MA 491: Introduction to Mathematical Modeling

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Module 1

1) Answer the following for a Markov chain on five states with probabilities:

$$p_{12} = 0.2,\ p_{15} = 0.2,\ p_{21} = 0.4,\ p_{23} = 0.4,\ p_{24} = 0.1,\ p_{25} = 0.1$$
$$p_{32} = 0.2,\ p_{34} = 0.3,\ p_{35} = 0.2,\ p_{43} = 0.5,\ p_{51} = 0.2,\ p_{54} = 0.8.$$

Values of p_{ii} must be inferred.

a) Draw a digraph for the Markov chain.

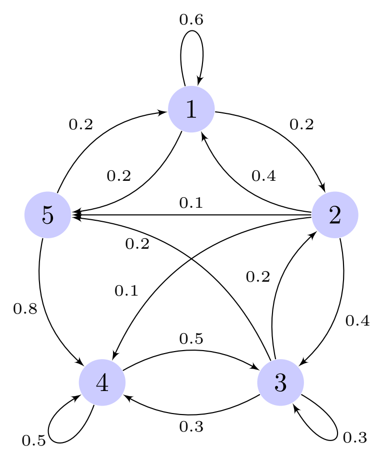
b) Calculate the transition probabilities for $t = 2$ and $t = 10$ (round to two decimal places).

c) What is the distribution at $t = 10$ if the initial distribution is uniform, i.e. all probabilities are initially the same?

d) Calculate increasingly large powers of the transition matrix. Do we suspect that A^n converges as n increases, and how do we interpret the limiting matrix if so?

The digraph of the Markov chain (including the probabilities of not changing state) is as follows

a)



b) To calculate transition probabilities we must first write the Markov chain in matrix form

$$A = \begin{bmatrix} 0.6 & 0.2 & 0 & 0 & 0.2 \\ 0.4 & 0 & 0.4 & 0.1 & 0.1 \\ 0 & 0.2 & 0.3 & 0.3 & 0.2 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0.2 & 0 & 0 & 0.8 & 0 \end{bmatrix}$$

The transition probabilities for $t = 2$ and $t = 10$ are then A^2 and A^{10} respectively. This is due to the fact that $P(j|i, t = n) = [A^n]_{ij}$.

$$t = 2 : \begin{bmatrix} 0.48 & 0.12 & 0.08 & 0.18 & 0.14 \\ 0.26 & 0.16 & 0.17 & 0.25 & 0.16 \\ 0.12 & 0.06 & 0.32 & 0.42 & 0.08 \\ 0.00 & 0.10 & 0.40 & 0.40 & 0.10 \\ 0.12 & 0.04 & 0.40 & 0.40 & 0.04 \end{bmatrix}$$

$$t = 10 : \begin{bmatrix} 0.15 & 0.09 & 0.30 & 0.35 & 0.10 \\ 0.15 & 0.09 & 0.31 & 0.36 & 0.10 \\ 0.14 & 0.09 & 0.31 & 0.36 & 0.10 \\ 0.13 & 0.09 & 0.31 & 0.36 & 0.10 \\ 0.14 & 0.09 & 0.31 & 0.36 & 0.10 \end{bmatrix}$$

c) The vector $v = [0.2, 0.2, 0.2, 0.2, 0.2]$ represents a uniform initial distribution. The distribution at $t = 10$ is then

$$vA^{10} = [0.2, 0.2, 0.2, 0.2, 0.2] \begin{bmatrix} 0.15 & 0.09 & 0.30 & 0.35 & 0.10 \\ 0.15 & 0.09 & 0.31 & 0.36 & 0.10 \\ 0.14 & 0.09 & 0.31 & 0.36 & 0.10 \\ 0.13 & 0.09 & 0.31 & 0.36 & 0.10 \\ 0.14 & 0.09 & 0.31 & 0.36 & 0.10 \end{bmatrix} = [0.14, 0.09, 0.31, 0.36, 0.10]$$

