2) Show that if (ω^*, γ^*) is optimal to

$$E^{\hat{d}} = \max \frac{\sum_{m=1}^{M} \gamma_m Y_{m\hat{d}}}{\sum_{n=1}^{N} \omega_n X_{n\hat{d}}} \text{ such that } \frac{\sum_{m=1}^{M} \gamma_m Y_{md}}{\sum_{n=1}^{N} \omega_n X_{nd}} \leq 1 \text{ for all } d,$$

then so is any positive scalar multiple of (ω^*, γ^*).

If (ω^*, γ^*) is optimal, then

$$E^{\hat{d}} = \frac{\gamma_1^* Y_{1,\hat{d}} + \gamma_2^* Y_{2\hat{d}} + ... + \gamma_M^* Y_{M\hat{d}}}{\omega_1^* Y_{1,\hat{d}} + \omega_2^* Y_{2\hat{d}} + ... + \omega_N^* Y_{N\hat{d}}} \text{ and } \frac{\gamma_1^* Y_{1d} + \gamma_2^* Y_{2d} + ... + \gamma_M^* Y_{Md}}{\omega_1^* Y_{1,d} + \omega_2^* Y_{2d} + ... + \omega_N^* Y_{Nd}} \leq 1 \text{ for all } d.$$

Substituting $\alpha(\omega^*, \gamma^*)$, where α is a positive scalar, yields

$$E^{\hat{d}} = \frac{\alpha\gamma_1^* Y_{1,\hat{d}} + \alpha\gamma_2^* Y_{2\hat{d}} + ... + \alpha\gamma_M^* Y_{M\hat{d}}}{\alpha\omega_1^* Y_{1,\hat{d}} + \alpha\omega_2^* Y_{2\hat{d}} + ... + \alpha\omega_N^* Y_{N\hat{d}}} \text{ and } \frac{\alpha\gamma_1^* Y_{1d} + \alpha\gamma_2^* Y_{2d} + ... + \alpha\gamma_M^* Y_{Md}}{\alpha\omega_1^* Y_{1,d} + \alpha\omega_2^* Y_{2d} + ... + \alpha\omega_N^* Y_{Nd}} \leq 1 \text{ for all } d,$$

$$E^{\hat{d}} = \frac{\alpha\left(\gamma_1^* Y_{1,\hat{d}} + \gamma_2^* Y_{2\hat{d}} + ... + \gamma_M^* Y_{M\hat{d}}\right)}{\alpha\left(\omega_1^* Y_{1,\hat{d}} + \omega_2^* Y_{2\hat{d}} + ... + \omega_N^* Y_{N\hat{d}}\right)} \text{ and } \frac{\alpha\left(\gamma_1^* Y_{1d} + \gamma_2^* Y_{2d} + ... + \gamma_M^* Y_{Md}\right)}{\alpha\left(\omega_1^* Y_{1,d} + \omega_2^* Y_{2d} + ... + \omega_N^* Y_{Nd}\right)} \leq 1 \text{ for all } d.$$

Upon canceling α, this becomes

$$E^{\hat{d}} = \frac{\gamma_1^* Y_{1,\hat{d}} + \gamma_2^* Y_{2\hat{d}} + ... + \gamma_M^* Y_{M\hat{d}}}{\omega_1^* Y_{1,\hat{d}} + \omega_2^* Y_{2\hat{d}} + ... + \omega_N^* Y_{N\hat{d}}} \text{ and } \frac{\gamma_1^* Y_{1d} + \gamma_2^* Y_{2d} + ... + \gamma_M^* Y_{Md}}{\omega_1^* Y_{1,d} + \omega_2^* Y_{2d} + ... + \omega_N^* Y_{Nd}} \leq 1 \text{ for all } d,$$

The same result for the case of (ω^*, γ^*) above. Hence, any positive scalar multiple of (ω^*, γ^*) is optimal.

