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DEPARTMENT OF INFORMATION TECHNOLOGY

COURSE CODE: BIT 1203

COURSE TITLE: BASIC DISCRETE MATHEMATICS

Instructional Material for BBIT - distance learning

Course outline

Purpose To prepare a learner to think logically in readiness for the programming courses in BIT

Objectives By the end of the course unit a learner shall be able to:

- To appreciate basic principles of Boolean algebra, logic, set theory permutations and combination and graph theory.
-

Course Content

- Boolean algebra:
  - Axiomatic definition of Boolean algebra
  - Proposition and proposition functions
  - Truth values and truth tables
- Logic; Predicate logic; Propositional logic; Logical reasoning; Conceptualizing variables; Open sentence; Truth sets, propositions; Truth values and table; Logical equivalence; Negation of statements; Conjunctions; Disjunctions; Tautology, contradiction etc; Contapositive, converse, inverse.
- Set theory; Set algebra, recussive definition sets; Orderings; Relations partially ordered sets; Lattices; Conceptualizing elements, finite and infinite; Universal empty and disjoint, subsets; Venn diagram union; Intersection; Complement; Difference; Number elements and logical arguments; Sets of sets; The power set and Cartesian product
- Mathematical inductions; Proof by induction; Contrapositive and contradiction.
- Permutations and combinations
- Graph theory; Directed and undirected graphs; Sub graphs, circuits; Paths; Cycles; Coactively; Adjacency and incidence matrices; Elements of transport network.

Assessments A learner is assessed through ;

- Continuous Assessment Tests (CATs) (30%)
- End of semester examination (70%)
- Total = (100%)

Required text books

Sharma J., Discrete Mathematics, McMillan

Abraham K. & Baker TP, Discrete Mathematics for Computer Scientists and Mathematicians, Prentice Hall

Other support materials

Various application manuals and journals

CHAPTER 1

BOOLEAN ALGEBRA

Learning objectives:

By the end of the chapter a student shall be able to know:

a) Axiomatic definition of Boolean
b) laws of Boolean algebra
c) Proposition and proposition functions
d) Truth values and truth tables

1.1 Axioms

Mathematical theories distinguish between axioms and theorems.

Axioms cannot be proven, but are accepted as facts.

Theorems can be proven from axioms by using logical inference.

An example of using axioms

A Boolean logic is a set with operators: and (∧), or (∨), not (A),

the elements "true" and "false" and the axioms:

$A \wedge (B \wedge C) = (A \wedge B) \wedge C \qquad A \vee (B \vee C) = (A \vee B) \vee C$

associativity

$A \wedge B = B \wedge A \qquad A \vee B = B \vee A$

commutativity

$A \vee (A \wedge B) = A \qquad A \wedge (A \vee B) = A$

absorption

$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C) \quad A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$

distributivity

$A \vee A = 1$ $A \wedge A = 0$

complements

De Morgan's law is a theorem which can be proven from the axioms.

Definition of Boolean algebra

A Boolean algebra B consists of a set S containing distinct elements 0 and 1, binary operators **+** and **.** on S, and a unary operator ' on S satisfying the following laws.

a) Associative laws: $(x + y) + z = x + (y + z)$
$(x \, . \, y) \, . \, z = x \, . \, (y \, . \, z)$ for all x, y, z ∊ S.
b) Commutative laws:
$x + y = y + x$, $x \, . \, y = y \, . \, x$ for all x, y ∊ S.
c) Distributive laws:
$x \, . \, (y + z) = (x . y) + (x \, . \, z)$
$x + (y \, . \, z) = (x + y) \, . \, (x + z)$ for all x, y, z ∊ S.
d) Identity laws:
$x + 0 = x$, $x \, . \, 1 = x$ for all x ∊ S.
e) Complement laws:
$x + x' = 1$, $x \, . \, x' = 0$ for all x ∊ S.

If B is a Boolean algebra, we write B = (**S**, +, ., ', **0**, **1**).

EXAMPLE

In a Boolean algebra,
if $x + y = 1$ and $xy = 0$, then $y = x'$

Proof.

$$\begin{aligned} Y &= y1 \\ &= y(x + x') \\ &= yx + yx' \\ &= 0 + yx' \\ &= xx' + yx' \\ &= x'x + x'y \\ &= x'(x + y) \\ &= x' 1 \\ &= x' \end{aligned}$$

Let B = (**S**, +, ., ', **0**, **1**) be a Boolean algebra. The following properties hold.

a) Idempotent laws:

$x + x = x, \qquad xx = x \qquad$ for all $x \in S$.

b) Bound laws:

$x + 1 = 1, \qquad x0 = 0 \qquad$ for all $x \in S$.

c) Absorption laws:

$x + xy = x, \qquad x(x + y) = x \qquad$ for all $x, y \in S$.

d) Involution law:

$(x')' = x \qquad$ for all $x \in S$.

e) 0 and 1 laws:

$0' = 1, \qquad 1' = 0.$

f) De Morgan's laws for Boolean algebras:

$(x + y)' = x'y', \qquad (xy)' = x' + y' \qquad$ for all $x, y \in S$.

Proof of (a), (b), and (c) above.

a) $x = x + 0$
$= x + (xx')$
$= (x + x)(x + x')$
$= (x + x)1$
$= x + x$

b) $x + 1 = (x + 1)1$
$= (x + 1)(x + x')$
$= x + 1x'$
$= x + x'1$
$= x + x'$
$= 1$
$= x0 + 0$
$= x0 + xx'$
$= x(0 + x')$
$= x(x' + 0)$
$= xx'$
$= 0$

c) $x + xy = x1 + xy$
$= x(1 + y)$
$= x(y + 1)$
$= x1$
$= x$

Proving de Morgan’s law

$A \vee B = (A \vee B) \wedge$

1

$= (A \vee B) \wedge ((A \wedge B) \vee A \wedge B)$

$= (A \wedge A \wedge B) \vee (A \wedge A \wedge B) \vee (B \wedge A \wedge B) \vee (B \wedge A \wedge B)$

$= 0 \vee (A \wedge A \wedge B) \vee 0 \vee (B \wedge A \wedge B)$

$= (A \wedge A \wedge B) \vee (B \wedge A \wedge B)$

$= ((A \vee B) \wedge A \wedge B)$

$= ((A \vee B) \wedge A \wedge B) \vee 0$

$= ((A \vee B) \wedge A \wedge B) \vee (A \wedge B \wedge A \wedge B)$

$= A \wedge B \wedge ((A \vee B) \vee (A \wedge B))$

$= A \wedge B \wedge ((A \vee B) \vee (A \wedge B) \vee (A \wedge B))$

$= A \wedge B \wedge (((A \vee (A \wedge B)) \vee (B \vee (A \wedge B)))$

$= A \wedge B \wedge (A \vee B \vee B \vee A)$

$=A \wedge B \wedge 1=A \wedge B$

1.2 Proposition and proposition functions

A sentence that is either true or false, but not both, is called a proposition.
Example:

a) The only positive integers that devide 7 are 1 and 7 itself
b) Earth is the only planet in the universe that has life
c) All cows eat meat

Sentence that is not proposition
Buy two tickets for friday cinema

Note: Propositions are the basic building blocks of any theory of logic.

Definition : Let p and q be propositions.
The conjunction of p and q, denoted $p \wedge q$, is the proposition

p and q.

The disjunction of p and q, denoted $p \vee q$, is the proposition

p or q

Propositions such as $p \wedge q$ and $p \vee q$ that result from combining propositions are called **compound propositions.**

Example:

If p: $1 + 1 =3$,
q: A decade is 10 years,

then the conjunction of p and q is

$$p \wedge q: 1 + 1 = 3 \text{ and a decade is 10 years.}$$

The disjunction of p and q is

$$p \vee q: 1 + 1 = 3 \text{ or a decade is 10 years.}$$

The truth values of propositions such as conjunctions and disjunctions can be described by **truth tables.** The truth table of proposition P made up of the individual propositions p1,......,pn lists all possible combinations of truth values for p1,....,pn, T denoting true and F denoting false, and for each such combination lists the truth value of P.

Definition: The truth value of compound proposition $p \wedge q$ is defined by the truth table

Example:

P	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

If p: $1 + 1 = 3$,
q: A decade is 10 years,

then p is false, q is true, and the conjunction

$$p \wedge q: 1 + 1 = 3 \text{ and a decade is 10 years}$$

is false.

Definition: The truth value of compound proposition $p \vee q$ is defined by the truth table

Example:

P	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

If p: 1 + 1 = 3,
q: A decade is 10 years,

then p is false, q is true, and the disjunction

p ∨ q: 1 + 1 = 3 or a decade is 10 years

is true.

Definition: The negation of p, denoted p, is the proposition

not p.

The truth value of the proposition is defined by the truth table

p	**p**
T	F
F	T

1.3 Chapter Review Questions

a) What is a proposition?
b) What is a truth table?
c) What is the disjunction of *p* and *q*? How is it denoted?
d) Give the truth table for the negation of *p*.
e) which of these inferences are valid?

The days are becoming longer.

The nights are becoming shorter if the days are becoming longer.

Hence, the nights are becoming shorter.

The earth is spherical implies that the moon is spherical.

The earth is not spherical.

Hence, the moon is not spherical.

The new people in the neighbourhood have a beautiful boat.

They also have a nice car.

Hence, they must be nice people.

All dogs are carnivorous.

Some animals are dogs.

Therefore, some animals are carnivorous.

SUGGESTED FURTHER READING

Ritchard Johnsonbaugh, Discrete Mathematics 6^{th} edition, pg 1-6

CHAPTER 2

LOGIC

Learning objectives:

By the end of the chapter a student shall be able to know:

a) Predicate logic.

b) Propositional logic Logical reasoning

c) Conceptualizing variables open sentence

d) Truth values and table Logical equivalence

e) Negation of statements conjunctions Disjunctions

f) Tautology, contradiction etc contrapositive, converse, inverse

2.1 Propositional and Predicate Logic - Syntax & Semantics

Propositional logic: *"and, or, not" and variables.*

Propositional logic does not have quantifiers:

"All poodles are dogs."

"There is at least one black swan."

"Only supervisors are allowed to fill in the form."

"No person can solve this problem."

These sentences cannot be expressed in propositional logic.

Predicate logic

Quantifiers: *all, at least one*

First order logic: quantifiers for individuals.

"All squirrels eat nuts."

Second order logic: quantifiers for predicates.

S = "squirrels eat nuts"

"All S involves chewing."

Third order logic: ...

Syntax and semantics

Predicate logic is a formal language (like a programming language) with rules for syntax (i.e. how to write expressions) and semantics (i.e. how to formalise the meaning of expressions).

Syntax: well-formed formulas

logical symbols: and, or, not, all, at least one, brackets,

variables, equality (=), true, false

predicate and function symbols

(for example, Cat(x) for "x is a Cat")

term: variables and functions

(for example, Cat(x))

formula: any combination of terms and logical symbols

(for example, “Cat(x) and Sleeps(x)”)

sentence: formulas without free variables

(for example, “All x: Cat(x) and Sleeps(x)”)

Semantics: meaning

The meaning of a term or formula is a set of elements. The meaning of a sentence is a truth value.

The function that maps a formula into a set of elements is called an interpretation.

An interpretation maps an intensional description (formula/sentence) into an extensional description (set or truth value).

Validity

A sentence is ... satisfiable if it is true under at least one interpretation.

valid if it is true under all interpretations. invalid if
it is false under some interpretation. contradictory
if it is false under all interpretations.

