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DEPARTMENT OF MATHEMATICS

COURSE CODE: BMA1106

COURSE TITLE: FOUNDATION MATHEMATICS

Instructional manual for BBM – Distance Learning

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COURSE OUTLINE

FOUNDATION MATHEMATICS - BBM: 112
Credit Hours: 3 Credits
Contact Hours: 45 Hours

PURPOSE: the learner to acquire Basic mathematical concepts in order to prepare for the advanced courses and acquiring positive attitude, knowledge and skills which will be relevant to her/his life in the university.

COURSE OBJECTIVES:

By the end of the course the learners should be able to:-

(a) Develop a positive towards learning mathematics
(b) Perform mathematical operations and manipulations with confidence, speed and accuracy
(c) Think and reason precisely, logically and critically in any given situation
(d) Develop investigative skills in mathematics
(e) Differentiate any given function
(f) Apply the techniques of differentiation to determine the optimum values and rate of change
(g) Apply differentiation in dynamics and higher derivatives
(h) Define integration as a reverse of differentiation
(i) Define, state and use the techniques of integration to any function

COURSE CONTENT

NUMBERS

Numbers: Natural numbers, integers, rational numbers, real numbers, Surds- a rational number , rationalizing denominator, Sets and sets operations

Logarithms: Indices – exponents laws, Logarithms – laws of logarithms, change of base, applications of logarithms and exponential to conversion of non- linear form, exponential and logarithmic functions

ALGEBRA

Equations and inequalities: Simple linear equation –linear inequalities, straight line graphs, algebraic fraction, simultaneous linear equation, Quadratic equations – expansion, factorization, completing square, formula and discriminant, simultaneous equations, factor\remainder and its application to solutions of higher degree, disguised quadratic and linear equations involving exponentials and logarithms

Expansions: Permutations and combinations, Binomial expansions and their applications

Functions: Mapping- domain, range and co-domain, Composite function, inverse function, Transforming of graphs of a function.,

Matrices: Matrix algebra, Definition of a matrix, operations with matrices, matrix method of solving linear simultaneous equations, transition matrices

DIFFERENTIONS

Derivative of functions; Derivatives of polynomials, logarithms and exponentials

Different ion rules: Sum, product, quotient and chain rules

Optimum: Maximum, minimum and point of inflexion, Increasing and decreasing function.
Anti –differentiation: Definite and indefinite integrals using inspection , summation
Techniques of integration: Substitutions, partial fractions, integration by parts

EVALUATION
Assignments and CATS 30%; Final examination - 70%; Total -100%

Teaching/ Learning methods: lectures and tutorials, group discussion, demonstration and assignments

Recommended text books

a) Pure mathematics 1 by backhouse
b) Basic mathematics by Brittinger and Keedy
c) Finite mathematics by Howard L. Rolf

CHAPTER ONE: NUMBERS

Introduction

The word "number" came from the Latin word "numerus" which mean "to apportion", to assign, to allot, and to take. Therefore a number is something that measured the size of an allotment.

Numeral is any symbol that can represent a number. E.g. five is a number represented by numeral 5 or v.

There are various types of numbers – Natural numbers, Integers, Rational, Real and complex numbers.

In this course, we will just define complex numbers; however, other numbers systems will be studied in length by looking at its compositions, the algebra of real numbers properties in operation of real numbers and sets of real numbers.

Man started with counting numbers 1, 2, 3, 4, 5…10, 11, 12……………denoted by IN

These numbers are usually referred to as natural numbers or positive integers or positive whole numbers. They developed negative integers including zero ……………..-7, -6, -5, -4 -3 -2 -1, 0. When we combine the positive and negative integers we have the combination called integers or whole numbers. -7, -6, -5, -4 -3 -2 -1, 0 1, 2, 3, 4,5, 6 ,7,8 9, 10, 11, 12 ………., denoted by Z

Further development of fractions as part of a whole which is written as $\frac{numerator}{denominator}$ gives proper and improper fractions as the denominator can be less than numerator or greater than numerator respectively.

17, -4, $\frac{-2}{3}$, $\frac{3}{4}$, 0, 0.66, $-5\frac{3}{8}$, $1\frac{5}{7}$ ……………

These numbers are called Rational numbers, denoted by Q

It is interesting to observe that all rational numbers can be expressed as terminating and recurring decimals whereas some cannot be expressed as $\frac{a}{b}$, where a and b are integers are called irrational numbers examples being $\sqrt{2}, 1+\sqrt{3},$ π, e etc

The combination of rational and irrational are called real numbers, denoted by IR

A skilled mathematician should be able to use real numbers using arithmetic rules

SURDS

You know that $\sqrt{16} = 4$ and that $\sqrt{\frac{1}{9}} = \frac{1}{3}$. These are examples of rational numbers.

However, $\sqrt{2}$ cannot be expressed as a fraction of two integers. $\sqrt{2}$, is an example of an irrational number. Roots such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, ……………. are called surds. The solutions to mathematical problems often contain surds. You should use your calculator to work out a decimal equivalent of the surds, but the decimals goes on and on. An answer rounded to, say, three decimal places is not accurate. It is accurate in surd form.

Some properties of surds

- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
- $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$
- $a\sqrt{c} \pm b\sqrt{c} = (a \pm b)\sqrt{c}$

Note that a, b and c are integers.

Examples

Simplify each of these quantities.

a) $\sqrt{48}$ b) $3\sqrt{50} + 2\sqrt{18} - \sqrt{32}$

(a) To simplify √48, we rewrite the quantity in its largest square factor of 48

$$48 = 16 \times 3 = 4^2 \times 3$$

There,

$$\sqrt{48} = \sqrt{4^2 \times 3} = 4\sqrt{3}$$

Similarly

b) $3\sqrt{50} + 2\sqrt{18} - \sqrt{32} = 3\sqrt{25\times 2} + 2\sqrt{9\times 2} - \sqrt{16\times 2}$ (25, 9 and 16 are the largest square factors of 50, 18 and 32)

$$= 15\sqrt{2} + 6\sqrt{2} - 4\sqrt{2}$$

$$= 17\sqrt{2}$$

When surds appear in the denominator of a fraction, is usual to eliminate them. This is called rationalising the denominator.

For example, to rationalize the fraction $\dfrac{3}{2\sqrt{7}}$ multiply its numerator and denominator by $\sqrt{7}$:

$$\frac{3}{2\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{3\sqrt{7}}{2\times 7} = \frac{3\sqrt{7}}{14}$$ (You can obtain equivalent fraction by multiplying the numerator and denominator by the same amount)

To rationalize a fraction $\dfrac{1}{a \pm \sqrt{b}}$, multiply its numerator and denominator by a ∓ √b

Express the fraction $\dfrac{2+\sqrt{3}}{1-\sqrt{3}}$

Multiply numerator and denominator by 1 + √3 gives

$$\frac{(2+\sqrt{3})(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})} = \frac{2+2\sqrt{3}+\sqrt{3}+3}{1-3}$$

$$= \frac{5+3\sqrt{3}}{-2}$$

$$= -\frac{5}{2}-\frac{3}{2}\sqrt{3}$$

Exercise 1

1. a) $\sqrt{112}$ b) $\sqrt{98}$ c) $\sqrt{75}+2\sqrt{27}$ d) $\sqrt{32}+\sqrt{128}-\sqrt{200}$

2. Simplify the following expression

   a) $(1 + \sqrt{3})(1 - \sqrt{3})$

   b) $(-5 + \sqrt{7})(5 - \sqrt{7})$

   c) $(\sqrt{2} + 5\sqrt{3}) (\sqrt{2} - 5\sqrt{3})$

   d) $(4\sqrt{2} - 7\sqrt{3}) (5\sqrt{2} + 6\sqrt{3})$

2. Express $\dfrac{\sqrt{(64)}-\sqrt{18}}{\sqrt{16}+\sqrt{18}}$, in the form a + b$\sqrt{2}$, where a and b are integers.

3. (a) (i) Express $\dfrac{1}{\sqrt{32}}$ as a power of 2.

   (ii) Express $(64)^{\frac{1}{x}}$ as powers of 2

   (iii) Hence solve the equation $\dfrac{(64)^{\frac{1}{x}}}{2^x} = \dfrac{1}{\sqrt{32}}$

4. a) $\dfrac{10\sqrt{7}}{\sqrt{5}}$

   b) $\dfrac{4\sqrt{45}}{5\sqrt{8}}$

5. Rationalize the following surds

   a) $\dfrac{1}{2-\sqrt{3}}$

   b) $\dfrac{3+\sqrt{24}}{2+\sqrt{6}}$

Real Numbers

A real number is any number that can be written as a decimal. Real numbers constitute the rational and irrational numbers.

Rational Numbers

The root of the word rational is "ratio". The number $\frac{2}{3}$ is a ratio of 2 or 3. In general, a rational number is a number of the form $\frac{m}{n}$, where m and n are integers, terminating decimals and recurring decimals. This is because the set of each numbers above can be divided by a non-zero integer.

e.g. $3 = \frac{3}{1}$

$-4 = \frac{-4}{1}$

$2.12 = \frac{212}{100}$

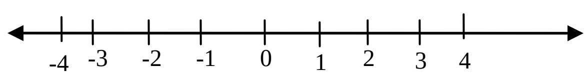
$0.579 = \frac{579}{1000}$

$0.3333\ldots = \frac{1}{3}$

$0.2727\ldots = \frac{3}{11}$

Activity

Convert the recurring decimal 0.324324… to a fraction. Integers constitute both the negative whole numbers. They are represented in a number line as follows.



The positive whole numbers are also called the counting or natural numbers.

Complex Numbers

Its important to note that, its possible to take the root of any nonnegative real number. However, for negative numbers some are possible, while other are not.

e.g. $\sqrt[3]{8} = -2$ since $(-2)^3 = -8$

$\sqrt[3]{-1} = -1$ Since $(-1)^3 = -1$

But consider $\sqrt{-1}$ we let $i = \sqrt{-1}$. Then $i^2 = -1$ But there is no such real number i, since for any real number r, $r^2 \geq 0$ consequently, $i\sqrt{-1}$ is not a real number.

The number i is the prototype imaginary number. Any number of the form $r\sqrt{-1} = ri$, where r is a real number, is a pure imaginary number.
The collection of numbers of the form $a+bi$, where a and b are real numbers forms the set of complex numbers.

e.g. $2+3i$

$\frac{-3}{4}+5i$

$\sqrt{-4}$

$-\pi-\frac{7}{6}i$

In the complex number, the six operations f addition, subtraction, multiplication, division by non zero, exponentiation and extraction of roots are always possible.

e.g. solve $\sqrt{-16}$

we let $-16 = 16\times 1$

Therefore $\sqrt{-16} = \sqrt{16\times -1}$

$= \sqrt{16}\times\sqrt{-1}$

but we know $\sqrt{-1} = i$ and $\sqrt{16} = 4$

$= 4i$

Example

Consider the following subset of real numbers.

$\left\{-7, -\sqrt{6}, 2, {}^{2}\!/_{5}, 0, \sqrt{2}, \pi\right\}$

List the numbers in this set that are

a) Natural numbers
b) Integers
c) Rational numbers
d) Irrational numbers

Solution

a) $\{2, 7\}$

b) $\{-7, 2, 0, 7\}$

c) $\left\{-7, 2, {}^{3}\!/_{5}, 0, -{}^{1}\!/_{4}, 7\right\}$

d) $\left\{\sqrt{6}, \sqrt{2}, \pi\right\}$

Basic Notation in Real numbers

The language of algebra is a language of symbols. We consider some of the notations used for the operations and relations defined on the real numbers

If a and b represent real numbers, there are four basic arithmetic operations that can be performed.

Addition $a+b$

Subtraction $a-b$

Multiplication $ab, a \cdot b, a(b), a \times b$

Division $a \div b, a/b$

Special notation is also used to express various relationships between numbers.

$a = b$ a is equal to b

$a \neq b$ a is not equal to b

$a < b (or\ b > a)$ a is less than b (or b is greater than a)

$a \leq b (or\ b \geq a)$ a is less than or equal to b (or b is greater than or equal to a)

Note: the relationship < and ≥ called order relationships, are defined by the following rules:

$a < b$ means b – a is positive

$a \leq b$ means b – a is nonnegative

Exercises2

1. Determine which numbers in the set are
   a) Natural numbers
   b) Integers
   c) Rational numbers
   d) Irrational numbers
      i). $-8, -8/3, 4, 3/4, \sqrt{3}, 0, 1$
      ii). $9, -12, -13/2, \sqrt{4}, 4.21, 1/3, \pi.$
      iii). $13, -12, \sqrt{9}, \sqrt{7}, 5/4$
2. Plot the following numbers as points on a real line.
   a). 3 b) -5 c) $3/4$ d) $-3/4$ e) $1\,1/2$ f). $8/4$
3. Determine whether each of the following is true or false.
   a) $6 < 2$ b) $0 < 2$ c) $5 \geq 6$ d) $3 \leq 3$ e) $4 \neq 4$
4. Show that the rational number 216/99 has a group of two digits that repeat in its decimal expansion.

Properties of Real Numbers

The basis operations of the number system are addition and multiplication. Each of these operations is called a binary operation because it pairs any two real numbers with a third real number.

The properties of these operations in real numbers all stem from a few basic statements, called *axioms* or *postulates,* statements that are assumed to be true.

i). **Commutative properties** allow you to add or multiply numbers in any order:

$a+b=b+a$ $\quad$ $5+3=3+5=8$

$ab=ba$ $\quad$ $2.5=5.2=10$

ii). **Associative properties** allow you to change the way in which group quantities to be added or multiplied.

$(a+b)+c=a+(b+c)$ $\quad$ $3+4+1$ $\quad$ $=3+(4+1)$ $\quad$ $=(3+4)+1$

$(ab)c \quad =a(bc)$ $\quad$ $3.4.6$ $\quad$ $=3(4.6)$ $\quad$ $=(3.4)6=72$

Note There are no associative properties for either subtraction or division.
For example $(9-6)-1=9-(6-1)$ is **wrong.**

iii). **Distribute properties** tie together, multiplication and addition (or subtraction).

$a(b+c) \quad =ab+ac.$ $\quad$ $4(3+2)=4.3+4.2=20$

$a(b-c) \quad =ab-ac$

iv). **Identify properties** single out two special numbers 0 and 1.

$0+a \quad =a$ Identify property of addition

$1.a \quad =a$ Identify property of multiplication

v). **Inverse properties**

- Every number a has a unique opposite or negative $-a$ as such that $a+(-a)=0$ (inverse property for addition)
- Every number $a\neq 0$ has a unique reciprocal $1/a$ such that $a.1/a=1$ (inverse property for multiplication).

Equality

Equality of two expressions means that both expressions are same for same quantity.

Properties of Equality

1. Reflective : $a=a$
2. Symmetric : if $a=b$, then $b=a$
3. Transitive: if $a=b$ and $b=c$, then $a=c$

Algebra Of Sets Of Real Numbers:

Given two sets *A* and *B*, the set consisting of all those elements which belong to both
A and *B* is called the intersection of *A* and *B*, denoted by $A\cap B$.

e.g. if $A=\{2,4,6,8,10,12\}$ and

$B=\{3,6,9,12\}$

then $A\cap B\{6,12\}$

The set consisting of those element which belong to A or B, or both, is called the union of A and B and is denoted by $A \cup B$

Thus $A \cup B = \{2,3,4,6,8,9,10,12\}$

Exercise:

Given that $A = \{1,2,3,3,4,5\}$, list the members of the following sets

$(a)\{x^2 : x \in A\}$ $(b)\{1/2 : x \in A\}$ $(c)\{x/2 :< 10\}$

Definition

The **Universal set,** denoted by ε, represent all the set of all elements under consideration.

The **complement** of a set A is a set of all the elements of the universal set ε which are not contained in a given set A, and is denoted by A'.

e.g. $\varepsilon = \{1,2,3,4,5,6\}$, $= \{4,5,\} \Rightarrow x^1 = \{1,2,3,6\}$

The notation $n(A)$ is used to denote the number of elements in set A. E.g. If $A = \{1,2,3\}$, $B\{2,9,10,11\}$

then $n(A) = 3$, $n(B) = 4$ $n(A \cap B) = 1$ $n(A \cup B) = 8$

Notice that

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Exercise3

1. Given that ε is the set of opposite integers less than 100 and the set A and B are subsets of ε. A is the set of multiple of 2 and B is the set of multiples of 7.
   a. List the member of $A, B, A \cap B, A \cup B$.
   b. Write down the value of $n(A) = n(B), n(A \cup B), n(A \cap B)$ and verify that $n(A \cup B) = n(A) + n(B), n(A \cap B)$

   List down A' and $B', A' \cap B', A' \cup B'$
2. State (a) the additive inverse (b) the multiplicative inverse (if it exists) for each of the following numbers.

   $(\text{i}) - 6$ $(ii) - 2/3$ $(\text{iii}) 0.2$ $(\text{iv}) - 2000$ $(\text{v}) - (-1)$

   (iv). 0 (No multiplicative inverse)
3. Find the values of x for which the following expressions do not exist.

   (i). $\frac{2x+5}{x^2 - x - 6}$ (ii). $\frac{x}{2x+5}$ (iii). $\frac{x}{x^2 - 5}$ (iv). $\frac{1}{x^2 - 3}$ (v) $\frac{1}{x^2 + 8x + 15}$

Definition of open interval

$(a,b) = \{x : a < x < b\}$

The numbers a and b are called the end points of the interval. The graph of the open interval (a, b) consist of all points on a co-ordinate line the lie between the points corresponding to a and b.

NB

A bracket is used instead of parenthesis to include the end points. If $a < b$, then closed interval denoted by $[a,b]$ and half open intervals denoted by $[a,b]$ or $[a,b]$ are define as follows.

$[a,b] = \{x : a \le x \le 6\}$

$[a,b] = \{x : a < x \le b\}$

We can also employ certain infinite intervals.

$(-\delta, a) = \{x : -\infty < x < a\}$

$(-\infty, a) = \{x : x < a\}$

$(a, \infty) = \{x : x > a\}$

$(a, \infty) = \{x :\ge a\}$

The set $\Re$ of real numbers is sometimes denoted as $(-\infty, \infty)$

Definition:

If a $\in \Re$ then absolute value of *a*, denoted by $|a|$

Is defined as

$$|a| = \begin{cases} a \text{ if } a \ge 0 \\ -a \text{ if } a < 0 \end{cases}$$

e.g. $|3| = 3, |-3| = -|3| = 3$

$\left|\sqrt{2} - 2\right| = -\left(\sqrt{2} - 2\right) = 2 - \sqrt{2}$

It can be shown that $|a| = |-a|, \forall a \in \Re$

We use the concept of absolute value to define the distance between any two points on the co-ordinate line.

Definition

Let *a* and *b* be the co-ordinates of two points *A* and *B*, respectively on a co-ordinate line *l*. The distance between *A* and *B* denoted *d*(*A*, *B*) is $d(A, B) = |b - a|$.

It is also called the length of the line segment

Note that

$d(B, A) = |a - b|$ and $|b - a| = |a - b|$

$\Rightarrow d(A, B) = d(B, A)$

$d(0, A) = |a - 0| = a$

Example

Let *A* and *B* have co-ordinates –5, –3, find *d*(*A*, *B*)

Solution

$d(A, B) = |-3 - (-5)| = |-3 + 5| = |2| = 2$

Properties of absolute values

$x \geq 0$; $\quad x = 0 \quad$ iff $x = 0$

$xy = xy$

$x/y = x/y, y \neq 0$

The triangular inequalities hold

i.e. $|x + y| \leq |x| + |y|$

$x^2 = |x|^2$,

$|x - y| \leq |x| - |y|$

$|x| < a$ iff $-a < x < a$

$|x| > a$, iff $x > a$ or $x < -a$

$|x| = a$, iff $x = a$ or $x = -a$

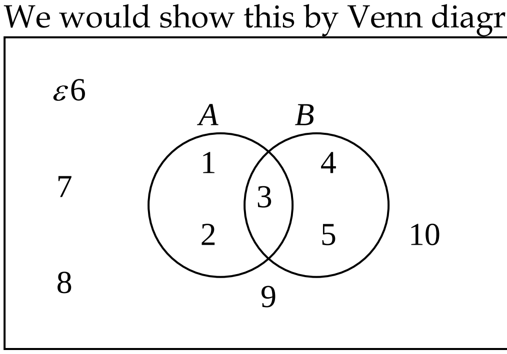
Venn Diagrams

Sets and set operations can conveniently be presented in diagrams. The basic idea is to represent the universal set by a rectangle and other sets by circles within the rectangle.

The Venn diagrams typically serve to describe relationships among two or more sets.

Consider the various ways two sets could be depicted. In general, *A* and *B* could be such that $A \cap B = \phi$, $A = B$, $A \subset B$, $B \subset A$ or they have some, but not all elements in common. Each of these could be shown separately, but convection dictates that set *A* and *B* be illustrated as overlapping. For example, if $\varepsilon = \{1, 2, 3, \ldots, 10\}$, $A = \{1, 2, 3\}$, $B = \{4, 3, 5\}$.

We would show this by Venn diagram as



NB

i). Elements in the intersecting region is the set of intersection of *A* and *B*, $A \cap B$.

ii). All elements in the two circles is the set of union of *A* and *B*; $A \cup B$.

iii). Elements outside the circles is the set of compliment of the union of A and B; $(A \cup B)$.

iv). Element outside circle A is the compliment of A; A'.

Example

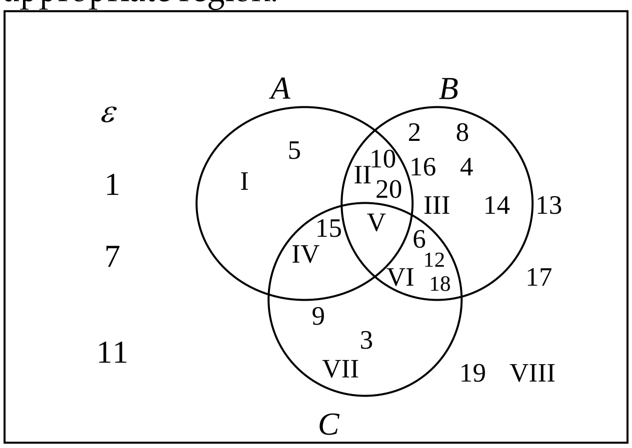
Let $\varepsilon = \{1, 2, 3, \ldots, 20\}$

$A = \{5, 10, 15, 20\}$

$B = \{2, 4, 6, 8, 10, 12, 14, 18, 20\}$.