This means that there is a 14% chance of terminating in state 1, 9% chance in state 2, 31% chance in state 3, 36% chance in state 4, and 10% chance in state 5 after 10 time steps. **d)**

$$A^{11} = \begin{bmatrix} 0.15 & 0.09 & 0.30 & 0.36 & 0.10 \\ 0.14 & 0.09 & 0.31 & 0.36 & 0.10 \\ 0.14 & 0.09 & 0.31 & 0.36 & 0.10 \\ 0.14 & 0.09 & 0.31 & 0.36 & 0.10 \\ 0.14 & 0.09 & 0.31 & 0.36 & 0.10 \end{bmatrix}, \ A^{12} = \begin{bmatrix} 0.14 & 0.09 & 0.31 & 0.36 & 0.10 \\ 0.14 & 0.09 & 0.31 & 0.36 & 0.10 \\ 0.14 & 0.09 & 0.31 & 0.36 & 0.10 \\ 0.14 & 0.09 & 0.31 & 0.36 & 0.10 \\ 0.14 & 0.09 & 0.31 & 0.36 & 0.10 \end{bmatrix}$$

The transition matrix converges quickly after $t = 10$. The identical rows of the limiting matrix can be interpreted as the chains equilibrium vector, which describes the probabilities of being in each state as $t \to \infty$. The limiting matrix is then

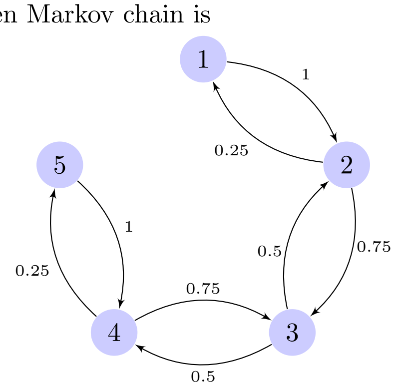
$$\hat{A} = \begin{bmatrix} 0.14 & 0.09 & 0.31 & 0.36 & 0.10 \\ 0.14 & 0.09 & 0.31 & 0.36 & 0.10 \\ 0.14 & 0.09 & 0.31 & 0.36 & 0.10 \\ 0.14 & 0.09 & 0.31 & 0.36 & 0.10 \\ 0.14 & 0.09 & 0.31 & 0.36 & 0.10 \end{bmatrix}$$

3) Draw a digraph for the Markov chain with the transition matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0.25 & 0 & 0.75 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.75 & 0 & 0.25 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Is this chain regular or ergodic? Interpret the transition probabilities as t increases.

The digraph representing the given Markov chain is



Some transition probabilities as t increases are

$$A^2 = \begin{bmatrix} 0.250 & 0.000 & 0.750 & 0.000 & 0.000 \\ 0.000 & 0.625 & 0.000 & 0.375 & 0.000 \\ 0.125 & 0.000 & 0.750 & 0.000 & 0.125 \\ 0.000 & 0.375 & 0.000 & 0.625 & 0.000 \\ 0.000 & 0.000 & 0.750 & 0.000 & 0.250 \end{bmatrix},\ A^5 = \begin{bmatrix} 0.000 & 0.531 & 0.000 & 0.469 & 0.000 \\ 0.133 & 0.000 & 0.750 & 0.000 & 0.117 \\ 0.000 & 0.500 & 0.000 & 0.500 & 0.000 \\ 0.117 & 0.000 & 0.750 & 0.000 & 0.133 \\ 0.000 & 0.469 & 0.000 & 0.531 & 0.000 \end{bmatrix},\ A^{10} = \begin{bmatrix} 0.125 & 0.000 & 0.750 & 0.000 & 0.125 \\ 0.000 & 0.500 & 0.000 & 0.500 & 0.000 \\ 0.125 & 0.000 & 0.750 & 0.000 & 0.125 \\ 0.000 & 0.500 & 0.000 & 0.500 & 0.000 \\ 0.125 & 0.000 & 0.750 & 0.000 & 0.125 \end{bmatrix}.$$

$$A^{15} = \begin{bmatrix} 0.000 & 0.500 & 0.000 & 0.500 & 0.000 \\ 0.125 & 0.000 & 0.750 & 0.000 & 0.125 \\ 0.000 & 0.500 & 0.000 & 0.500 & 0.000 \\ 0.125 & 0.000 & 0.750 & 0.000 & 0.125 \\ 0.000 & 0.500 & 0.000 & 0.500 & 0.000 \end{bmatrix},\ A^{20} = \begin{bmatrix} 0.125 & 0.000 & 0.750 & 0.000 & 0.125 \\ 0.000 & 0.500 & 0.000 & 0.500 & 0.000 \\ 0.125 & 0.000 & 0.750 & 0.000 & 0.125 \\ 0.000 & 0.500 & 0.000 & 0.500 & 0.000 \\ 0.125 & 0.000 & 0.750 & 0.000 & 0.125 \end{bmatrix}$$