5) Show that

$$\begin{aligned} E^{\hat{d}} &= \max \sum_{m=1}^{M} \gamma_m Y_{m\hat{d}} \\ &\text{such that} \\ &\sum_{m=1}^{M} \gamma_m Y_{md} - \sum_{n=1}^{N} \omega_n X_{nd} \leq 0, \\ &\sum_{n=1}^{N} \omega_n X_{n\hat{d}} = 1, \\ &\omega \geq 0, \\ &\gamma \geq 0, \end{aligned}$$

is feasible by constructing γ_m and ω_n that satisfy the constraints. You may assume X_{md} and Y_{md} are strictly positive.

Let

$$\omega_n = \begin{cases} \frac{1}{X_{1\hat{d}}}, & n = 1 \\ 0, & n \neq 1 \end{cases}$$

$$\gamma_m = \begin{cases} \frac{\min_d X_{1d}}{X_{1\hat{d}} \max_d Y_{1d}}, & m = 1 \\ 0, & m \neq 1 \end{cases}$$

Thus, ω_n and γ_n are both positive. Notice, since $\frac{Y_{1d}}{\max_d Y_{1d}} \leq 1$ by definition of max and all Y_{md} are non-negative, and $\min_d X_{1d} - X_{1d} \leq 0$ by the definition of minimum,

$$\sum_{n=1}^{N} \omega_n X_{n\hat{d}} = \frac{1}{X_{1\hat{d}}} X_{1\hat{d}} = 1, \text{ and}$$

$$\sum_{m=1}^{M} \gamma_n Y_{md} - \sum_{n=1}^{N} \omega_n X_{nd} = \frac{\min_d X_{1d}}{X_{1\hat{d}}} \left(\frac{Y_{1d}}{\max_d Y_{1d}} \right) - \frac{1}{X_{1\hat{d}}} X_{1d}$$

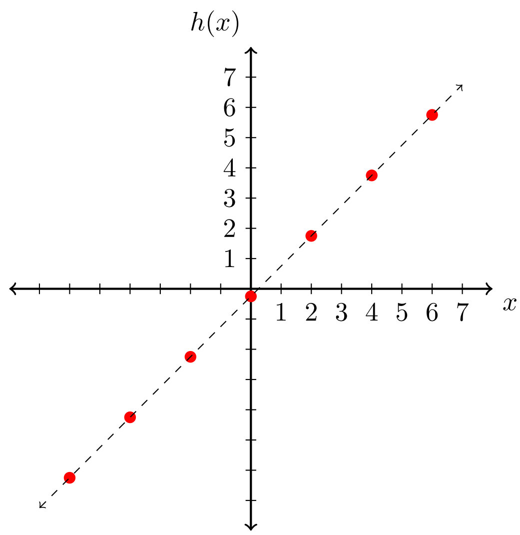
$$\leq \frac{\min_d X_{1d}}{X_{1\hat{d}}} - \frac{X_{1d}}{X_{1\hat{d}}} = \frac{1}{X_{1\hat{d}}} \left(\min_d X_{1d} - X_{1d} \right) \leq \frac{1}{X_{1d}}(0) = 0.$$

Thus, the constraints of (2) are met by the chosen ω_n and γ_m.

6) Set $h(x) = \min_y y^2 + y + x$. Sketch a graph of h and argue that it is differentiable.

Tabulating some values of $h(x)$ and plotting gives:

x	y
-6	-25/4
-4	-17/4
-2	-9/4
0	-1/4
2	7/4
4	15/4
6	23/4



To show that $h(x)$ is differentiable, we will rewrite the optimization problem as

$$h(x) = \beta^2 + \beta + x, \text{ where } \beta \text{ satisifies, } \frac{\partial}{\partial y}(y^2 + y + x)\big|_\beta = 0.$$

We can recognize this as a minimum because $\frac{\partial^2}{\partial y^2} = 2$, which is strictly positive for all x and y meaning $h(x)$ is strictly concave up, and β satisfies a minimum of $h(x)$. Solving the right equality for β results in

$$\frac{\partial}{\partial y}(y^2 + y + x)\big|_\beta = 2\beta + 1 = 0,$$

$$\beta = -\frac{1}{2}.$$

Substituting $\beta = -\frac{1}{2}$ into $h(x)$ gives

$$h(x) = \beta^2 + \beta + x,$$

$$= \left(-\frac{1}{2}\right)^2 - \frac{1}{2} + x,$$

$$= -\frac{1}{4} + x,$$

which is known to have continuous derivatives, and is hence differentiable.

