M}, \ x, y \in \mathcal{D} \}$$

$\mathcal{M}_w'$ is a $*$-invariant weakly-closed unital subspace of $\mathcal{B}(\mathcal{H})$.

For the following definitions, $\mathcal{M}$ need only be a $*$-invariant subset $\mathcal{C}(\mathcal{D}, \mathcal{H})$

$\hookrightarrow$ linear $\mathcal{D} \longrightarrow \mathcal{H}$ such that $\mathcal{D}(X) \cap \mathcal{D}(X^*) \supseteq \mathcal{D}$ (not necessarily leaving $\mathcal{D}$ invariant)

$$\mathcal{M}_\sigma' := \{ S \in \mathcal{C}(\mathcal{D}, \mathcal{H}) : \langle Ax, Sy \rangle = \langle S^* x, A^* y \rangle, \ \forall A \in \mathcal{M}, \ x, y \in \mathcal{D} \}$$

$$\mathcal{M}_c' := \mathcal{M}_\sigma' \cap \mathcal{L}^+(\mathcal{D})$$

$$\mathcal{M}_{w\sigma}'' := \{ A \in \mathcal{C}(\mathcal{D}, \mathcal{H}) : \langle CAx, y \rangle = \langle Cx, A^* y \rangle, \ \forall C \in \mathcal{M}_w', \ x, y \in \mathcal{D} \}$$

$$\mathcal{M}''_{\sigma\sigma} = \{ A \in C(\mathcal{D}, \mathcal{H}) \mid \langle Sx, Ay \rangle = \langle A^* x, S^* y \rangle \quad \forall S \in \mathcal{M}'_\sigma, \ x, y \in \mathcal{D} \}$$

$$\mathcal{M}''_{cc} = \{ A \in \mathcal{L}^+(\mathcal{D}) : AS = SA \quad \forall S \in \mathcal{M}'_c \}$$

Lemma:

(1) $\mathcal{M}'_\sigma$ strongly $*$-closed subspace

(2) $\mathcal{M}'_c$ is an Op^*-algebra on $\mathcal{D}$

(3) $\mathcal{M}''_{w\sigma}$ is a strongly $*$-closed $*$-invariant subspace containing $\mathcal{M} \cup \mathcal{M}''_{ww}$

(4) $\mathcal{M}''_{\sigma\sigma}$ is a strongly $*$-closed $*$-invariant subspace containing $\mathcal{M}$.

(5) If $\mathcal{M}\mathcal{D} = \mathcal{D}$, $\mathcal{M}''_{cc}$ is an Op^*-algebra on $\mathcal{D}$ containing $\mathcal{M}$.

If $\mathcal{M}'_w$ is an algebra, then

$$\overline{\mathcal{M}}^{\tau_s^*} \subseteq \mathcal{M}''_{\sigma\sigma} \subseteq \mathcal{M}''_{w\sigma} = \overline{\mathcal{M}''_{ww}}^{\tau_s^*}$$

$$\mathcal{M}''_{cc} \subseteq \mathcal{M}''_{wc} = \overline{\mathcal{M}''_{ww}}^{\tau_s^*} \cap \mathcal{L}^+(\mathcal{D})$$

Analogue of commutant lifting:

Lemma: Let $A \geq 0$ be self-adjoint on $\mathcal{H}$. Then $\mathcal{D}(\sqrt{A}) \supseteq \mathcal{D}(A)$ and $A = \sqrt{A} \cdot \sqrt{A}$

Proof:

- For any Borel function $f : \mathbb{R} \to \mathbb{C}$, $\mathcal{D}(f(A)) = \{ \psi \in \mathcal{H} : f \in L^2(\mathbb{R}, d\mu_\psi) \}$

Suppose $\varphi : (\mathcal{M}, \mathcal{D}_0) \longrightarrow (\mathcal{A}, \mathcal{D})$ is UCP with Stinespring triple $(V, \pi, \mathcal{D}_1)$:

- $\mathcal{D}_1 \subseteq \mathcal{H}_1$, dense subspace
- $\pi : \mathcal{M} \longrightarrow \mathcal{L}^+(\mathcal{D}_1)$, $*$-rep $(\mathcal{B} = \pi(\mathcal{M}))$
- $V \in \mathcal{L}(\mathcal{D}[\tau_{\mathcal{A}}], \mathcal{D}_1[\tau_{\mathcal{B}}])$
- $\varphi(x) = V^* \pi(x) V$

