Homework 3

See homework submission guidelines on Canvas. Please submit all computer codes with comments online.

Problem 1

Solve numerically the one dimensional unsteady transport equation:

$$\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(\Gamma \frac{\partial \phi}{\partial x}\right)$$

Here, ϕ is a specific property, U is the convective velocity, and Γ is the diffusion coefficient. The domain size is $L = 40$ with $5 \le x \le 45$.
Boundary conditions are $\phi(5,t) = 0$ and $\phi(45,t) = 0$.
The initial condition at $t = 10$ is $\phi(x,10) = (0.4\pi)^{-0.5}\exp\left(-2.5(x-10)^2\right)$.

Consider two cases:
Case 1: $U = 1$ and $\Gamma = 0.01$ with the analytical solution

$$\phi(x,t) = (4\pi\Gamma t)^{-0.5}\exp\left(-(x-Ut)^2/(4\Gamma t)\right)$$

Case 2: $U = 1$ and $\Gamma = 0$ with the analytical solution

$$\phi(x,t) = (0.4\pi)^{-0.5}\exp\left(-2.5(x-Ut)^2\right)$$

For the above problem please do the following:

1. Write down the simplified finite difference schemes for the 1D unsteady transport equation for the following three integration schemes: Euler explicit, Euler implicit and Crank-Nicholson. Choose an appropriate spatial discretization. Employ TDMA to solve for implicit terms.

2. Plot ϕ vs. x at $t = 20, 30, 40$ comparing the numerical solution obtained using the three integration schemes to the analytical solution for Case 1. Repeat above but with Case 2. What do you observe for Case 2 and why? For a given integration scheme and case, plot all three time snap shots on a single graph.

3. For Case 1, plot ϕ vs t at x = 15 and x = 25. Use N = 500 grid points and a time step of dt = 0.01, compare the time evolution obtained for the three integration schemes to the analytical solution. Which method matches closest to the analytical and why? I should clearly be able to see this in the two plots you should provide - one for x = 15 and one for x = 25.

4. For Case 1, choose 5 different grid sizes (decreasing h by 2 each time) and plot the error vs the resolution on a loglog plot. Evaluate the spatial convergence rate. Does it match theoretical spatial convergence rate? (Choose any time integration scheme.)

5. For Case 1, choose 5 different time steps (decreasing *dt* by 2 each time) and plot the error vs time step on a single loglog plot comparing the temporal convergence curves of each scheme. Evaluate the temporal convergence rate for each scheme. Do they match the theoretical temporal convergence rate?

6. Discuss the stability of the different schemes for Case 1 and Case 2. If need be, find the maximum time step that leads to a stable solution.

7. Discuss the computational cost of each method.