$C = \{3, 6, 9, 12, 15, 18\}$.

Draw a Venn diagram, label the regions and place each element in the appropriate region.



Note that region V, which represents $(A \cap B) \cap C$ is empty. i.e. $(A \cap B) \cap C = \phi$

Region VIII represents $((A \cup B) \cup C)' = \{1, 7, 11, 13, 17, 19\}$.

Exercise 4

In the 1992 summer Olympic games, 37 countries won gold medals, 44 won silver medals, 54 won bronze medals, 30 won both gold and silver medals, 33 won both gold and bronze medals, 36 won silver and bronze medals and 28 won gold, silver and bronze medals.

Required

Represent this information using Venn's diagram and find;

a) How many countries won only gold medals?
b) How many countries won only silver medals?
c) How many countries won only bronze medals?
d) How many countries won gold and silver medals but no bronze medals?
e) How many countries won gold and bronze medals but no silver medals?
f) How many countries won silver and bronze medals but no gold medals?

Suggested References

1. Backhouse, J.K. et.al, (2005) Pure Mathematics Volume1, Longman, pg 22-24
2. Blitzer, Robert. (2004) College Algebra, Prentice Hall pg 1 -35

CHAPTER TWO: EQUATIONS AND INEQUALITIES

Linear Equations:

An algebraic expression is a symbolic mathematical statement. A symbolic statement that equates two expressions is called an equation. (i.e equation have equal symbol that separate the two expressions).

Linear equations:

Linear equation are equation of order one. i.e The highest power of the unknown variable is one.

For example $x+3=17, \quad 2m-5=25, \quad \frac{n}{4}-3=8$ are Linear equations

$x^2+4=8, \quad 2y^3-y=7$, are not linear equations.

Solving linear equations:

Solving an equation involves determining the set of all real numbers for which the equation is true.

Remark:

Linear equations have exactly one solution value for its variables.

Since an equation states equality of two things, what is done on one side is also done on the other side.

Example 1

Solve the equation $8x-6=5x+9$

Solution

Add 6 to both sides, $\quad 8x-6+6=5x+9+6$

$$8x=5x+15$$

Subtract $5x$ from both sides

$$px-5x=5x-5x+15$$

$$3x=15$$

Divide both sides by 3

$$\frac{3x}{3}=\frac{15}{3}$$

i.e. $x=5$

The reason for adding 6 to both sides and subtracting $5x$ from both sides is to get rid of -6 on the Left Hand Side (LHS) and $5x$ on the (RHS). A simpler equation is obtained in which the L.H.S contains only the unknown and the RHS contains the known. The LHS of the equation $3x=15$ is divided by 3 to give x and the RHS is also divided by 3.

Example 2

Solve the equation

$3x+34-8x=11-9x-13$

Solution

$3x+34-8x=11-9x-13$

Simplify both sides

$$3x+8x+34=11-13-9x$$

i.e. $-5x+34=-2-9x$

Add $9x$ to both sides

$$-5x+34+9x=-2-9x+9x$$

$$4x+34=-2$$

Subtracting -34 on both sides

$$4x+34-34=-2-34$$

$$4x=-36$$

Dividing both sides by 4

$$\frac{4x}{4}=-\frac{36}{4}$$

Therefore $x=-9$

Example 3

Solve $\frac{2}{5}y+\frac{3}{4}=10+\frac{y}{2}$

Solution

LCM of 5, 4 and 2 = 20

Multiplying both sides by 20

$$20\left(\frac{2}{5}y+\frac{3}{4}\right)=20\left(10+\frac{y}{2}\right)$$

$$8y+15=200+10y$$

Subtracting $10y$ from both sides

$$8y+16-10y=200+10y-10y-2y+15=200$$

Subtracting 15 on both sides

$$-2y+15-15=200-15$$

$$-2y=185$$

Dividing both sides by -2

$$\frac{-2y}{-2}=\frac{185}{-2}$$

Therefore $y=-92\frac{1}{2}$

Note should there be a term like $1\frac{1}{2}x$ in the equation, always write it in the improper fraction form, as $\frac{3}{2}x$ and then proceed as in the above example.

Exercise 5

Solve the following equations

1. $x+113=153$
2. $x+8=-12$
3. $x-17=37$
4. $\frac{x}{8}=6$
5. $24x=72$
6. $\frac{x}{3}+3=3$
7. $24-8=p-3$
8. $3\frac{1}{2}+2\frac{1}{4}x=17\frac{1}{2}$
9. $1\frac{4}{5}x+\frac{3}{5}=1\frac{1}{5}x+3$
10. $\frac{12}{x}-\frac{1}{2}=1\frac{1}{3}+\frac{3}{x}$

Equations involving brackets

Recall that

i). $a+(b+c)=a+b+c$

ii). $a+(b-c)=a+b-c$

iii). $a-(b+c)=a-b-c$

iv). $a-(b-c)=a-b+c$

Use these rules to remove bracket before solving equations involving bracket.

Example 4

Solve the equation $9x-(4x-3)=11+2(2x-1)$

Solution

Removing brackets

$$9x-4x+3=11+4x-2$$

Simplifying both sides

$$5x+3=4x+9$$

Subtracting $4x$ and 3 from both sides

$$5x-4x=9-3$$

Therefore $x=6$

Example 5

Solve the equation $\frac{5x+2}{4}-\frac{3}{2}=\frac{7x-1}{3}$

Solution

Multiply both sides by 12 (LCM of 4, 2 and 3)

$$12\left(\frac{5x+2}{4}-\frac{3}{2}\right)=12\left(\frac{7x-1}{3}\right)$$

i.e. $3(5x+2)-12\times\frac{3}{2}=4(7x-1)$

$$3(5x+2)-18=4(7x-1)$$

Removing brackets

$$15x+6-18=28x-4$$

Simplifying LHS

$$15x-12=28x-4$$

Subtracting $28x$ from both sides and adding 12 to both sides

$$15x-28x=-4+12$$

i.e. $-13x=8$

Dividing both sides by -13

$$\frac{-13x}{-13}=\frac{8}{-13}$$

$$x=\frac{-8}{13}$$

Exercise: 6

Solve the following equations

1. $4d+(5-d)=7$
2. $12m+(1-7m)=23$
3. $3(5x-1)=4(3x+2)$
4. $\frac{3}{2}y-\frac{14-3}{5}=\frac{y-4}{4}$
5. $\frac{3x+1}{2}=\frac{4x-3}{3}+3$
6. $\frac{x-1}{7}-1=\frac{5x-1}{5}$

Problems leading to simple equations

To solve a word problem in which a number is to be bond

i). Introduce a letter to stand for a number to be found (unknown)

ii). Form an equation involving this letter by expressing the given information in symbols instead of words.

iii). Solve the equation to get the required number

Example 6

The sum two numbers is 120 and their difference is 18. Find the two numbers.

Solution

Let the smaller number be x then, the large number is $x+18$ sum of the two number is $x+(x+18)$

$$x+(x+18)=120$$
$$x+x+18=120$$
$$2x+18=120$$
$$2x=102$$
$$x=51$$

Thus, the smaller number is 51 and the larger number is $51+18=69$.

Example 7

When 55 is added to a certain number and the sum is divided by 3, the result is 4 times the original number. What is the original number?

Solution

Let the number be x

Adding 55 and dividing the sum by 3 gives

$$\frac{x-55}{3}=4x$$

Therefore $x+55=12x$

$$55=12x-x$$
$$55=11x$$
$$x=5$$

Thus, the number is 5

Exercise 7

1. When I double a number and add 17, the result is 59 what is the number?
2. When a number is added to 4 times itself, the result is 30. Find the number.
3. A rectangle is 3 times as long as it is wide. The total length round its boundary is $56cm$. Find its length and width.
4. Find a number such that when it is divided by 3 and 2 added, the result is 17.
5. A profit of ksh126, 000 is shared among 3 business partners, Ann, Bett and Charles. Charles gets ksh6000 more than Bett while Ann gets twice as much as Charles. Find how much each gets?
6. A man is 30 years old while his daughter is 4. In how many years time will the daughter be half the age of her father?

Simultaneous Linear Equations

Pairs of equations with two unknown variables, Consider the equation

$$3x+2y=12$$
$$4x-2y=2$$

Each equation contains unknown quantity x and y. The solutions of the equations are the values of x and y which satisfy both equations simultaneously. Equations such as these are called simultaneous equations. To solve simultaneous equations, we look for a pair of numbers which satisfies the two equations at the same time.

Solutions of simultaneous equations by elimination method

Solve the simultaneous equations

$3x+2y=12$ (1)

$4x-2y=2$ (2)

Solution

Since the LHS of an equation is always equal to the RHS, then $(3x+2y)$ and 12 in equation (1) can be used interchangeably. Also RHS, $(4x-2)$ and 2 in equation (2) represent the same quantity

Therefore, we can combine the given equation thus

$$\begin{aligned}(3x+2y)+(4x-2y)&=12+2\\ 3x+2y+4x-2y&=14\\ 7x&=14\\ x&=2\end{aligned}$$

In any of the original equations, we can use 2 instead of x i.e. we can substitute 2 for x.

Using equation 1

$$3x+2y=12$$

Become

$$\begin{aligned}3\times2+2y&=12\\ 6+2y&=12\\ 2y&=6\\ y&=3\end{aligned}$$

The solution of the simultaneous equations are therefore $x=2$ and $y=3$

Note: When we added equation 1 and 2, the variable y was eliminated. We remained with a simple equation with one unknown.

Example 1

Solve the equations

$2x+4y=-12$ (1)

$5x+4y=33$ (2)

Solution

If we subtract equation (2) from equation (1) we get a simple equation in one unknown. Thus, $(2x+4)-(5x+4y)=-12-(-33)$

$$2x+4y-5x-4y=-12+33$$
$$-3x=21$$
$$x=-7$$

Substituting for x to solve for y

Thus, $2x+4y=-12$ becomes

$$2x-7+4y=-12$$
$$-14+4y=-12$$
$$4y=2$$
$$y=\frac{1}{2}$$

This method of getting rid of one of the variables by addition or subtractions is known as elimination method.

<u>Exercise 8</u>

Solve the simultaneous equations

1. $3x-y=8$
   $x+y=4$
2. $3x-2y=0$
   $x-2y=-4$
3. $6x+4y=24$
   $7x-4y=2$

More pair of simultaneous equations

Consider the equation

$3x-2y=8$

$x+5y=-3$

In this pair we cannot eliminate any variable by simple addition or subtraction. We must first make the coefficient of x or y the same in both cases.

Solution

$3x-2y=8$ (1)

$x+5y=-3$ (2)

Leave (1) as it is $3x-2y=8$ (1)

Multiply (2) by 3 $3x+15y=-9$ (3)

Subtract (3) from (1) $-17y=17$

$y=-1$

Substituting for x in (1)

$3x-2(-1)=8$

$3x+2=8$

$3x=6$

$x=2$

Example 1

Solve the simultaneous equations

$3x+4y=0$ (1)

$2y-3y=1$ (2)

Solution

Multiply (1) by 2 $6x+8y=20$ (3)

Multiply (2) by 3 $6x-9y=3$ (4)

Subtract (4) from 3 $17y=17$

$y=1$

Substitute $y=1$ in

$3x+4x1=10$

$3x+4=10$

$3x=6$

$x=2$

To solve simultaneous equation by elimination method

1. Decide which variable to eliminate.
2. Make the coefficient of the variable the same in both equations.
3. Eliminate the variables by addition as subtraction as is appropriate.
4. Solve for the remaining variable
5. Substitute your value from (4) above in any of the original equations to solve for the other variable.

Exercise 9

Solve the simultaneous equations

1. $3x+2y=16$
   $2x-y=6$
2. $2x+3y=27$
   $3x-2y=13$
3. $x=5-2y$
   $5x+2y=1$
4. $x+y=0$
   $2y-3x=10$
5. $3x-4y=-5$
   $2x+y=6$
6. $2x-7y=-10$
   $9y+5x=6$

Solution of simultaneous equations by substitution method

Considers the equations

$3x-5y=23$ (1)

$x - 4y = 3$ (2)

Using equation (2) add 4y to both sides

$x - 4y + 4y = 3 + 4y$

$x = 3 + 4y$ (3)

In equation (3) x is said to be expressed in terms of y

In equation (1) use $(3 + 4y)$ in place of x

$3x - 5y = 23$

Becomes $3(3 + 4y) - 5y = 23$

i.e. $9 + 12 - 5y = 23$

$9 + 7y = 23$

$7y = 14$

$y = 2$

Substitute in $y = 2$ equation 3

$x = 3 + 4(2)$

$x = 3 + 8$

$x = 11$

This method of solving simultaneous equations is called substitution method

Example 1

$$\frac{x+y}{3} - \frac{x-y}{4} = \frac{2}{3} \quad (1)$$

$$\frac{2x-3}{3} - \frac{2y+3}{4} = \frac{19}{12} \quad (2)$$

Solution

Multiply (1) by 12 to remove the denominators

$4x + 4y - 3x + 3y = 8$

$x + 7y = 8$ (3)

Multiply (2) by 12 to remove the denominators

$8x - 12 - 6y - 9 = -19$

$8x - 6y = 2$ (4)

Using (3) express x in terms of y

$x = 8 - 7y$ (5)

Substitute (5) in (4) to eliminate x

$8(8 - 7y) - 6y = 2$

$64 - 56y - 6y = 2$

$62y = 62$

$y = 1$

Substitute $y = 1$ in (5)

$x = 8 - 7(1)$

$x = 1$

To solve simultaneous equations by substitution method

1. First decide which variable is easier to eliminate.
2. Using the simpler of the two equations express the variable to be eliminated in terms of the other.
3. Using the other equations, substitute the equivalent for the variable to be removed.
4. Solve for the remaining variable.
5. By substitution solve for the other unknown.
6. By substitution, check whether your solutions satisfy the equations

Exercise 10

Use a suitable method to solve the following pairs of simultaneous equation

1. $a+b=3$
   $4a-3b=5$
2. $4x-3y=1$
   $x-4=2y$
3. $4m-n=-3$
   $8m+3n=4$
4. $5q+2p=10$
   $3q+7q=29$
5. $2x-4y+10=0$
   $3x+y-6=0$
6. $\frac{2y}{5}-\frac{2}{3}=\frac{8}{3}$
   $y=2(z+1)$
7. $\frac{a-1}{2}-\frac{b+1}{5}=\frac{1}{5}$
   $\frac{a+b}{3}=b-1$

Forming & solving simultaneous equations

Consider the following situations

Two years ago, a man was 7 times as old as his son. In 3 years time, he will be only 4 times as old as son.

Solution

If the present age of the man is *m* and the present are of the son is 5years then two years ago, the man's age was (m – 2) years and the son was (5-2) years.

Therefore $m-2=7(5-2)$ (1)

In 3 years time the man's age will be $m+3$ and the son's age will be $5+3$

Therefore $m+3=4(5+3)$ (2)

Equation (1) and (2) can be simplified as

$m=75-12$ (3)

$m=45+9$ (4)

Since the two equations are equal then,

$7s-12=45+9$

$3s=21$

$s=7$

Substituting 7 for s in (3)

$m = 7 \times 7 - 12$

$= 49 - 12$

$= 37$

The present age for man =37 years

The present age for son = 7 years

Example 1

A two digit number is such that its value equals 4 times the sum of its digits. If 27 is added to the number, the result is equal to the value of the number obtained when the digits are interchanged what is the number.

Solution

Let the tens digit be x and ones digit be y.

Therefore the value of the number is $10x + y$ and the sum of the digits is $x + y$

$10x + y = 4(x + y)$

$10x + y = 4x + 4y$

$6x = 3y$

$6x - 3y = 0$

i.e. $2x - y = 0$ (1)

The value of the number formed by interchanging the digits is $10y + x$

Therefore $10x + y + 27 = 10y + x$

$9x - 9y + 27 = 0$

$x - y = -3$ (2)

Subtract (2) from (1) x =3

Substituting x = 3 in (1)

$(2 \times 3) - y = 0$

$6 - y = 0$

$y = 6$

The original number is 36.

Exercise 11

1. The sum of two numbers is 10 and their difference is 6. Find the numbers.
2. Mary is one year older than June, and their ages add up to 15. Find the age of each girl.
3. A bag contains sh.5 coins and sh.10 coins. There are 14 coins in total and their value is sh.105. Find the number of each type of coin.
4. Two numbers are such that twice the larger numbers differs from twice the smaller number by 4. The sum of the two numbers is 17. Find the numbers.
5. The cost of 3 sheep and 2 goats is sh.7,200. If 4 sheep and a goat costs sh.7,600. Find the cost of two goats and a sheep.

Quadratic Equations

Earlier on, we studied linear equations and their solutions. Now, we shall consider the solution of an equation that contains the unknown to the second power. Such an equation is called a quadratic equation.

Quadratic equations can be written inform of $ax^2+bx+c=0$ where a, b and c are constants and $a \neq 0$

The form $ax^2+bx+c=0$ is called the standard quadratic form, the terms in the LHS are arranged in descending powers of the variable. On the LHS, there is only the zero.

When solving quadratic equations, it is necessary to write the equation in standard form. The solution of a quadratic equation in the standard form is the value of the variable that makes the LHS equal to zero, hence, forming a true statement.

Solving quadratic equations

Consider the equation $x^2+5x+6=0$. In factor form the equation becomes $(x+2)(x+3)=0$

This equation states that the product of $(x+2)$ and $(x+3)$ is zero.

We know that

If a and b are real numbers and $a \times b = 0$ then either $a=0$ or $b=0$ or $a=b=0$

If an equation has the value (as a root) then $x-1$ must be factors of the LHS of the equation.

By the property if $(x+2)(x+3)=0$ then either $x+2=0$ or $x+3=0$

Since each of these factors is linear then we use the method of solving linear equations

if $x+3=0$ and if $x+2=0$

Then $x=-3$ then $x=-2$

Example 1

Solve equation

(a) $(3x+4)(2x-1)=0$

(b) $x(x-5)=0$

Solution

(a) $(3x+4)(2x-1)=0$

For the product to be equal to zero, then one or other factors must be equal to 0

If $3x+4=0$ or if $2x-1=0$ thus $3x=-4$

Then $2x=1$

Therefore, $x=\frac{-4}{3}$; $x=\frac{1}{2}$

Therefore the solution of the equation $(3x+4)(2x-1)=0$ or $x=\frac{-4}{3}$ or $x=\frac{1}{2}$

(b) $x(x-5)=0$ This equation x and $x-5$ either $x=0$ or $x-5=0$ then $x=5$

Therefore $x=0$ or 5 is the solution to the equation.

Note; The solutions of quadratic equations are also known as its roots.

Solving quadratic equation by factor method

We have just seen that if the LHS of an equation is the product of linear factors and the RHS equals zero, there will be one solution for each linear factors. Thus, we can solve some standard quadratic equations of the form $a\times^2+bx+c=0$ by factoring the LHS of the expression into linear factors

In general, to solve a quadratic equation by factor method, we use the following procedure

1. Write the equation in standard quadratic form
2. Factorize the LHS completely
3. Set each of the factors containing the variable equal to zero and solve the resulting equations
4. Check the solution by substituting in the original equation. Substitute each solution at a time.

Note that a quadratic equation must have two roots

Example1

Solve the quadratic equation $x^2-2x-8=0$

Solution

$x^2-2x-8=0$

$=(x-4)(x+2)=x^2-2x-8$

Thus, $x^2-2x-8=0$ becomes $(x-4)(x+2)=0$

$\Rightarrow$ Either $x-4=0$ or $x+2=0$

i.e. $x=4$ or $x=-2$

The solution of $x^2-2x-8=0$ is $x=4$ or $x=-2$

Example 2

Solve the equation in standard form $4x^2=20x-25$

Solution

Write the equation in standard form

$4x^2=20x-25 \Rightarrow 4x^2-20x+25=0$

$(2x-5)^2=0$ Factorizing the LHS

$2x-5=0 \Rightarrow x=\frac{2}{5}$ Setting the repeated factors equal to zero once

This is an example of repeated root and should be written twice

Exercise 12

Solve the following equations

1. $x^2+3x+2=0$
2. $x^2+6x-16=0$
3. $x^2=-11x-10$
4. $x^2-14x=15$
5. $x^2=27-6x$
6. $b^2-7b=-10$
7. $x^2+4x=0$
8. $2y^2-18=0$
9. $6x^2-5x+1=0$
10. $9x^2+20=-27x$
11. $2x^2-7x-9=0$
12. $x(x+3)=-2$

Solution of quadratic equation by completing the squares

Consider the quadratic equation $x^2+4x-12=0$

On adding 12 to both sides, we have $x^2+4x=12$

Making the expression on LHS a perfect square, by adding 4 on both sides, we get

$x^2+4x+4=16$

Factorizing the LHS gives

$(x+2)^2=16$

Taking square root of both sides gives

$x+2=\sqrt{16}$

$x+2=\pm 4$

$x=-2\pm 4$

$x=2 \text{ or } -6$

Example 1

Solve $x^2+5x+1=0$ using completing the squire method

Solution

Subtracting 1 from both sides gives $x^2+5x=-1$

Making the LHS a * square gives

$$x^2+5x+\left(\frac{5}{2}\right)=-1+\left(\frac{5}{2}\right)^2$$

$$x^2+5x+\frac{25}{4}=\frac{25}{4}-1$$

$$x^2+5x+\frac{25}{4}=\frac{21}{4}$$

Factoring the LHS gives $\left(x+\frac{5}{2}\right)=\frac{21}{4}$

Taking square root gives

$x+\frac{5}{2}=\pm\sqrt{\frac{21}{4}}$

$x=-\frac{5}{2}\pm 4.583$

$=-\frac{0.417}{2}$ or $-\frac{9.583}{2}$

Therefore x= -0.2085 or x - 4.792

Note: The method of completing the square enables us to solve quadratic equation which cannot be solved by factorization.