from which an obvious pattern arises (within some slight rounding, which dissapears in the limit at $t \to \infty$) once n becomes sufficiently large. The given Markov chain is not regular, but it is ergodic. The obvious pattern shows that there is no such n such that A^n has all positive elements, making the chain irregular. There is, however, some value of n for each $A_{i,j}$ such that $A^n_{i,j}$ is positive, making the chain ergodic, or connectable. The irregularity of the chain means there is not a positive chance of moving from any state to any other state in the same n time steps, but its ergodic nature means you can reach any state from any other state for some n. This chain also has no equilibrium as clearly shown in the pattern above.

5) Let A be a matrix with every row being a probability vector. Show that

$$\min_i \vec{y}_i \leq \min_i [A\vec{y}]_i \leq \max_i [A\vec{y}]_i \leq \max_i \vec{y}_i,$$

where y is an arbitrary vector of appropriate length.

Let A be a transition matrix with every row being a probability vector. And let $\vec{y}$ be an arbitrary vector of appropriate length. First note that

$$[A\vec{y}]_k = \sum_j A_{kj}\vec{y}_j \leq \sum_j A_{kj} \max_i \vec{y}_i = \max_i \vec{y}_i \sum_j A_{k,j} = \max_i \vec{y}_i$$

$$[A\vec{y}]_k \leq \max_i \vec{y}_i$$

Also note that

$$[A\vec{y}]_k = \sum_j A_{kj}\vec{y}_j \geq \sum_j A_{kj} \min_i \vec{y}_i = \min_i \vec{y}_i \sum_j A_{k,j} = \min_i \vec{y}_i$$

$$[A\vec{y}]_k \geq \min_i \vec{y}_i$$

Also by the definitions of minimum and maximum,

$$\min_i [A\vec{y}]_i \leq \max_i [A\vec{y}]_i,$$

holds. By the transitive property

$$\min_i \vec{y}_i \leq \min_i [A\vec{y}]_i \leq \max_i [A\vec{y}]_i \leq \max_i \vec{y}_i$$

7) You are tired and working, yet again, through the wee hours of the morning to complete homework. An overly exuberant friend approaches you with a request for help. Your friend is trying to calculate the equilibrium vector of a regular Markov chain and is confident that one of the following is an eigenvector of A^T for the eigenvalue of one,

$$(2, 4, 6, 9),\ (1, 0, 1, 2),\ (2, 1, 6, 0),\ \text{or}\ (2, 4, 6, 9).$$

You opaquely scan the vectors through a fog of caffeinated steam, do a quick calculation, and claim to the know the answer. What is the equilibrium vector?

The regularity of A requires that the equilibrium vector of A has all positive elements. The fundamental theorem for regular Markov chains tells us that the equilibrium vector of a regular Markov chain, A, is an eigenvector of A^T for an eigenvalue of one. If $\vec{a}$ is an eigenvector of A^T for the eigenvalue of one, then so is

$k\vec{a}$, a linear multiple of this eigenvector. From the given eigenvectors, the only one whose elements can be made strictly positive via a linear multiple is

$$\vec{a} = (-2, -4, -6, -9)$$

Simply multiplying this vector by -1 yields a vector of all positive elements. Scaling the resulting vector by the inverse of the row sum yields a vector which has an element sum of one, which is required of a probability vector.

$$\hat{a} = -\frac{1}{21}(-2, -4, -6, -9) = \left(\frac{2}{21}, \frac{4}{21}, \frac{6}{21}\frac{9}{21}\right) = (0.095, 0.190, 0.286, 0.429)$$

8) Prove the second statement of Theorem 2. Hint, consider the null space of I-Q and use part 1).