7) Assuming that the maximums and minimums exists, show that

$$\min_x \max_y F(x,y) \geq \max_y \min_x F(x,y)$$

where $F(x,y)$ is a real valued function of the vectors x and y.

Let us first define

$$A(x) \equiv \max_y F(x,y).$$

As a result,

$$A(x) \geq F(x,y)$$

for all x and y. It also holds that

$$\min_x A(x) \geq \max_y F(x,y)$$

for all x. Implying

$$\min_x A(x) \geq \max_y \min_x F(x,y)$$

$$\min_x \max_y F(x,y) \geq \max_y \min_x F(x,y)$$

$\square$

12) Add a fourth DMU to the problem shown in Figure 1 so that optimal λ vectors for DUMs two and three can have more than one positive component. What characteristic of this fourth DMU is mandatory to have this property?

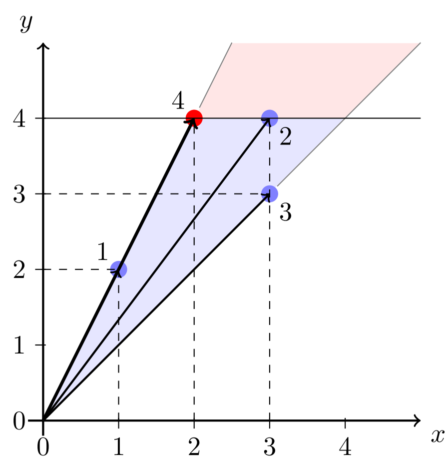


Figure 1: DEA Model From Notes With Additional DMU 4

Notice that, since DMU one and DMU four are linearly dependent, DMUs two and three can use any linear combination of DMUs one and four, which are efficient since they cannot be "pushed" left any more, to maintain the same $y-$value. Thus, the λ vector can have more than one positive component. It is required that DMU four is efficient, or else only one component, λ_1, is in the vector of optimal λ's, because adding a linear combination of λ_4 would make the result inefficient. If DMU one becomes inefficient, then only λ_4 will be in the optimal λ's. Thus, it is a requirement that DMU four is a scalar multiple of DMU one.

13) Consider the DEA problem with five DMUs, each have a single input and output as noted by

$$X = [9, 4, 3, 6, 5] \text{ and } Y = [7, 3, 2, 6, 6].$$

Sketch the DMUs on a graph with the horizontal axis being the input and the vertical axis being the output. Be sure to label each DMU as a point on this graph. Calculate the efficiency scores of each DMU for both the original and updated models.