Exercise 13

Use completing the square method to solve the following quadratic equations

1. $x^2+2x-1=0$
2. $x^2-8x+13=0$
3. $x^2+5x+3=0$
4. $x^2-\frac{1}{2}x-\frac{1}{3}=0$
5. $x^2-5x+2=0$

If the coefficient of $x^2 \neq 0$ i.e $a \neq 1$, proceed as below.

Solve the equation

$2x^2+4x+1=0$

Solution

Subtracting 1 from both sides $2x^2+4x=-1$

Making the coefficient of x^2 one by dividing through by 2, $x^2+2x=-\frac{1}{2}$

Completing the square in LHS $x^2+2x+1=-\frac{1}{2}+1$

Factoring the LHS

$(x-1)^2=\frac{1}{2}$

$x+1=\pm\sqrt{\frac{1}{2}}$

$x=-1\pm\sqrt{0.5}$

$=-1\pm 0.071$

Therefore $x=0.2929$ or – 1.7071 to 4 significant figures

Exercise 14

1. $2x^2+3x-7=0$
2. $5x^2+6x-3=0$
3. $6x^2-5x-4=0$
4. $(x+1)(x+3)=13$
5. $3x^2+7x-4=0$
6. $4x^2+12x-9=0$
7. $2x(x+1)=4$

The quadratic formula

Consider the general quadratic equation $ax^2+bx+c=0$ where $a\neq 0$ subtracting c from both sides $ax^2+bx=-c$

Dividing trough by a $a, x^2+\frac{b}{a}x=-\frac{c}{a}$

Complete the square on HLS by adding $\left(\frac{b}{2a}\right)^2$ to both sides

$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$ But

$$\left(\frac{b}{2a}\right)^2-\frac{c}{a}=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$=\frac{b^2-4ac}{4a^2}$$

Therefore $x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$

Factoring the LHS gives $\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$

Taking squire roots of both sides gives

$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$

It therefore follows that $x=\frac{-b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$

Thus $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

This is the solution to the general quadratic equation and is known as the quadratic formula.

Example

Use quadratic formula to solve $2x^2-5x-3=0$

Solution

Comparing this equation to the general quadratic equation $ax^2+bx+c=0$, we get $a=2,\quad b=-5$ and $c=-3$

Substituting in quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-5\pm\sqrt{25-4(2)(-3)}}{4}$$

$$=\frac{5\pm\sqrt{49}}{4}$$

$$=\frac{5\pm7}{4}$$

$$=\frac{12}{4}\text{ or }\frac{-2}{4}$$

Therefore $x=3$ or $x=-\frac{1}{2}$

Exercise 15

Use the quadratic formula to solve the following equations

1. $x^2+7x+3=0$
2. $2x+7-7x^2=0$
3. $4d^2+7d+3=0$
4. $1-3x-3x^2=0$
5. $9p^2+24p+16=0$
6. $6k^2+9k+1=0$

Formation of quadratic equations

Consider the following examples

Example 1

Peter travels to his uncle's home, 30km away from his place. He cycles for two third of the journey before the bicycle develops mechanical problems and he has to push it for the rest of the journey. If his cycling speed is 10km/hr faster than his walking speed and he completes the journey in 3 hours and 20 minutes. Determine his cycling speed.

Solution

Let Peter's cycling speed be $x$km/hr. Then, his walking speed is (x – 10)km/hr.

Time taken in cycling $=\left(\frac{2}{3}\times30\right)\div x$

$$=\frac{20}{x}\text{hr}$$

Time taken in walking (30 – 20) ÷ (x – 10)

$$=\frac{10}{x-10}\text{hr}$$

Total time $=\left(\frac{20}{x}+\frac{10}{x+10}\right)$hr

Therefore, $\frac{20}{x}+\frac{10}{x-10}=3\frac{1}{3}$

$$\frac{20}{x}+\frac{10}{x-10}=\frac{10}{3}$$

$$60\left(x-10\right)+30\left(x\right)=10x\left(x-10\right)$$

$$10x^2-190x+600=0$$

$$x^2-19+60=0$$

$$x=\frac{19\pm\sqrt{361-240}}{2}$$

$$x=15 \text{ or } x=4$$

If his cycling speed is 4km/hr, then his walking speed is (4-10) km/hr = - 6km which is not unrealistic. Therefore his cycling is 15km/hr.

Example 2

A group of young men decided to raise sh.480,000 to a business to a start a business. Before the actual payment was made, four of the members pulled out and each of those remaining had to pay an additional sh.20,000. Determine the original number of the members.

Solution

Let the original number be x. Originally each had to pay $\frac{480000}{x}$ after withdraw each had to pay $\frac{480000}{x-4}$.

Now, $\frac{480000}{x}+20,000=\frac{480,000}{x-4}$

Simplifying, we get

$\frac{480}{x}+20=\frac{480}{x-4}$. Taking the LCM $x\left(x-4\right)$

$$480\left(x-4\right)+20x\left(x-4\right)=480x$$

$$480x-1920+20x^2-80x=480x$$

$$20x^2-80x-1920=0$$

$$x^2-4x-96=0$$

$$x=\frac{4\pm\sqrt{16+384}}{2}$$

$$=\frac{4\pm 20}{2}$$

$= 12$ or -8

Thus since we are dealing with measurements -8 is inadmissible hence 12 was the original number.

Equations leading to quadratic equations

Many equations involving fractions eventually lend to quadratic equations.

Example

Solve the equation $\frac{3}{2x+1}+\frac{4}{5x-1}=2$

Solution

Multiply each term by the LCM of the denominators in order to remove the fractions.

LCM of denominators $=(2x+1)(5x-1)$

Therefore $(2x+1)(5x-1)\left[\left(\frac{3}{2x+1}\right)+\frac{4}{5x-1}\right]=2(2x+1)(5x-1)$

$$(5x-1)\times 3+(2x+1)\times 4=(2x+1)(5x-1)\times 2$$

Multiply out and simplify

$$3(5x-1)+4(2x+1)=2(2x+1)(5x-1)$$

$$15x-3+8x+4=2(10x^2-2x+5x-1)$$

$$23x+1=20x^2+6x-2$$

Rearranging the equation into standard quadratic form

$$20x^2-17x-3=0$$

$$x=\frac{17\pm\sqrt{289-4(-60)}}{40}$$

$$=\frac{17\pm\sqrt{529}}{40}$$

$$=\frac{17\pm 23}{40}=1 \quad \text{or} \quad \frac{-6}{40}=\frac{-3}{20}$$

Exercise16

Solving the following equations

1. $\frac{x+2}{x-2}=\frac{x+3}{x-9}$
2. $\frac{3x+5}{6x+5}=x-1$
3. $2-\frac{1}{x}=x$
4. $x-\frac{9}{x}=0$

5. $x - 9 = \dfrac{72}{x - 8}$

6. $\dfrac{x + 10}{x - 5} = \dfrac{7x}{x - 5}$

7. $\dfrac{x - 1}{2} + \dfrac{x + 3}{4} = \dfrac{1}{x - 1}$

8. $\dfrac{2}{y} + 1 = \dfrac{y - 2}{3}$

9. $\dfrac{y - 3}{4} - \dfrac{1}{y} = \dfrac{4}{3} - y$

10. $\dfrac{1}{x - 1} + 3 = \dfrac{1}{x + 2} - \dfrac{1}{4}$

Linear Inequalities

In this section, we are concerned with statements which involve the following symbols.

- $>$ Meaning 'greater than' e.g. 6>2 means 6 is greater than 2
- $<$ Meaning 'less than' e.g. 3<7 means 3 is less than 7.
- $\geq$ Meaning 'greater than or equal to'
- $\leq$ Meaning 'less than or equal to'

Statements containing these symbols are called inequalities or in equations. For example $x < 2, x \geq 2, y \leq 2, y > 10$ are inequalities.

A statement such as $x > 2$ means all numbers that are greater than 2; which is a range of values. Just as we represent individual numbers on a number line, we can also represent such a range of numbers on a number line as shown in the following examples.

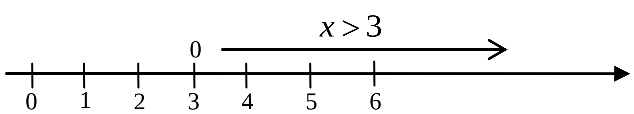
Example 1

Illustrate each of the following on a number line

(a). $x > 3$ (b). $x \geq 3$ (c). $x < -2$ (d). $x \leq -2$

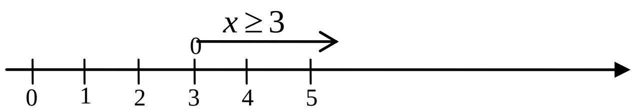
Solution

(a). $x > 3$



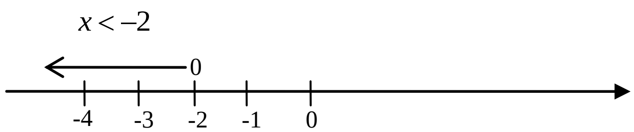
In the figure above, the number 3 is not included in the list of numbers to the right of 3. The heavy arrow shows that the values of x go on without end. The open dot 0 is used to indicate that 3 is not included.

(b). $x \geq 3$



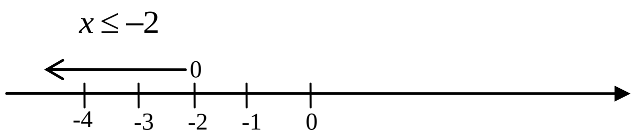
The number 3 is included in the list of numbers required. The closed dot is used to show that 3 is part of the list.

(c). $x < -2$



The number – 2 is not included.

(d). $x \leq -2$



The number – 2 is included.

Compound statements

Sometimes, two simple inequalities may be combined into one compound statement such as $a < x < b$. This statement means that $a < x$ and $x < b$ or $x > a$ and $x < b$.

Example

Write the following pairs of simple inequality statements as compound statements and illustrate them on a number line.

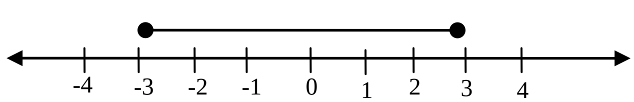
a) $x \leq 3, \quad x > -3$

b) $x > -1$ and $x < 2$

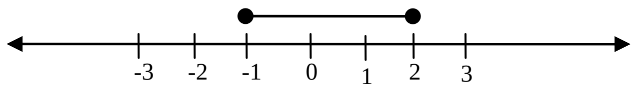
Solution

a) $x \leq 3, \quad x > -3$ becomes $-3 < x \leq 3$

Therefore x lies between – 3 and 3 and 3 is included.



b) $x > -1$ and $x < 2$ becomes $-1 < x < 2$ therefore x lies between – 1 and 2.



Solutions of linear inequalities in one unknown
Solving an inequality means obtaining all the possible values of the unknown which make the statement true. This is done in much the same way as solving an equation.

Example 1
Solve the following inequalities
(a). $x - 3 < 7$ (b). $x + 5 > 11$
Solution
(a). $x - 3 < 7$
$\Rightarrow x - 3 + 3 < 7 + 3$ (Adding 3 to both sides)
$\Rightarrow x < 10$ is the solution of the inequality $x - 3 < 7$
(b). $x + 5 > 11$
$\Rightarrow x + 5 - 5 > 11 - 5$ (Subtracting 5 from both sides)
$\Rightarrow x > 6$
NB; Adding or subtracting the same number from both sides of an inequality does not change it.
Example 2
Solve the following inequalities
(a). $3x - 4 \geq 5$ (b). $\frac{1}{4}x + 5 \leq 14$
Solution
(a). $3x - 4 \geq 5$
$\Rightarrow 3x - 4 + 4 \geq 5 + 4$
$\Rightarrow 3x \geq 9$
$\Rightarrow \frac{3x}{3} \geq \frac{9}{3}$ (Dividing both sides by 3)
$\Rightarrow x \geq 3$
(b). $\frac{1}{4}x + 5 \leq 14$
$$\Rightarrow \frac{1}{4}x + 5 - 5 \leq 14 - 5$$
$$\Rightarrow \frac{1}{4} \leq 9$$
$$\Rightarrow \frac{1}{4}x \times 4 \leq 9 \times 4$$
$$x \leq 36$$

NB: Multiplying or dividing both sides of an inequality by the same positive number does not change it

Multiplication and division of inequalities by negative numbers

We know that 8 < 10. Consider multiplying both sides of this inequality by any negative number, say – 4,

LHS = $8\times -4 = -32$

RHS = $10\times -4 = -40$

We know that – 32 is greater than – 40. i.e. $-4\times 8 > 10\times -4$.

Thus, the inequality R reversed.

Similarly $8\div -4 = -2$ and $10\div -4 = -2.5$

But we know that

$-2 > -2.5$

Thus, the inequality is reversed.

In general, multiplying or dividing both sides of an inequality by a negative number reverses the inequality sign.

Example 1

Solve the inequality $3-2x\geq 15$

Solution

$3-2x\geq 15$

$\Rightarrow 3-2x-3\geq 15-3$

$\Rightarrow -2x\geq 12$

$\Rightarrow \frac{-2x}{2}\geq \frac{12}{-2}$

$\Rightarrow x\leq -6$

Solving simultaneous inequalities

Inequalities that must be satisfied at the same time are called **simultaneous inequalities**.

Example 4

Solve the following pair of simultaneous inequalities

$3-x<5, \qquad 2x-5<7$

Solution

$3-x<5,$

$\Rightarrow -x<2$

$\Rightarrow x>-2$ (i)

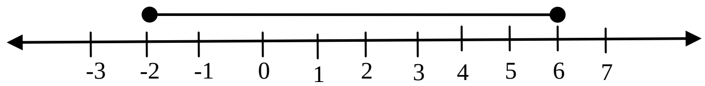
Also $2x-5<7$

$2x<12$

$x<6$ (ii)

Combining (i) and (ii) we have $-2<x<6$. Thus, x lies between -2 and 6.

This is represented on a number line as



Example 5

Solve the inequality $3x-2<10+x<2+5x$

Solution

Split the inequality into two simultaneous inequalities as

$3x-2<10+x$ (i)

And $10+x<2+5x$ (ii)

Solve each separately

$3x-2<10+x$

$\Rightarrow 3x-x<10+2$

$\Rightarrow 2x<12$

$\Rightarrow x<6$ (iii)

$10+x<2+5x$

$\Rightarrow 10-2<5x-x$

$\Rightarrow 8<4x$

$\Rightarrow 2<x$ (iv)

Combining (iii) and (iv) we get $2<<x<6$

Exercise 18

A). Solve the following inequalities and represent the solutions on number lines

1. $x+4>11$
2. $2x-8\leq 4$
3. $3>4x-2$
4. $7\leq 5x+12$
5. $\frac{1}{3}x-3>4$
6. $4m-3<7m$
7. $2-2p>13-3p$
8. $\frac{1}{3}q>2-4q$
9. $\frac{1}{9}r<5$
10. $2(1+x)+3(x-2)\geq 25$

B). Solve the following simultaneous inequality and represent each solution on a number line

1. $2x<10, \quad 5x\geq 15$
2. $x+7<0, \quad x-2>-10$
3. $4x-33<-1, \quad -2<3x+1$
4. $2x-5<22\leq 5x-6$

5. $3x-4<8+x<2+7x$

$6x+2<3x+8<27x-1$

Inequality Involving Fractions

e.g. Solve the inequality

$$\frac{2x-7}{x-5}\le 3$$

Solution

$\frac{2x-7}{x-5}-3\le 0$ - Standard form

$$\frac{2x-7-3x+15}{x-5}\le 0$$

$$\frac{-x+8}{x-5}\le 0$$

Critical values: $x=5, x=8$

Test interval: $(-\infty,5),(5,8),(8,\infty)$

Test: is $\frac{-x+8}{x-5}\le 0$?

Solution interval:

$(-\infty,5),(8,\infty)$

Exercise 19

1. Solve the inequality $\frac{x+1}{x+3}\le 2$ and represent the solutions graphically.[Solution interval is (-5, -3)]
2. Solve $(x+2)(x-1)(x-5)>0$ and represent the solutions graphically.
3. The braking distance, *d* (in feet) of a car traveling *v* mph is approximated by $d=v+\left(v2/20\right)$. Determine those velocities that result in braking distance of less than 75 feet.
4. Solve the inequalities below
   - i). $\frac{x^2-x-2}{x^2-4x+3}\ge 0$
   - ii). $5x-c>11$
   - iii). $0<2-3/4\,x\le 1/2$
   - iv). $|x-10|<0.05$
   - v). $|5-3x|<7$
   - vi). $(2x+4)>0$
   - vii). $\frac{x+4}{2x-1}<3$
   - viii). $x^3-x^2-4x+4\ge 0$

ix). $\frac{x+6}{x+1} < 2$

x). $\frac{5}{x-6} > \frac{3}{x+2}$

5. A rectangle with a perimeter of 100m is to have an area of at least 500m². Within what bounds must the length of the rectangle lie?
6. Use inequality notation to denote each of the given statements
   i). x is non negative
   ii). x is at least 200

Inequalities Involving Polynomials of Degree n

We shall use the following theorem to solve such equations

Theorem:

Let $a_n x^n + \ldots + a_1 x_1 + a_0$ be a polynomial. If the real numbers c and d are successively the solutions of the polynomial are positive or all values are negative.

If we choose any number k such that $c < k < d$, and if the value of the polynomial is positive for $x = k$, then the polynomial is positive for every x in (c, d). Similarly, if the polynomial at $x = k$ a test value for the interval (c, d)

Examples

1. Solve $2x^2 - x < 3$

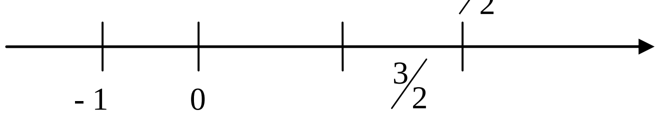
Solution:

$2x^2 - x - 3 < 3$ Standard form

$(x+1)(2x-3) < 0$ Factored form

From the factored form, the equation

$2x^2 - x - 3 = 0$ Has solutions -1 and $3/2$ called critical numbers.



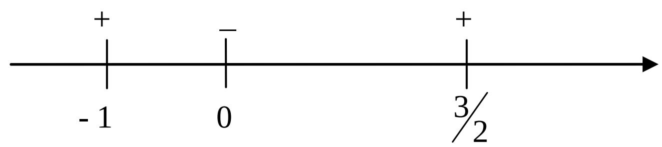
These points divide the axis into three parts

$(-\infty,-1),\left(-1,3/2\right)\left(3/2,\infty\right)$, referred to as the test intervals.

We test the sign of polynomial $2x^2 - x - 3$ in each interval using an appropriate test value.

Interval	$(-\infty,-1)$	$\left(-1,3/2\right)$	$\left(3/2,\infty\right)k$
K	-2	0	2

Test value at k	7	-3	3
Sign of polynomial in the interval	+	-	+



Thus the solutions $2x^2 - x - 3 < 0$, are the real numbers in the interval $\left(-1, \frac{3}{2}\right)$

2. Solve the quadratic inequality $x^2 < x + 6$

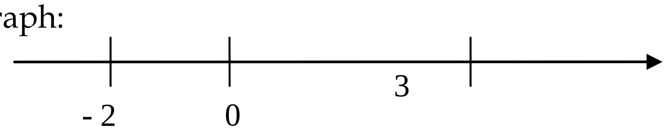
Solution

$x^2 - x - 6 < 0$ - Standard form

$(x-3)(x+2) < 0$ - Factor

Critical numbers $x = 3, x = -2$

Graph:



Test interval $(-\infty, -2), (-2, 3)(3, \infty)$,

Test: is $(x-3)(x+2) < 0$?

Interval	$(-\infty, -2)$	$(-2, 3)$	$(3, \infty)$
Test value	-3	0	4
Test sign	+	-	+

The inequality is satisfied only by the middle test interval.
Therefore the true solution test is the interval (-2, 3)

3. Solve the cubic inequality
$2x^3 + 5x^2 \geq 12x$

Solution

$2x^3 + 5x^2 - 12x \geq 0$ Standard form

$x(2x^2 + 5x - 12) \geq 0$

$x(2x-3)(x+4) \geq 0$

Critical numbers: $x = -4, x = 0, x = \frac{3}{2}$

Test intervals $(-\infty,-4),(-4,0)\left(0,\tfrac{3}{2}\right)\left(\tfrac{3}{2},\infty\right)$

Test: is $x(2x-3)(x+4)\geq 0$

Solution intervals: $[-4,0],\left[\tfrac{3}{2},\infty\right]$

Suggested References

1. Backhouse, J.K. et.al, (2005) Pure Mathematics Volume1, Longman, pg 512-521
2. Blitzer, Robert. (2004) College Algebra, Prentice Hall pg 36-174

CHAPTER THREE: FUNCTIONS

Definition 1

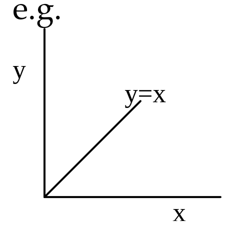
Let A and B be sets. A function f from A to B is a rule that assigns exactly one element of B to each element of A.

If b is the unique element of B assigned by the function f to the element a of A, we write $f(a) = b$. i.e. if f is a function from A to B write $f : A \to B$.

There are generally 4 ways to represent a function:

i. Verbally: by a description in words e.g. $p(t)$ is the population of the world at time t.
ii. Algebraically – by a formula e.g. $f(x) = x^2$
iii. Visually – by a graph

e.g.



iv. Numerically – by a table of values

e.g..

