Exploring the Usefullness of Preconditioners for Solving Nearly Circulant Systems of Linear Equations

CSE6644 Project Proposal

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Introuction: D. Mitsotakis recently developed a novel matrix splitting iterative method for solving systems of near circulant system of linear equations that showed improved convergence rates over GMRES [1]. For our project, we plan to implement the existing algorithm without preconditioning, and then develop or combine existing near circulant preserving preconditioners to accelerate convergence and compare our results to PGMRES. In order to avoid time consuming forays into hardware optimization, number of iterations will be the quantity of interest in our convergence goals.

Existing Method: Solving a generic problem $\mathbf{A}x = \mathbf{b}$, D. Mitsotakis' method takes advantage of 2 main elements. First, it considers a generic matrix splitting method $\mathbf{A} = \mathbf{M} - \mathbf{N}$ with solution updates $\mathbf{x}_{k+1} = \mathbf{M}^{-1}(\mathbf{N}\mathbf{x}_k + \mathbf{b})$, but in this case it chooses $\mathbf{A}$ to be "nearly" circulant such that $\mathbf{M}$ is the circulant part of the matrix and $\mathbf{N}$ is a sparse "pertubation" derived from the Sherman-Morrison-Woodbury (SMW) decomposition $\mathbf{N} = \mathbf{U}\mathbf{V}^T$. Secondly, it takes advantage of the diagonalization of $\mathbf{M}$ using the fourier transform (implemented with an FFT), $\mathbf{M} = \mathcal{F}\mathbf{D}\mathcal{F}^{-1}$, enabling a efficient calculation of $\mathbf{M}^{-1}$.

Research Question: Can we find a suitable preconditioner? While generic preconditioners may be suitable to accelerate the convergence of this new method, if they result in destruction of the near circularity of the system, the matrix $\mathbf{N}$ could become dense after preconditioning. To answer this question we will first examine how generic preconditioners affect the structure of $\mathbf{N}$. If generic preconditioners are suitable, we will move forward with them, but if they are not, then we will do further research to either find or construct a suitable preconditioner.

Research Question: Coupling this pre-conditioner to the matrix splitting method, can we demonstrate improved performance? This method was challenged by a mass matrix generated by a finite element space of cubic splines when benchmarked against the popular GMRES algorithm. Not only will we identify other problems where this algorithm performs more poorly, but we will also investigate how the pre-conditioner may improve convergence to support a broader suite of problems with this algorithm.

Tools and Datasets: To implement the algorithms we plan to use Julia as our primary programming language. Additionally, when parallelizing the algorithm, we will benchmark it on the Georgia Tech PACE Phoenix cluster. To compare our algorithm with GMRES, we will used the test problems given in [1], which include a finite element problem, a finite difference problem, and an extension to saddle point problems.

Project Goals:

- Find or develop a suitable preconditioner that maintains the near circulant nature of the given problem.
- Implement this new algorithm and apply these preconditioner/s to accelerate convergence.
- Compare this new preconditioned approch to PCGMRES (while hopefully maintaing superior convergence in some cases).
- (Stretch) Make efficiency optimizations to our implementation of this new algorithm and GMRES so that iteration time can be included in our convergence goals.
- (Long Shot) Parallelize our preconditioned algorithm to produce convergence results for larger systems.

References

[1] D. Mitsotakis. *On iterative methods based on Sherman-Morrison-Woodbury splitting.* 2023. arXiv: `2305.10968`.