Let us first label our $X's$ and $Y's$ with indices n and m as

n	1	2	3	4	5
X	9	4	3	6	5
m	1	2	3	4	5
Y	7	3	2	6	6

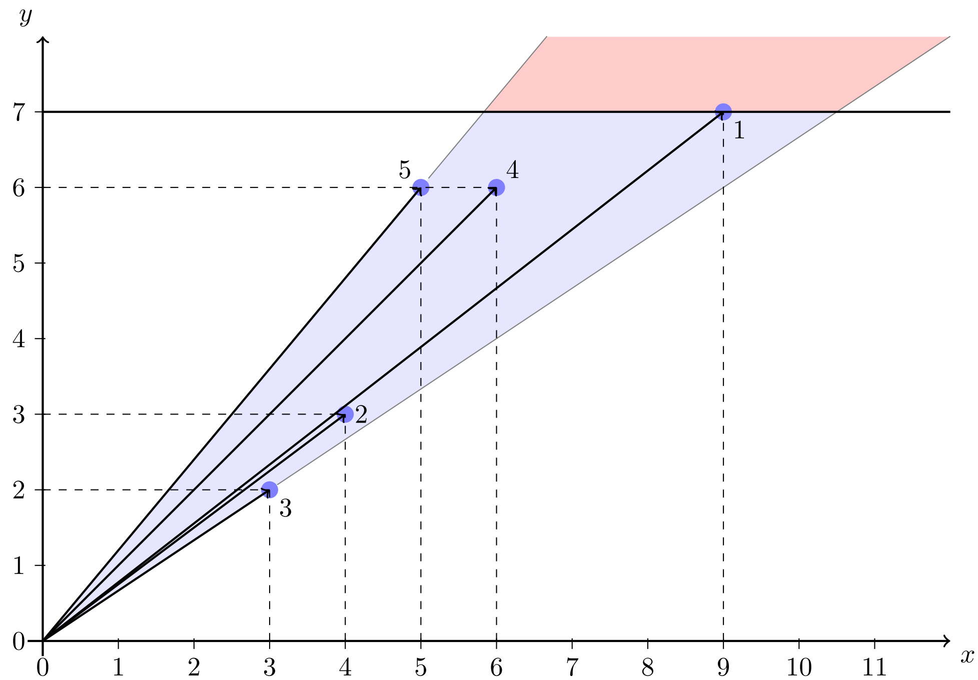


Figure 2: Geometric Representation of Original DEA Model

In this original model, $E^5 = 1$ by inspection. The constraints of the associated optimization problem to find the other E^d are the components of

$$D\lambda = \begin{bmatrix} 9 & 4 & 3 & 6 & 5 \\ 7 & 3 & 2 & 6 & 6 \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix},$$

$$= \begin{pmatrix} 9 \\ 7 \end{pmatrix}\lambda_1 + \begin{pmatrix} 4 \\ 3 \end{pmatrix}\lambda_2 + \begin{pmatrix} 3 \\ 2 \end{pmatrix}\lambda_3 + \begin{pmatrix} 6 \\ 6 \end{pmatrix}\lambda_4 + \begin{pmatrix} 5 \\ 6 \end{pmatrix}\lambda_5,$$

with $\lambda_i \geq 0$. In this model, efficiency is limited by DMU 5, so the optimal λ is $(0,0,0,0,\lambda_5)^T$. The efficiency for DMU 4 is hence found by solving

$$\begin{pmatrix} 5 \\ 6 \end{pmatrix}\lambda_5 = \begin{pmatrix} 6\theta_4 \\ 6 \end{pmatrix}$$

for θ_4. Similarly, solving

$$\begin{pmatrix} 5 \\ 6 \end{pmatrix}\lambda_5 = \begin{pmatrix} 3\theta_3 \\ 2 \end{pmatrix}, \ \begin{pmatrix} 5 \\ 6 \end{pmatrix}\lambda_5 = \begin{pmatrix} 4\theta_2 \\ 3 \end{pmatrix}, \ \begin{pmatrix} 5 \\ 6 \end{pmatrix}\lambda_5 = \begin{pmatrix} 9\theta_1 \\ 7 \end{pmatrix}$$

for θ_3, θ_2 and θ_3 yields the efficiencies of DMUs one, two, and three. The efficiencies of each DMU are then

DMU	1	2	3	4	5
Efficiency	$\frac{35}{54}$	$\frac{5}{8}$	$\frac{5}{9}$	$\frac{5}{6}$	1

Interpreting these results yields no ranking of DMUs, but only that DMUs one, two, three, and four are inefficient while DMU five is efficient using this metric.