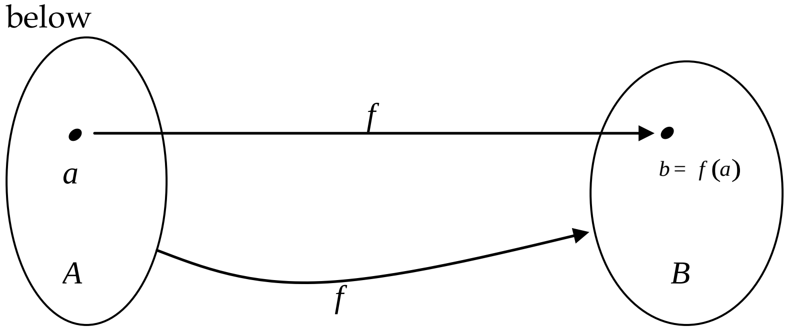
x	1	2	3
x^2	1	4	9

Definition 2

If f is a function from A to B then A is called the Domain of f and B the Codomain of f.

If $f(a) = b$, "b" is called the image of "a" and "a" is called the pre-image of "b".

The set of all images of the elements of A is called the range. Consider the figure below



The function f maps A to B.

i.e. $f : A \rightarrow B$
A is the domain of f
B is the codomain of f.
A is the pre-image of b.
B is the image of a
$f(a) = b$ is the range i.e. set of all images of elements of set A, more precisely all the elements in B constitute the range.

Example 1
The squaring function from the et of integers to the set of integers assigns to each integer number x its square x^2.
i.e. $f(x) = x^2$ e.g. $f(3) = 3^2 = 9$, $f(4) = 4^2 = 16$ etc
The domain of the function $f(x) = x^2$ is the set of all integers. The codomain is the set of all positive integers. (Since all squares are positive). The range is the set of all positive integers which are perfect squares.
(i.e. 0, 1, 4, 9, 16, …)

Example 2
For the function $f(x) = 2x^2 + 3x - 1$ evaluate

i. $f(2)$
ii. $f(a)$
iii. $f(a+h)$

Solution

i. $f(2) = 2(2)^2 + 3(2) - 1 = 8 + 6 - 1 = 13$

ii. $f(a) = 2a^2 + 3a - 1$

iii. $f(a+h) = 2(a+h)^2 + 3(a+h) - 1$

$$= 2(a^2 + 2ah + h^2) + 3(a+h) - 1$$

$$= 2a^2 + 4ah + 2h^2 + 3a + 3h - 1$$

Addition and multiplication of functions
Definition
Let f_1 and f_2 be functions from set A to the set of real numbers. Then the sum $f_1 + f_2$ and the product $f_1 f_2$ are also functions from the set A to the set of reals defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \text{ [sum]}$$

and

$$(f_1 f_2)(x) = f_1(x) f_2(x) \text{ [product]}$$

Example

Let f_1 and f_2 be functions from the set of real numbers to the set of real numbers i.e. From $\mathbb{R}$ to $\mathbb{R}$ such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. Find $f_1 + f_2$ and $f_1 f_2$

Solution

$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$

$(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4$

Exercise 20

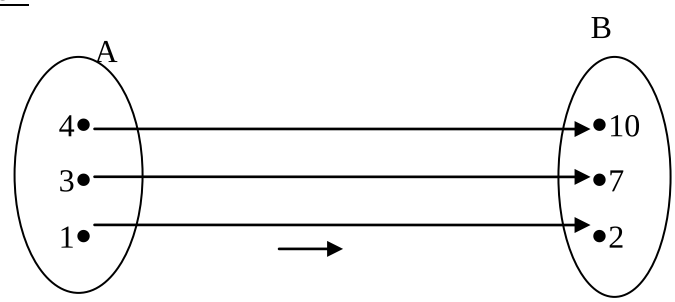
If $f(x) = x^2 + 1$ and $g(x) = x + 2$ are functions from the set of real numbers to the set of real numbers i.e. from $\mathbb{R}$ to $\mathbb{R}$. Find $f + g$ and fg.

One To One And Onto Functions

Definition 1 – ONE TO ONE FUNCTION

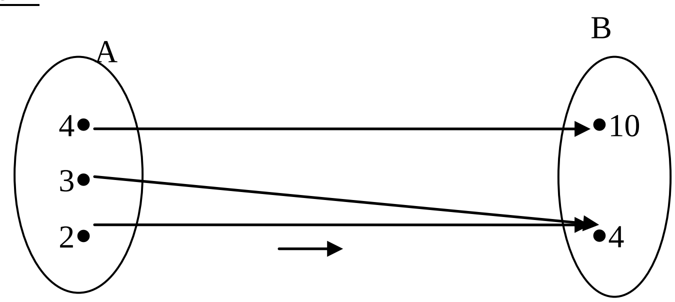
A function *f* is said to be one to one or injective if and only if $f(x) = f(y)$ implies that $x = y$ for all *x* and *y* in the domain of *f*. In other words a function within domain *A* is called a one to one function if no two elements of *A* have the same image in the codomain of *B*. Consider the function *f* and *g* below

Figure I



f is one to one since all the elements in *A* have unique image in *B*.

Figure II

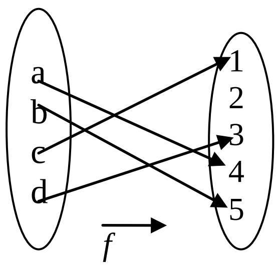


g is not one to one since 3 and 2 in *A* have the same image 4 in *B*.

Example 1

Determine whether the function *f* from the set $\{a, b, c, d\}$ to the set $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$ and $f(d) = 3$ is one to one.

Solution



f is one to one since every element in the domain has a unique image.

Example 2

Determine whether the function $f(x) = x + 1$ for $x = 0, 1, 2, 3$ is one to one.

Solution

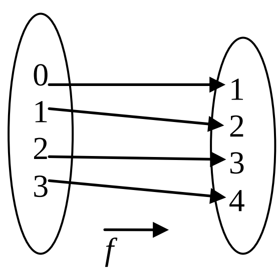
$f(x) = x + 1$

$f(0) = 0 + 1 = 1$

$f(1) = 1 + 1 = 2$

$f(2) = 2 + 1 = 3$

$f(3) = 3 + 1 = 4$



The function is one to one.

Exercise 21

1. Determine whether the function $f(x) = x^3$ from $\mathbb{R}$ to $\mathbb{R}$ is one to one. Explain.
2. Is the function $f(x) = 3x + 4$ from the set of integers to integers one to one? Why?.

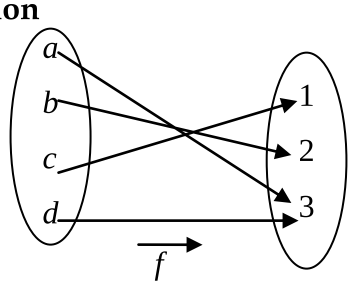
Definition 2 (ONTO FUNCTION)

A function *f* from A to B is called <u>onto</u> or subjective if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. i.e for <u>onto</u> functions every member of the codomain is the image of some element of the domain.

Example

Let *f* be a function from the set $\{a, b, c, d\}$ to the set $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$ and $f(d) = 3$. Is *f* an onto function?

Solution



f is an onto function since all the three elements of the codomain are images of elements in the domain.

Note: for onto function, every element in the codomain must have a match in the domain.

Exercise 22

With an explanation determine whether the following functions from the set of integers to integers are one to one or onto.

1. $f(x) = x^2$
2. $f(x) = x + 1$

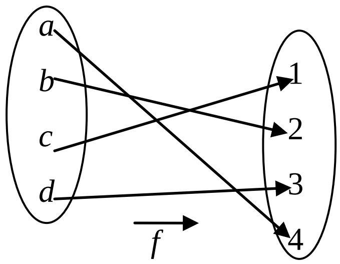
Definition 3 One To One Correspondence

A function *f* is called a one to one correspondence or bijective if it is both one to one and onto. In other words a bijection is both an injection and a surjection.

Example

Let *f* be the function from the set $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$ and $f(d) = 3$. Is *f* a one to one and onto function? Why?

Solution



f is a one to one and onto function.
Reasons

1. Its one to one since all the elements in the domain takes on distinct values in the codomain.
2. Its onto since all the four elements in the codomain are images of elements in the domain. Therefore *f* is a bijection.

Exercises 23
Determine whether each of the functions below from the set of real numbers to the set of real numbers i.e. from $\mathbb{R}$ to $\mathbb{R}$ is a one to one correspondence. Explain.

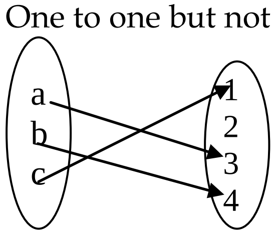
i. $f(x) = 2x + 1$

ii. $f(x) = x^2 + 1$

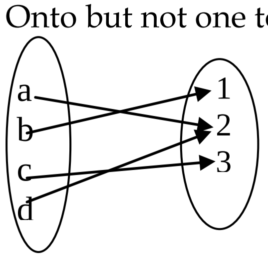
iii. $f(x) = x^3$

Summary of different types of correspondences

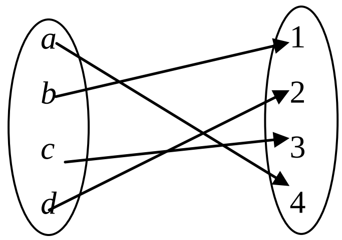
i. One to one but not onto



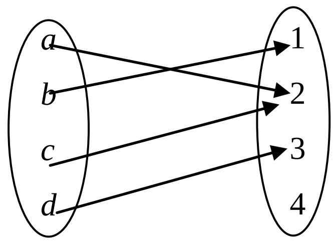
ii. Onto but not one to one



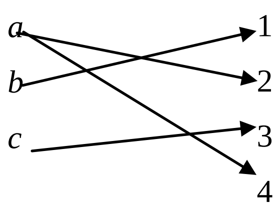
iii. One to one and onto



iv. Neither one to one nor onto



v. Not a function



Composition of Functions

Definition

If f and g are functions of a variable x, then the composition of the functions f and g denoted fog is defined by

$$fog(x) = f(g(x))$$

And the domain of fog is given by

$$Dom(fog) = \{(X, g(x)) : x \in X \text{ and } g(x) \in Dom\ f\}$$

Example

Let f and g be the functions from the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of?

i. f and g.
ii. g and f.

Solution

i. $(fog)x = f(g(x)) = f(3x + 2)$

$$= 2(3x + 2) + 3 = 6x + 7$$

ii. $gof(x) = g(f(x)) = g(2x + 3)$

$$= 3(2x+3)+2 = 6x+11$$

Example 2

If $f(x) = x^2$ and $g(x) = x+1$, find fog.

Solution

$fog = f(g(x)) = f(x+1) = (x+1)^2$

Remarks

fog And gof are not equal hence the commutative property does not hold for composition of functions.

Exercise 24

Determine *fog* and *gof* given that

i. $f(x) = x^3$ and $g(x) = x^{\frac{1}{3}}$

ii. $f(x) = \dfrac{x-1}{x+1}$ and $g(x) = \dfrac{1}{x}$

iii. $f(x) = \sqrt{x}$ and $g(x) = x^2+1$

Inverse functions

Definition The functions *f* and *g* are said to be inverses of each other iff

i. $f(g(x)) = x$ For every *x* in the domain of *g*.

ii. $g(f(x)) = x$ For every *x* in the domain of *f*.

If *g* is the inverse of *f* then we denote *g* by f^{-1}, thus $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

Example 1

Show that the functions $f(x) = 2x+3$ and $g(x) = \frac{1}{2}(x-3)$ are inverses of each other.

Solution

$$f(g(x)) = f\left(\frac{1}{2}(x-3)\right) = 2\left(\frac{1}{2}(x-3)\right)+3 = x$$

and

$$g(f(x)) = g(2x+3) = \frac{1}{2}((2x+3)-3) = x$$

Hence inverse

Example 2

Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$ and $f(c) = 1$. Find f^{-1}.

Solution

The inverse function f^{-1} reverses the correspondence given f such that $f^{-1}(1) = c$, $f^{-1}(2) = a$ and $f^{-1}(3) = b$.

Remarks: A function has an inverse if it's a one to one correspondence.

Exercise 25

1. Show that the functions $f(x) = x^3$ and $g(x) = x^{\frac{1}{3}}$ are inverses of each other.
2. Show that the functions f and g are inverses of each other by showing that $f(g(x)) = x$.
   i. $f(x) = x^3 - 8$ and $g(x) = \sqrt[3]{x - 8}$
   ii. $f(x) = \dfrac{x - 5}{2x + 3}$ and $g(x) = \dfrac{3x + 5}{1 - 2x}$
3. Find the inverse of the functions
   i. $f(x) = 3x - 2$
   ii. $f(x) = \dfrac{5 - 3x}{2}$

Graphs Of Function

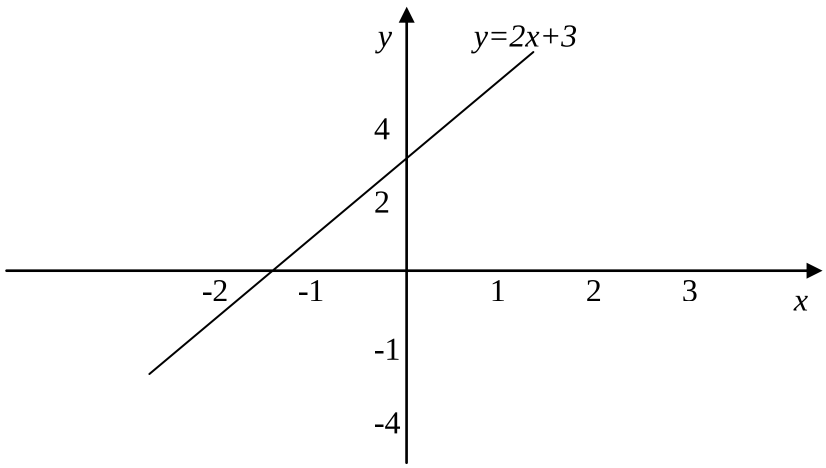
A graph of a function f is a set of all points $(x, f(x))$ in a co-ordinate plane such that x is in the domain of f.

Finding Domain and Range from Graph of function

Examples

1. Sketch the graph of the function $f(x) = 2x + 3$

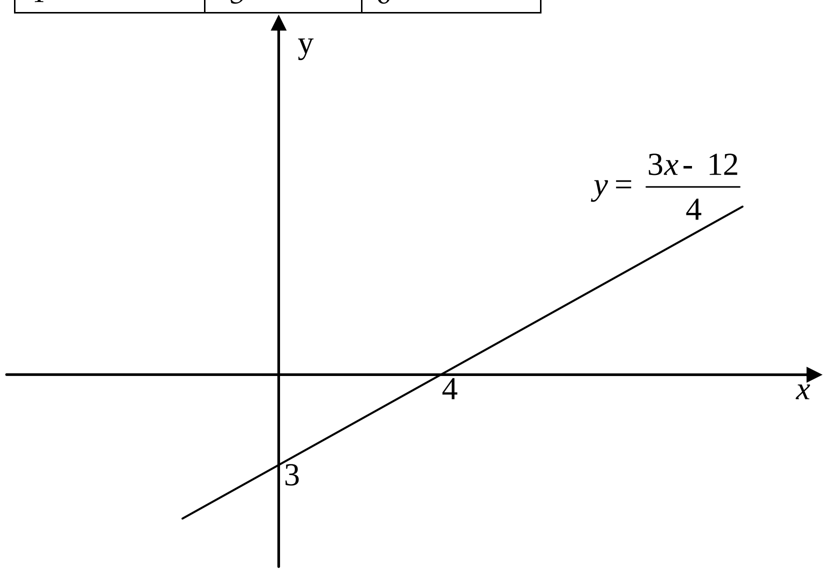
x	0	-1.5	2
$f(x) = y$	3	0	7



$\Rightarrow$ the function is linear with gradient = 2 and y intercept = 3. Domain and range is the set of all $\mathbb{R}$

2. Graph the function $f(x) = \dfrac{3x-12}{4}$

X	0	4
Y	-3	0



Remarks

It can be shown that two lines are perpendicular if the product of their slopes is -1. e.g. $2x + y + 3 = 0$ And $2x + -4y + 7 = 0$ are perpendicular because,

$2x + y + 3 = 0$ Can be written as $y = -2x - 3$ hence gradient is -2

$2x + -4y + 7 = 0$ can be written as $y = \dfrac{2x+3}{4}$, hence gradient is $\dfrac{2}{4} = \dfrac{1}{2}$.

The product of their gradients are $-2\times\frac{1}{2}=-1$

Examples

Find the equation of a line passing through (-2, 3) and (4, 5).

Solution

Gradient $\frac{\Delta y}{\Delta x}=\frac{-5-3}{4+2}=\frac{-8}{6}$

Choose any point (x, y) on the line

$\therefore$ gradient $=\frac{-8}{6}=\frac{y-3}{x+2}\Rightarrow y-3=\frac{-8}{6}(x+2)$

3. Find a line through the point (1, 5) which is perpendicular to the line though the points (-2, 3) and (4, 2).

Solution

The line through (-2, 3) and (4, 2) has gradient of $-\frac{1}{6}$. The gradient of the perpendicular line =6. Thus the required line is of slope 6 through $(1,5)\Rightarrow y-5=6(x-1)$

Aids for Sketching Graphs of Functions

Intercepts, symmetry and equation recognition can aid us to sketch the graph of functions.

Symmetry

Definition

a. A function is **even** if its graph is symmetric with respect to Y-axis.

b. A function is **odd** if its graph is symmetric with respect to the Origin.

Example

Determine whether the following functions are **even, odd** or neither and hence sketch their graphs.

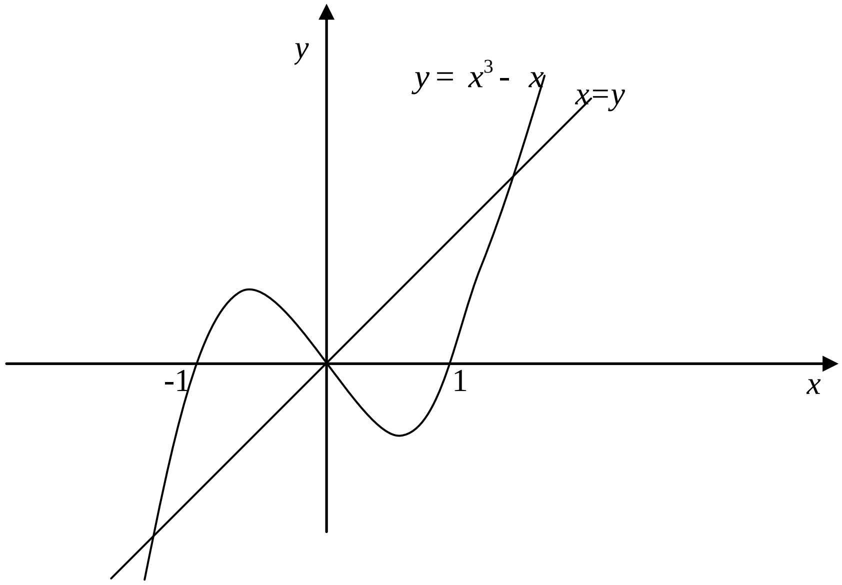
a. $g(x)=x^3-x$

b. $h(x)=x^2+1$

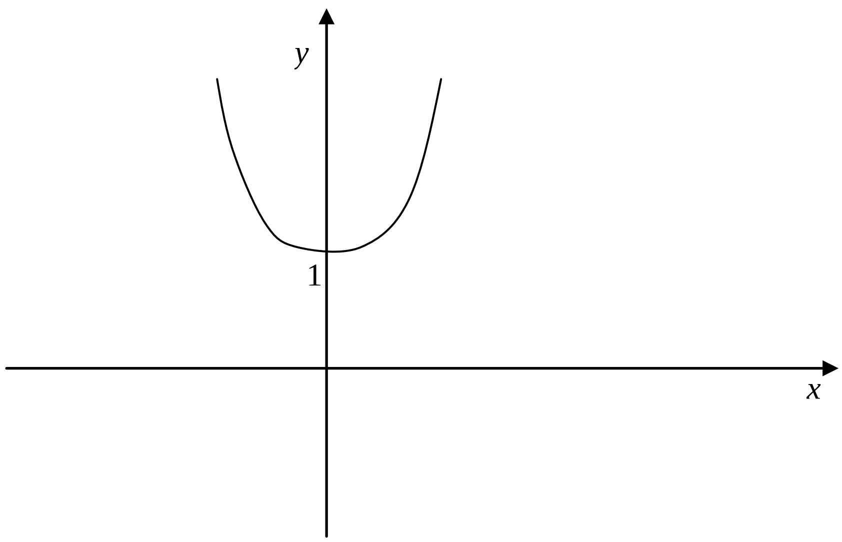
c. $f(x)=x^3-1$

Solution

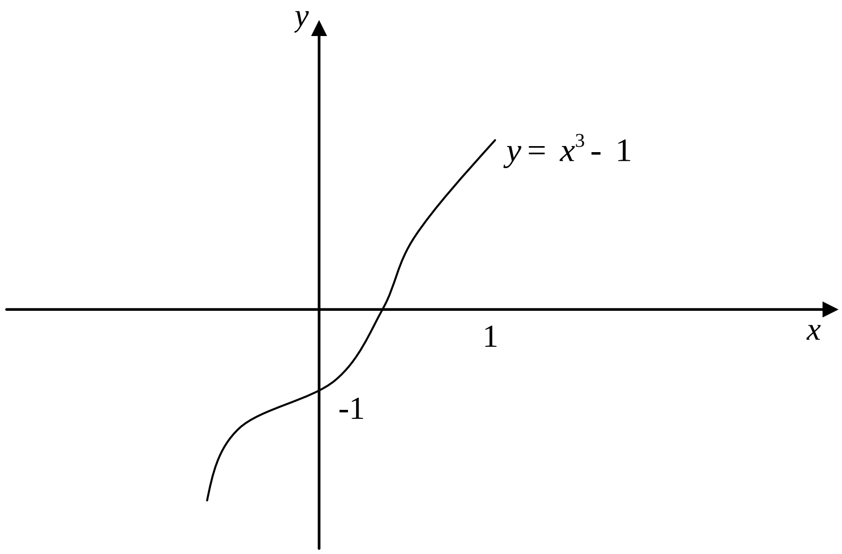
a. $g(x)=x^3-x$ is odd since $g(-x)=-x^3+1=-g(x)$



