ME422: Finite Elements 1

Homework Set #1

Ben Wilfong
CM 570

The point types are:

2	2	2	2	2	2	2	2	2	1	1	1	0	0	0	0	0	0	0
4	3	3	3	3	3	3	3	3	3	3	3	1	1	0	0	0	0	0
4	3	3	3	3	3	3	3	3	3	3	3	3	3	1	1	0	0	0
4	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	1	0	0
4	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	1	0	0
4	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	1	0
4	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	1	0
4	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	1
4	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	7
6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	8

The boundary condition on the curved edge is

$$\nabla T(x,y,t)\cdot n = -1 \quad \rightarrow \quad \frac{\partial T(x,y,t)}{\partial x}n_x + \frac{\partial T(x,y,t)}{\partial y}n_y = -1 \quad \rightarrow \quad \frac{\partial T_{j,k,i}}{\partial x}n_x + \frac{\partial T_{j,k,i}}{\partial y}n_y = -1$$

At points of type 1 using one sided differences this is:

$$\frac{n_x}{2h}\left(T_{j,k-2,i} - 4T_{j,k-1,i} + 3T_{j,k,i}\right) + \frac{n_y}{2h}\left(-3T_{j,k,i} + 4T_{j+1,k,i} - T_{j+2,k,i}\right) = -1$$

$$\frac{3n_x}{2h}T_{j,k,i} - \frac{3n_y}{2h}T_{j,k,i} = -1 + \frac{n_x}{2h}(4T_{j,k-1,i} - T_{j,k-2,i}) + \frac{n_y}{2h}(T_{j+2,k,i} - 4_{j+1,k,i})$$

$$T_{j,k,i} = \left(\frac{3n_x - 3n_y}{2h}\right)^{-1}\left(-1 + \frac{n_x}{2h}(4T_{j,k-1,i} - T_{j,k-2,i}) + \frac{n_y}{2h}(T_{j+2,k,i} - 4_{j+1,k,i})\right)$$

At points of type 7 using one sided differences this is:

$$\frac{n_x}{2h}\left(T_{j,k-2,i} - 4T_{j,k-1,i} + 3T_{j,k,i}\right) + \frac{n_y}{2h}\left(-4T_{j,k,i} + 4T_{j+1,k,i}\right) = -1$$

$$\frac{3n_x}{2h}T_{j,k,i} - \frac{4n_y}{2h}T_{j,k,i} = -1 + \frac{n_x}{2h}(4T_{j,k-1,i} - T_{j,k-2,i}) - \frac{n_y}{2h}T_{j+1,k,i}$$

$$T_{j,k,i} = \left(\frac{3n_x - 4n_y}{2h}\right)^{-1}\left(-1 + \frac{n_x}{2h}(4T_{j,k-1,i} - T_{j,k-2,i}) - \frac{n_y}{2h}T_{j+1,k,i}\right)$$

At points of type 8 using one sided differences this is:

$$\frac{n_x}{2h}\left(T_{j,k-2,i} - 4T_{j,k-1,i} + 3T_{j,k,i}\right) = -1$$

$$T_{j,k,i} = -\frac{2h}{n_x} + 4T_{j,k-1,i} - T_{j,k-2,i}$$

The 2 dimensional discretization of the wave equation on an equally spaced grid is:

$$u_{i,j}^{k+1} = 2u_{i,j}^{k} - u_{i,j}^{k-1} + r\left(u_{i+1,j}^{k} + u_{i-1,j}^{k} + u_{i,j+1}^{k} + u_{i,j-1}^{k} - 4u_{i,j}^{k}\right)$$

where $h = \Delta x = \Delta y$ and

$$r = \left(\frac{H\Delta t^2}{\rho h^2}\right).$$

Substituting $u(x, y, t_n) = g(\xi)^n e^{I\xi xy}$ where $I = \sqrt{-1}$ yields:

$$g(\xi)^{k+1}e^{I\xi h^2 ij} = 2g(\xi)^k e^{I\xi h^2 ij} - g(\xi)^{k-1}e^{I\xi h^2 ij} + rg(\xi)^k\left(e^{I\xi h^2(i+1)j} + e^{I\xi h^2(i-1)j} + e^{I\xi h^2 i(j+1)} + e^{I\xi h^2 i(j-1)} - 4e^{I\xi h^2 ij}\right).$$

Dividing by $g(\xi)^{k-1}$ on both sides yields:

$$g(\xi)^2 e^{I\xi h^2 ij} = 2g(\xi)e^{I\xi h^2 ij} - e^{I\xi h^2 ij} + rg(\xi)\left(e^{I\xi h^2(i+1)j} + e^{I\xi h^2(i-1)j} + e^{I\xi h^2 i(j+1)} + e^{I\xi h^2 i(j-1)} - 4e^{I\xi h^2 ij}\right).$$

Dividing by $e^{I\xi h^2 ij}$ on both sides yields:

$$g(\xi)^2 = 2g(\xi) - 1 + rg(\xi)\left(e^{I\xi h^2} + e^{-I\xi h^2} + e^{I\xi h^2} + e^{-I\xi h^2} - 4\right),$$

$$g(\xi)^2 = 2g(\xi) - 1 + rg(\xi)\left(2e^{I\xi h^2} + 2e^{-I\xi h^2} - 4\right),$$

$$0 = -1 + rg(\xi)\left(4\cos(\xi h^2) - 4 + \frac{2}{r}\right) - g(\xi)^2$$