Goal: Can we define a $*$-homomorphism $\rho : \varphi(\mathcal{M})_c' \longrightarrow \pi(\mathcal{M})_c'$ as in Takesaki? If $y \in \varphi(\mathcal{M})_c'$, define $\rho(y)$ on $\mathcal{M} \otimes \mathcal{D}$ by $\rho(y)(\sum_i x_i \otimes \psi_i) = \sum x_i \otimes (y(\psi_i))$. If $\xi \in \mathcal{M} \otimes \mathcal{D}$ satisfies $\langle \xi \xi \rangle_\varphi = 0$ (Stinespring inner product $\langle a \otimes \psi, b \otimes \xi \rangle_\varphi = \langle \varphi(b^* a)\psi, \xi \rangle_{\mathcal{H}}$).

- Let then $\xi = \sum_i x_i \otimes \psi_i$ with $x_i \in \mathcal{M}$, $\psi_i \in \mathcal{D}$, and $0 = \langle \xi, \xi \rangle_\varphi$
$= \sum \langle \psi_i, \varphi(x_i^* x_j) \psi_j \rangle = \langle (\psi_1, \dots, \psi_n), [\varphi(x_i^* x_j)] (\psi_1, \dots, \psi_n) \rangle_{\mathcal{H}^n} \geq 0$
in $M_n(\mathcal{A})$, symmetric.

- Let A be the Friedrichs extension of $[\varphi(x_i^* x_j)]$ (non-negative self-adjoint operator on $\mathbb{C}^n \otimes \mathcal{H}$). By the lemma, $\mathcal{D}(\sqrt{A}) \supseteq \mathcal{D}(A)$, $\sqrt{A} \cdot \sqrt{A} = A$.
Also, $\mathcal{D}(A) \supseteq \mathcal{D}([\varphi(x_i^* x_j)]) \supseteq \mathcal{D}^n$, $0 = \langle [\psi_j], \sqrt{A} \cdot \sqrt{A} [\psi_j] \rangle$
$= \| \sqrt{A} \cdot [\psi_j] \|^2$ implies $\sqrt{A} \cdot [\psi_j] = 0$, $[\varphi(x_i^* x_j)] \cdot [\psi_j] = A [\psi_j]$
$= \sqrt{A} \cdot \sqrt{A} \cdot [\psi_j] = 0$.

- Given $y \in \varphi(\mathcal{M})'_c$, then $\| p(y) \xi \|_\varphi^2 = \langle p(y) \xi, p(y) \xi \rangle$
$= \sum \langle y \psi_i, \varphi(x_i^* x_j) y \psi_j \rangle = \langle (1 \otimes y)[\psi_j], [\varphi(x_i^* x_j)] (1 \otimes y) [\psi_j] \rangle$

$\in \mathcal{D}^n$ as $1 \otimes \varphi(\mathcal{M})'_c$ is an Op*-algebra on $\mathbb{C}^n \otimes \mathcal{D}$.

<u>Check:</u> Op*-algebra, let $\alpha \otimes \psi \in \mathbb{C}^n \otimes \mathcal{D}$, then

$1 \otimes y(\alpha \otimes \psi) = \alpha \otimes \underbrace{y\psi}_{\in \mathcal{D}}$. $(1 \otimes y)^* = 1 \otimes y^*$ which has $\mathcal{D}$ in domain.

<u>Check:</u> $1 \otimes \varphi(M)_c' \subseteq M_n(\varphi(M))_c'$. Let $1 \otimes y \in 1 \otimes \varphi(M)_c'$ and take $[\varphi(m_{ij})] = [a_{ij}] \in M_n(\varphi(M)) \subseteq M_n(A)$. Then one has that