Example: “All x: Cat(x) and Sleeps(x)”

If this is interpreted on an island which only has one cat that always sleeps, this is satisfiable.

Since not all cats in all interpretations always sleep, the sentence is not valid.

Reasoning (Charles Sanders Peirce)

Induction: Every dog I know has ears. =⇒ Dogs have ears.

(from specific examples to general)

Deduction: Dogs bark. Bobby is a dog. =⇒ Bobby barks.

(from general to specific)

Abduction: Sherlock Holmes: the murderer was left-handed.
Smith is left-handed. =⇒ Smith is the murderer.
(based on shared attributes)

2.2 More Complex Sentences

We need to apply operators to construct more complex sentences from atoms.

Negation:

¬ applied to an atom negates the atom:

¬ **loves(kermit, voiceof(misspiggy))**

"Kermit does not love Miss Piggy's voice"

Conjunction:

∧ combines two conjuncts:

loves(misspiggy, kermit) ∧ loves(misspiggy, voiceof(kermit))

"Miss Piggy loves Kermit and Miss Piggy loves Kernit's voice"

Notice it is not correct syntax to write in logic

loves(misspiggy, kermit) ∧

voiceof(kermit)

because we have tried to conjoin a sentence (truth valued) with an object. Logic operators must apply to truth-valued sentences.

Disjunction:

∨ combines two disjuncts:

loves(misspiggy, kermit) ∨ loves(misspiggy, voiceof(kermit))

"Miss Piggy loves Kermit or Miss Piggy loves Kermit's voice"

Implication:

⇒ combines a condition and conclusion

$\text{loves(misspiggy, voiceof(kermit))} \Rightarrow \quad \text{loves(misspiggy, kermit)}$

"If Miss Piggy loves Kermit's voice then Miss Piggy loves Kermit"

The language we have described so far contains atoms and the connectives $\neg$, $\wedge$, $\vee$ and $\Rightarrow$. This defines the syntax of propositional Logic. It is normal to represent atoms in propositional logic as single upper-case letters but here we have used a more meaningful terminology for the atoms that extends easily to Predicate Logic.

2.3 Propositional logic

Every logic comprises a (formal) language for making statements about objects and reasoning about properties of these objects. This view of logic is very general and actually we will restrict our attention to mathematical objects, programs, and data structures in particular. Statements in a logical language are constructed according to a predefined set of formation rules (depending on the language) called syntax rules.

One might ask why a special language is needed at all, and why English (or any other natural language) is not adequate for carrying out logical reason-ing. The first reason is that English (and any natural language in general) is such a rich language that it cannot be formally described. The second reason, which is even more serious, is that the meaning of an English sentence can be ambiguous, subject to different interpretations depending on the context and implicit assumptions. If the object of our study is to carry out precise rigor-ous arguments about assertions and proofs, a precise language whose syntax can be completely described in a few simple rules and whose semantics can be defined unambiguously is required.

Another important factor is conciseness. Natural languages tend to be verbose, and even fairly simple mathematical statements become exceedinglylong (and unclear) when expressed in them. The logical languages that we shall define contain special symbols used for abbreviating syntactical constructs.

A logical language can be used in different ways. For instance, a language can be used as a deduction system (or proof system); that is, to construct proofs or refutations. This use of a logical language is called proof theory. In this case, a set of facts called axioms and a set of deduction rules (inference rules) are given, and the object is to determine which facts follow from the axioms and the rules of inference. When using logic as a proof system, one is not concerned with the meaning of the statements that are manipulated, but with the arrangement of these statements, and specifically, whether proofs or refutations can be constructed. In this sense, statements in the language are viewed as cold facts, and the manipulations involved are purely mechanical, to the point that they could be carried out by a computer. This does not mean that finding a proof for a statement does not require creativity, but that the interpetation of the statements is irrelevant. This use of logic is similar to game playing. Certain facts and rules are given, and it is assumed that the players are perfect, in the sense that they always obey the rules. Occasionally, it may happen that following the rules leads to inconsistencies, in which case it may be necessary to revise the rules.

However, the statements expressed in a logical language often have an intended meaning. The second use of a formal language is for expressing statements that receive a meaning when they are given what is called an in-terpretation. In this case, the language of logic is used to formalize properties of structures, and determine when a statement is true of a structure. This use of a logical language is called model theory. One of the interesting aspects of model theory is that it forces us to

have a precise and rigorous definition of the concept of truth in a structure. Depending on the interpretation that one has in mind, truth may have quite a different meaning. For instance, whether a statement is true or false may depend on parameters. A statement true under all interpretations of the parameters is said to be valid. A useful (and quite reasonable) mathematical

assumption is that the truth of a statement can be obtained from the truth (or falsity) of its parts (substatements). From a technical point of view, this means that the truth of a statement is defined by recursion on the syntactical structure of the statement. The notion of truth that we shall describe (due to Tarski) formalizes the above intuition, and is firmly justified in terms of the concept of an algebra

The two aspects of logic described above are actually not independent, and it is the interaction between model theory and proof theory that makes logic an interesting and effective tool. One might say that model theory and proof theory form a couple in which the individuals complement each other.

In propositional logic, there are atomic assertions (or atoms, or propositional letters) and compound assertions built up from the atoms and the logical connectives, and , or , not, implication and equivalence. The atomic facts are interpreted as being either true or false. In propositional logic, once the atoms in a proposition have received an interpretation, the truth value of the proposition can be computed. Technically, this is a consequence of the fact that the set of propositions is a freely generated inductive closure.

Certain propositions are true for all possible interpretations. They are called tautologies. Intuitively speaking, a tautology is a universal truth. Hence, tautologies play an important role.

For example, let “John is a teacher,” “John is rich,” and “John is a rock singer” be three atomic propositions. Let us abbreviate them as A,B,C.

Consider the following statements:

“John is a teacher”;

It is false that “John is a teacher” and “John is rich”;

If “John is a rock singer” then “John is rich.”

We wish to show that the above assumptions imply that

It is false that “John is a rock singer.”

This amounts to showing that the (formal) proposition

(*) (A and not(A and B) and (C implies B)) implies (not C)

is a tautology. Informally, this can be shown by contradiction. The statement (*) is false if the premise (A and not(A and B) and (C implies B)) is true and the conclusion (not C) is false. This implies that C is true. Since C is true, then, since (C implies B) is assumed to be true, B is true, and since A is assumed to be true, (A and B) is true, which is a contradiction, since not(A and B) is assumed to be true.

Of course, we have assumed that the reader is familiar with the semantics and the rules of propositional logic, which is probably not the case. In this chapter, such matters will be explained in detail.

Syntax of Propositional Logic

The Language of Propositional Logic

Propositional formulae (or propositions) are strings of symbols from a count-

able alphabet defined below, and formed according to certain rules
(The alphabet for propositional formulae) This alphabet

consists of:

(1) A countable set PS of proposition symbols: $P0$,$P1$,$P2$ ...;

(2) The logical connectives: $\wedge$ (and), $\vee$ (or), $\supset$ (implication), $\neg$ (not),

and sometimes $\equiv$ (equivalence) and the constant $\perp$ (false);

(3) Auxiliary symbols: “(” (left parenthesis), “)” (right parenthesis).

The set of propositional formulae (or propositions) is defined as

the inductive closure of a certain subset of the alphabet of

definition under certain operations defined below.

Definition: The Propositional formulae is the set of propositional

formulae (or propositions) is the inductive closure of the set $PS \cup \{\perp\}$ under

the functions $C_\neg$, $C_\wedge$, $C_\vee$, $C_\supset$ and $C_\equiv$, defined as follows: For any two strings

A, B over the alphabet of definition.

$$C_\neg(A) = \neg A,$$

$$C_\wedge(A, B) = (A \wedge B),$$

$$C_\vee(A, B) = (A \vee B),$$

$$C_\supset(A, B) = (A \supset B) \text{ and}$$

$$C_\equiv(A, B) = (A \equiv B).$$

The above definition is the official definition of proposition as an inductive closure, but is a bit formal. For that reason, it is often stated less formally as follows:

The set P ROP of propositions is the smallest set of strings over the alphabet of definition 3.2.1, such that:

(1) Every proposition symbol Pi is in P ROP and $\perp$ is in P ROP ;

(2) Whenever A is in P ROP , $\neg A$ is also in P ROP ;

(3) Whenever A, B are in P ROP , $(A \vee B)$, $(A \wedge B)$, $(A \supset B)$

(A ≡ B) are also in P ROP .

(4) A string is in P ROP only if it is formed by applying the rules

(1),(2),(3).

The official inductive definition of P ROP will be the one used in proofs.

2.4 Semantics of Propositional Logic

We have defined the syntax of propositional Logic. However, this is of no use without talking about the meaning, or semantics, of the sentences. Suppose our logic contained only atoms; e.g. no logical connectives. This logic is very silly because any subset of these atoms is consistent; e.g. beautiful(misspiggy) and ugly(misspiggy) are consistent because we cannot represent ugly(misspiggy) ⇒ ¬ beautiful(misspiggy) So we now need a way in our logic to define which sentences are true.

Model: A *model* is a subset of the atoms defined for our language and contains exactly those atoms that are true. So all atoms in a model M are true and all atoms not in M are false.

Example: Models Define Truth

Suppose a language contains only one object constant misspiggy and two relation constants ugly and beautiful. The following models define different facts about Miss Piggy.

M=∅ : In this model Miss Piggy is neither ugly nor beautiful. M={ugly(misspiggy)}: In this model Miss Piggy is ugly and not beautiful. M={beautiful(misspiggy)}: In this model Miss Piggy is beautiful and not ugly. M={ugly(misspiggy), beautiful(misspiggy)}: In this model Miss Piggy is both ugly and beautiful. The last statement is intuitively wrong but the model selected commits the truth of the atoms in the language.

Compound Sentences

So far we have restricted our attention to the semantics of atoms: an atom is true if it is a member of the model M; otherwise it is false. Extending the semantics to compound sentences is easy. *Notice* that in the definitions below p and q do not need to be atoms because these definitions work recursively until atoms are reached.

Conjunction:

$p \wedge q$ is true in M iff p and q are true in M individually.

So the conjunct

loves(misspiggy, kermit) $\wedge$ **loves(misspiggy, voiceof(kermit))**

is true only when both

Miss Piggy loves Kermit; and
Miss Piggy loves Kermit's voice

Disjunction:

$p \vee q$ is true in M iff at least one of p or q is true in M.

So the disjunct

loves(misspiggy, kermit) $\vee$ **loves(misspiggy, voiceof(kermit))**

is true whenever

Miss Piggy loves Kermit;
Miss Piggy loves Kermit's voice; or
Miss Piggy loves both Kermit and his voice.

Therefore the disjunction is weaker than either disjunct and the conjunction of these disjuncts.

Negation:

$\neg p$ is true in M iff p is not true in M.

Implication:

$p \Rightarrow q$ is true in M iff p is not true in M or q is true in M.

We have been careful about the definition of $\Rightarrow$. When people use an implication $p \Rightarrow q$ they normally imply that p causes q. So if p is true we are happy to say that $p \Rightarrow q$ is true iff q is true. But if p is false the causal link causes confusion because we can't tell whether q should be true or not. Logic requires that the connectives are truth functional and so the truth of the compound sentence must be determined from the truth of its component parts. Logic defines that if p is false then $p \Rightarrow q$ is true regardless of the truth of q.