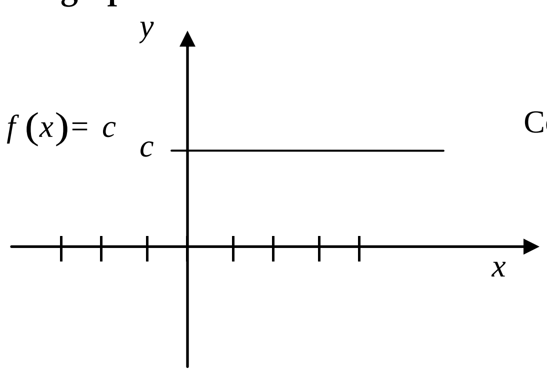
b. $h(x) = x^2 + 1$ is even since $h(-x) = x^2 + 1 = h(x)$



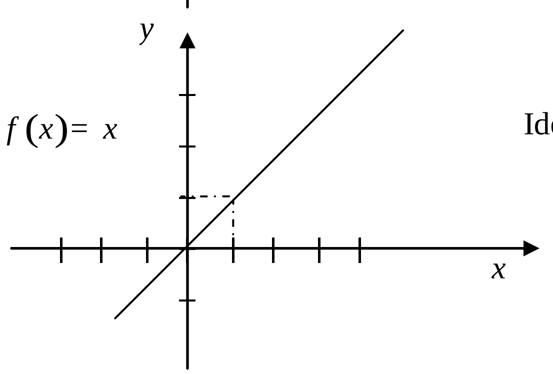
c. The function is neither even nor odd. $H(x) = x^3 - 1$



The graphs of some Basic Functions that Occur Frequently in Application

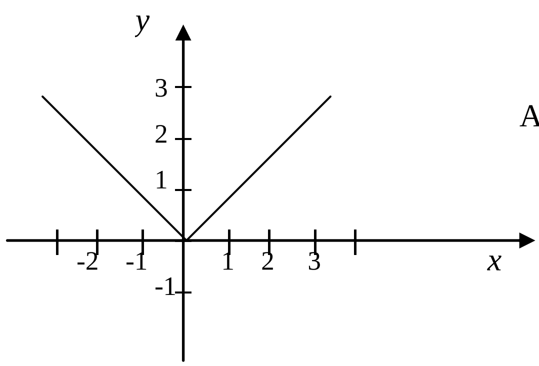


Constant Function



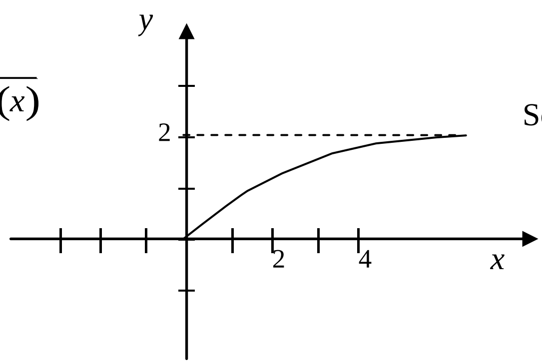
Identity Function

$f(x) = |x|$



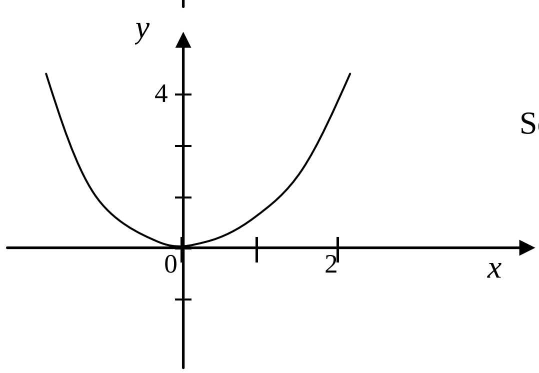
Absolute value Function

$f(x) = \sqrt{(x)}$



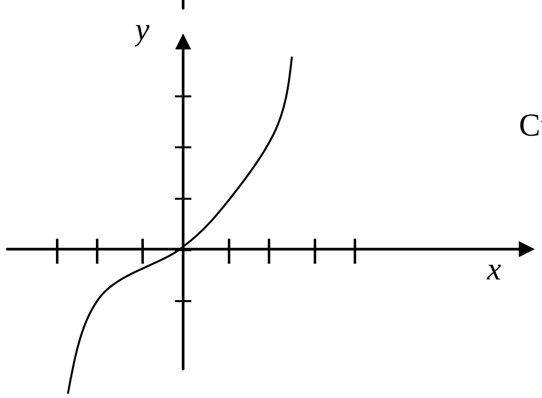
Square root Function

$f(x) = x^2$



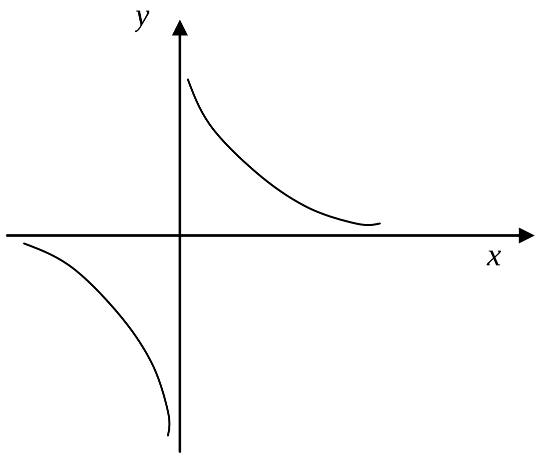
Squaring Function

$f(x) = x^3$

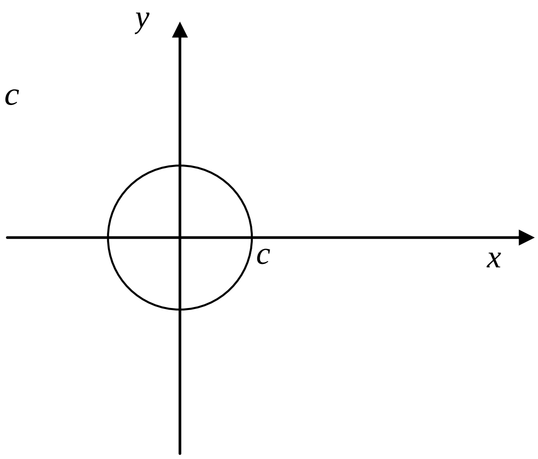


Cubing Function

$f(x) = \frac{1}{x}$



$x^2 + y^2 = c$



Shifting, Reflecting and Stretching Graphs

Many functions have graphs that are similar transformations of familiar graphs shown above.

Horizontal and Vertical Shifts:

Let $C \in \mathbb{R}^+$. Vertical and horizontal shifts in the graph of $y = f(x)$ are represented as follows:

a. Vertical shift *c* units upward: $h(x) = f(x) + c$

b. Vertical shift *c* units downward: $h(x) = f(x) - c$

c. Horizontal shift *c* units to the right: $h(x) = f(x - c)$

d. Horizontal shift *c* units to the left: $h(x) = f(x + c)$

Some graphs can be obtained from a combination of vertical and horizontal shifts:

Example

Use the graph of $f(x) = x^2$ to sketch the graph of each of the following

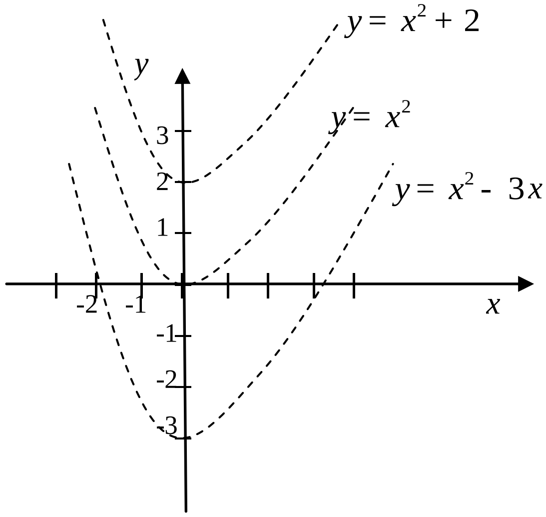
a. $h(x) = x^2 + 2$

b. $g(x) = x^2 - 3$

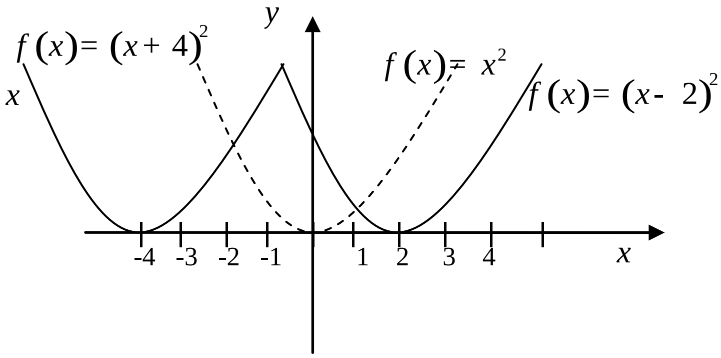
c. $f(x) = (x - 2)^2$

d. $k(x) = (x + 4)^2$

e. $f(x)$

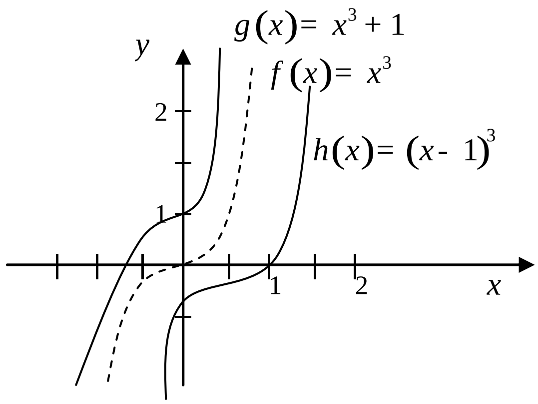


$(c)(d)k(x) = (x + 4)^2$

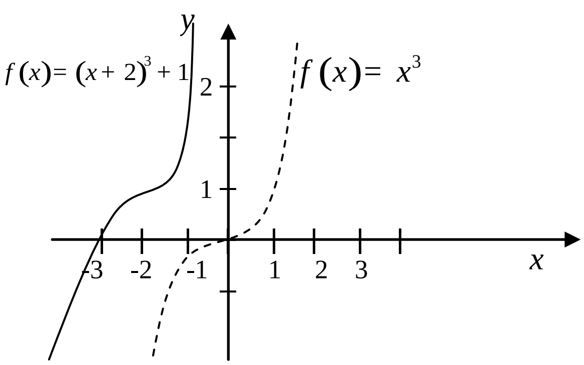


1. Use the graph of $f(x) = x^3$ to sketch the graphs of each of the following functions
   a. $g(x) = x^3 + 1$
   b. $h(x) = (x - 1)^3$
   c. $k(x) = (x + 2)^3 + 1$

(a, b)



c.



Reflections in the Co-ordinate Axes:

Reflections in the co-ordinate axes, of the graph $y = f(x)$ are represented as follows.

Reflection in the x-axis: $h(x) = -f(x)$

Reflection in the y-axis: $h(x) = f(-x)$

Examples

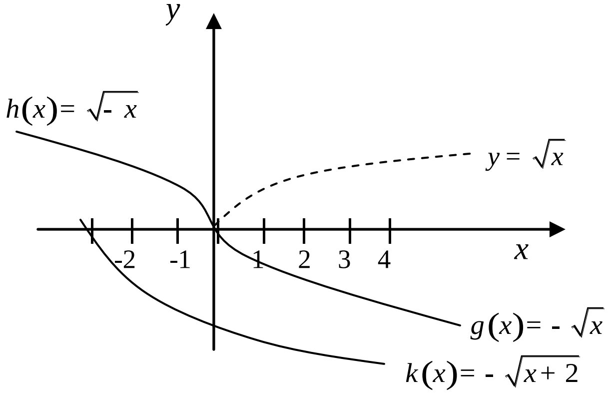
Sketch a graph of each of the following functions

a. $g(x) = -\sqrt{x}$

b. $h(x) = -\sqrt{-x}$

c. $k(x) = -\sqrt{x+2}$

Solution



Let $f(x) = \sqrt{x}$ \ $k(x) = -\sqrt{x+2} = -f(x+2)$**.**

The graph *f* is first, a left of two units, followed by a reflection in the X-axis.

Non-rigid transformations

The shifts and reflections are called rigid transformations because the basic shape of the graph is unchanged, only the positions change Non-rigid transformations are those that cause distortion.

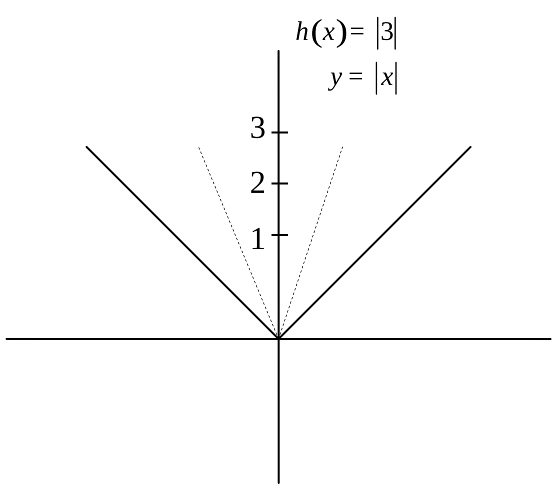
e.g. A non-rigid transformation of the graph $y = f(x)$ is represented by $y = c\,f(x)$ where the transformations is a vertical sketch if $c > 1$ and a vertical shrink if $0 < c < 1$.

Example

Sketch a graph of each of the following

a) $h(x) = 3|x|$

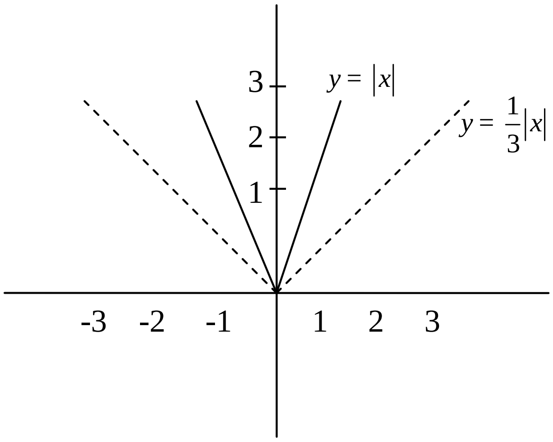
b) $g(x) = \frac{1}{3}|x|$



a)

$h(x) = 3|x| = 3f(x) \Rightarrow$ Vertical stretch (multiply each value of y by 3)

b) $g(x) = \frac{1}{3}f(x) \Rightarrow$ h is a vertical shrink of f.



Exercise

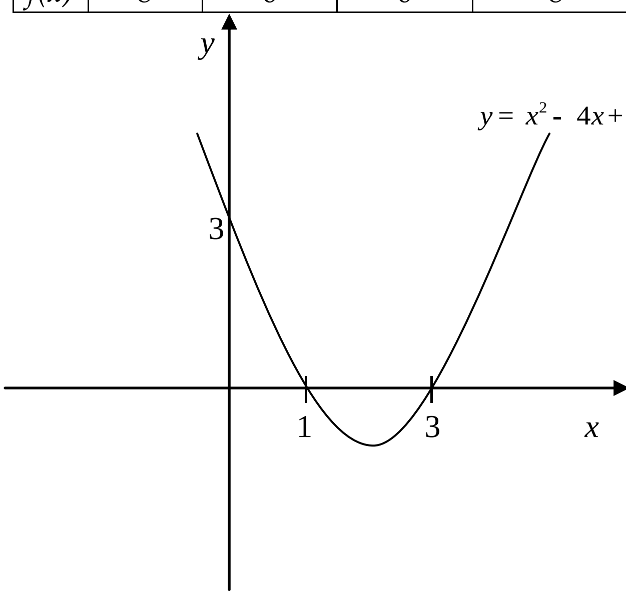
Graph the function $f(x) = x^2 - 4x + 3$

Solution

Select arbitrary values of the variable

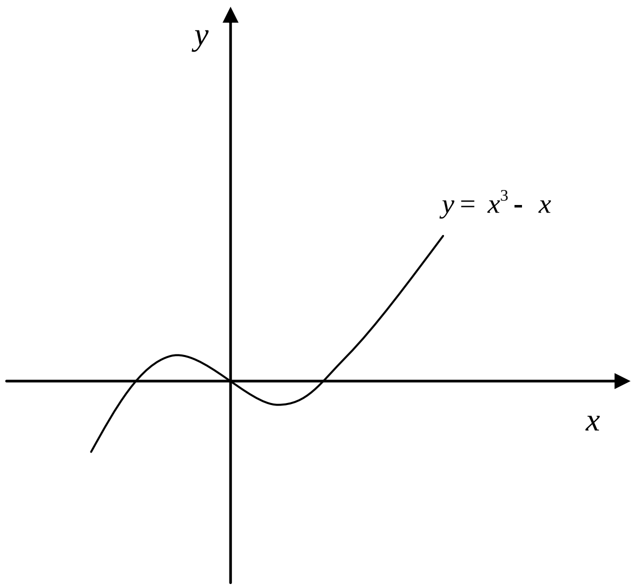
Compute the corresponding values of the function and arrange in tabular form.

x	-1	1	3	5
$f(x)$	8	0	0	8



Graph the equation $y = x^3 - x$

X	-2	$-\frac{3}{2}$	-1	0	1	2
Y	-6	$-\frac{15}{5}$	0	0	0	6



Remarks

1. The graph of a function of the form $f(x) = ax^2 + bx + c\,(a \neq 0)$ is called a parabora. If $a > 0$, the parabola opens upwards $\cup$. If $a < 0$, the parabola opens downwards $\cap$.

The lowest or highest point of a parabola is known as a vertex and is given by $\left(-b/2a,\ C - b2/4a\right)$.

The axis of symmetry is the vertical line $x = -b/2a$.

2. In the graph of $f(x) = \frac{a}{x}$, $x = 0$ does not belong to the domain of the function. As $x \to 0$, $f(x) \to \infty$. We describe this behaviour by saying that $\frac{a}{x}$ has a vertical asymptote at $x = 0$.

Graph of $\frac{a}{x} \to x-$ axis as $x \to \infty$. Thus we say that the *x*-axis i.e. Line *y*=0 is the horizontal asymptote to $\frac{a}{x}$.

Examples

i. $y = \dfrac{x}{(x-1)(x-1)}$ has vertical asymptote at *x*=1 and the line *y*=0 is a horizontal asymptote

ii. $y = \dfrac{x^3 - 2x + 1}{x^2}$ has a vertical asymptote $x = 0$.
There is no horizontal asymptote, since the degree of the numerator is larger than the degree of the denominator.

3. $y = \dfrac{x^2 - x}{x - 1}$ Does not have asymptote at $x = 1$ since the numerator is also 0 at $x = 1$.

This function is not defined at $x = 1$ but has the same value as the function x " member[1] 1.

Exercise 26

1. Graph the following equations. Find the co-ordinates of the vertex and the equation of the axis of symmetry.
   a. $y = x^2$
   b. $y = 4 - x^2$
2. Graph the equations $y = x^2 - 2x$ and $y = x^2 + 2x$ on the same axes.
3. Graph the following functions. Find the vertical and horizontal asymptotes
   a. $y = x^3$
   b. $y = x^3 - 1$
   c. $y = x^3 + x^2 + x$
   d. $y = \dfrac{4}{x}$
   e. $y = \dfrac{1}{x - 1}$
   f. $y = 1 + \dfrac{1}{x}$
   g. $y = x^4 - x^2 + 1$

Suggested Reference

1. Backhouse, J.K. et.al, (2005) Pure Mathematics Volume1, Longman, pg 22-54
2. Blitzer, Robert. (2004) College Algebra, Prentice Hall pg 175-278

CHAPTER FOUR: LOGARITHMIC AND EXPONENTIAL NOTATION

Exponent

Recall $2^3 = 2\times2\times2\}$ 3 factors

$3^5 = 3\times3\times3\times3\times3\}$ 5 factors and so on

Thus $^4 = a\times a\times a\times a$

The raised numeral is called an exponent or power and *a* is the base

In general of $n = a^x$

$= a\times a\times\ldots\times a\}$ x factors

Where x is a positive integer, a^x is the exponential form on n, where a is the base and x the exponent or power (also called index)

Laws Of Indices

i). $a^m\times a^n = a^{m+n}$

e.g. $2^3\times2^5$

$$=\underbrace{(2\times2\times2)}_{\text{(3 factors)}}\times\underbrace{(2\times2\times2\times2\times2)}_{\text{(5 factors)}}$$

$$=\underbrace{2\times2\times2\times2\times2\times2\times2\times2}_{\text{(3 + 5) same factors}}$$

$= 2^8$

Therefore $2^3\times2^5 = 2^{3+5}$

$= 2^8$

ii). $a^m\div a^n = a^{m-n}$

e.g. simplify $3^7\div3^4$

$$=\frac{\overbrace{3\times3\times3\times3\times3\times3\times3}^{\text{7 factors}}}{\underbrace{3\times3\times3\times3}_{\text{4 factors}}}$$

$$=\underbrace{3\times3\times3}_{\text{(7-4) factors}}$$

$= 3^{7-4}$

$= 3^3$

iii). $(a^x)^y = a^{x\times y} = a^{xy}$

e.g. simplify $(2^2)^3$

$= 2^2\times2^2\times2^2$

From (i) we have

$= 2^{2+2\times2} = 2^6$

But $2+2+2=2\times 3$

So in evaluating $\left(2^2\right)^3$, we have in fact multiplied the indices together

iv). Consider $3^4 \div 3^4$ using (ii)

$$3^4 \div 3^4 = 3^{4-4} = 3^0$$

Using factor form

$$3^4 \div 3^4 = \frac{3^4}{3^4} = \frac{3\times 3\times 3\times 3}{3\times 3\times 3\times 3} = 1$$

Therefore $3^0 = 1$

In general any non-zero number raised to power zero, equals 1 i.e. $a^0 = 1$ where $a \neq 0$

v). Consider $x^2 \div x^5$

From (ii) $x^2 \div x^5 = x^{2-5} = x^{-3}$ using factor method

$$\frac{\cancel{x}\times\cancel{x}}{\cancel{x}\times\cancel{x}\times x\times x\times x} = \frac{1}{x^3}$$

Therefore $x^{-3} = \frac{1}{x^3}$

i.e. $a^{-x} = \frac{1}{a^x}$ where $a \neq 0$

vi). From *v* above it can be proved that $\left(\frac{a}{b}\right)^{-x} = \left(\frac{b}{a}\right)^x$

vii). $(ab)^x = a^x b^x$

viii). $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

ix). $a^{\frac{1}{n}} = \sqrt[n]{a}$

Note $\sqrt[n]{a}$ means the n^th root of *a* and *n* is called the order of the root

x). If $x^m = x^n \Rightarrow m = n$ and if $x^m = y^m \Rightarrow x = y$

Exercise27

1. Simplify

a). $5^8 \times 5^5$ b). $p^3 \times p^4 \times p^5$ c). $2^{18} \div 2^{15}$ d). $\frac{a^6}{a^4}$

e). $x^5 + x^8$ f). 100^0 g). $\sqrt{9x^2y^4}$ h). $4x^7y^2 \times 2xy^3z^2$

i). $\left(5^3bc^2\right)^2$ j). $\frac{35a^7b^{12}c^3}{5a^5b^4c^2}$

2. If $A = 27x^4y^3z^4$ and $B = 3x^2yz^2$ find (a). AB (b). $A \div B$ (c). A^2
3. Solve for x in $9^{x+1} + 3^{2x+1} = 36$
4. Solve for x and y in (a). $2^{2x+y} = 8$ and $3^{x-y} = 1$
5. Solve the equation $9^x \times 3^{(2x-1)} = 3^{15}$
6. If $\left(\frac{1}{27}\right)^m \times 81^{-n} = 243$, express m in terms of n.