$$
\begin{aligned}
\langle [a_{ij}]\xi, (1 \otimes y)\eta \rangle &= \langle (1 \otimes y^*)[a_{ij}]\xi, \eta \rangle \\
&= \langle (1 \otimes y^*)(\textstyle\sum_{ij} E_{ij} \otimes a_{ij})\xi, \eta \rangle \\
&= \langle \textstyle\sum_{ij} E_{ij} \otimes y^* a_{ij} \xi, \eta \rangle \\
&= \textstyle\sum_{ij} \langle E_{ij} \otimes y^* a_{ij} \xi, \eta \rangle \\
&= \textstyle\sum \langle E_{ij}\xi_1, \eta_1 \rangle \cdot \langle y^* a_{ij} \eta_1, \eta_2 \rangle \\
&= \textstyle\sum \langle E_{ij}\xi_1, \eta_1 \rangle \cdot \langle y^* \xi_2, a_{ij}^* \eta_2 \rangle \\
&= \textstyle\sum \langle \xi_1, E_{ij}\eta_1 \rangle \cdot \langle \xi_2, y a_{ij}^* \eta_2 \rangle \\
&= \textstyle\sum \langle \xi, E_{ij} \otimes y a_{ij}^* \eta \rangle
\end{aligned}
$$

$= \sum \langle \xi, (1 \otimes y)(E_{ij} \otimes a_{ij}^*)\eta \rangle$

$= \langle (1 \otimes y^*)(\xi), (\sum E_{ij} \otimes a_{ij})^* \eta \rangle$

Going back to the above;

symmetric

$$= \langle [\varphi(x_i^* x_j)]^{\cancel{*}} [\psi_j], (1 \otimes y)^* (1 \otimes y)[\psi_j] \rangle = 0.$$

So $p(y)$ induces a well defined linear map on $\mathcal{D}_1 = M \otimes \mathcal{D}/N$.

$*$-rep?

$p(y_1)p(y_2) = p(y_1 y_2)$? 1)

$p(y) \in \pi(M)_c'$ 2)

$p(y^*) = p(y)^*$ 3)

$p : \varphi(M)_c' \longrightarrow \pi(M)_c'$

~~Assume that 2) holds.~~ $\langle \xi, p(y_1 y_2)\eta \rangle$

$$= \langle \xi, \sum x_i \otimes (y_1 y_2)\psi_i \rangle = \langle \xi, p(y_1)\left(\sum x_i \otimes y_2 \psi_i\right) \rangle$$

$$= \langle \xi, p(y_1)p(y_2)\eta \rangle, \quad \xi \in \mathcal{H} \supseteq \mathcal{D} \ni \eta, \text{ so } p(y_1 y_2) = p(y_1)p(y_2)$$

on $\mathcal{D}$.

3) $\langle \xi, p(w^*)\eta \rangle = \sum \langle x_i \otimes \xi_i, y_j \otimes w^*\eta_j \rangle$

$$= \sum \langle \varphi(y_j^* x_i)\xi_i, w^*\eta_j \rangle$$

$$= \sum \langle w(\xi_i), \varphi(y_j^* x_i)^*\eta_j \rangle$$

$$= \sum \langle \varphi(y_j^* x_i) w(\xi_i), \eta_j \rangle$$

$$= \sum \langle x_i \otimes w\xi_i, y_j \otimes \eta_j \rangle$$

$$= \langle p(w)(\xi), \eta \rangle$$

$$= \langle \xi, p(w)^*\eta \rangle$$

2) $p(w) \in \pi(\mathcal{M})_c'$

$$\langle \pi(m)\xi, p(w)\eta \rangle = \sum \langle mx_i \otimes \xi_i, y_j \otimes w\eta_j \rangle$$
$$= \sum \langle \varphi(y_j^*(mx_i))\xi_i, w\eta_j \rangle$$
$$= \sum \langle \varphi(y_j^*(mx_i)) w^*\xi_i, \eta_j \rangle$$
$$= \sum \langle mx_i \otimes w^*\xi_i, y_j \otimes \eta_j \rangle$$
$$= \sum \langle p(w)^*(x_i \otimes \xi_i), \pi(m)^*(y_j \otimes \eta_j) \rangle$$

Don't these just commute?

Is $p : \varphi(\mathcal{M})_c' \longrightarrow \pi(\mathcal{M})_c'$ uw-cts?