So both of the following implications are true (provided you believe pigs do not fly!):

$\mathrm{fly(pigs)} \Rightarrow \mathrm{beautiful(misspiggy)}$

$\mathrm{fly(pigs)} \Rightarrow \neg \mathrm{beautiful(misspiggy)}$

Example: Implications and Models

In which of the following models is

$\mathrm{ugly(misspiggy)} \Rightarrow \neg \mathrm{beautiful(misspiggy)}$ true?

M=$\emptyset$ Miss Piggy is not ugly and so the antecedent fails. Therefore the implication holds. (Miss Piggy is also not beautiful in this model.)

M={$\mathrm{beautiful(misspiggy)}$}

Again, Miss Piggy is not ugly and so the implication holds.

M={$\mathrm{ugly(misspiggy)}$}

Miss Piggy is not beautiful and so the conclusion is valid and hence the implication holds.

M={$\mathrm{ugly(misspiggy), beautiful(misspiggy)}$}

Miss Piggy is ugly and so the antecedent holds. But she is also beautiful and so $\neg$ $\mathrm{beautiful(misspiggy)}$ is not true. Therefore the conclusion does not hold and so the implication fails in this (and only this) case.

2.5 Truth Tables

Truth tables are often used to calculate the truth of complex propositional sentences. A truth table represents all possible combinations of truths of the atoms and so contains all possible models. A column is created for each of the atoms in the sentence, and all combinations of truth values for these atoms are assigned one per row. So if there are n atoms then there are n initial columns and 2^n rows. The final column contains the truth of the sentence for each combination of truths for the atoms. Intervening columns can be added to store intermediate truth calculations. Below are two
sample truth tables:

p	q	$p \Rightarrow q$		p	q	r	$p \Rightarrow q$	$(p \Rightarrow q) \vee r$
T	T	T		T	T	T	T	T
T	F	F		T	T	F	T	T
F	T	T		T	F	T	F	T
F	F	T		T	F	F	F	F
				F	T	T	T	T
				F	T	F	T	T
				F	F	T	T	T
				F	F	F	T	T

Equivalence: Two sentences are *equivalent* if they hold in exactly the same models.

Therefore we can determine equivalence by drawing truth tables that represent the sentences in the various models. If the initial and final columns of the truth tables are identical then the sentences are equivalent. Examples of equivalences include:

$p \Rightarrow q$ and $\neg p \vee q$

$p \Rightarrow q$ and $\neg q \Rightarrow \neg p$

$p \vee q$ and $q \vee p$ ($\vee$ is commutative)

$(p \wedge q) \wedge r$ and $p \wedge (q \wedge r)$ ($\wedge$ is associative)

Unlike $\wedge$ and $\vee$, $\Rightarrow$ is not commutative:

loves(misspiggy, voiceof(kermit)) $\Rightarrow$ **loves(misspiggy, kermit)**

is very different from

loves(misspiggy, kermit) $\Rightarrow$ **loves(misspiggy, voiceof(kermit))**

Similarly $\Rightarrow$ is not associative.

2.6 Syntax of Predicate Logic

Propositional logic is fairly powerful but we must add variables and quantification to be able to reason about objects in atoms and express properties of a set of objects without listing the atom corresponding to each object.

We shall adopt the Prolog convention that variables have an initial capital letter. (This is contrary to many Mathematical Logic books where variables are lower case and constants have an initial capital.)

When we include variables we must specify their scope or quantification. The first quantifier we want is the *universal* quantifier ∀ (*for all*).

$$\forall \; X.\mathbf{loves}(misspiggy, X)$$

This allows X to range over all the objects and asserts that Miss Piggy loves each of them. We have introduced one variable but any number is allowed:

$$\forall \; XY.\mathbf{loves}(X, Y)$$

Each of the objects love all of the objects, even itself! Therefore ∀ XY. is the same as ∀ X. ∀ Y. Quantifiers, like connectives, act on sentences. So if Miss Piggy loves all cute things (not just Kermit!) we would write

$$\forall \; C.[\mathbf{cute}(C) \rightarrow \mathbf{loves}(misspiggy, C)]$$

rather than

$$\mathbf{loves}(misspiggy, \forall \; C.\mathbf{cute}(C))$$

because the second argument to **loves** must be an object, not a sentence.

When the world contains a finite set of objects then a universally quantified sentence can be converted into a sentence without the quantifier; e.g. $\forall \; X.\mathbf{loves}(misspiggy, X)$ becomes

$\mathbf{loves}(misspiggy, misspiggy) \wedge \mathbf{loves}(misspiggy, kermit) \wedge$
$\mathbf{loves}(misspiggy, animal) \wedge \ldots$

Contrast this with the infinite set of positive integers and the sentence

$$\forall \; N.[\mathbf{odd}(N) \vee \mathbf{even}(N)]$$

The other quantifier is the *existential* quantifier ∃ (*there exists*).

$$\exists \; X.\mathbf{loves}(misspiggy, X)$$

This allows X to range over all the objects and asserts that Miss Piggy loves (at least) one of them. Similarly

$$\exists \; XY.\mathbf{loves}(X, Y)$$

asserts that there is at least one loving couple (or self-loving object).

Note: We shall be using First Order Predicate Logic where quantified variables range over object constants only. We are defining Second Order Predicate Logic if we allow quantified variables to range over functions or predicates as well; e.g.

> $\exists \; X.\mathbf{loves}(misspiggy, X(kermit))$ includes $\mathbf{loves}(misspiggy, \mathbf{voiceof}(kermit))$
>
> $\exists \; X.X(misspiggy, kermit)$ (there exists some relationship linking Miss Piggy and Kermit!)

2.7 Semantics of First Order Predicate Logic

Now we must deal with quantification.

$\forall$:

$\forall X.p(X)$ holds in a model iff $p(z)$ holds for all objects z in our domain.

$\exists$:

$\exists X.p(X)$ holds in a model iff there is some object z in our domain so that $p(z)$ holds.

Example: Available Objects affects Quantification

If misspiggy is the only object in our domain then

$$\text{ugly(misspiggy)} \Rightarrow \neg \text{ beautiful(misspiggy)}$$ is equivalent to

$$\forall X.\text{ugly}(X) \Rightarrow \neg \text{ beautiful}(X)$$

If there were other objects then there would be more atoms and so the set of models would be larger; e.g. with objects misspiggy and kermit the possible models are all combinations of the atoms ugly(misspiggy), beautiful(misspiggy) ugly(kermit), beautiful(kermit). Now the 2 sentences are no longer equivalent.

1. Although, every model in which $\forall X.\text{ugly}(X) \Rightarrow \neg \text{ beautiful}(X)$ holds, $\text{ugly(misspiggy)} \Rightarrow \neg \text{ beautiful(misspiggy)}$ also holds
2. there are models in which $\text{ugly(misspiggy)} \Rightarrow \neg \text{ beautiful(misspiggy)}$ holds, but $\forall X.\text{ugly}(X) \Rightarrow \neg \text{ beautiful}(X)$ does not hold; e.g. M = {ugly(kermit), beautiful(kermit)}.

   What about M = {ugly(misspiggy)}, beautiful(misspiggy)?

2.8 Biconditional proposition

Definition: If p and q are propositions, the compound proposition p if and only if q is called a biconditional proposition and is denoted

$$p \leftrightarrow q$$

P	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example: The statement

$$1 < 5 \text{ if and only if } 2 < 8$$

can be written symbolically as

$$p \leftrightarrow q$$

if we define

p: 1 < 5, q: 2 < 8.

since both p and q are true, the statement $p \leftrightarrow q$ is true.

2.9 Logical equivalence

Definition: Suppose that the compound propositions P and Q are made up of the proposition p1,........,p2. We say that P and Q are logically equivalent and write

$$P \equiv Q,$$

provided that given any truth values of p1,....,pn, either P and Q are both true or P and Q are both false.

Example: By writing the truth tables for $P = \overline{p \vee q}$ and $Q = p \wedge q$, we can verify that given any truth values of p and q, either P and are both true or P and Q are both false:

P	q	$\overline{p \vee q}$	$p \wedge q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

Thus P and Q are logically equivalent.

2.10 Contrapositive, converse, inverse

Definition: The contrapositive (or transposition) of the conditional proposition

$p \to q$ is the proposition $q \to p$.

Notice the difference between the contrapositive and the converse. The converse of a conditional proposition merely reverses the roles of p and q, whereas the contrapositive reverses the roles of p and q and negates each of them.

Example:

Write the proposition

If $1 < 4$, then $5 > 8$

symbolically. Write the converse and the contrapositive both symbolically and in words. Find the truth value of each proposition.

If we define

$$p: 1 < 4, \qquad q: 5 > 8,$$

then the given proposition may be written symbolically as

$$p \to q.$$

The converse is

$$q \to p$$

or, in words,

if $5 > 8$, then $1 < 4$.

The contrapositive is

$$q \to p$$

or, in words,

If 5 is not greater than 8, then 1 is not less than 4.

We see that $p \to q$ is false, $q \to p$ is true, and $q \to p$ is false.

An important fact is that a conditional proposition and its contrapositive are logically equivalent.

Proof: The conditional proposition $p \to q$ and its contrapositive $q \to p$ are logically equivalent.

P q	$p \to q$	$p \to q$
T T	T	T
T F	F	F
F T	T	T
F F	T	T

shows that $p \to q$ and its contrapositive $q \to p$ are logically equivalent.

2 .11 Tautology

A tautology is a logical statement in which the conclusion is equivalent to the premise. More colloquially, it is formula in propositional calculus which is always true

If p is a tautology, it is written $\models p$. A sentence whose truth table contains only 'T' is called a tautology. The following sentences are examples of tautologies:

where $\wedge$ denotes AND, $\equiv$ denotes "is equivalent to," ! denotes NOT, $\vee$ denotes OR, and $\Rightarrow$ denotes implies.

$A \wedge B \equiv\, !\,(!\,A \vee !\,B)$
$A \vee B \equiv\, !\,A \Rightarrow B$
$A \wedge B \equiv\, !\,(A \Rightarrow !\,B)$

2.12Chapter review Questions

Are these sentences satisfiable, valid, invalid or contradictory? If a sentence is satisfiable or invalid, provide an interpretation which makes it true (or false).

i. 1+1=1

ii. A ∩ B = not(not A ∪ not B)

iii. All x: ToBe(x) or not ToBe(x)

CHAPTER 3

SET THEORY

Learning objectives:

By the end of the chapter a student shall be able to know:

a) Set algebra

b) Recursive definition sets

c) Orderings

d) Relations partially ordered sets Conceptualizing elements, finite and infinite Universal empty and disjoint, subsets

e) Venn diagram union Intersection Complement Difference

f) Number elements and logical arguments

g) Sets of sets

h) The power set and Cartesian product

3.1 ALGEBRA OFSETS

Set Algebra and Proofs Involving Sets

There are a lot of rules involving sets; you'll probably become familiar with the most important ones
simply by using them a lot. Usually you can check informally (for instance, by using a Venn diagram)

whether a rule is correct; if necessary, you should be able to write a proof. In most cases, you can give a
proof by going back to the definitions of set contructions in terms of elements.