Logarithms

If $a^m = b$ then the logarithm of b to base a is m.

The exponents of a is the logarithm of b to base a i.e. $a^m = b \Rightarrow \log_a b = m$

Laws Of Logarithms

i). If A and B are positive real numbers, then $\log_b(AB) = \log_b A + \log_b B$

Proof

Let $\log_a A = m$ and $\log_a B = n$

$\Rightarrow a^m = A$ and $a^n = B$

$AB = a^m \times a^n = a^{m+n}$

$= AB = a^{m+n}$

$\Rightarrow \log_a(AB) = m + n$

But $\log_a A = m$ and $\log_a B = n$

Therefore $\log_a(AB) = \log_a A + \log_a B$

ii). Similarly it can be proved that $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$

iii). $(\log_a A)^p = p\log_a A$

iv). From $a^0 = 1$ then $\log_a 1 = 0$ where $a \neq 0$

v). If we recall that $a^1 = a$ then $\log_a a = 1$ where $a \neq 0$

Example

1. Solve for x given $\log_3 9 = x$

Solution

$\log_3 9 = x$

$\Rightarrow 3^x = 9$

$3^x = 3^2$

Therefore $x = 2$

2. Solve for x given

$\log_x 8 = 3$

$\Rightarrow x^3 = 8$

$x^3 = 2^3$

$\Rightarrow x = 2$

Common and natural logarithms

The systems of logarithms with 10 as the base are called the common logarithms, i.e. $\log_{10} A$.These imply that if 10 where the base is not explicitly written it's understood to be 10.

e.g. log 2 is understood to be $\log_{10} 2$.

Common logarithms are applied in computation. Since all positive real numbers can be expressed as standard number. i.e. as products of 10.

e.g.	1.2	=	1.2×10^0	
	12	=	1.2×10^1	
	120	=	1.2×10^2	
	0.12	=	1.2×10^{-1}	etc

By the use of common logarithms tables and applying the laws of logarithms computation is made easier. (This content is omitted here. Student are advised to read on how to compute using logarithm tables).

Another system of logarithm is the logarithm with an irrational number as the base. This irrational number is exponential (e = 2.71828…). This system is commonly used in advanced mathematic such as calculus and classical mechanics.

Note in your calculator, the key **log** gives common logarithms and **ln** stands for 'natural logarithms'.

Exercise 28

Solve for x given

i). $\log_{\frac{2}{3}} 2\tfrac{1}{4} = x$

ii). $\frac{1}{2}\log(x+1) = -2$

iii). Evaluate $\frac{\log 75 + \log 9 + \log 5}{\log 5 + \log 45}$

iv). Find y if $\log_2 y - 2 = \log_2 92$

Suggested References

1. Backhouse, J.K. et.al, (2005) Pure Mathematics Volume1, Longman, pg 173-211
2. Blitzer, Robert. (2004) College Algebra, Prentice Hall pg 373-435

CHAPTER FIVE: MATRICES

Definition: If m and n are positive integer, then $m \times n$ **matrix** (read "m by n") is a rectangular array of numbers.

$$\begin{pmatrix} a_{11} & a_{12}, \ldots, a_{1n} \\ a_{21} & a_{22}, \ldots, a_{2n} \end{pmatrix} \quad \text{e.g.} \begin{pmatrix} 5 & 4\ldots & 6 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 8 & 7 & 11 \end{pmatrix}$$

In which each entry a_{ij}, of the matrix is real number. An $m \times n$ matrix has m vows (horizontal lines) and n columns (vertical lines).

The entry in the i^{th} and j^{th} column is denoted by the double subscript notation a_{ij}. We call i the **row subscript** because it gives the position in horizontal line, and j the column subscript because it gives the position in the vertical lines.

A matrix having rows and columns is said to be of order $m \times n$, if $m = n$, the matrix is said to be square matrix of order n.

Example of matrices

a) Order $1 \times 4 \rightarrow \left[1, -3, 0, \frac{1}{2}\right]$

b) Order $2 \times 2 \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

c) Order $2 \times 3 \begin{bmatrix} -1 & 0 & 5 \\ 2 & 1 & -4 \end{bmatrix}$

d) Order $3 \times 2 \rightarrow \begin{bmatrix} 5 & 0 \\ 2 & -2 \\ -7 & 4 \end{bmatrix}$

A matrix that has only one row is a row **matrix**, and matrix that has only one column is a **column matrix.**

Operations With Matrices

Matrix Addition

If $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ are matrices both of order $m \times n$, then their sum is the $m \times n$ matrix given by

$A+B=\left[a_{ij}+b_{ij}\right]$

NB Then sum of two matrices of different order is undefined

Examples

a) $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ and $B=\begin{pmatrix}1 & 3\\ -1 & 2\end{pmatrix}$

$$A+B=\begin{pmatrix}-1 & 2\\ 0 & 1\end{pmatrix}+\begin{pmatrix}1 & 3\\ -1 & 2\end{pmatrix}=\begin{pmatrix}-1+1 & 2+3\\ 0+(-1) & 1+2\end{pmatrix}=\begin{pmatrix}0 & 5\\ -1 & 3\end{pmatrix}$$

b) $A=\begin{bmatrix}1\\ -3\\ 2\end{bmatrix}\quad B=\begin{bmatrix}-1\\ 3\\ 2\end{bmatrix}$

$$\therefore A+B=\begin{bmatrix}1\\ -3\\ 2\end{bmatrix}+\begin{bmatrix}-1\\ 3\\ 2\end{bmatrix}=\begin{bmatrix}1+(-1)\\ -3+3\\ 2+2\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 4\end{bmatrix}$$

c) $$A+B=\begin{bmatrix}1 & 2 & 0\\ 1 & 4 & 1\\ 2 & 1 & 4\end{bmatrix}+\begin{bmatrix}2 & 1 & 0\\ 4 & 0 & 1\\ 1 & 2 & 3\end{bmatrix}=\begin{bmatrix}1+2 & 2+1 & 0+0\\ 1+4 & 4+0 & 1+1\\ 2+1 & 1+2 & 4+3\end{bmatrix}=\begin{bmatrix}3 & 3 & 0\\ 5 & 4 & 2\\ 3 & 3 & 7\end{bmatrix}$$

Matrix subtraction

If $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ are matrix both of order $m\times n$ then their difference is the $m\times n$ matrix given by $A-B=\left[a_{ij}-b_{ij}\right]$

Examples

a. $A=\begin{bmatrix}1\\ 3\\ 2\end{bmatrix}\quad B=\begin{bmatrix}0\\ 1\\ 2\end{bmatrix}\quad A-B=\begin{bmatrix}1+-0\\ 3+-1\\ 2+-2\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 0\end{bmatrix}$

b. $A=\begin{pmatrix}-1 & 2\\ 0 & 1\end{pmatrix}\quad B=\begin{pmatrix}1 & 3\\ -1 & 3\end{pmatrix}\quad A-B=\begin{bmatrix}-1 & -1 & 2-3\\ 0 & (-1) & 1-3\end{bmatrix}=\begin{pmatrix}-2 & -1\\ 1 & -2\end{pmatrix}$

c. $A=\begin{bmatrix}1 & 2 & 0\\ 2 & 1 & 4\end{bmatrix}\quad B=\begin{bmatrix}2 & 1 & 0\\ 4 & 0 & 1\end{bmatrix}\quad \therefore A-B=\begin{bmatrix}1-2 & 2-1 & 0-0\\ 1-4 & 4-0 & 1-1\\ 2-1 & 1-2 & 4-3\end{bmatrix}$

$$=\begin{bmatrix}-1 & 1 & 0\\ -3 & 4 & 0\\ 1 & -1 & 1\end{bmatrix}$$

Matrix product

If $A = \left[c_{ij}\right]$ is an $m \times n$ matrix and $B = \left[b_{ij}\right]$ in an $n \times p$ matrix, then the **product** $A \times B$ is the $m \times p$ matrix

$A \times B = \left[c_{ij}\right]$

Where $c_{ij} = a_{1i}b_{1j} + a_{i2}b_{2j} + \ldots a_{2in}b_{nj}$

The definition indicate a **row by column**, where the entries c_{ij} in the i^{th} row and j^{th} column of the product AB which is obtained by multiplying the entries in the i^{th} row of A by the corresponding entries in the i^{th} column of B and then adding the result.

Example

We consider matrix A of 3×2 and B of 2×1.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$3 \times 2 \qquad 2 \times 1 \qquad 3 \times 1$$

$$AB = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} (1\times 2) & + & (2\times 3) \\ (2\times 2) & + & (1\times 3) \\ (3\times 2) & + & (0\times 3) \end{bmatrix} = \begin{bmatrix} 2+6 \\ 4+3 \\ 6+0 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 6 \end{bmatrix}$$

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \quad B\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}, A \times B = \begin{bmatrix} (2\times 3)+ & (1\times 1) & (2\times 2)+ & (1\times 0) \\ (0\times 3)+ & (1\times 1) & (0\times 2)+ & (0\times 0) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 4 \\ 1 & 0 \end{bmatrix}$$

$$2 \times 2$$

$$A = [1, \quad -2 \quad -3] \quad B = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad AB = \left[(1\times 2)+ \quad (-2\times 1)+ \quad (3\times 1)\right] = [1]$$

$$3 \times 1 \qquad 1 \times 3 \qquad 1 \times 1$$

$$A = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \qquad B = [1, \quad -2 \quad -3]$$

$$3 \times 1 \qquad 1 \times 3$$

$$AB = \begin{bmatrix} 2\times1 & 2x-2 & 2x-3 \\ -1\times1 & -1\times-2 & -1\times-3 \\ 1\times1 & -1x-2 & 1\times-3 \end{bmatrix} = \begin{bmatrix} 2 & -4 & -6 \\ -1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}$$

Identity matrix
Definition

The $n\,x\,n$ matrix that consists of 1's on its main diagonal 0's elsewhere in the identity matrix of order n and is denoted by

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \ldots 0 \\ 0 & 1 & 0 & \ldots 0 \\ 0 & 0 & 1 & \ldots 0 \\ 0 & 0 & 0 & \ldots 0 \end{bmatrix} \text{ e.g. } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3×3

$0\,0\,0\ldots1$

Note that and identity matrix most is square. If A is a $n\,x\,n$ matrix, then the identity matrix has the property.

$Ax_{in} = A \quad I_n \times A = A$

e.g. (i) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}$

(ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ 1 & 3 & 6 \\ 2 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 \\ 1 & 3 & 6 \\ 2 & 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 \\ 1 & 3 & 6 \\ 2 & 5 & 2 \end{bmatrix}$

Determinants of matrix

Each square matrix Z has an associated number called Determinant of A, denoted by $|A|$ or Det$A$. This notation should not be confused with absolute value of real numbers. If A is a square matrix of order $1(1\times1)$ then A has only one element thus $A = [a_{11}]$ and we define $|A| = a_{11}$. If A is a square matrix of order 2 i.e. (2×2) then

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Definition

Then the determinant of A is defined by

$|A| = a_{11}a_{22} - a_{21}a_{12}$

Another notation that can be used is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21} - a_{12}$$

Example

Find $|A|$ if $A = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$

Solution

$$|A| = \begin{vmatrix} 2 & -1 \\ 4 & -3 \end{vmatrix} = (2)(-3)(-4)(-1) = -6 + 4 = -2$$

Inverse of matrix

In this section we consider the multiplicative inverse of a value x which is x^{-1}. We know that $x^{-1} = 1$. Then we define a multiplicative inverse of a matrix in a similar way.

Defn: Let A be a square matrix of orders n(i.e $n \times n$). If there exists a matrix A^{-1} such that $AA^{-1} = I_n = A^{-1}A$ then A^{-1} is the inverse of A

Example

Find the inverse matrix $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$

Solution

$$\underset{A}{\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}} \underset{A^{-1}}{\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}} = \underset{I_2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$AA^{-1} = \begin{bmatrix} (1)x_{11} + (4)x_{21} & (1)x_{11} + (4)x_{22} \\ (-1)x_{21} + (-3)x_{21} & (-1)x_{12} + (-3)x_{22} \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow x_{11} + 4x_{21} = 1 \quad x_{12} + 4x_{21} = 0$

$-x_{11} + 3x_{21} = 0 \quad -x_{12} + 3x_{22} = 1$

When you these simultaneous equations you will get

$x_{11} = -3 \qquad x_{21} = 1 \qquad x_{12} = -4 \qquad x_{22} = 1$

$$A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

This method seems tedious and requires a lot of time, but there is an alternative method which takes less time.

Consider a matrix A of order 2(i.e. 2×2) given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible (i.e. has an inverse) if $ad - bc \neq 0$, then,

$$A^1 = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

1) Consider $A = \begin{pmatrix} 1 & 4 \\ -1 & 3 \end{pmatrix}$

   First we find determinant of $A = ad - bc = (1)(-3)(-4)(-1)$

$$= -3 + 4 = 1$$

   Therefore $A^{-1} = \frac{1}{1}\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$

2) Consider $A = \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$ find A^{-1}

$$|A| = ad - bc = (3)(2) - (-1)(-2) = 6 - 2 = 4$$

$$\therefore A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{4} & \frac{1}{4} \\ \frac{2}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Systems of equations

A system of two equation in x and y is any equation in those variables. A solution of this system is an ordered pair that satisfies each equation in the system. When you find the set of all solutions you are solving the system of equation.

There are several methods to solve systems of equation. These methods are applied depending on the complexity of the system of equation and the number of variable in them. The following are the methods.

- Method of substitution
- Graphical method
- Method of elimination
- Use of matrix methods.(Inverse and row echelon methods)

Recall we discussed elimination and substitution method in chapter two. But let us remind ourselves with few examples.

Method of substitution

This method is applied especially when working with two simple equation having two variables.

Examples

1) Solve the following system of equations using substitution method.

   a. $x+y=4$
   $x-y=2$

   b. $x^2-x-y=1$
   $-x+y=-1$

Solution

a. $x+y=4$
$x-y=2$

We first decide to solve for either x or y and then equate that variable to the other then we produce a single variable equation.

We take $x+y=4$ $\quad\therefore y=4-x$

Therefore substitute y in the other equation with $x+4$

$x-y=2 \Rightarrow x-(4-x)=2$

$$x-4+x=2$$
$$2x-4=2 \Rightarrow 2x=2+4=6$$

$2x=6 \Rightarrow x={}^{6}\!/\!_{2}=3$ $\quad x=3$

Since $y=4-x \Rightarrow y=4-3 \quad \therefore y=1$

Thus the solution in the ordered pair $(x,y)=(3,1)$

b. $x^2-x-y=1$
$-x+y=-1$

We take $-x+y=-1$ to obtain a substitution expression

$\therefore y=x-1$

Therefore In the equation $x^2-x-y=1$

$$x^2-x-(x+1)=1$$
$$x^2-x-(x+1)=1$$
$$x^2-2x=0$$
$$\therefore x\ (x-2)=0$$

$x=0$ or $x-2=0 \quad x=2$

If x= 0 then $y=x-1=0-1=-1$

Solution $(x,y)=(0,-1)$

If $x=2$ then $y=x- \Rightarrow y=2-1=1$

Solution $(x, y) = (2,1)$

The two solutions $(0-1)$ and $(2,1)$ are the solution of the above system of equations.

Examples

2) A total of \$12,000 is inverted in two funds paying 9% and 11% simple interest. If the year interest is \$1180, how much of the \$12,000 is inverted at each rate.

Solution

$9\% = 9/100 = 0.09, \quad 11\% = 11/100 = 0.11$

We take x ,y variables to represent the simple interest rates.

$\therefore 0.09x + 0.11y = 1180$

$x + y = 12,000$

Therefore substitute for $y \Rightarrow y = 12,000 - x$

We consider $0.09x + 0.11y = 1180$

$0.09x + 0.11(12,000 - x) = 1180$

$0.09x + 1320 - 0.11x = 1180$

$0.09x - 0.11x = 1180 - 1320$

$-0.02 = -140 \qquad \therefore = -140/-0.02 = 7000$

Since $y = 12000 - x$

$= 12000 - 7000$

$= 5000$

The solution is an ordered pair (7000, 5000)

Method of elimination

The key step in this method is to obtain coefficients that differ only in the sign so that by adding the two equations this variable is eliminated.

Example

a) Solve the system of linear equation

$3x + 2y = 4$

$5x - 2y = 8$

Solution

In this question note that the coefficient of y differ only by sign

$$\begin{array}{l} \therefore 3x + 2y = 4 \\ \underline{+5x - 2y = 8} \\ 8x + 0 = 12 \end{array} \qquad \therefore x = 12/8 = 3/2$$

We take one equation $3x + 2y = 4$ and substitute $x = 3/2$

$3\left(3/2\right) + 2y = 4$

$\therefore \frac{9}{2} + 2y = 4 \Rightarrow 4\frac{1}{2} + 2y = 4$

$2y = 4 = 4\frac{1}{2} \Rightarrow 2y = -\frac{1}{2}$

$\therefore y = -\frac{1}{4}$

The solution is an order pair $\left(\frac{3}{2}, -\frac{1}{4}\right)$

b) Solve the system of linear equation

$2x - 3y = -7$

$3x + y = -5$

Solution

In this equation we multiply the second equation with a value 3 in order for the coefficient of y to be the same.

$\Rightarrow 2x - 3y = -7 \qquad \Rightarrow 2x - 3y = -7$

$3(3x + y = -5) \qquad 9x + 3y = -15$

$\therefore 2x - 3y = -7$

$$\begin{array}{r} + \underline{9x + 3y = -15} \\ 11x + 0 = -22 \end{array} \qquad \therefore 11x = -22$$

$x = -2$

By taking $2x - 3y = -7$ we can solve for y

$\Rightarrow 2(-2) - 3y = -7$

$-4 - 3y = -7 \quad \Rightarrow -3y = -7 + 4 = -3$

$-3y = -3$

$y = \frac{-3}{-3} = 1$

The solutions is an ordered pair $(-2,1)$

Solve $\quad 5x + 3y = 9$

$2x - 4y = 14$

Solution

The first equation multiplied by 4 and the second on by 3

$\Rightarrow 4(5x + 3y = 9) \quad \Rightarrow 20x + 12y = 36$

$3(2x - 4y = 14) \quad 6x - 12y = 42$

$20x + 12y = 36$

$$\begin{array}{r} \underline{6x - 12y = 42} \\ 26x + 0 = 78 \end{array} \qquad \therefore x = \frac{78}{26} = 3$$

Therefore we take $2x - 4y = 14$ and substitute $x = 3$

$2(3) - 4y = 14 \Rightarrow 6 - 4y = 14$

$-4y = 14 - 6 = 8 \qquad \therefore y = \frac{8}{-4} = -2$

The solution is an ordered pair $(3, -2)$

Matrix Methods

We use matrix when the number of variables is 2 or greater. These methods enable us to solve a system of equation with many variable easily where the other two methods cannot and even if they can, to provide the solution it will consume a lot of time.

Inverse matrix method

If the coefficient matrix A of a square system in invertible (has an inverse matrix) the system has a unique solution, which is given by.

$AX = B$ Multiply by inverse if A

$A^{-1}AX = A^{-1}B$

$I_n X = A^{-1}B$

$X = A^{-1}B$

Example

Consider a system of linear equation

$x + 4y = 2$

$-x - 3y = 1$

We solve using inverse matrix method.