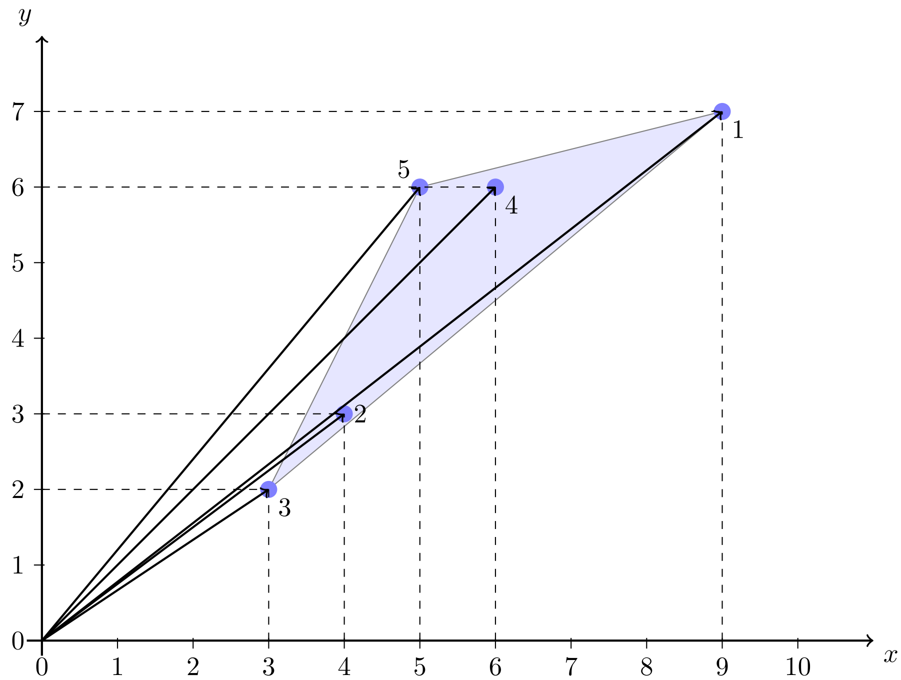


Figure 3: Geometric Representation of Modified DEA Model

In this updated model, E^1, E^3, and E^5 are all 1 by inspection. Similarly to the original model, the constraints of the associated optimization problem to find the other E^d are the components of

$$D\lambda = \begin{bmatrix} 9 & 4 & 3 & 6 & 5 \\ 7 & 3 & 2 & 6 & 6 \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix},$$

$$= \begin{pmatrix} 9 \\ 7 \end{pmatrix}\lambda_1 + \begin{pmatrix} 4 \\ 3 \end{pmatrix}\lambda_2 + \begin{pmatrix} 3 \\ 2 \end{pmatrix}\lambda_3 + \begin{pmatrix} 6 \\ 6 \end{pmatrix}\lambda_4 + \begin{pmatrix} 5 \\ 6 \end{pmatrix}\lambda_5,$$

with $\lambda_i \geq 0$. However, in this updated model DMUs one, three, and five all contribute to the efficient frontier. The optimal values of both DMUs two and four will lie on the frontier at or between DMUs one, three and five, so using this updated model, the optimal λ is $(0, 0, \alpha, 0, 1-\alpha)^T$. The efficiency of DMU four is hence found by solving

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix}\alpha + \begin{pmatrix} 5 \\ 6 \end{pmatrix}(1-\alpha) = \begin{pmatrix} 6\theta_4 \\ 6 \end{pmatrix}.$$

for θ_4. Similarly, the efficiency of DMU two can be found by solving

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix}\alpha + \begin{pmatrix} 5 \\ 6 \end{pmatrix}(1-\alpha) = \begin{pmatrix} 4\theta_2 \\ 3 \end{pmatrix}.$$

for θ_2. The efficiencies of each DMU are then

DMU	1	2	3	4	5
Efficiency	1	$\frac{7}{8}$	1	$\frac{5}{6}$	1

Like before, interpreting these results yields no ranking of DMUs, but only that DMUs two and four are inefficient while DMUs one, three, and five are efficient using this metric.