Sets under the operations of union, intersection, and complement satisfy various laws or identities:

Laws of the algebra of sets

a) Idempotent laws

$A \cup A = A$, $A \cap A = A$

b) Associative laws

$(A \cup B) \cup C = (A \cup B) \cup C$, $\quad (A \cap B) \cap C = A \cap (B \cap C)$

c) Commutative laws

$A \cup B = B \cup A$, $\quad A \cap B = B \cap A$

d) Distributive laws

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $\quad A \cap (B \cap C) = (A \cap B) \cup (A \cap C)$

e) Identity laws

$A \cup \Phi = A$, $\quad A \cap \square = A$

$A \cup \square = \square$, $\quad A \cap \Phi = \Phi$

f) Involution laws

$(A^c)^c = A$

g) Complement laws

$A \cup A^c = \square$, $\quad A \cap A^c = \Phi$

$\square^c = \Phi$, $\quad \Phi = \square$

h) DeMorgan's laws

$(A \cup B)^c = A^c \cap B^c$, $\quad (A \cap B) = A^c \cup B^c$

Example. (Distributivity) Let A, B, and C be sets. Prove that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

If X and Y are sets, $X = Y$ if and only if for all x, $x \in X$ if and only if $x \in Y$.

Let x be an arbitrary element of the universe.

$x \in A \cap (B \cup C) \leftrightarrow$	$x \in A \wedge x \in (B \cup C)$	Definition of $\cap$
$\leftrightarrow$	$x \in A \wedge (x \in B \vee x \in C)$	Definition of $\cup$
$\leftrightarrow$	$(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$	Distributivity of $\wedge$ over $\vee$
$\leftrightarrow$	$(x \in A \cap B) \vee (x \in A \cap C)$	Definition of $\cap$
$\leftrightarrow$	$x \in (A \cap B) \cup (A \cap C)$	Definition of $\cup$

I've shown that

$$x \in A \cap (B \cup C) \leftrightarrow x \in (A \cap B) \cup (A \cap C).$$

By definition of set equality, this proves that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

The idea of the proof was to reduce everything to statements about elements.

Then I used logical rules
to manipulate the element statements.

Note: It's also true that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Example. (DeMorgan's Law) Let A and B be sets. Prove that

$$A \cup B = A \cap B \qquad A \cap B = A \cup B.$$

and

I'll just prove the first statement; the second is similar. This proof will illustrate how you can work with
complements. I'll use the logical version of DeMorgan's law to do the proof.

Let x be an arbitrary element of the universe.

$x \in A \cup B \leftrightarrow x \in A \cup B$
Definition of complement

$\leftrightarrow \sim (x \in A \cup B)$ Definition of $\in$

$\leftrightarrow \sim (x \in A \vee x \in B)$ Definition of $\cup$

$\leftrightarrow \sim (x \in A) \wedge \sim (x \in B)$ DeMorgan's law

$\leftrightarrow (x \in A) \wedge (x \in B)$ Definition of $\in$

$\leftrightarrow (x \in A) \wedge (x \in B)$ Definition of complement

$\leftrightarrow \quad x \in A \cap B$ Definition of $\cap$

Therefore, $A \cup B = A \cap B$.

Example: Set Algebra and Proofs Involving Sets

There are a lot of rules involving sets; you'll probably become familiar with the most important ones simply by using them a lot. Usually you can check informally (for instance, by using a Venn diagram) whether a rule is correct; if necessary, you should be able to write a proof. In most cases, you can give a proof by going back to the definitions of set contructions in terms of elements.

Example. (Distributivity) Let A, B, and C be sets. Prove that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

If X and Y are sets, X = Y if and only if for all x, $x \in X$ if and only if $x \in Y$.

Let x be an arbitrary element of the universe.

$x \in A \cap (B \cup C) \leftrightarrow x \in A \wedge x \in (B \cup C)$ Definition of $\cap$

$\leftrightarrow x \in A \wedge (x \in B \vee x \in C)$ Definition of $\cup$

$\leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$ Distributivity of $\wedge$ over $\vee$

$\leftrightarrow (x \in A \cap B) \vee (x \in A \cap C)$ Definition of $\cap$

$\leftrightarrow x \in (A \cap B) \cup (A \cap C)$ Definition of $\cup$

I've shown that

$$x \in A \cap (B \cup C) \leftrightarrow x \in (A \cap B) \cup (A \cap C).$$

By definition of set equality, this proves that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

The idea of the proof was to reduce everything to statements about elements. Then I used logical rules to manipulate the element statements.

Note: It's also true that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Example. (DeMorgan's Law) Let A and B be sets. Prove that

$$A \cup B = A \cap B \qquad A \cap B = A \cup B.$$

and

I'll just prove the first statement; the second is similar. This proof will illustrate how you can work with

complements. I'll use the logical version of DeMorgan's law to do the proof.

Let x be an arbitrary element of the universe.

$x \in A \cup B \quad \leftrightarrow \quad x \in A \cup B$

/ Definition of complement

$\leftrightarrow \ \sim (x \in A \cup B)$ Definition of $\in$ /

$\leftrightarrow \ \sim (x \in A \vee x \in B)$ Definition of $\cup$

$\leftrightarrow \ \sim (x \in A) \wedge \sim (x \in B)$ DeMorgan's law

$\leftrightarrow \ (x \in A) \wedge (x \in B)$ Definition of $\in$

$\leftrightarrow \ (x \in A) \wedge (x \in B)$ Definition of complement

$\leftrightarrow \ x \in A \cap B$ Definition of $\cap$

Therefore, $A \cup B = A \cap B$.

Example: Let A and B be sets. Prove that $A \cap B \subset A$.

This example will show how you prove a subset relationship.

By definition, if X and Y are sets, $X \subset Y$ if and only if for all x, if $x \in X$, then $x \in Y$.

Take an arbitrary element x. Suppose $x \in A \cap B$ (conditional proof). I want to show that $x \in A$.

$x \in A \cap B$ means that $x \in A$ and $x \in B$, by definition of intersection. But $x \in A$ and $x \in B$ implies

$x \in A$ (decomposing a conjunction), and this is what I wanted to show. Therefore, $A \cap B \subset A$.

By the way, you usually don't write the logic out in such gory detail. The proof above could be shortened

to:

$x \in A \cap B$ means that $x \in A$ and $x \in B$, so in particular $x \in A$. Therefore, $A \cap B \subset A$.

The "in particular" substitutes for decomposing the conjunction.

The procedure I've followed is so common that it's worth pointing it out.

> To prove a subset relationship (an inclusion) $X \subset Y$, take an arbitrary
>
> element of X and prove that it must be in Y.

In the next example, I'll need the following facts from logic. First, $P \vee \sim P$ is a tautology:

P	$\sim P$	$P \vee \sim P$
T	F	T
F	T	T

Also, $P \wedge$ (a tautology) $\leftrightarrow P$:

P	a tautology	$P \wedge$ (a tautology)
T	T	T
F	T	F

In effect, this means that I can drop tautologies from "and" statements. I'll just call this "Dropping tautologies" in the proof below.

Example. Prove that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

$x \in (A - B) \cup (B - A) \leftrightarrow$

$x \in (A - B) \lor x \in (B - A) \leftrightarrow$ Definition of union

$(x \in A \land x \in B) \lor (x \in B \land x \in A)$
$\leftrightarrow$

Definition of complement

$[x \in A \lor (x \in B \land x \in A)] \land [x \in B \lor (x \in B \land x \in A)] \leftrightarrow$

Distributivity

$(x \in A \lor x \in B) \land (x \in A \lor x \in A)] \land (x \in B \lor x \in B) \land (x \in B \lor x \in A) \leftrightarrow$

Distributivity

$(x \in A \lor x \in B) \land (x \in B \lor x \in A) \leftrightarrow$

Dropping tautologies

$(x \in A \lor x \in B) \land (\sim x \in B \lor \sim x \in A) \leftrightarrow$ Definition of “not in”

$(x \in A \lor x \in B) \land \sim (x \in B \land x \in A) \leftrightarrow$ DeMorgan

$(x \in A \cup B) \land \sim (x \in A \cap B) \leftrightarrow$ Definition of union and intersection

$x \in (A \cup B) - (A \cap B)$ Definition of complement

Therefore, $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

Example: Let A be a set. Prove that

$$A \cup \varnothing = A \quad \text{and } A \cap \varnothing = \varnothing .$$

This example will show how you can deal with the empty set.

To prove $A \cup \varnothing = A$, let x be an arbitrary element of the universe. First, by definition of $\cup$,

$$x \in A \cup \varnothing \leftrightarrow (x \in A) \vee (x \in \varnothing).$$

I'll show that $[(x \in A) \vee (x \in \varnothing)] \leftrightarrow (x \in A)$. To prove $P \leftrightarrow Q$, I must prove $P \rightarrow Q$ and $Q \rightarrow P$.

First, if $x \in A$, then $(x \in A) \vee (x \in \varnothing)$ (constructing a disjunction).

Next, suppose $(x \in A) \vee (x \in \varnothing)$. The second statement $x \in \varnothing$ is false for all x, by definition of $\varnothing$. But

the $\vee$-statement is true by assumption, so $x \in A$ must be true by disjunctive syllogism. This proves that if

$(x \in A) \vee (x \in \varnothing)$, then $x \in A$.

This completes my proof that $[(x \in A) \vee (x \in \varnothing)] \leftrightarrow (x \in A)$. So

$$\begin{aligned} x \in A \cup \varnothing &\leftrightarrow (x \in A) \vee (x \in \varnothing) && \text{Definition of } \cup \\ &\leftrightarrow x \in A && \text{Proved above} \end{aligned}$$

Therefore, $A \cup \varnothing = A$.

To prove that $A \cap \varnothing = \varnothing$, I must prove that for all x, $x \in A \cap \varnothing$ if and only if $x \in \varnothing$.

As usual, x be an arbitrary element of the universe. To prove $x \in A \cap \varnothing$ if and only if $x \in \varnothing$, I must prove that the two implications

$$(x \in A \cap \varnothing) \rightarrow x \in \varnothing \quad \text{and} \quad x \in \varnothing \rightarrow (x \in A \cap \varnothing)$$

are true. I'll do this by showing that, in each case, the antecedent (i.e. the "if" part of the statement) is false — since by basic logic, if P is false, then $P \rightarrow Q$ is true.

For the first implication, consider the statement $x \in A \cap \varnothing$. By definition of intersection,

$$x \in A \cap \varnothing \leftrightarrow (x \in A \wedge x \in \varnothing).$$

Now $x \in \varnothing$ is false, by definition of the empty set. Therefore, the conjunction $x \in A \wedge x \in \varnothing$ is also false.

3.2 Recursive Definition

Recursive definition of sets have two parts, a basis step and recursive step. In the basis step, primitive elements (at least one) are specified. In the recursive step, rules for generating new elements in the set from those already known to be in the set are provided.

Example: Consider a set of multiples of 3.
Let A denote this set. The recursive definition to define A is as:

i. $3 \in A$
ii. if $x \in A$ then $x + 3 \in A$

We can enumerate elements of A as {3, 6, 9, 12, 15, ...}

Example: Let B be the set defined recursively as follows:

i. $2 \in B$
ii. if $n \in B$, then $n^2 \in B$. Describe the set by listing method.

Here B is defined recursively. By basis step, $2 \in B$.
Let $n = 2$, then by recursive step, $4 \in B$.
Now let $n = 4$, then $16 \in B$. Continuing in the similar way, we get
B = {2, 4, 16, 256, 65536, ...}

Example: Let **C** = {a, aa, ba, aaa, aba, baa, bba, ...}.

C consists of words from symbols from {a, b} that end in letter **a**.

Define **C** recursively as,

<u>solution</u>

C can be defined recursively as,

i. $a \in C$
ii. if $x \in C$, then $ac \in C$ and $bx \in C$ (concatenation operation)

Step (ii) is recursive step in which string concatenation operation is used to genarate member strings of C from existing member strings.