$\{a_\alpha\}_\alpha \longrightarrow 0$ <u>iff</u> $\sum_n \langle a_i \xi_n, \eta_n \rangle \longrightarrow 0$

- Let $\{a_\alpha\}_\alpha \subseteq \varphi(\mathcal{M})_c'$ go to zero (ultraweakly on $\mathcal{D}$), wts that $\{p(a_i)\}_i \longrightarrow 0$ ultraweakly on $\mathcal{M} \otimes \mathcal{D}/\mathcal{N}$.
- $\sum \langle p(a_\alpha)(\sum x_i \otimes \xi_i), \sum y_j \otimes \eta_j \rangle = \sum\sum \langle x_i \otimes a_\alpha \xi_i, y_j \otimes \eta_j \rangle$

$$= \sum\sum \langle \varphi(y_j^* x_i) a_\alpha \xi_i, \eta_j \rangle = \sum\sum \langle a_\alpha \xi_i, \varphi(y_j^* x_i)^* \eta_j \rangle \longrightarrow 0?$

Is $\pi : \mathcal{M} \longrightarrow \mathcal{L}^+(\mathcal{D}_1)$ ultraweakly cts?

- $\{m_\alpha\}_\alpha \subseteq \mathcal{M}$ converging to $0 \in \mathcal{M}$.

$$\sum \langle \pi(m_\alpha)(\textstyle\sum x_i \otimes \xi_i), \sum_j y_j \otimes \eta_j \rangle$$
$$= \sum\sum \langle \pi(m_\alpha)(x_i \otimes \xi_i), y_j \otimes \eta_j \rangle$$
$$= \sum\sum \langle m_\alpha x_i \otimes \xi_i, y_j \otimes \eta_j \rangle$$
$$= \sum\sum \langle \varphi(y_j (m_\alpha x_i)^*) \xi_i, \eta_j \rangle$$
$$= \sum \langle \varphi(y_j (m_\alpha x_i)^*) \xi_i, \eta_j \rangle \longrightarrow 0$$

since φ is uw-cts.

$$\left.\begin{array}{l} p(\omega) V = V\omega, \quad \omega \in . \quad V : \mathcal{D} \longrightarrow \mathcal{D}_1, \quad V(\psi) = 1 \otimes \psi. \\ p(\omega) V(\psi) = 1 \otimes (\omega\psi) = V(\omega\psi). \end{array}\right\}$$

Is $\mathcal{E}^c$ well-defined, $V^* V = I$.

$V : \mathcal{D} \longrightarrow \mathcal{D}_1, \quad V^* : \mathcal{D}_1^* \longrightarrow \mathcal{D}^*$

$V\psi = 1 \otimes \psi, \quad V^* f(\psi) = f(V\psi)$

$= f(1 \otimes \psi)$

. $\mathcal{E}^c : \mathcal{E}(\mathcal{M})_c' \longrightarrow \mathcal{L}^+(\mathcal{D})$. $\mathcal{D}_1 \subseteq \mathcal{D}_1^*$ via $2(\xi) = \langle \xi, \cdot \rangle$

1) Is $V^*(\mathcal{D}_1) \subseteq \mathcal{D}$? $\sum \langle V^*(x_i \otimes \psi_i), \xi \rangle = \sum \langle x_i \otimes \psi_i, V(\xi) \rangle$

$= \sum \langle x_i \otimes \psi_i, 1 \otimes \xi \rangle$ .... $= \sum \langle \mathcal{E}(x_i)\psi_i, \xi \rangle = \langle \underbrace{\sum \mathcal{E}(x_i)\psi_i}_{\in \mathcal{D}}, \xi \rangle$

2) Is $V^*V = I_{\mathcal{D}}$? $V^*V\psi = V^*(1 \otimes \psi)$. $\langle V^*V\psi, \xi \rangle = \langle V\psi, V\xi \rangle$

$= \langle \mathcal{E}(1)\psi, \xi \rangle = \langle \psi, \xi \rangle$ ✓

3) Hence if $R : \mathcal{N} \longrightarrow \pi(\mathcal{M})_c'$, $R = P|_{\mathcal{N}}$. We have $\mathcal{E}^c(R)(w)$

$= V^* p(w) V = V^*V w = w$.

$$\langle ASx, \varphi \rangle = \langle Ax, S^* \varphi \rangle \qquad A \in \mathcal{B}(S)$$

$$\underline{\langle Sx, A\varphi \rangle} = \langle A^* x, S^* \varphi \rangle \quad — \quad A \in \mathcal{B}(S)$$

$$\langle ASx, \varphi \rangle = \langle Sx, A^* \varphi \rangle$$

$$= \langle Ax, S^* \varphi \rangle$$