We rearrange in order to get the coefficients

Therefore $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $x = \begin{bmatrix} x \\ y \end{bmatrix}$

$\therefore x = A^{-1}B$

We find $A^{-1} \Rightarrow A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\det A = (1)(-3) - (-1)(4) = -3 + 4 = 1$

$A^{-1} = \frac{1}{1}\begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix}$

$\therefore x = A^{-1}B$

$= \begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (-3 \times 2) + (-4 \times 1) \\ (1 \times 2) + (1 \times 1) \end{bmatrix} = \begin{bmatrix} -6 - 4 \\ 2 + 1 \end{bmatrix} = \begin{bmatrix} -10 \\ 3 \end{bmatrix}$

Hence x = -10 and y = 3

Example 2

Use matrix method to solve the following pair of simultaneous equation

$3a + 2b = 2$

$4a - b = 5$

$$\begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 12 \\ 5 \end{bmatrix}$$

The inverse of the coefficient matrix is $\frac{-1}{11} x \begin{pmatrix} -1 & -2 \\ -4 & 3 \end{pmatrix}$

Pre-multiplying both sides of matrix equation by the inverse

$$\frac{-1}{11}\begin{pmatrix} -1 & -2 \\ -4 & 3 \end{pmatrix}\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{-1}{11}\begin{pmatrix} -1 & -2 \\ -4 & 3 \end{pmatrix}\begin{bmatrix} 12 \\ 5 \end{bmatrix}$$

$$\frac{-1}{11}\begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{-1}{11}\begin{bmatrix} -22 \\ -33 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Therefore $a = 2$ and $b = 3$

Exercise 29

Use matrix method to solve the following pairs of simultaneous equations

1. $-2x + 3y = 3$
   $x + 3y = 4$
2. $2c + 3d = -2$
   $3c + d = 4$
3. $2p + 3q = 15$
   $7q + 5p = -13$
4. $3x + y = 8$
   $-y + 2x = -3$
5. $2x + y = 7$
   $4x + 3y = 17$
6. $3x + 3y = 5$
   $2x + 6y = 7$
7. $3x + 4y = -5$
   $2y - x = 9/2$

Suggested References

1. Backhouse, J.K. et.al, (2005) Pure Mathematics Volume1, Longman, pg 508-510
2. Blitzer, Robert. (2004) College Algebra, Prentice Hall pg 508-580

CHAPTER SIX: PERMUTATIONS AND COMBINATIONS

Arrangements

This chapter aims at teaching a method of approach to certain problems involving arrangements and selections.

Example 1; From a pack of playing cards, the Ace, King Queen, Jack and Ten of Spades are taken. In how many ways can three of these five cards be placed in a row from left to right?

The first card can be any one of the five, Viz:

A; K; Q; J; 10

When the first card has been placed, there are four cards left to choose from, and so the possible ways of placing the first cards are:

AK,	AQ,	AJ,	A10;
KA,	KQ,	KJ,	K10;
QA,	QK,	QJ,	Q10;
JA,	JK,	JQ,	J10;
10A,	10K,	10Q,	10J;

Thus, for each of the 5 ways of choosing the first, there are 4 ways in which the second card may be chosen; therefore there are 5×4 (i.e. 20) ways of choosing the first two cards.

Now for each of the 20 ways of placing the first two cards, there are 3 cards left to choose from (e.g. if the first two cards were AK, third could be Q, J or 10); therefore there are 20×3 ways of placing the third card.

Thus, three cards chosen from the Ace, King, Queen, Jack, and Ten of Spades may be placed in a row from left to right in 60 different ways.

Example 2; Three schools have teams of six or more runners in a cross- county race. In how many ways can the first six places be taken by the three schools, if there are dead heats?

First it should be made clear that there is no question of the individuality of the runners, but only which school each of the first six runners belongs to.

The first place can be taken by any of the 3 schools.

When the first runner has come in, the second place can be taken by any of the 3 schools, so the first two places can be taken in 3×3, or 3^2, ways.

Similarly, the third place can be taken by any of the 3 schools, so the first three places can be taken in $3^2 \times 3$ or 3^3, ways.

Continuing the argument for the fourth, fifth and sixth places, it follows that the first six places may be taken in 3^6, or 729, ways by the three schools.

Example 3; How many even numbers, greater than 2000, can be formed with the digits 1, 2, 4, 8, if each digit may be used only once in each number?

If the number is greater than 2000, the first digit can be chosen in 3 ways (Viz 2, 4, or 8).
Then, whichever has been chosen to be the first digit, there are 2 ways in which the last digit may be chosen, in order to make the number even. Thus, there are 3×2 ways of choosing the first, last and last digits.
When the first and last digits have been chosen, there are 2 digits, either of which may be the second digit of the number. Thus there are $3\times2\times2$ ways of choosing the first, last and second digit.
Now, when three digits have been chosen, there is only 1 left to fill the remaining place, and so there are $3\times2\times2\times1$, i.e. 12 even numbers greater than 2000 which may be formed from the digits 1, 2, 4, 8, without repetitions.
The following table is useful for showing the argument briefly

Position of digit	First	Last	Second	Third
Number of possibilities	3	2	2	1

It is to be understood, in this and later tables, that the choice is made in the order of the first line.

Exercise 30

1) Ten boys are running a race. In how many ways can the first three places be filled, if there are no dead heats?
2) How many five- figure odd numbers can be made from the digit 1, 2, 3, 4, 5, if no digit is repeated
3) There are sixteen books on a shelf. In how many ways can these be arranged if twelve of them are volumes of a history, and must be kept together, in order?
4) Three letters from the word RELATION are arranged in a row. In how many ways can this be done? How many of these contain exactly one vowel?
5) Four men and their wives sit on a bench. In how many ways can they be arranged if
    a) There is no restriction,
    b) Each man sits next to his wife?

The Factorial Notation

There are times when a problem on arrangements leads to an answer involving a product of more factors than it is convenient to write down. The next example shows how this may arise.

Example 4 In how many ways can the cards of one suit, from a pack of playing cards, be placed in a row?

Position of card in row	First	Second	…	Twelfth	Thirteenth
Number of possibilities	13	12	…	2	1

The table abbreviates the type of argument used in the last three examples, and it leads to the conclusion that the cards of one suit can be placed in a row in
$13\times12\times11\times10\times9\times8\times7\times6\times5+4\times3\times2\times1$ Ways
To shorten the answer the product could be evaluated, giving 6 227 020 800; but it is easier to write.
13!
(Which is read, 'factorial thirteen', or by some, 'thirteen shriek'!). Thus,
$7!=7\times6\times5+4\times3\times2\times1=5040$ And similarly for any other positive integer
The factorial notation will be used freely in this chapter and the reader should become thoroughly used to it.

Example 5 (a) Evaluate $\frac{9!}{2!7!}$,

(b) Write $40\times39\times38\times37$ in factorial notation

a) Written in full,

$$\frac{9!}{2!7!}=\frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{2\times1\times7\times6\times5\times4\times3\times2\times1}$$

$$=\frac{9\times8}{2\times1}$$
$$=36$$

b) $$40\times39\times38\times37=40\times39\times38\times37\times\frac{36\times35\times\ldots\times2\times1}{36\times35\times\ldots\times2\times1}$$
$$=\frac{40!}{36!}$$

Exercise 31

1. Evaluate

   a) $3!$, b). $5!$, c). $\frac{6!}{(3!)^2}$

2. Express in factorial notation:

   a). $6\times5\times4$, b). $n(n-1)(n-2)$, c). $\frac{7\times6\times5}{3\times2\times1}$, d). $\frac{(n+1)n(n-1)}{3\times2\times1}$

3. Express in factors:

   a). $20!+21!$, b). $15!+4(14!)$, c) $n!+2(n-1)!$

4. Simplify:

a). $\frac{15!}{11!4!}+\frac{15!}{12!3!}$, b. $\frac{16!}{9!7!}+\frac{2\times16!}{10!6!}+\frac{16!}{11!5!}$, c). $\frac{n!}{(n-r)!r!}+\frac{n!}{(n-r+1)!(r-1)!}$

Permutations

In example 4, it was found that 13 playing cards could be placed in a row in 13! ways. If we consider n unlike objects placed in a row, using the same method,

Position of object in row	1st	2nd	…	(n-1)th	nth
Number of possibilities	N	$n-1$	…	2	1

We find that they may be arranged in n! ways.
The arrangements of the n objects are called **permutations**. Thus
ABC, ACB, BCA, BAC, CAB, CBA, is the 3! Permutations of the three letter A, B, C.
Again, in example 1, it was found that 3 cards chosen from 5 unlike cards could be arranged in 60 ways. This might be expressed by saying that there are 60 permutations of 3 cards chosen from 5 unlike cards.
A permutation is an arrangement of a number of objects in a particular order. In practice, the order may be in space, such as from left to right in a row; or it may be in time, such as reaching the winning post in a race, or dialing on a telephone.
How many permutations are there of r objects chosen from n unlike objects?
The method is indicated in the table below

Order of choice of object	1st	2nd	3rd	$\cdots$	$(r-1)^{th}$	r^{th}
Number of possibilities	n	$(n-1)$	$(n-2)$	$\cdots$	$(n-r+2)$	$(n-r+1)$

Thus there are

$n(n-1)(n-2)\ldots(n-r+2)(n-r+1)$

Permutations of the objects, But

$$n(n-1)(n-2)\ldots(n-r+2)(n-r+1)$$

$$=\frac{n(n-1)(n-2)\ldots(n-r+2)(n-r+1)\times(n-r)\ldots2\times1}{(n-r)\ldots2\times1}$$

$$=\frac{n!}{(n-r)!}$$

Therefore there are $n!(n-r)!$ permutations of r objects chosen from n unlike objects, if r is less than n.
(We have already found that there are $n!$ permutations of n unlike objects.)

Example 6; There are 20 books on a shelf, but the red covers of two of them clash, and they must not be put together. In how many ways can the books be arranged?
This is best tackled by finding out the number of ways in which the two books are together, and subtracting this from the number of ways in which the 20 books can be arranged if there is no restriction.
Suppose the two red books are tied together, then there are 19 objects, which can be arranged in 19! ways. Now if the order of the two red books is reversed, there

will again be 19! Arrangements; so that there are $2\times 19!$ ways of arranging the books with the red ones next to each other
With no restriction, 20 books can be arranged in 20! Ways; therefore the number of arrangements in which the red books are not together is
$20! - 2\times 19! = 18\times 19!$

Example 7; in how many ways can 8 people sit at a round table?
Since the table is round, the position of people relative to the table is of no consequence. Thus, supposing they sit down, and then all move one place to the left, the arrangement is still the same.
Therefore one person may be considered to be fixed, and the other 7 can then be arranged about him or her in 7! ways. Thus there are 5040 ways in which 8 people can sit at a round table.

Example 8; In how many ways can the letters of the word BESIEGE be arranged?
First, give the three E's suffixes: $BE_1SIE_2GE_3$. Then, treating the E's as different; the 7 letters may be arranged in 7! Ways.
Now, in every distinct arrangement, the 3 E's may be rearranged amongst themselves in 3! ways, without altering the positions of the B, S, I, or G; for instance, SEIBEEG would have been counted 3! Times in the 7! Arrangements as
$SE_1IBE_2E_3G,\ \ SE_2IBE_3E_1G,\ \ SE_3IBE_1E_2G,$
$SE_1IBE_3E_2G,\ \ SE_2IBE_1E_3G,\ \ SE_3IBE_2E_1G,$
Therefore the number of distinct arrangements of the letters in BESIEGE is $7!/3! = 840$.
In the next exercise there are some examples which are best tackled from first principles, like the next example.

Example 9 How many even numbers, greater than 50,000, can be formed with the digits $3, 4, 5, 6, 7, 0,$ without repetitions?
Compared with example 3, 12.1 there are two extra difficulties: the number can have either 5 or 6 digits, and the number cannot begin with 0. Therefore the problem is split up into four parts:
(1). Numbers with 5 digits, the first digit being even.

Position of digit in number	1st	5th	2nd	3rd	4th
Number of possibilities	1	2	4	3	2

This gives $1\times 2\times 4\times 3\times 2 = 48$ possibilities.

(2) Numbers with 5 digits, the first digit being odd.

Position of digit in number	1st	5th	2nd	3rd	4th
Number of possibilities	2	3	4	3	2

This gives $2\times 3\times 4\times 3\times 2 = 144$ possibilities.

(3) Numbers with 6 digits, the first digit being even

Position of digit in number	1st	6th	2nd	3rd	4th	5th
Number of possibilities	2	2	4	3	2	1

This gives $2 \times 2 \times 4 \times 3 \times 2 \times 1 = 96$ possibilities

(4) Numbers with 6 digits, the first digit being odd

Position of digit in number	1st	6th	2nd	3rd	4th	5th
Number of possibilities	3	3	4	3	2	1

This gives $3 \times 3 \times 4 \times 3 \times 2 \times 1 = 216$ possibilities

Therefore the total number of possibilities is 48 + 144 + 96 + 216 = 504

Exercise32

1. Seven boys and two girls are to sit together on a bench. In how many ways can they arrange themselves so that the girls do not sit next to each other?
2. There is room for ten books on a bedside table, but there are fifteen to choose from. Of these, however, a Bible and a book of ghosts stories must go at the end. In how many ways can the books be arranged?
3. I a crocket team, the captain has settled the first four places in the batting order, and has decided that the four bowlers will occupy the last four places.
4. How many numbers of five digits can be made from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, when each number contains exactly one even digit and no digit more than once?
5. Six natives and two foreigners are seated in a compartment of a railway carriage with four seats either side. In how many ways can the passengers seat themselves if
    a. The foreigners do not sit opposite each other.
    b. The foreigners do not sit next to each other.

Combinations

In the last section, attention was given to permutations, where the order of set of objects was of importance; but in other circumstances, the order of selection is irrelevant. If for instance, eight tourists find there is only room for five of them at a hotel, they will be chiefly interested in which five of them stay there, rather than in any order of arrangement.

When a selection of objects is made with no regard being paid to order, it is referred to as a **combination**. Thus, ABC, ACB, CBA, are different permutations, but they are the same combination of letters.

Example 10 *In how many ways can* 13 *cards be selected from a pack of* 52 playing cards?

First of all, suppose that thirteen cards from the pack are laid on a table in an order from left to right. From the last section, it follows that this can be done in 52!/59! Ways.

Now each combination of cards can be arranged in 13! Ways, therefore the number of permutations $= 13! \times$ (the number of combinations)

Therefore $\frac{52!}{39!} = 13! \times$(the number of combinations)

Therefore the number of combinations of 13 cards chosen from a pack of playing cards is 52!/(39!13!)

In how many ways can r objects be chosen from n unlike objects?

it was shown above that there are $n!/(n-r)!$ permutations of r objects chosen from n unlike objects.

Now each combination of r objects can be arranged in r! ways, therefore the number of permutations $= r! \times$ (the number of combinations)

Therefore $\frac{n!}{(n-r)!} = r! \times$(the number of combinations)

Hence the number of combinations of r objects chosen from n unlike objects is $\frac{n!}{(n-r)!r!}$

For brevity, the number of combinations of r objects chosen from n unlike objects is written ${}^nC_r$ thus

$$ {}^nC_r = \frac{n!}{(n-r)!r!} $$

${}^nC_r$ is also sometimes written as ${}_nC_r$ and $\binom{n}{r}$

Example 11 A mixed hockey team containing 5 men and 6 women is to be chosen from 7 men and 9 women. In how many ways can this be done?

Five men can be selected from 7 men in ${}^7C_5$ ways, and 6 women can be selected from 9 women in ${}^9C_6$ ways.

Now for each of the ${}^7C_5$ ways of selecting the men, there are ${}^9C_6$ ways of selecting the women, therefore there are ${}^7C_5 \times {}^9C_6$ ways of selecting the team.

$$ \begin{aligned} {}^7C_5 \times {}^9C_6 &= \frac{7!}{2!5!} \times \frac{9!}{3!6!} \\ &= 21 \times 84 \end{aligned} $$

Therefore the team can be chosen in 1764 ways.

Exercise 33

1. Evaluate: (a) ${}^{10}C_2$, (b) ${}^{6}C_4$ (c). ${}^{7}C_3$ (d) ${}^{9}C_5$ (e) ${}^{8}C_4$.

   Express in factors: (f) ${}^{n}C_2$, (g) ${}^{n}C_3$ (h). ${}^{n}C_{n-2}$ (i) ${}^{n+1}C_2$ (j). ${}^{n+1}C_{n-1}$.
2. Nine people are going to travel in two taxis. The larger has five seats, and the smaller has four. In how many ways can the party be split up?
3. Two punts each hold six people. In how many ways can a party of six boys and six girls divide themselves so that there are equal numbers of boys and girls in each punt?
4. Four people are to play bridge and four others are to play whist. Find the number of ways in which they may be chosen if eleven people are available.
5. A committee of ten is to be chosen from nine men and six women. In how many ways can it be formed if at least four women are to be on the committee?

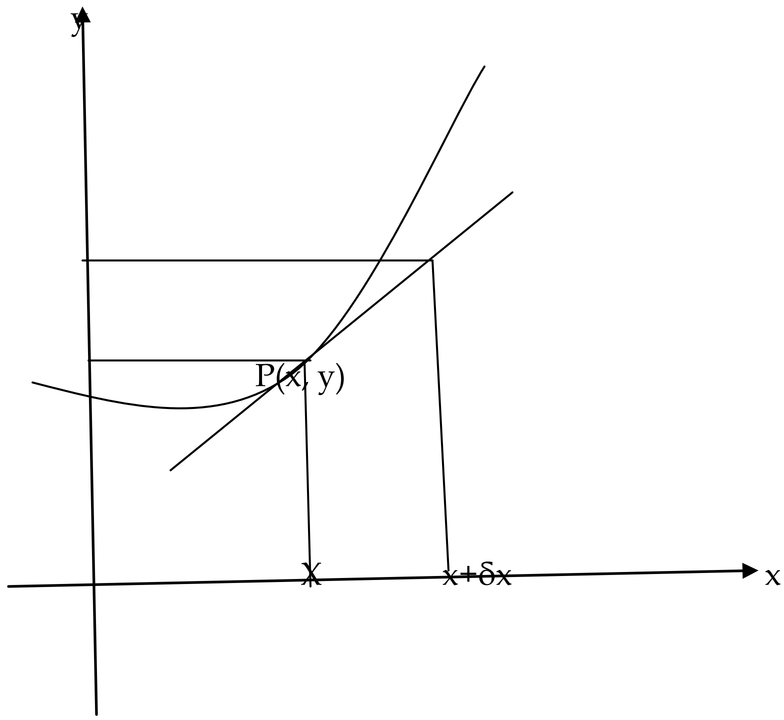
Suggested References

1. Backhouse, J.K. et.al, (2005) Pure Mathematics Volume1, Longman, pg 243-254, pg 275-287
2. Blitzer, Robert. (2004) College Algebra, Prentice Hall pg 673-689
3. Sharon, L. Senk, et.al Advanced Algebra Second Edition, University of Chicago school of Mathematics. Pg 835 -860

CHAPTER SEVEN : CALCULUS

GRADIENT OF A CURVE

The gradient of a curve with equation y = f(x) at any point on the curve is defined as the gradient of the tangent to the curve at this point.



The gradient of the chord PQ is

$$y + \delta y = f(x + \delta x)$$

$$y = f(x)$$

$$\frac{y + \delta y - y}{x + \delta x - x} = \frac{f(x + \delta x) - f(x)}{x + \delta x - x}$$

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$\frac{dy}{dx} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

The symbol $\frac{dy}{dx}$ is called the derivative, or the differential coefficient of y with respect to x.

$$\frac{dy}{dx} = f'(x)$$

Where f ′(x) is often called derived function of f.

The procedure used to find $\frac{dy}{dx}$ from y is called differentiating y with respect to x.

Example differentiate $2x^2 + 3x - 4$ using first principles

Solution

$$y = 2x^2 + 3x - 4$$

$$y + \delta y = f(x + \delta x) = 2(x + \delta x)^2 + 3(x + \delta x) - 4$$

$$y + \delta y = f(x + \delta x) = 2x^2 + 4x\delta x + \delta x^2 + 3x + 3\delta x - 4$$

$$y + \delta y - y = f(x + \delta x) - f(x) = 2x^2 + 4x\delta x + \delta x^2 + 3x + 3\delta x - 2x^2 - 3x + 4$$

$$\delta y = f(x + \delta x) - f(x) = 4x\delta x + \delta x^2 + 3\delta x$$

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x} = \frac{4x\delta x + \delta x^2 + 3\delta x}{\delta x}$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} 4x + 3$$

$$\frac{dy}{dx} = 4x + 3$$

In general $f(x) = x^n$

Then $\frac{dy}{dx} = nx^{n-1}$ for all rational n

Notice that if y = a, a constant, you can write this as $y = ax^0$ and $\frac{dy}{dx} = (a \times 0)x^{-1} = 0$

The derivative of a constant is always zero.

Example

Find $\frac{dy}{dx}$ for each of these functions.

a) $y = x^3$ b) $y = 6x^4$ c) $y = 6x^{-5}$

When $y = x^3$

a) $\frac{dy}{dx} = 3x^{3-1}$

$\frac{dy}{dx} = 3x^2$

b) $\frac{dy}{dx} = 6 \times 5x^{4-1}$

$\frac{dy}{dx} = 30x^3$

c) $\frac{dy}{dx} = -5 \times 6x^{-5-1}$

$\frac{dy}{dx} = -30x^{-6} = -\frac{30}{x^6}$

Sum or difference of two functions

To find the derivative of a function that has more than one term, you need to differentiate each function in turn.

$$y = f(x) \pm g(x) \text{ then } \frac{dy}{dx} = f'(x) \pm g'(x)$$

This applies to the sum or difference of any number of functions.

Find f′(x) for each of these functions

(a) $4x^2 + 2x^{-4} - 5$

(b) $\frac{2}{\sqrt{x}} + \frac{3}{x^5} - 6x^{\frac{2}{3}} + x$

Solutions

a) $\frac{dy}{dx} = 2 \times 4x^{2-1} + -4 \times 2x^{-4-1}$

$\frac{dy}{dx} = 8x - 8x^{-5}$

b) $\frac{dy}{dx} = 2 \times \frac{-1}{2} x^{-1/2-1} + 3 \times -5x^{-5-1} + x^{1-1}$

$\frac{dy}{dx} = -x^{3/2} - 15x^{-6} + 1$

A function may not be given in the form ax^n. in this case, you must manipulate the expression for y and write it as a sum of functions, each in the form ax^n.

Example Find $\frac{dy}{dx}$ for each of these functions.