Example: A set S is defined recursively as follows. Find at least five elements of S.

i. $1 \in S$
ii. if $x \in S$ then $2x \in S$.

Using above recursive definition we get S as S = {1, 2, 4, 8, 16, 32, ...}

3.3 Ordering

We define set as an unordered collection of distinct objects. The order in which the set elements are listed does not matter. The sequence has no relevance. Hence, the sets {a, b,c} and {b,c, a} both represent the same set.

The ordered set is defined as ordered collection of distinct objects. {3, 6, 7, 8, 9}

Week days = {Sun, Mon, Tue, Wed, Thurs, Frid, Sat}.

Enumerated data type set in many programming languages are examples of ordered sets.

Example: Let S be any collection of sets. The relation ⊆ of set inclusion is a partial ordering of S, specifically,
A ⊆ A for any set A; if A ⊆ B and B ⊆ A then A = B; and if
A ⊆ B and B ⊆ C then A ⊆ C

Example: Consider the set N of positive integers. We say "a divides b ", written a/b, if there exists an integer c such that ac =b. For example, 2/4, 3/12, 7/21, and so on. This relation of divisibility is partial ordering of N.

Example: Consider the set Z of integer.

Define aℝb if there is a positive integer r such that b = $a^{r.}$
For instance, 2ℝ8 since 8 = 2^3. Then ℝ is a partial ordering of Z.

3.4 Lattices

Let L be a nonempty set closed under two binary operations called meet and join, denoted respectively by ∧ and ∨.

Then L is called a lattice if the following axioms hold where a, b, c are elements in L:

i. Commutative law:

a ∧ b = b ∧ a
a ∨ b = b ∨ a

ii. Associative law:

$(a \wedge b) \wedge c = a \wedge (b \wedge c)$
$(a \vee b) \vee c = a \vee (b \vee c)$

iii. Absorption law:

$a \wedge (a \vee b) = a$
$a \vee (a \wedge b) = a$

Example: Let L be a lattice. Then

i. $a \wedge b = a$ if and only if $a \vee b = b$.

Proof:
Suppose $a \wedge b = a$. Using the absorption law in the first step we have:

$$b = b \vee (b \wedge a) = b \vee (a \vee b) = b \vee a = a \vee b.$$

Now suppose ab =b. Again
using the absorption law in the first step we have:
$a = a \wedge (a \vee b) = a \wedge b$

Thus ab =a if and only if $a \vee b = b$.

Finite and Infinite sets

A set is a collection of objects. If a set is finite and not too large, we can describe it by listing the elements in it. For example, the equation

$$A = \{1, 2, 3, 4\}$$

describes set A made up of the four elements 1, 2, 3, and 4. A set is described by its elements and not by any particular order in which the elements might be listed.

If a set is a large finite set or an infinite set, we can describe it by listing a property necessary for membership. For example, the equation

$$B = \{x | x \text{ is a positive, even integer}\}$$

describes the set B made up of all positive, even integers; that is, B consists of the integers 2, 4, 6, and so on. The vertical bar "|" is read "such that."

The set with no elements is called the empty set (or null or void) set and is denoted □. Thus □ = {}.

Suppose that X and Y are sets. If every element of X is an element of Y, we say that X is a subset of Y and write $X \subseteq Y$.

Example: If

$C = \{1, 3\}$ and $A = \{1, 2, 3, 4\}$, then C is a subset of A.

Any set X is a subset of itself, since any element in X is in X. If X is a subset of Y and X does not equal Y, we say that X is a proper subset of Y. The empty set is a subset of every set. The set of all subsets (proper or not) of a set X, denoted $\mathcal{P}(X)$, is called the power set of

X. Example: If A = {a, b, c}, the members of $\mathcal{P}(A)$ are

$\varnothing$, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}.

All but {a, b, c} are proper subsets of A. For this example,

$$|A| = 3, \qquad |\mathcal{P}(A)| = 2^3 = 8$$

Given two sets X and Y, there are various ways to combine X and Y to form a new set. The set

$$X \cup Y = \{ x \mid x \in X \text{ or } x \in Y \}$$

is called the **union** of X and Y. The union consists of all elements belonging to either X or Y (or both).

The set

$$X \cap Y = \{ x \mid x \in X \text{ or } x \in Y \}$$

is called the **intersection** of X and Y. The intersection consists of all elements belonging to both X and Y.

Sets X and Y are **disjoint** if $X \cap Y = \varnothing$. A collection of sets S is said to be **pairwise disjoint** if whenever X and Y are distinct sets in S, X and Y are disjoint.

The set

$$X - Y = \{ x \mid x \in X \text{ or } x \notin Y \}$$

is called the **difference** (or **relative complement**). The difference X -Y consists of all elements in X that are not in Y.

Example: If A = {1, 3, 5} and B = { 4, 5, 6}, then

$$A \cup B = \{1, 3, 4, 5, 6\}$$
$$A \cap B = \{5\}$$
$$A - B = \{1, 3\}$$
$$B - A = \{4, 6\}.$$

Example: The sets

$$\{1, 4, 5\} \text{ and } B = \{2, 6\}$$

are disjoint. The collection of sets

$$S = \{\{1, 4, 5\}, \{2, 6\}, \{3\}, \{7, 8\}\}$$

is pairwise disjoint.

Sometimes we are dealing with sets all of which are subsets of a set □. This set □ is called a **universal** set or a **universe.** The set □ must be explicitly given or inferred from the context. Given a universal set □ and a subset X of □, the set □ – X is called the **complement** of X and is written

Example: Let A ={1, 3, 5}. If □ , a universal set, is specified as □ = {1, 2, 3, 4, 5}, then $\bar{A}$ = {7, 9}. The complement obviously depends on the universe in which we are working.

The **union** of two sets A and B is the set of elements, which are in A **or** in B **or** in both. It is denoted by A ∪ B and is read 'A union B'

Example :

Given U = {1, 2, 3, 4, 5, 6, 7, 8, 10}

X = {1, 2, 6, 7} and Y = {1, 3, 4, 5, 8}

Find $X \cup Y$ and draw a Venn diagram to illustrate $X \cup Y$.

Solution:

$X \cup Y$ = {1, 2, 3, 4, 5, 6, 7, 8} ← 1 is written only once.

If $X \subset Y$ then $X \cup Y = Y$. We will illustrate this relationship in the following example.

Example:

Given U = {1, 2, 3, 4, 5, 6, 7, 8, 10}

X = {1, 6, 9} and Y = {1, 3, 5, 6, 8, 9}

Find $X \cup Y$ and draw a Venn diagram to illustrate $X \cup Y$.

Solution:

$X \cup Y$ = {1, 3, 5, 6, 8, 9}

The complement of the set $X \cup Y$ is the set of elements that are members of the universal set U but are not in $X \cup Y$. It is denoted by $(X \cup Y)'$

Example:

Given: U = {1, 2, 3, 4, 5, 6, 7, 8, 9}

X = {1, 2, 6, 7} and Y = {1, 3, 4, 5, 8}

a) Draw a Venn diagram to illustrate $(X \cup Y)'$

b) Find $(X \cup Y)'$

Solution:

a) First, fill in the elements for $X \cap Y$ = {1}

Fill in the other elements for X and Y and for U

Shade the region outside $X \cup Y$ to indicate $(X \cup Y)'$

b) We can see from the Venn diagram that

$(X \cup Y)'$ = {9}

Or we find that $X \cup Y$ = {1, 2, 3, 4, 5, 6, 7, 8} and so

$(X \cup Y)'$ = {9}

Example:

Given U = {x : $1 \leq x \leq 10$, x is an integer}, A = The set of odd numbers, B = The set of factors of 24 and C = {3, 10}.

a) Draw a Venn diagram to show the relationship.

b) Using the Venn diagram or otherwise, find:

i) $(A \cup B)'$ ii) $(A \cup C)'$ iii) $(A \cup B \cup C)'$

Solution:

A = {1, 3, 5, 7, 9}, B = {1, 2, 3, 4, 6, 8} and C = {3, 10}

a) First, fill in the elements for $A \cap B \cap C$ = {3}, $A \cap B$ {1, 3},

$A \cap C$ = {3}, $B \cap C$ = {3} and then the other elements.

b) We can see from the Venn diagram that

i) $(A \cup B)'$ = {10}

ii) $(A \cup C)' = \{2, 4, 6, 8\}$

iii) $(A \cup B \cup C)' = \{\ \}$

Venn diagrams provide pictorial views of sets. In venn diagram, a rectangle depicts a universal set. Subsets of the universal set are drawn as circles. The inside of a cirle represents the members of that set.

3.5 Sets

Sets are simply collections of items. A set may contain your favorite even numbers, the days of the week, or the names of your brothers and sisters. The items contained within a set are called elements, and elements in a set do not "repeat".

The elements of a set are often listed by roster.

A roster is a list of the elements in a set, separated by commas and surrounded by French curly braces.

Let set A be the numbers 3, 6, 9.

$A = \{3, 6, 9\}$ (in roster notation)

$3 \in A$

The symbol, $\in$, is read "is an element of".

$5 \notin A$ (5 is not in set A)

Let B = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

A = {3, 6, 9}

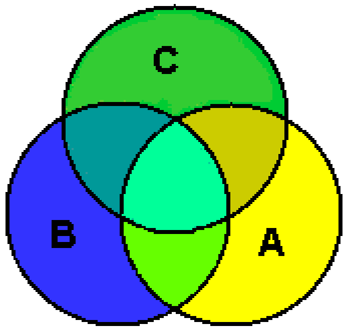
Set A is a subset of set B, since every element in set A is also an element of set B. The notation is: $A \subset B$

The empty set is denoted with the symbol: $\varnothing = \{\}$

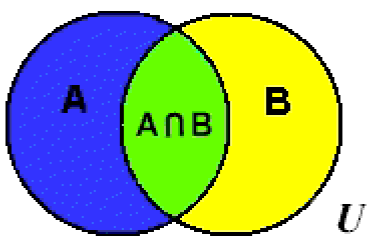
Sets are often represented in pictorial form with a circle containing the elements of the set.

Such a depiction is called a Venn Diagram.

A Venn diagram is a drawing, in which circular areas represent groups of items usually sharing common properties. The drawing consists of two or more circles, each representing a specific group or set. This process of visualizing logical relationships was devised by John Venn (1834-1923).



Each Venn diagram begins with a rectangle representing the universal set. Then each set of values in the problem is represented by a circle. Any values that belong to more than one set will be placed in the sections where the circles overlap.
The universal set is often the "type" of values that are solutions to the problem. For example, the universal set could be the set of all integers from -10 to +10, set A the set of positive integers in that universe, set B the set of integers divisible by 5 in that universe, and set C the set of elements -1, - 5, and 6.



The Venn diagram at the left shows two sets A and B that overlap. The universal set is U. Values that belong to both set A and set B are located in the center region labeled $A \cap B$ where the circles overlap. This region is called the "intersection" of the two sets.
(Intersection, is only where the two sets intersect, or overlap.)

The notation $A \cup B$ represents the entire region covered by both sets A and B (and the section where they overlap). This region is called the "union" of the two sets. (Union, like marriage, brings all of both sets together.)

If we cut out sets A and B from the picture above, the remaining region in U, the universal set, is labeled , $(A \bigcup B)^c$ and is called the complement of the union of sets A and B.

A complement of a set is all of the elements (in the universe) that are NOT in the set.

