ME 7751: Intro to CFD

Homework Set #1

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1. Part A - Numerical Method

The one dimensional steady state transport equation

$$U\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(\Gamma \frac{\partial \phi}{\partial x}\right) + Q$$

can be discretized with central differences as

$$U\frac{\phi_{i+1}-\phi_{i-1}}{2\Delta x} - \underbrace{\frac{\left(\Gamma\frac{\partial\phi}{\partial x}\right)_{i+1/2} - \left(\Gamma\frac{\partial\phi}{\partial x}\right)_{i-1/2}}{\Delta x}}_{b} = Q_i. \tag{1}$$

The first derivatives in the numerator of b can be written as

$$\left(\Gamma\frac{\partial\phi}{\partial x}\right)_{i+1/2} = \Gamma_{i+1/2}\frac{\phi_{i+1}-\phi_i}{\Delta x} \quad \text{and} \quad \left(\Gamma\frac{\partial\phi}{\partial x}\right)_{i-1/2} = \Gamma_{i-1/2}\frac{\phi_i-\phi_{i-1}}{\Delta x}. \tag{2}$$

Substituting eq. (2) into eq. (1) and rearanging gives

$$\begin{aligned}
U\frac{\phi_{i+1}-\phi_{i-1}}{2\Delta x} - \frac{\Gamma_{i+1/2}\frac{\phi_{i+1}-\phi_i}{\Delta x} - \Gamma_{i-1/2}\frac{\phi_i-\phi_{i-1}}{\Delta x}}{\Delta x} &= Q_i \\
U\frac{\phi_{i+1}-\phi_{i-1}}{2\Delta x} - \frac{\Gamma_{i+1/2}\phi_{i+1} - (\Gamma_{i+1/2}+\Gamma_{i-1/2})\phi_i + \Gamma_{i-1/2}\phi_{i-1}}{(\Delta x)^2} &= Q_i \\
\left(\frac{U}{2\Delta x} - \frac{\Gamma_{i+1/2}}{(\Delta x)^2}\right)\phi_{i+1} + \left(\frac{\Gamma_{i+1/2}+\Gamma_{i-1/2}}{(\Delta x)^2}\right)\phi_i - \left(\frac{U}{2\Delta x} + \frac{\Gamma_{i-1/2}}{(\Delta x)^2}\right)\phi_{i-1} &= Q_i.
\end{aligned} \tag{3}$$

Equation (3) can be cast as the system of linear equations

$$\underbrace{\begin{bmatrix}
1 & & & \\
-\left(\frac{U}{2\Delta x}+\frac{\Gamma_{1/2}}{(\Delta x)^2}\right) & \frac{\Gamma_{3/2}+\Gamma_{1/2}}{(\Delta x)^2} & \frac{U}{2\Delta x}-\frac{\Gamma_{3/2}}{(\Delta x)^2} & \\
\ddots & \ddots & \ddots & \\
-\left(\frac{U}{2\Delta x}+\frac{\Gamma_{N-3/2}}{(\Delta x)^2}\right) & \frac{\Gamma_{N-1/2}+\Gamma_{N-3/2}}{(\Delta x)^2} & \frac{U}{2\Delta x}-\frac{\Gamma_{i-1/2}}{(\Delta x)^2} & \\
 & & & 1
\end{bmatrix}}_{\text{A}}
\underbrace{\begin{pmatrix}\phi_0\\ \phi_1\\ \vdots\\ \phi_{N-1}\\ \phi_N\end{pmatrix}}_{\Phi}
=
\underbrace{\begin{pmatrix}1\\ Q_1\\ \vdots\\ Q_{N-1}\\ 0\end{pmatrix}}_{b}$$

for a discretization at $\{x_0,\ x_1,\ \dots\ x_N\} = \{0,\ \Delta x,\ 2\Delta x, \dots,\ N\Delta x\}$. The boundary conditions are given by $\phi_{-1/2} = 1$ and $\phi_{N+1/2} = 0$.

For the cases in which $U,\ \Gamma$, and Q are constant, the solution is found via the following steps:

(1) Populate the matrix A
(2) Populate the vector b
(3) Solve $A\Phi = b$ for the vector Φ using TDMA

When $U,\ \Gamma$, or Q depend on the value of ϕ the problem becomes nonlinear and an iterative approach must be taken. For nonlinear problems, the solution is found via the following steps:

(1) Make an initial guess for the vector $\Phi^{(0)}$
(2) Calculate $U(\Phi^{(n)}),\ \Gamma(\Phi^{(n)})$, and $Q(\Phi^{(n)})$
(3) Populate the matrix A
(4) Populate the vector b
(5) Solve $A\Phi^{(n+1)} = b$ for the vector $\Phi^{(n+1)}$ using TDMA
(6) Repeat steps 2 though 5 until convergence

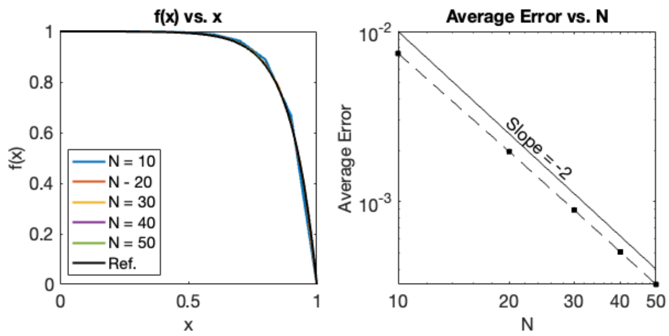

FIGURE 1. Solution and average error for $N = 10, 20, 30, 40, 50$

For this problem I will use a discretization of $\phi(x) = 1 - x/L$ as an initial guess.

The tridiagonal matrix A is stored in an $N + 1 \times 3$ to take advantage of it's sparsity. The lower, center, and upper diagonal are stored in $A_{1:N,0}$, $A_{0:N,1}$, and $A_{0:N-1,2}$. The tridiagonal linear solve using this data structure is given in algorithm 1.

Algorithm 1 Tridiagonal Matrix Algorithm

Given: A ▷ $N + 1 \times N + 1$
Given: x ▷ $N + 1 \times 1$
Given: b ▷ $N + 1 \times 1$

for $i = 1 : N$ **do** ▷ In place LU factorization
  $w = A_{i,0}/A_{i-1,1}$
  $A_{i,0} = A_{i,0} - wA_{i-1,1}$
  $A_{i,1} = A_{i,0} - wA_{i-1,2}$
  $A_{i,0} = w$
end for

$y_0 = b_0$ ▷ Forward Substitution
for $i = 1 : N$ **do**
  $y_i = b_i - A_{i,0)y_{i-1}}$
end for

$x_N = y_N/A_{N,1}$ ▷ Backward Substitution
for $i = N - 1 : -1 : 0$ **do**
  $x_i = \frac{y_i - A_{i,2}x_{i+1}}{A_{i,1}}$
end for

2. PART B - LINEAR PROBLEMS

Figure 1 shows the solution, average error, and runtime for the case $U = 1$, $\Gamma = 0.1$, and $Q = 0$ with $N = 10,\ 20,\ 30,\ 40,\ 50$. The error decreases with increasing N as expected. The plot of average error versus N shows that the method is second-order-accurate. This is because the error decreases with a slope of two in loglog space. This order of accuracy is expected given that central differences were used to discretize the derivatives.

3. PART C - NON-LINEAR PROBLEMS