(a) $y = (x+3)^2$ (b) $y = \sqrt{x}(x^2 - 1)$ (c) $y = \frac{x^3 + 6}{x}$

a) To differentiate $y = (x+3)^2$, first expand the bracket:

$Y = (x+3)(x+3)$

$= x^2 + 6x + 9$

$\therefore \frac{dy}{dx} = 2x + 6$

b) $y = \sqrt{x}(x^2 - 1) = x^{1/2}(x^2 - 1)$

So: $y = x^{5/2} - x^{1/2}$

$\therefore \therefore \frac{dy}{dx} = \frac{5}{2} x^{3/2} - \frac{1}{2} x^{-1/2}$

You can factorise this expression for $\frac{dy}{dx}$ by taking out $\frac{1}{2} x^{-1/2}$

This gives $\frac{dy}{dx} = \frac{1}{2} x^{-1/2} (5x^2 - 1)$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} (5x^2 - 1)$

c) when $$y = \frac{x^3+6}{x} = \frac{x^3}{x} + \frac{6}{x}$$

$$y = x^2 + 6x^{-1}$$

$$\frac{dy}{dx} = 2x - 6x3^{-2}$$
$$= 2x^{-2}(x^3 - 3)$$
$$= \frac{2(x^3-3)}{x^2}$$

Example b) and c) show how you can manipulate the derivative to obtain a more mathematically tidy result. Note that any form of the correct derivative is acceptable. However, is useful to understand such manipulation since many examination questions ask the derivative in a particular form.

THE SECOND DERIVATIVE

Differentiating function gives you another function, the derived function or first derivative. You can differentiate this function to get second derivative of the original function.

The derivative $\frac{dy}{dx}$, that is $\frac{d}{dx}(\frac{dy}{dx})$, is denoted by $\frac{d^2y}{dx^2}$ and is called the second derivative of y with respect to x.

Example

Give that f(x) = x + $\frac{1}{x}$, find f′(x) and f′′ (x).

Therefore, f′ (x) = 1 $-x^{-2}$

$= 1- \frac{1}{x^2}$

The second derivative

f′′(x) = $2x^{-3}$

$= \frac{2}{x^3}$

TANGENTS AND NORMALS TO THE CURVE

The derivative is defined as the gradient to the curve at the given point P; this is the gradient of the tangent to the curve at the given point.
The normal is perpendicular to the tangent so the product of their gradients equal to negative one.

Example

To find the gradient of the curve $f(x) = x^2 + \frac{1}{x}$ at point P (1, 2)

To find the gradient of the curve, first find f′(x):

$$f'(x) = 2x - \frac{1}{x^2}$$

The gradient to the curve at point P (1, 2)

Then, $f'(1) = 2(1) – 1 = 1$

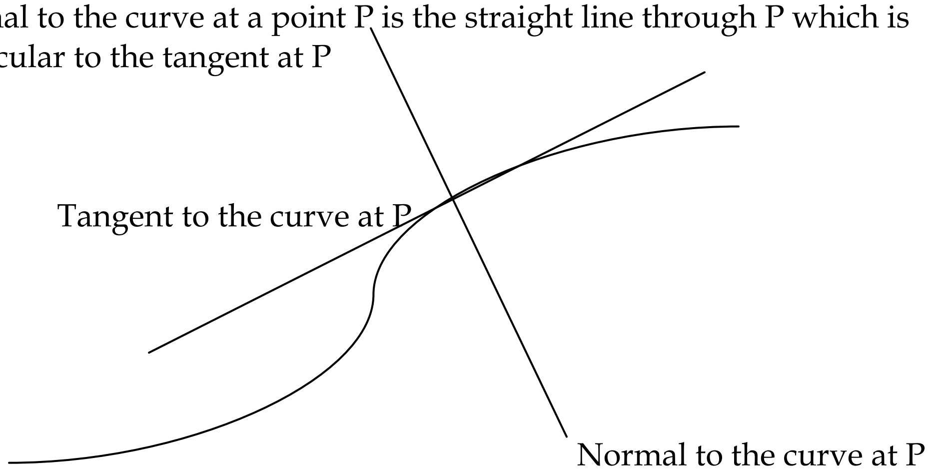
The gradient of the tangent is 1

The equation of the tangent to the curve at point P is $y - 2 = 1(x - 1)$

The equation is $y = 2x + 1$

Equation of the normal

The normal to the curve at a point P is the straight line through P which is perpendicular to the tangent at P



Since the tangent and normal are perpendicular to each other, if the gradient of the tangent is m, the gradient of the normal is -1/m.

Example

Find equation of the normal to the curve $y = 3x^2 + 7x - 2$ at point where $x = -1$.

When $x = -1$

$Y = 3(-1)^2 + 7(-1) - 2 = -6$

Therefore, P has coordinate P (-1, -6)

When $y = 3x^2 + 7x - 2$, then $\frac{dy}{dx} = 6x + 7$.

At point P (-1 -6)

$$\frac{dy}{dx} = 6(-1) + 7 = 1$$

The gradient of the tangent is 1. Therefore, the gradient of the normal is $-\frac{1}{1} = -1$

The equation of the normal at P is

$$y – (-6) = -1(x - (-1))$$

The equation of the normal to the curve at the point

P (-1, -6) is $y = -x - 7$

MAXIMUM, MINIMUM AND POINT OF INFLEXION

Appoint in a curve at which the gradient is zero, where $\frac{dy}{dx}$ =0 is called a stationary point. At stationary point the point, the tangent to the curve is horizontal and the curve is flat.

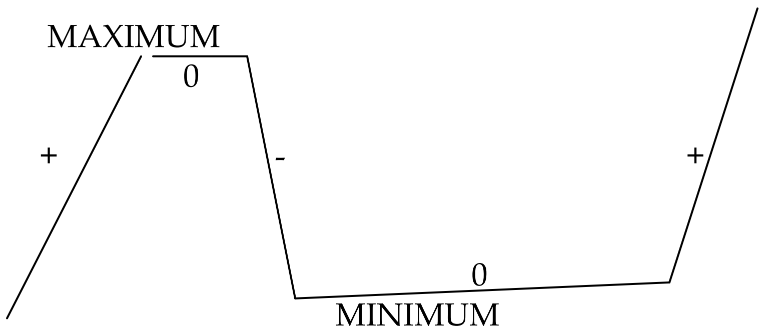
There three types of stationary point and you know how to distinguish one from another.

<u>Minimum point</u>

In this case, the gradient of the curve is negative to the left of P. to the right of point P, the gradient of the curve is positive.

<u>Maximum point</u>

In this case, the gradient of the curve is positive to the left of P. To the right of point P, the gradient of the curve is negative.



In case f point of gradient has the same sign each side of the stationary sides. A point of inflexion which has zero gradients is called a horizontal point of inflexion. This distinguishes it from other points of inflexion which has non-zero gradient.

Second derivative and stationary points

At all stationary points on y = f(x) the gradient of the curve is zero i.e. $\frac{dy}{dx} = 0$

At the point P, the gradient of the derived function is Zero i.e. $\frac{dy}{dx} = 0$

At the maximum point and minimum point, the gradient is zero i.e. $\frac{dy}{dx} = 0$

Distinguishing between maximum and minimum

$\frac{dy}{dx}$ Means how y is changing as x increases or the gradient is changing, as x increases.

$\frac{d^2y}{dx^2} < 0$, means P is a maximum

$\frac{d^2y}{dx^2} = 0$, means P is a point of inflexion

$\frac{d^2y}{dx^2} > 0$, means P is a minimum

Example

A firm has analysed their operating conditions, prices and costs and have developed the following functions.

Revenue £R = 800Q – 4Q^2

Cost £C = Q^2 + 10Q +30, where Q is the number of units sold

The firm wishes to maximize profit and wishes to know

i. What quantity should be sold?
ii. At what price?
iii. What will be amount of profit?

The profit is measured when marginal cost is equal to marginal revenue

$\frac{dR}{dQ}$ =800 -8Q (marginal revenue)

$\frac{dC}{dQ}$ = 2Q +10 (marginal cost)

Marginal cost = Marginal revenue

800 – 8Q = 2 Q +10

790 = 10Q

Q = 79

Total revenue = £800x 79 – £4 x 79^2

=£63200 – £24964

= £38236

ii) The price per unit $\frac{38236}{79} = 484$ each

Total profit = Revenue – cost

= £38236 – (79^2 +10x 79 +30)

= £38236 -7061

=£31175

INTEGRATION

Integration can be regarded as the reverse of differentiation. The applications involves finding the sum or total costs and areas under the curves

$\frac{dax^n}{dx}$ = n x^{n-1}

Therefore the integral involves adding one and dividing by the new power n+1

$$\int ax^n dx = \frac{ax^{n+1}}{n+1}, n \neq -1$$

Exercise 35

1. Given $y = x^{1/3} - x^{-1/3}$

   a) Find $\frac{dy}{dx}$

   b) Evaluate $\int_1^8 y\,dx$

2.

Function f(x) = y	$\frac{2}{3}x^6$	$\frac{-3}{4x^4}$	$\frac{2}{3}x^6 + e^{3x}$	$5x^{\frac{6}{5}} - \ln(2x^3 - 3)$	$x^{-7}\left(e^x - 5\right)^3$	$\frac{x^3 + 5}{3\sqrt{x}}$	$e^{4x} \ln x$
Derived function f′(x) $= \frac{dy}{dx}$							

3. The profit, y hundred euro, generated from the sale, x thousand, of a certain product is given by the formula $y = 72x + 3x^2 - 2x^3$. Calculate how many items should be sold in order to maximize the profit, and determine that the maximum profit.
4. Integrate each of these functions with respect to x

a) $3x^{5/6} + \frac{5}{x^3} - \frac{6}{x}$

b) $e^{5x} - 2x^{2/3}$

5. Find y as a function of x given that $\frac{dy}{dx} = \frac{5}{x^2} - 4, (x \neq 0)$ and that y = -12 when x = 5.

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1. Backhouse, J.K. et.al, (2005) Pure Mathematics Volume1, Longman, pg 55-172
2. Frank Ayres, Jr. and Elliott Mendelson (1999) Calculus (fourth Edition), Schaum's outline Series pg79-97, pg102-107,115-128
3. Robert Ellis and Denny Gulick, (1978) Calculus with Analytic Geometry, Harcourt Brace Jovanovich, Inc. pg 140-177,259-318

SAMPLE EXAMINATION PAPER

Mt Kenya University

SCHOOL OF APPLIED SOCIAL SCIENCES
DEPARTMENT OF BUSINESS STUDIES
BACHELOR OF BUSINESS MANAGEMENT
BBM 112: FONDATION MATHEMATICS

INSTRUCTIONS

- *Answer Question ONE and any other Two Questions*
- ***TIME : 2 HOURS***

QUESTION ONE

a) i). Give two examples of a set of numbers that are real numbers. (2 marks)

ii). Define a rational number. (1 mark)

b) i). List the members of the set of real numbers for which the expression

$\frac{1}{(x-1)(x-2)(x-3)}$ does not exist. (1 mark)

ii). Explain the term function, image of a function and the domain of a function. (3 marks)

iii).Given that $f(x)=\sqrt{(x+3)}$ and $g(x)=\sqrt{(16-x^2)}$. Find $\left(\frac{f}{g}\right)x$ and Dom $\left(\frac{f}{g}\right)x$ (3 marks)

c) i). Find y if $\log_2 y-2=\log_2 92$ (3 marks)

ii). Solve for x without using tables or a calculator $\frac{\log 75+\log 9+\log 5}{\log 5+\log 45}$ (3 marks)

d) Solve the following equations

i). $8y-3x=12$
$5y-2x=7$ (3 marks)

ii). $3-x-2x^2=0$ (3 marks)

e) Solve the inequality $1-\frac{3x}{2}\geq x-4$ and represent the solution graphically. (3 marks)

f) i). Evaluate $\dfrac{20!}{3!\,8!}$ (2 marks)

ii). Write $50\times49\times48\times47\times46$ in factorial form. (2 marks)

iii). Using an example of matrix A and B, what does the phrase "matrix multiplication is not commutative" mean. (1 mark)

QUESTION TWO

a) i). Derive the quadratic formula by solving the equation $ax^2+bx+c=0$ where a, b, and c are real numbers and $a\neq0$. (Hint use completing square method)

(4 marks)

ii). Use the formula derived above to solve the equation $2x^2+7x-15=0$.

(3 marks)

b) i). Find the inverse of the matrix M where $M=\begin{pmatrix}3 & 2\\ 2 & 5\end{pmatrix}$ and hence solve the matrix equation $MX=C$ in which $X=\begin{pmatrix}x\\ y\end{pmatrix}$ and $C=\begin{pmatrix}10\\ 3\end{pmatrix}$

(5 marks)

ii). Given that $A=\begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 3 & 2 & 2\end{bmatrix}$ $B=\begin{bmatrix}2 & 0 & -1\\ 1 & 1 & -1\\ -4 & -1 & 3\end{bmatrix}$ find $(AB)^T$ (4 marks)

c) Solve the inequalities below

$\dfrac{x^2-x-2}{x^2-4x+3}\geq0$ (4 marks)

QUESTION THREE

a) i). Given that $f(x)=5x+1$ and $g(x)=x^2$. Express the composite function *fog* and *gof* in their simplest form possible and find $f(g(-2))$ and $g(f(3))$

(6 marks)

ii). Find the inverse of the function $f(x)=\sqrt{(2x-3)}$ (4 marks)

b) The third term of a geometric sequence is 9 and 6th term is 243. Find the first term and the common ratio. (5 marks)

c) Given $A=\{1, 2, 3, 4, 5\}$, $B=\{2, 4, 6, 8, 10\}$

$C=\{3, 4, 5, 6, 7\}$, $\varepsilon=\{1, 2, 3, \ldots, 12\}$ find

i). $A\cup B$

ii). $A\cap B$

iii). A'

iv). $(B\cup C)'$

v). $(A \cap C)'$ (5 marks)

QUESTION FOUR

a) A sewing machine valued at sh.25, 000 can be bought by cash at a discount of 10% or by installments whereby a deposit of sh.3,000 is paid followed by 15 monthly installments of sh.1,500 each. Find.

i). The cash price of the machine. (1 mark)

ii). The hire purchase price of the machine. (2 marks)

iii). The carrying charge. (2 marks)

iv). The interest rate of hire purchase buying. (3 marks)

b) If \$1700 is invested at 7.8% compounded quarterly, find the amount compounded at the end of 10 years in Ksh. given 1\$ = 72.00. (4 marks)

c) Given that $\log_{10} 2 = 0.301$, $\log_{10} 5 = 0.699$ and $\log_{10} 3 = 0.4771$. Find

i). $\log_{10} 50$ (2 marks)

ii). $\log_{10} 30$ (3 marks)

d) In how many ways can the letters of the

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SCHOOL OF APPLIED SOCIAL SCIENCES
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BACHELOR OF BUSINESS MANAGEMENT
MARCH 2010
BBM 112: FONDATION MATHEMATICS

INSTRUCTIONS: ANSWER QUESTIONS ONE (COMPULSORY) AND ANY OTHER TWO QUESTIONS

QUESTION ONE (COMPULSORY) 30 MARKS

(a) Solve this equation x

$$5^{4x-1} = 7^{x+2}$$ (5 Marks)

(b) Given A = $\begin{pmatrix} 0 & -1 & 2 \\ 1 & -1 & -3 \\ -2 & 3 & 5 \end{pmatrix}$ B = $\begin{pmatrix} 4 & 11 & 5 \\ 1 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

(i). Find **AB** (5marks)

(ii). Hence solve the simultaneous equations

$4x + 11y + 5z = 2$
$X + 4y + 2z = 1$
$X + 2y + z = 4$ (5 marks)

(c) A rocket is fired vertically upwards; its height **h** metres after t seconds is given by

$$\mathbf{h} = 98t - 4.9t^2$$

(i). Find an expression in terms of t the rate of change of height (3 Marks)

(ii). Find the maximum height reached (5 Marks)

(d) It is given that $\frac{dy}{dx} = x^3 + 2x - 1$ and that $y=13$ when $x=2$, find y in terms of x

(6 Marks)

QUESTION TWO (20 MARKS)

The function f is defined by $f(x) = \frac{x^2}{1-x}$, $x \neq 1$

(a) Find f^1 (x) and $f^{11}(x)$ (8 marks)

(b) Find the values of x for which $f^1(x) = 0$ (4 Marks)

(c) Use the second derivative to these values of x minimizes f(x) (2 Marks)

(d) Find the coordinates of the turning point distinguishing them (6 Marks)

QUESTION THREE (20 MARKS)

(a) Find the values of a and b if x-1 and x-2 are both factors of x^3+ax^2+bx-6 (6 marks)

(b) Evaluate without using mathematical devices $2\log_6 3 + \log_6 12 + \log_6 8 - \log_6 24$ (4 marks)

(c) How many different arrangement can be made by taking (i) five (ii) all of the letters of the word NUMBERS (4 Marks)

(d) Consider the expansion of $(3X^2 - \frac{1}{X})^9$

(i). How many terms are there in this expansion (2 marks)

(ii). Find the constant term in this expansion (4 marks)

QUESTION FOUR (20 MARKS)

(a) Solve the following pair of simultaneous equations, using matrices

X+2y=2

4y-3x=7 (6 marks)

(b) If A = $\begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 1 \\ 3 & 4 & 2 \end{pmatrix}$ B= $\begin{pmatrix} 1 & -1 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

(i). Find AB (5 Marks)

(ii). Find BA (5 Marks)

(iii). Find A+B (4 Marks)

<u>QUESTION FIVE (20 MARKS</u>

(a) If $\frac{dy}{dx}$ = 2-x^2 and y=3 when x=2, find y in terms of x (6 Marks)

(b) Find the area of the region bound by the curve y = x^2+1, the ordinates x=1, x=2 and x axis (6 marks)

(c) By rotating a quadrant of the circle $x^2 + y^2 = r^2$ about the axis of x, find a formula for the volume of a sphere (8 Marks)

Mt Kenya University

UNIVERSITY EXAMINATIONS 2009
SCHOOL OF APPLIED SOCIAL SCIENCES
DEPARTMENT OF BUSINESS STUDIES
BACHELOR OF BUSINESS MANAGEMENT
UNIVERSITY EXAMINATION APRIL 2010
BBM 112: FOUNDATION MATHEMATICS

INSTRUCTIONS: ANSWER QUESTIONS ONE (COMPULSORY) AND ANY OTHER TWO QUESTIONS

QUESTION ONE (COMPULSORY) 30 MARKS

(a) Express $\frac{8-3\sqrt{2}}{4+3\sqrt{2}}$ in the form a+b$\sqrt{2}$, where a and b are integers. [4marks]

(b) A committee of 5 people is to be selected from 6 men and 4 women. Find

(i) The number of different ways in which the committee can be selected [2marks]

(ii) The number of these selections with more women than men [3marks]

(c)

(i) Given that $\log_9 x = a\log_3 x$, find a [1mark]

(ii) Given that $\log_{27} y = b\log_3 y$ find b [1mark]

(iii) Hence solve, for x and y, the simultaneous equations

$$6\log_9 x + 3\log_{27} y = 8$$

$$\log_3 x + 2\log_9 y = 2$$ [5marks]

(d) The function f and g are defined, for $x \in \Re$, by

f: x $\rightarrow 3x-2$

$g : x \rightarrow \frac{7x-a}{x+1}$, where x ≠ -1 and a is positive.

(i) Obtain the expression of f^{-1} and g^{-1} [3marks]

(ii) Determine the value of a for which $f^{-1}g(4) = 2$ [4marks]

(e) It is given that A = $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ and that A +A^{-1} – KI = O, where I is the identity matrix and O is the zero matrix. Evaluate k [3marks]

(f) Evaluate $\int_{1}^{4} (6x - 3\sqrt{x})\, dx$ [4marks]

QUESTION TWO 20 MARKS

(a) $f(x) = 3x^3 - 2x^2 + kx + 9$

Given that when f(x) is divided by x+2 there is a remainder of -35

i) find the value of the constant k, [2marks]

ii) find the remainder when f(x) is divided by (3x -2) [3marks]

(b)

i) express $\frac{1}{\sqrt{32}}$ as a power of 2 1mark]

ii) express $(64)^{\frac{1}{x}}$ as a power of 2 [2 marks]

iii) hence or otherwise solve the equation

$$\frac{(64)^{\frac{1}{x}}}{2^x} = \frac{1}{\sqrt{32}}$$ [5 marks]

(c) (i) Expand $(1 + 3x)^8$ in ascending powers of x up and including the term in x^3. You should simplify each coefficient in your expansion [4marks]

(ii) Use your series, together with a suitable value of x which you should state, to estimate the value of $(1.003)^8$ giving your answer to 8 significant figures. [3marks]

QUESTION 3 (20 MARKS)

(a) a group operates a chain of filling stations in each of which are employed cashiers, attendants and mechanics as shown below

Types of filling stations			
	Large	Medium	Small
Cashiers	4	2	1
Attendants	12	6	3
Mechanics	6	4	2

The numbers of filling stations are

	Nairobi	Thika
Large	7	3
Medium	5	8
Small	12	4

How many of the various types of staff are employed in Nairobi and Thika [6marks]

(b) if A = $\begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

(i) Find A +B [3 Marks]

(ii) Find AB [5 marks]

(iii) Hence solve the simultaneous equations

x +3y +3z =3

x +4y +3z = 8

x +3y +4z = 2 [6Marks]

QUESTION FOUR (20 MARKS)

Given that y = $\frac{x^2}{2-x}$

(a) Show that $\frac{dy}{dx} = \frac{(4-x)x}{(2-x)^2}$ [4 marks]

(b) Find $\frac{d^2y}{dx^2}$ [4 marks]

(c) Find the coordinates of the two points on the curve where the gradient of the curve is zero stating their nature.

[8 marks]

(d)Find the equation of the tangent at the point (1.5, 4.5) [4 marks]

QUESTION FIVE (20 MARKS)

(a) Evaluate (i) $\int (\frac{1+x-4x^3}{x^3})dx$ [3 marks]

(ii) $\int (x^2 + 2 + \frac{1}{x^2})dx$ [3 marks]

(iii) $\int_{-2}^{-1} x^2(x-1)(x-2)dx$ [4 marks]

(b) If $\frac{dy}{dx} = \frac{3x-2}{x^3}$, find y in terms of x, when it passes through the point (1, 1)

[5marks]

(c) Find the area enclosed by x-axis, x=4 and x =5 of the curve y =x^2 (3- x)

[5 marks]