NOTE*: The complement of a set can be represented with several differing notations.

The complement of set A can be written as

$$A^c \text{ or } A' \text{ or } \overline{A} \text{ or } \sim A$$

* A statement from the NY SED says that students should be familiar with all notations for complement of a set.

The SED Glossary shows the first two notations, while the SED Sample Tasks show the third.

Example:

Let U (the universal set) = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} (a subset of the positive integers)

A = {2, 4, 6, 8}

B = {1, 2, 3, 4, 5} $A \bigcup B = \{1, 2, 3, 4, 5, 6, 8\}$ Union - ALL elements in BOTH sets $A \bigcap B = \{2, 4\}$ Intersection - elements where sets overlap $A^c = \{1, 3, 5, 7, 9, 10\}$ Complement - elements NOT in the set

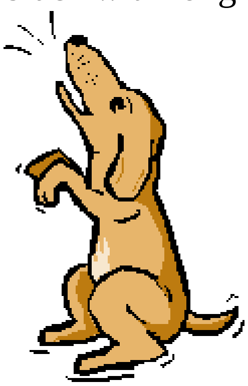
$B^c = \{6, 7, 8, 9, 10\}$

** One of the most interesting features of Venn diagrams is the areas or sections where the circles overlap one another -- implying that a sharing is occurring.This ability to represent a "sharing of conditions" makes Venn diagrams useful

tools for solving complicated problems. Consider the following example:

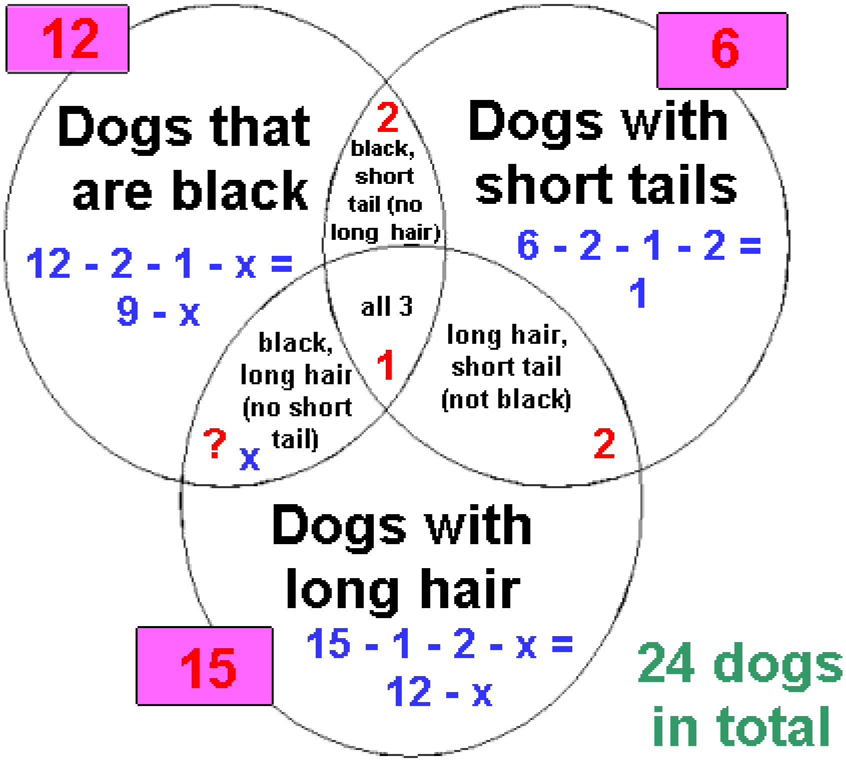
Example:

Twenty-four dogs are in a kennel. Twelve of the dogs are black, six of the dogs have short tails, and fifteen of the dogs have long hair. There is only one dog that is black with a short tail and long hair. Two of the dogs are black with short tails and do not have long hair. Two of the dogs have short tails and long hair but are not black. If all of the dogs in the kennel have at least one of the mentioned characteristics, how many dogs are black with long hair but do not have short tails?



Solution:

Draw a Venn diagram to represent the situation described in the problem.
Represent the number of dogs that you are looking for with **x**.



Notice that the number of dogs in each of the three categories is labeled OUTSIDE of the circle in a colored box. This number is a reminder of the total of the numbers which may appear anywhere inside that particular circle.

After you have labeled all of the conditions listed in the problem, use this OUTSIDE box number to help you determine how many dogs are to be labeled in the empty sections of each circle.

Once you have EVERY section in the diagram labeled with a number or an expression, you are ready to solve the problem.

Add together EVERY section in the diagram and set it equal to the total number of dogs in the kennel (24). Do NOT use the OUTSIDE box numbers.

$9 - x + 2 + 1 + 1 + 2 + x + 12 - x = 24$

$27 - x = 24$

$x = 3$ (There are 3 dogs which are black with long hair but do not have a short tail.)

The union of two sets A and B is the set of elements, which are in A or in B or in both. It is denoted by $A \cup B$ and is read 'A union B'

Example :

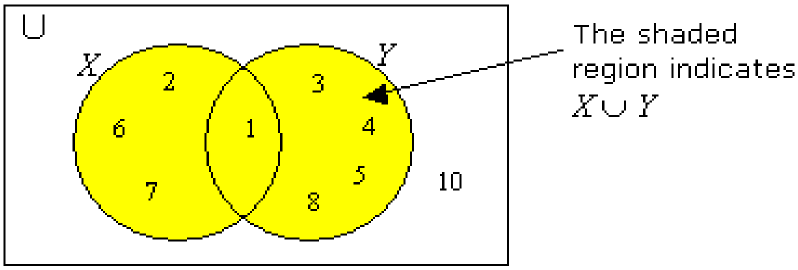
Given U = {1, 2, 3, 4, 5, 6, 7, 8, 10}

$X = \{1, 2, 6, 7\}$ and $Y = \{1, 3, 4, 5, 8\}$

Find $X \cup Y$ and draw a Venn diagram to illustrate $X \cup Y$.

Solution:

$X \cup Y = \{1, 2, 3, 4, 5, 6, 7, 8\}$ ← 1 is written only once.



If $X \subset Y$ then $X \cup Y = Y$. We will illustrate this relationship in the following example.

Example:

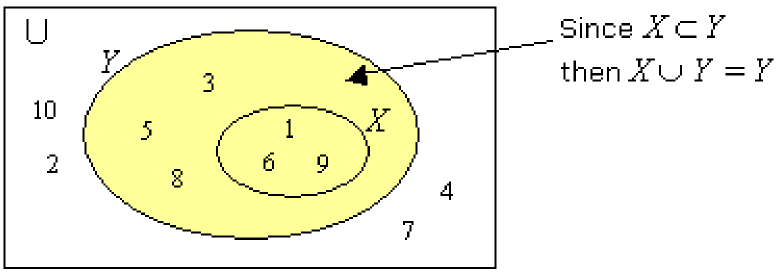
Given U = {1, 2, 3, 4, 5, 6, 7, 8, 10}

$X = \{1, 6, 9\}$ and $Y = \{1, 3, 5, 6, 8, 9\}$

Find $X \cup Y$ and draw a Venn diagram to illustrate $X \cup Y$.

Solution:

$X \cup Y = \{1, 3, 5, 6, 8, 9\}$



The complement of the set $X \cup Y$ is the set of elements that are members of the universal set U but are not in $X \cup Y$. It is denoted by $(X \cup Y)'$

Example:

Given: U = {1, 2, 3, 4, 5, 6, 7, 8, 9}

X = {1, 2, 6, 7} and Y = {1, 3, 4, 5, 8}

a) Draw a Venn diagram to illustrate $(X \cup Y)'$

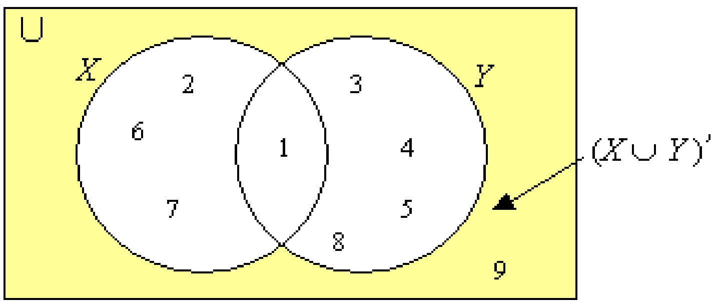
b) Find $(X \cup Y)'$

Solution:

a) First, fill in the elements for $X \cap Y$ = {1}

Fill in the other elements for X and Y and for U

Shade the region outside $X \cup Y$ to indicate $(X \cup Y)'$



b) We can see from the Venn diagram that

$(X \cup Y)'$ = {9}

Or we find that $X \cup Y$ = {1, 2, 3, 4, 5, 6, 7, 8} and so

$(X \cup Y)'$ = {9}

Example:

Given U = {x : $1 \le x \le 10$, x is an integer}, A = The set of odd numbers, B = The set of factors of 24 and C = {3, 10}.

a) Draw a Venn diagram to show the relationship.

b) Using the Venn diagram or otherwise, find:

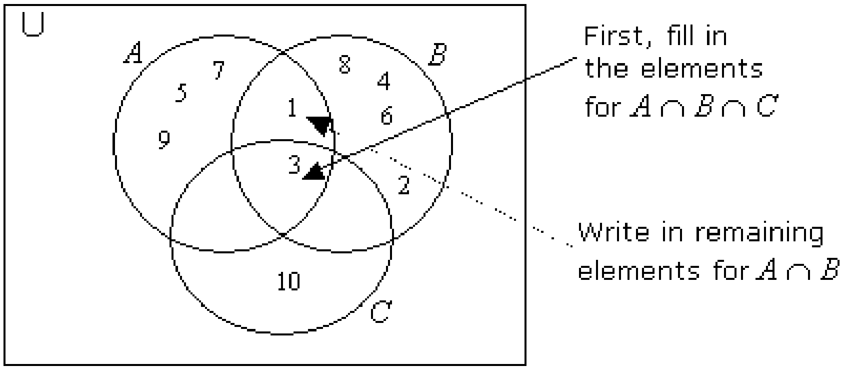
i) $(A \cup B)'$ ii) $(A \cup C)'$ iii) $(A \cup B \cup C)'$

Solution:

A = {1, 3, 5, 7, 9}, B = {1, 2, 3, 4, 6, 8} and C = {3, 10}

a) First, fill in the elements for $A \cap B \cap C = \{3\}$, $A \cap B$ $\{1, 3\}$,

$A \cap C = \{3\}$, $B \cap C = \{3\}$ and then the other elements.



b) We can see from the Venn diagram that

i) $(A \cup B)' = \{10\}$

ii) $(A \cup C)' = \{2, 4, 6, 8\}$

iii) $(A \cup B \cup C)' = \{\ \}$

An ordered pair of elements, written (a, b), is considered distinct from the **ordered pair** (b, a), unless, of course, a = b. To put it another way, (a, b) = (c, d) if and only if a = c and b = d. If X and Y are sets, we let X * Y denote the set of all ordered pairs (x, y) where x ∈ X and x ∈ Y. We call X * Y the **cartesian product** of X and Y.

Example: If X = {1, 2, 3} and Y = {a, b}, then

X * Y = {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)}
Y * X = {(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)}
X * X = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}
Y * Y = {(a, a), (a, b), (b, a), (b, b)}.

