ME 7751: Intro to CFD

Homework Set #2

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1. Part A: Discretization as a System of Linear Equations

Suppose that the two-dimensional steady state heat equation

$$\lambda\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+Q=0 \tag{1}$$

is discretized onto $\{x_0, x_1, \ldots, x_N\}$ and $\{y_0, y_1, \ldots, y_N\}$ and that T_{ij} is the temperature at coordinate (x_i, y_j). The second order derivatives in the x- and y-directions can be approximated with the second order central discretizations:

$$\frac{\partial T_{ij}}{\partial x} \approx \frac{T_{i-1,j}-2T_{i,j}+T_{i+1,j}}{(\Delta x)^2} \qquad \frac{\partial T_{ij}}{\partial y} \approx \frac{T_{i,j-1}-2T_{ij}+T_{i,j+1}}{(\Delta x)^2}.$$

These approximations can be substituted into eq. (1) yielding:

$$\lambda\left(\frac{T_{i-1,j}+T_{i+1,j}-4T_{i,j}+T_{i,j-1}+T_{i,j+1}}{2h^2}\right)=-Q_{i,j},$$

assuming that $h=\Delta x=\Delta y$. The Dirichelet boundary conditions on the left and right of the domain are given by $T_{0,j}=0$ and $T_{N,j}=2y_j^3-3y_j^2+1$. The Neumann boundary conditions at the top and bottom of the boundry are given by

$$\begin{aligned} j=0: \frac{-3T_{i,0}+2T_{i,1}-T_{i,2}}{2h}&=0,\\ j=N: \frac{-3T_{i,N}+2T_{i,N-1}-T_{i,N-2}}{2h}&=0, \end{aligned}$$

where second order one-sided differences are used. This discretization can be cast as a block tridiagonal system of linear equations. The solution vector T is given by:

$$T=\begin{pmatrix} T_0\\ T_1\\ \vdots\\ T_N \end{pmatrix} \text{ where } T_{j\in 0:N}=\begin{pmatrix} T_{0,j}\\ T_{1,j}\\ \vdots\\ T_{N,j} \end{pmatrix}.$$

The right hand side b is given by:

$$b=\begin{pmatrix} b_0\\ b_1\\ \vdots\\ b_{N-1}\\ b_N \end{pmatrix} \text{ where } b_{j=0}=\begin{pmatrix} 0\\ 0\\ \vdots\\ 0\\ 0 \end{pmatrix}, b_{j\in 1:N-1}=\begin{pmatrix} 2y_j^2-3y_j^2+1\\ -Q_{1,j}/\lambda\\ \vdots\\ -Q_{N-1,j}/\lambda\\ 0 \end{pmatrix}, b_{j=N}=\begin{pmatrix} 0\\ 0\\ \vdots\\ 0\\ 0 \end{pmatrix}.$$

The matrix A is given by:

$$A=\begin{bmatrix} B_0 & C_0 & X & & \\ A_1 & B_1 & C_1 & & \\ & \ddots & \ddots & \ddots & \\ & & A_{N-1} & B_{N-1} & C_{N-1}\\ & & Y & A_N & B_N \end{bmatrix}. \tag{2}$$

The submatrices $A_j,\ B_j,\ C_j,\ X$, and Y are given by:

$$A_{j=1:N-1} = \begin{bmatrix} 1/2h^2 & & & & \\ & 1/2h^2 & & & \\ & & \ddots & & \\ & & & 1/2h^2 & \\ & & & & 1/2h^2 \end{bmatrix}, \quad A_{j=N} = \begin{bmatrix} 1/h & & & & \\ & 1/h & & & \\ & & \ddots & & \\ & & & 1/h & \\ & & & & 1/h \end{bmatrix},$$

$$B_{j=0} = \begin{bmatrix} -3/2h & & & \\ & -3/2h & & \\ & & \ddots & \\ & & & -3/2h^2 \end{bmatrix}, \quad B_{j=1:N-1} = \begin{bmatrix} 1 & & & & \\ 1/2h^2 & -2/h^2 & 1/2h^2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1/2h^2 & -2/h^2 & 1/2h^2 \\ & & & & 1 \end{bmatrix},$$

