Homework 2 (finite differences)

See homework submission guidelines on Canvas. Please submit all computer code with comments online.

Problem 1

The two-dimensional steady heat equation (Poisson's equation) is given by

$$\lambda\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+Q=0$$

Here, T is the temperature, λ is the thermal diffusivity, and Q is a source.
Take the physical domain size to be $[1 \times 1]$ with $Q = 4y^3 - 6y^2 + 2 - 6(1-x^2)(2y-1)$ and $\lambda = 1$. Fix the temperature at the right wall to be $T(1,y) = 0$ and the left wall to be $T(0,y) = 2y^3 - 3y^2 + 1$. Impose a zero flux condition at the top and bottom walls (i.e., Neumann boundary conditions $\partial T/\partial n = 0$). For the above conditions, the analytical solution is $T(x,y) = (1-x^2)(2y^3 - 3y^2 + 1)$.

a. Discretize the PDE using a central difference scheme to generate a matrix equation $[A]\{T\} = \{Q\}$. What is the size of the matrix $[A]$, assuming a square grid with N^2 nodes?

b. For each of the Jacobi, Gauss-Seidel, and SOR methods, express $T_{i,j}^{k+1}$, the temperature at node (i,j) and iteration $k+1$, in terms of the temperatures of its neighbors at either the current or past iterations. Note that this step allows you to avoid explicitly generating and storing the very sparse matrix $[A]$, which should not be done.

c. Using one of the methods, solve for the temperature with grids $[10,10]$, $[20,20]$, and $[40,40]$. State what condition was used for stopping the iterative solver. Plot the temperature contours for each grid on a single contour plot and compare to the analytical. Additionally, on a separate plot, compare the numerical solution to the analytical solution along the vertical centerline $T(0.5,y)$ for the three grids. Compute the error using an appropriate metric that considers all points in the domain.

d. Using one of the methods, evaluate the spatial convergence rate of the finite difference scheme using the grids corresponding to $[10,10]$, $[20,20]$, $[40,40]$, $[80,80]$, and $[160,160]$. Discuss the discretization of the flux boundary conditions and how it affects the overall spatial convergence rate.

e. Plot the residual as a function of the iteration count for Jacobi, Gauss-Seidel and SOR using the grid $[40,40]$. For SOR, investigate the effect of the relaxation factor and determine its optimal value.

f. Compare computational time obtained with different methods and evaluate the time complexity of the methods.

g. Solve the problem using a 2-level geometric multigrid method. Compare the computational time with other methods.