3.6 chapter Review Questions

Let the universe be the set □ = {1, 2, 3,.....,10}.
Let A = {1, 4, 7, 10}, B = {1, 2, 3, 4, 5}, and C = {2, 4, 6, 8}.
List the elements of each set.

a) A ∪ B
b) A ∩ B
c) A - B
d) B - A

A group of 191 students, of which 10 are taking French, business, and music; 36 are taking French and business; 20 are taking French and music; 18 are taking business and music; 65 are taking French; 76 are taking business; and 63 are taking music.

e) How many are taking French and music but not business?
f) How many are taking business and neither french nor music?
g) How many are taking French or business (or both)?
h) How many are taking none of the three subjects?

Let X = {1, 2} and Y = {a, b, c}. List the elements in each set. i. X*Y
ii. X*X
iii. Y*X
iv. Y*Y

Suggested further reading

Richard Johnsonbaugh, Discrete Mathematics 5th edition,page 55-60

CHAPTER 4

PERMUTATION AND COMBINATIONS

Learning objectives:

By the end of the chapter a student shall be able to know:

Permutation

Combinations

4.1 Permutations

BY THE PERMUTATIONS of the letters *abc* we mean all of their possible arrangements:

abc acb
bac bca
cab cba

There are 6 permutations of three different things. As the number of things (letters) increases, their permutations grow astronomically. For example, if twelve different things are permuted, then the number of their permutations is 479,001,600.

Now, this enormous number was not found by counting them. It is derived theoretically from the

Fundamental Principle of Counting:

If something can be chosen, or can happen, or be done, in m different ways, and, *after that has happened*, something else can be chosen in n different ways, then the number of ways of choosing both of them is $m \cdot n$.

For example, imagine putting the letters *a, b, c, d* into a hat, and then drawing two of them in succession. We can draw the first in 4 different ways: either *a* or *b* or *c* or *d*. After that has happened, there are 3 ways to choose the second. That is, to *each* of those 4 ways there correspond 3. Therefore, there are $4 \cdot 3$ or 12 possible ways to choose two letters from four.

ab means that *a* was chosen first and *b* second; *ba* means that *b* was chosen first and *a* second; and so on.

Let us now consider the total number of permutations of all four letters. There are 4 ways to choose the first. 3 ways remain to choose the second, 2 ways to choose the third, and 1 way to choose the last. Therefore the number of permutations of 4 different things is

$4 \cdot 3 \cdot 2 \cdot 1 = 24$

Thus the number of permutations of 4 different things taken 4 at a time is 4!. (See Topic 19.)

(To say *taken 4 at a time* is a convention. We mean, "4! is the number of permutations of 4 different things taken from a total of 4 different things.")

In general,

The number of permutations of n different things taken n at a time
is n!.

Example 1. Five different books are on a shelf. In how many different ways could you arrange them?

Answer. $5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$

Example 2. There are 6! permutations of the 6 letters of the word *square*.

a) In how many of them is *r* the second letter? _ *r* _ _ _ _

b) In how many of them are *q* and *e* next to each other?

Solution.

a) Let *r* be the second letter. Then there are 5 ways to fill the first spot. After that has happened, there are 4 ways to fill the third, 3 to fill the fourth, and so on. There are 5! such permutations.

b) Let *q* and *e* be next to each other as *qe*. Then we will be permuting the 5 units *qe, s, u a, r.*. They have 5! permutations. But *q* and *e* could be together as *eq*. Therefore, the total number of ways they can be next to each other is $2\cdot 5! = 240$.

Permutations of less than all

We have seen that the number of ways of choosing 2 letters from 4 is $4\cdot 3 = 12$. We call this

"The number of permutations of 4 different things taken 2 at a time."

We will symbolize this as ${}^4P_2$:

${}^4P_2 = 4\cdot 3$

The lower index 2 indicates the number of factors. The upper index 4 indicates the first factor.

For example, ${}^8P_3$ means "the number of permutations of 8 different things taken 3 at a time." And

For, there are 8 ways to choose the first, 7 ways to choose the second, and 6 ways to choose the third.

In general,

${}^nP_k = n(n-1)(n-2)\cdot\ \cdot\ \cdot$ to k factors

Factorial representation

We saw in the Topic on factorials,

5! is a factor of 8!, and therefore the 5!'s cancel.

Now, $8\cdot 7\cdot 6$ is ${}^8P_3$. We see, then, that ${}^8P_3$ can be expressed in terms of factorials as

$${}^8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!}$$

In general, the number of arrangements -- permutations -- of *n* things taken *k* at a time, can be represented as follows:

$${}^nP_k = \frac{n!}{(n-k)!} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .(1)$$

The upper factorial is the upper index of *P*, while the lower factorial is the difference of the indices.

Example 3.   Express $^{10}P_4$ in terms of factorials.

Solution. $\quad {}^{10}P_4 = \dfrac{10!}{6!}$

The upper factorial is the upper index, and the lower factorial is the difference of the indices. When the 6!'s cancel, the numerator becomes 10· 9· 8· 7.

This is the number of permutations of 10 different things taken 4 at a time.

Example 4.   Calculate ^nP_n.

Solution. $\quad {}^nP_n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = \dfrac{n!}{1} = n!$

^nP_n is the number of permutations of *n* different things taken *n* at a time -- it is the total number of permutations of *n* things: *n*!. The definition 0! = 1 makes line (1) above valid for all values of *k*: $k = 0, 1, 2, \ldots, n$.

Problem 1.   Write down all the permutations of *xyz*.

To see the answer, pass your mouse over the colored area.
To cover the answer again, click "Refresh" ("Reload").

xyz, *xzy*, *yxz*, *yzx*, *zxy*, *zyx*.

Problem 2.   How many permutations are there of the letters *pqrs*?

$$4! = 1\cdot 2\cdot 3\cdot 4 = 24$$

Problem 3.   a) How many different arrangements are there of the letters of the word *numbers*?

$$7! = 5{,}040$$

b) How many of those arrangements have *b* as the first letter?

Set *b* as the first letter, and permute the remaining 6. Therefore, there are 6! such arrangements.

c) How many have *b* as the last letter -- or in any specified position?

The same. 6!.

d) How many will have *n*, *u*, and *m* together?

Begin by permuting the 5 things -- *num*, *b*, *e*, *r*, *s*. They will have 5! permutations. But in each one of them, there are 3! rearrangements of *num*. Consequently, the total number of arrangements in which *n*, *u*, and *m* are together, is 3!· 5! = 6· 120 = 720.

Problem 4.   a) How many different arrangements (permutations) are there of the digits 01234?

$$5! = 120$$

b) How many 5-digit numbers can you make of those digits, in which the

b) first digit is not 0, and no digit is repeated?

Since 0 cannot be first, remove it. Then there will be 4 ways to choose the first digit. Now replace 0. It will now be one of 4 remaining digits. Therefore, there will be 4 ways to fill the second spot, 3 ways to fill the third, and so on. The total number of 5-digit numbers, then, is 4· 4! = 4· 24 = 96.

c) How many 5-digit *odd* numbers can you make, and no digit is
c) repeated?

Again, 0 cannot be first, so remove it. Since the number must be odd, it must end in either 1 or 3. Place 1, then, in the last position. _ _ _ _ 1. Therefore, for the first position, we may choose either 2, 3, or 4, so that there are 3 ways to choose the first digit. Now replace 0. Hence, there will be 3 ways to choose the second position, 2 ways to choose the third, and 1 way to choose the fourth. Therefore, the total number of odd numbers that end in 1, is $3 \cdot 3 \cdot 2 \cdot 1 = 18$. The same analysis holds if we place 3 in the last position, so that the total number of odd numbers is $2 \cdot 18 = 36$.

Problem 5.

a) If the five letters *a, b, c, d, e* are put into a hat, in how many different
a) ways could you draw one out? 5

b) When one of them has been drawn, in how many ways could you
a) draw a second? 4

c) Therefore, in how many ways could you draw two letters? $5 \cdot 4 = 20$

This number is denoted by ${}^5P_2$.

d) What is the meaning of the symbol ${}^5P_3$?

The number of permutations of 5 different things taken 3 at a time.

e) Evaluate ${}^5P_3$. $5 \cdot 4 \cdot 3 = 60$

Problem 5. Evaluate

a) ${}^6P_3 = 120$ b) ${}^{10}P_2 = 90$ c) ${}^7P_5 = 2520$

Problem 6. Express with factorials.

a) ${}^nP_k \quad \dfrac{n!}{(n-k)!}$ b) ${}^{12}P_7 \quad \dfrac{12!}{5!}$ c) ${}^8P_2 \quad \dfrac{8!}{6!}$ d) ${}^mP_0 \quad \dfrac{m!}{m!}$

4.2 Permutations and Combinations

Certain types of probability calculations involve dividing the number of outcomes associated with an event by the total number of possible outcomes. For simple problems it is easy to count the outcomes, but in more complex situations manual counting can become laborious or impossible.

Fortunately, there are formulas for determining the number of ways in which members of a set can be arranged. Such arrangements are referred to as **permutations** or **combinations**, depending on whether the order in which the members are arranged is a distinguishing factor.

The number of different orders in which members of a group can be arranged for a group of *r* members taken *r* at a time is:

$$(r)(r-1)(r-2)...(1)$$

This is more easily expressed as simply r!.

When order is a distinguishing factor, a group of n members taken r at a time results in a number of permutations equal to the first *r* terms of the following multiplication:

$$(n)(n-1)(n-2)...$$

This can be expressed as:

$${}_nP_r = n! / (n - r)!$$

In combinations, order is not a distinguishing factor:

$${}_nC_r = {}_nP_r / (r!) = n! / (n - r)!r!$$

For the special case of possible pairs in a group of *n* members, assuming order in a pair is not important, then:

$$r = 2$$

and the number of possible pairs is:

$$n(n - 1) / 2.$$

Example: *How many two-element subsets of {1,2,3,4} are there that do not contain the pair of elements 2 and 4 ?*

Solution: 4! / (2!)(2!) = 6, but the subset {2,4} is not to be counted, so the answer is 5.

Given *n* items taken *r* at a time, to find the number of combinations in which *x* particular items are not present, simply reduce *n* by *x* and solve as one would a normal combination problem.

Combinations of Groups

If Group A has *x* members, Group B has *y* members, and Group C has *z* members, there are (x)(y)(z) possible combinations assuming that one member from each of the three groups is used in each combination, and assuming that the order is not a distinguishing factor. In general, if more than one member is taken at a time from each group, the number of combinations is the product of ${}_nC_r$ (or ${}_nP_r$ if appropriate) associated with each particular group.

Permutaion

An ordering of objects, such as the names on the ballot, is called a **permutation**.

Example: There are six permutations of three elements. If the elements are denoted A, B, C, the six permutations are

ABC, ACB, BAC, BCA, CAB, CBA.

Theorem: There are n! Permutations of n elements.

e.g.

$n! = n(n-1)(n-2)........2.\ 1$

e.g.

$10! = 10.9.8.7.6.5.4.3.2.1 = 3,628,800$ permutations of 10 elements

Combinations

A selection of objects without regard to order is called a **combination**.

Example: In how many ways can we select a committee of three from a group of 10 distinct persons?

Since a committee is an unordered group of people, the answer is

$C(10, 3) = 10.9.8.7! / (10-3)!\ 3! = 10.9.8 / 3! = 120$

4.2 chapters Review Questions

a) How many permutations are there of a, b, c, d?
b) In how many ways can we select a chairperson, vice-chairperson, and recorder from a group of 11 persons?
c) In how many ways can we select a committee of four from a group of 12 persons?