$$B_{j=N} = \begin{bmatrix} -3/2h & & & & \\ & -3/2h & & & \\ & & \ddots & & \\ & & & -3/2h & \\ & & & & -3/2h \end{bmatrix}, \quad C_0 = \begin{bmatrix} 1/h & & & & \\ & 1/h & & & \\ & & \ddots & & \\ & & & 1/h & \\ & & & & 1/h \end{bmatrix},$$

$$C_{j=1:N-1} = \begin{bmatrix} 1/2h^2 & & & & \\ & 1/2h^2 & & & \\ & & \ddots & & \\ & & & 1/2h^2 & \\ & & & & 1/2h^2 \end{bmatrix}, \quad X = \begin{bmatrix} -1/2h & & & & \\ & -1/2h & & & \\ & & \ddots & & \\ & & & -1/2h & \\ & & & & -1/2h \end{bmatrix},$$

$$Y = \begin{bmatrix} -1/2h & & & & \\ & -1/2h & & & \\ & & \ddots & & \\ & & & -1/2h & \\ & & & & -1/2h \end{bmatrix}.$$

Writing the system in this way is much more compact than attempting to write the matrix A explicitly. The A_i matrices account for the $T_{i,j-1}$ contributions. The C_i matrices account for the $T_{i,j+1}$ contributions. The B_i matrices account for the local left and right neighbor contributions. X and Y account for the $T_{i,j+2}$ and $T_{i,j-2}$ contributions to the one-sided second-order boundary condition discretizations. This discretization assumes that at the corners of the domain the direchlet boundary condition is enforced because at these locations, the neuman boundary connditions are incompatible with the dirichlet boundary conditions. Each submatrix in A is of size $N+1 \times N+1$. This means that the matrix A has a size of $(N+1)^2 \times (N+1)^2$ assuming a square grid with N^2 nodes. The total number of nonzero elements is only $11(N+1)^2 - 2(N+1)$, making this matrix quite sparse. For the case $N = 100$, only 0.11% of the elements of A are nonzero. As N increases, this percentage continues to decrease.

2. Part B: Iterative Method Forumlation

Jacobi Iteration

- Bottom boundary:

$$T_{i,0}^{k+1} = -\frac{T_{i,2}^{k} - 2T_{i,1}^{k}}{3}$$

- Interior points:

$$T_{i,j}^{k+1} = \frac{2h^2 Q_{i,j}}{4\lambda} - \frac{1}{4}\left(T_{i-1,j}^{k} + T_{i+1,j}^{k} + T_{i,j-1}^{k} + T_{i,j+1}^{k}\right)$$

- Top boundary;

$$T_{i,N}^{k+1} = -\frac{T_{i,N-2}^{k} - 2T_{i,N-1}^{k}}{3}$$

Gauss-Seidel Iteration

- Bottom boundary:

$$T_{i,0}^{k+1} = -\frac{T_{i,2}^{k} - 2T_{i,1}^{k}}{3}$$

- Interior points:

$$T_{i,j}^{k+1} = \frac{2h^2 Q_{i,j}}{4\lambda} - \frac{1}{4}\left(T_{i-1,j}^{k+1} + T_{i+1,j}^{k} + T_{i,j-1}^{k+1} + T_{i,j+1}^{k}\right)$$

- Top boundary;

$$T_{i,N}^{k+1} = -\frac{T_{i,N-2}^{k+1} - 2T_{i,N-1}^{k+1}}{3}$$

Sucessive Over-relaxtion Iteration

- Bottom boundary:

$$T_{i,0}^{k+1} = (1-\omega)T_{i,0}^{k} - \omega\left(\frac{T_{i,2}^{k} - 2T_{i,1}^{k}}{3}\right)$$

- Interior points:

$$T_{i,j}^{k+1} = (1-\omega)T_{i,j}^{k} + \frac{2\omega h^2 Q_{i,j}}{4\lambda} - \frac{\omega}{4}\left(T_{i-1,j}^{k+1} + T_{i+1,j}^{k} + T_{i,j-1}^{k+1} + T_{i,j+1}^{k}\right)$$

- Top boundary;

$$T_{i,N}^{k+1} = (1-\omega)T_{i,j}^{k} - \omega\left(\frac{T_{i,N-2}^{k+1} - 2T_{i,N-1}^{k+1}}{3}\right)$$