Let us assume there is a vector $\vec{e} = (e_1, e_2, ..., e_n)$ such that $\vec{e}$ is a solution to

$$(I - Q)\vec{e} = 0$$

rearranging yields

$$\begin{aligned} (I - Q)\vec{e} &= 0 \\ I\vec{e} - Q\vec{e} &= 0 \\ \vec{e} - Q\vec{e} &= 0 \\ Q\vec{e} &= \vec{e} \end{aligned}$$

To satisfy $(I - Q)\vec{e} = 0$, $\vec{e}$ must be an eigenvector of Q for an eigenvalue of one. This means that

$$\begin{aligned} Q^n\vec{e} &= 1^n\vec{e}, \\ Q^n\vec{e} &= \vec{e} \\ 0\vec{e} &= \vec{e} \end{aligned}$$

For this to be true, $\vec{e}$ must be the zero vector. null$(I - Q) = \emptyset$ implies that $(I - Q)$ has full rank, and thus $(I - Q)^{-1}$ exists.

9) Suppose the transition matrix for a Markov chain is

$$\begin{bmatrix} 0.45 & 0.15 & 0.10 & 0.05 & 0.15 & 0.10 \\ 0.15 & 0.30 & 0.25 & 0.10 & 0.10 & 0.10 \\ 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 \\ 0.15 & 0.15 & 0.05 & 0.05 & 0.25 & 0.35 \\ 0.20 & 0.25 & 0.05 & 0.05 & 0.25 & 0.20 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix}$$

Suppose further that our initial distribution of the Markov chain is uniform over states one, two, four, and five. What percentages do we expect to terminate in states three and six, and what are the expected number of iterations before being absorbed (per starting state).

Reassigning states of the transition matrix in general form, so the absorbing states are first, yields

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0.10 & 0.10 & 0.45 & 0.05 & 0.15 & 0.15 \\ 0.05 & 0.35 & 0.15 & 0.05 & 0.25 & 0.15 \\ 0.05 & 0.20 & 0.20 & 0.05 & 0.25 & 0.25 \\ 0.25 & 0.10 & 0.15 & 0.10 & 0.10 & 0.30 \end{bmatrix}$$

This was found by switching the order of state 1 and state 3, and state 2 and state 6. The new order of the states will become state 3, state 6, state 1, state 4, state 5, state 2 for both rows and columns. From A we can extract

$$R = \begin{bmatrix} 0.10 & 0.10 \\ 0.05 & 0.35 \\ 0.05 & 0.20 \\ 0.25 & 0.10 \end{bmatrix}, \; Q = \begin{bmatrix} 0.45 & 0.05 & 0.15 & 0.15 \\ 0.15 & 0.05 & 0.25 & 0.15 \\ 0.20 & 0.05 & 0.25 & 0.25 \\ 0.15 & 0.10 & 0.10 & 0.30 \end{bmatrix}$$

The fundamental matrix $(I-Q)^{-1}$ is

$$(I-Q)^{-1} = \begin{bmatrix} 2.334 & 0.240 & 0.651 & 0.784 \\ 0.726 & 1.191 & 0.627 & 0.634 \\ 0.916 & 0.228 & 1.706 & 0.855 \\ 0.735 & 0.254 & 0.473 & 1.809 \end{bmatrix}$$

The row sums of the fundamental matrix, which represent the expected number of iterations before being absorbed, are

$$(4.009, 3.178, 3.705, 3.271)^T$$

Therefore, starting in state 1 is expected to have 4.009 iterations, state 2 is expected to have 3.271 iterations, state 4 is expected to have 3.178 iterations, and state 5 is expected to have 3.705 iterations before being absorbed.

The absorption probability matrix $(I-Q)^{-1}R$ is

$$(I-Q)^{-1}R = \begin{bmatrix} 0.474 & 0.526 \\ 0.322 & 0.678 \\ 0.402 & 0.598 \\ 0.562 & 0.438 \end{bmatrix}.$$

The above matrix can be interpreted to mean that if one starts in node 1, there is a 0..474 probability of being absorbed into state 3, and a 0.526 probability of being absorbed into state 6. Similarly, starting in node 2 yields termination in node 3 56.2% of the time and node 6 43.8% of the time. Starting in node 4 yields termination in node 3 32.3% of the time and node 6 67.8 % of the time. Starting in node 5 yields termination in node 3 40.2% of the time and node 6 43.8% of the time.

Tabulating these results yields

Starting State	Iterations Until Termination	State 3 Termination Probability	State 6 Termination Probability
1	4.009	47.4%	52.6%
2	3.271	56.2%	43.8%
4	3.178	32.2%	67.8%
5	3.705	40.2%	59.8%