1. Part A - Numerical Method

The one dimensional steady state transport equation

$$U\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(\Gamma \frac{\partial \phi}{\partial x}\right) + Q$$

can be discretized with central differences as

$$U\frac{\phi_{i+1}-\phi_{i-1}}{2\Delta x} - \underbrace{\frac{\left(\Gamma\frac{\partial \phi}{\partial x}\right)_{i+1/2} - \left(\Gamma\frac{\partial \phi}{\partial x}\right)_{i-1/2}}{\Delta x}}_{b} = Q_i. \tag{1}$$

The first derivatives in the numerator of b can be written as

$$\left(\Gamma\frac{\partial \phi}{\partial x}\right)_{i+1/2} = \Gamma_{i+1/2}\frac{\phi_{i+1}-\phi_i}{\Delta x} \quad \text{and} \quad \left(\Gamma\frac{\partial \phi}{\partial x}\right)_{i-1/2} = \Gamma_{i-1/2}\frac{\phi_i-\phi_{i-1}}{\Delta x}. \tag{2}$$

Substituting eq. (2) into eq. (1) and rearanging gives

$$\begin{aligned}
U\frac{\phi_{i+1}-\phi_{i-1}}{2\Delta x} - \frac{\Gamma_{i+1/2}\frac{\phi_{i+1}-\phi_i}{\Delta x} - \Gamma_{i-1/2}\frac{\phi_i-\phi_{i-1}}{\Delta x}}{\Delta x} &= Q_i \\
U\frac{\phi_{i+1}-\phi_{i-1}}{2\Delta x} - \frac{\Gamma_{i+1/2}\phi_{i+1} - (\Gamma_{i+1/2}+\Gamma_{i-1/2})\phi_i + \Gamma_{i-1/2}\phi_{i-1}}{(\Delta x)^2} &= Q_i \\
\left(\frac{U}{2\Delta x} - \frac{\Gamma_{i+1/2}}{(\Delta x)^2}\right)\phi_{i+1} + \left(\frac{\Gamma_{i+1/2}+\Gamma_{i-1/2}}{(\Delta x)^2}\right)\phi_i - \left(\frac{U}{2\Delta x} + \frac{\Gamma_{i-1/2}}{(\Delta x)^2}\right)\phi_{i-1} &= Q_i.
\end{aligned} \tag{3}$$

Equation (3) can be cast as the system of linear equations

$$\underbrace{\begin{bmatrix}
1 & & & & \\
-\left(\frac{U}{2\Delta x} + \frac{\Gamma_{1/2}}{(\Delta x)^2}\right) & \frac{\Gamma_{3/2}+\Gamma_{1/2}}{(\Delta x)^2} & \frac{U}{2\Delta x} - \frac{\Gamma_{3/2}}{(\Delta x)^2} & & \\
\ddots & \ddots & \ddots & & \\
& -\left(\frac{U}{2\Delta x} + \frac{\Gamma_{N-3/2}}{(\Delta x)^2}\right) & \frac{\Gamma_{N-1/2}+\Gamma_{N-3/2}}{(\Delta x)^2} & \frac{U}{2\Delta x} - \frac{\Gamma_{i-1/2}}{(\Delta x)^2} \\
& & & 1
\end{bmatrix}}_{\text{A}}
\underbrace{\begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_{N-1} \\ \phi_N \end{pmatrix}}_{\Phi}
=
\underbrace{\begin{pmatrix} 1 \\ Q_1 \\ \vdots \\ Q_{N-1} \\ 0 \end{pmatrix}}_{b}$$

for a discretization at $\{x_0,\ x_1,\ \ldots\ x_N\} = \{0,\ \Delta x,\ 2\Delta x, \ldots,\ N\Delta x\}$. The boundary conditions are given by $\phi_{-1/2} = 1$ and $\phi_{N+1/2} = 0$.

For the cases in which $U,\ \Gamma$, and Q are constant, the solution is found via the following steps:

(1) Populate the matrix A
(2) Populate the vector b
(3) Solve $A\Phi = b$ for the vector Φ using TDMA

When $U,\ \Gamma$, or Q depend on the value of ϕ the problem becomes nonlinear and an iterative approach must be taken. For nonlinear problems, the solution is found via the following steps:

(1) Make an initial guess for the vector $\Phi^{(0)}$
(2) Calculate $U(\Phi^{(n)}),\ \Gamma(\Phi^{(n)})$, and $Q(\Phi^{(n)})$
(3) Populate the matrix A
(4) Populate the vector b
(5) Solve $A\Phi^{(n+1)} = b$ for the vector $\Phi^{(n+1)}$ using TDMA
(6) Repeat steps 2 though 5 until convergence

For this problem I will use a discretization of $\phi(x) = 1 - x/L$ as an initial guess.

The tridiagonal matrix A is stored in an $N+1 \times 3$ to take advantage of it's sparsity. The lower, center, and upper diagonal are stored in $A_{1:N,0}$, $A_{0:N,1}$, and $A_{0:N-1,2}$. The tridiagonal linear solve using this data structure is given in algorithm 1.

Algorithm 1 Tridiagonal Matrix Algorithm

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Given: A                                   ▷ N+1 × N+1
Given: x                                   ▷ N+1 × 1
Given: b                                   ▷ N+1 × 1

for i = 1 : N do                           ▷ In place LU factorization
    w = A_{i,0}/A_{i-1,1}
    A_{i,0} = A_{i,0} - w A_{i-1,1}
    A_{i,1} = A_{i,0} - w A_{i-1,2}
    A_{i,0} = w
end for

y_0 = b_0                                  ▷ Forward Substitution
for i = 1 : N do
    y_i = b_i - A_{i,0)y_{i-1}
end for

x_N = y_N/A_{N,1}                          ▷ Backward Substitution
for i = N-1 : -1 : 0 do
    x_i = (y_i - A_{i,2} x_{i+1}) / A_{i,1}
end for
```

2. Part B - Linear Problems

3. Part C - Non-linear Problems