Suggested further reading

Richard Johnsonbaugh, Discrete Mathematics 5th edition,page 174- 18

CHAPTER 5

GRAPH THEORY

Learning objectives:
By the end of the chapter a student shall be able to know:

a) Directed and undirected graphs
b) Sub graphs
c) Paths
d) Cycles (circuit)
e) Adjacency and incidence vatices
f) Elements of transport network

5.1 Directed and undirected graphs

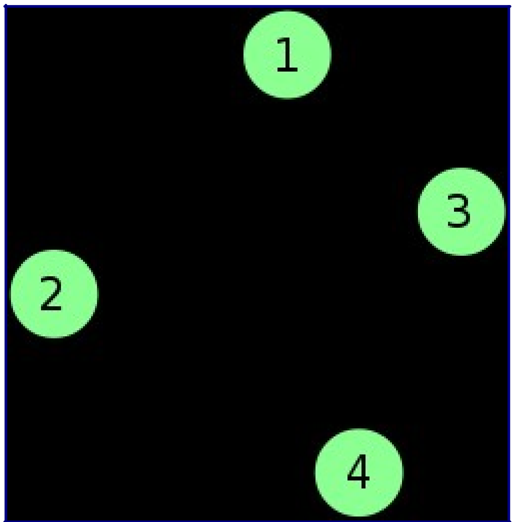
A graph (or undirected graph) G consists of a set V of vertices (or nodes) and a set E of edges (or arcs) such that each edge $e \in E$ is associated with an unordered pair of vertices. If there is a unique edge e associated with the vertices v and w, we write $e = (v, w)$ or $e = (w, v)$. In this context, (v, w) denotes an edge between v and w in an undirected graph and not an ordered pair.

A directed graph (or digraph) G consists of a set V of vertices (or nodes) and a set E of edges (or arcs) such that each edge $e \in E$ is associated with an ordered pair of vertices. If there is a unique edge e associated with the ordered pair (v, w) of vertices, we write $e = (v ,w)$, which denotes an edge from v to w.

An edge e in a graph (undirected or directed) that is associated with the pair of vertices v and w is said to be **incident** on v and w, and v and w are said to be **incident** on e and to be **adjacent vertices**

5.2 Paths:

If we start at a vertex v_0, travel along an edge to vertex v_1, travel along another edge to vertex v_2, and so on, and eventually arrive at vertex v_n, we call the complete tour a path from v_0 to v_n.



An undirected graph

In [graph theory] an [undirected graph] G has two kinds of incidence matrix: unoriented and oriented. The **incidence matrix** (or **unoriented incidence matrix**) of G is a p × q matrix (b_{ij}), where p and q are the numbers of vertices and edges respectively, such that b_{ij} = 1 if the vertex v_i and edge x_j are incident and 0 otherwise.

For example the incidence matrix of the undirected graph shown on the right is a matrix consisting of 4 rows (corresponding to the four vertices) and 4 columns (corresponding to the four edges):

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The **incidence matrix** of a directed graph D is a p × q matrix [b_{ij}] where p and q are the number of vertices and edges respectively, such that $b_{ij} = -1$ if the edge x_j leaves vertex v_i, 1 if it enters vertex v_i and 0 otherwise. (Note that many authors use the opposite sign convention.)

An **oriented incidence matrix** of an undirected graph G is the incidence matrix, in the sense of directed graphs, of any orientation of G. That is, in the column of edge e, there is one +1 in the row corresponding to one vertex of e and one −1 in the row corresponding to the other vertex of e, and all other rows have 0. All oriented incidence matrices of G differ only by negating some set of columns. In many uses, this is an insignificant difference, so one can speak of the oriented incidence matrix, even though that is technically incorrect.

The oriented or unoriented incidence matrix of a graph G is related to the adjacency matrix of its line graphL(G) by the following theorem:

$$A(L(G)) = B(G)^T B(G) - 2I_q$$

where A(L(G)) is the adjacency matrix of the line graph of G, B(G) is the incidence matrix, and I_q is the identity matrix of dimension q.

The integral cycle space of a graph is equal to the null space of its oriented incidence matrix, viewed as a matrix over the integers or real or complex numbers. The binary cycle space is the null space of its oriented or unoriented incidence matrix, viewed as a matrix over the two-element field.

Signed and bidirected graphs

The incidence matrix of a signed graph is a generalization of the oriented incidence matrix. It is the incidence matrix of any bidirected graph that orients the given signed graph. The column of a positive edge has a +1 in the row corresponding to one endpoint and a −1 in the row corresponding to the other endpoint, just like an edge in an ordinary (unsigned) graph. The column of a negative edge has either a +1 or a −1 in both rows. The line graph and Kirchhoff matrix properties generalize to signed graphs.

The definitions of incidence matrix apply to graphs with loops and multiple edges. The column of an oriented incidence matrix that corresponds to a loop is all zero, unless the graph is signed and the loop is negative; then the column is all zero except for ±2 in the row of its incident vertex.

Hypergraphs

Because the edges of ordinary graphs can only have two vertices (one at each end), the column of an incidence matrix for graphs can only have two non-zero entries. By contrast, a hypergraph can have multiple vertices assigned to one edge; thus, the general case describes a hypergraph.

[edit] Incidence structures

The **incidence matrix** of an incidence structure C is a p × q matrix $[b_{ij}]$, where p and q are the number of **points** and **lines** respectively, such that $b_{ij} = 1$ if the point p_i and line L_j are incident and 0 otherwise. In this case the incidence matrix is also a biadjacency matrix of the Levi graph of the structure. As there is a hypergraph for every Levi graph, and vice-versa, the incidence matrix of an incidence structure describes a hypergraph.

Finite geometries

An important example is a finite geometry. For instance, in a finite plane, X is the set of points and Y is the set of lines. In a finite geometry of higher dimension, X could be the set of points and Y could be the set of subspaces of dimension one less than the dimension of Y; or X could be the set of all subspaces of one dimension d and Y the set of all subspaces of another dimension e.

[edit] Block designs

Another example is a block design. Here X is a finite set of "points" and Y is a class of subsets of X, called "blocks", subject to rules that depend on the type of design. The incidence matrix is an important tool in the theory of block designs. For instance, it is used to prove the fundamental theorem of symmetric 2-designs, that the number of blocks equals the number of points.

5.3 Subgraphs:

A connected graph consists of one "piece," while a graph that is not connected consists of two or more "pieces." These "pieces" are called subgraphs

Definition: Let v and w be vertices in agraph G.

A simple path from v to w is apath from v to w with no repeated vertices.

A cycle (or circuit) is a path of nonzero length from v to v with no repeated edges

Directed and Undirected
Graphs

- A graph is a mathematical structure consisting of a set of vertices

and a set of edges connecting the vertices.

Formally: G = (V, E), where V is a set and $E \subseteq V \times V$.

- G = (V, E) undirected if for all v,

$w \in V : (v, w) \in E$

$\Longleftrightarrow (w, v) \in E.$

Otherwise directed.

A
directed
graph

G = (V, E) with vertex set

V = 0, 1, 2,

3, 4, 5, 6 and edge set

E = (0, 2), (0, 4), (0, 5), (1, 0), (2, 1), (2, 5), (3, 1), (3, 6), (4, 0), (4, 5), (6, 3), (6,
5) .

5.4 chapter Review questions

a) Define undirected graph
b) How many edges are incident on a vertex in an n-cube?

FURTHER READING

- **Diestel, Reinhard (2005), Graph Theory, Graduate Texts in Mathematics, 173 (3rd ed.), Springer-Verlag, ISBN 3-540-26183-4 .**
- **Coxeter,H.S.M. RegularPolytopes, (3rd edition, 1973), Dover edition, ISBN0-486-61480-8 (Section 9.2 Incidence matrices, pp. 166-171)**
- **Jonathan L Gross, Jay Yellen, Graph Theory and its applications, second edition, 2006 (p 97, Incidence Matrices for unidrect graphs; p 98, incide**nce matrices for digraphs)

Mt Kenya University

UNIVERSITY EXAMINATION 2009/2010

SCHOOL OF APPLIED AND SOCIAL SCIENCES

DEPARTMENT OF INFORMATION TECHNOLOGY

SEMSTER I EXAMINATION FOR BACHELOR FOR BUSINESS INFORMATION TECHNOLOGY

BIT 1201: BASIC DISCRETE MATHEMATICS

DATE: JULY 2010 **TIME: 2HRS**

INSTRUCTIONS

Answer Question One And Any Other Two

Q1. A) Draw the truth table for the following Boolean expression $Z=A.B+\bar{A}\bar{B}$ (5mks)

b) Simplify $\bar{A}.B+\bar{A}.B$ by using demorgans law and rules of Boolean algebra (4mks)

c) let A={1,2,3,4}

B={3,4,5,6,7}

C={2,3,5,7}

Find (i) AUB

(ii) AnC (4mks)

d) which of the following statements are true

(i) 1+2=3

(ii) I am ugly and you are clever

(iii) All those over 2 meters tall are over 200yrs old. (3mks)

e) In hockey a match can be won, draw or lost. If a team play 5 matches how many different sequences of results are possible? (4mks)

f)Using induction method prove that 1+2+3+4+…..+n=n(n+1)/2
(5mks)

g) 3 boys A,B and C are throwing a ball among themselves such that A always throws the ball to B, but B and C are just as likely to throw the ball to A as they are to each other. Illustrate the information given above in form of a graph and show the probabilities on the arcs (5mks)

Q2. A) complete the following truth table

p	q	-p	-q	-p∧-q	-pv-q	-pvq	-(p∧-q)

(10mks)

b) construct a switchingcicuit to meet the requirements of the Boolean expression Z=A.Q+A.P+A.P.Q. hence construct the truth table for this circuit.
(10mks)

Q3. A) design a circuit with four switches A,B,C and D according to the following table.

A	B	C	D
1	1	1	1
1	1	0	1
0	1	1	1
0	1	0	1

(10mks)

b) Verify that the proposition (p∧q)∧-(pvq) is a contradiction (5MKS)

c) Show that hv - (h∧m) is a tautology (5mks)

Q4. A) In a survey of 60 people, it was found that 25 read Nation, 26 read times, and 26 read fortune. Also 9 read both Nation and times, 8 read both times and fortune, and 8 read none of the magazines.

(i) Find the number of people who read all the 3 magazines
(ii) Draw the Venn diagram to represent the above information.
(iii) Determine the number of people who read exactly one magazine (14MKS)

(b) Show that A={2,3,4,5} is a proper subset of C={1,2,3,4,5,…..8,9} (3mks)

(c) Show that A={2,3,4,5} is not a subset of B={X:xεN, x is even} (3mks)

Q5 (a) Prove by induction that 1□ +2□ +3□ +……..+n□= ¼$n^2(n+1)^2$ (5mks)

(b) 9 players are available to play for a tennis team of 4 players. In how many ways can the team be selected if 2 of the players are brothers and must either both be included or both be excluded, and if two other players have recently quarreled and should not both play in the team (12mks)

(c) How many different signals, each consisting of eight flags hung in vertical line, can be formed with 4 indistinguishable red flags, 3 indistinguishable white flag, and a blue flag? (3mks)

Mt Kenya University

UNIVERSITY EXAMINATION 2010/2011

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF INFORMATION TECHNOLOGY

EXAMINATION FOR BACHELOR OF BUSINESS INFORMATION TECHNOLOGY

UNIT CODE: BBIT 1201 TITLE: DISCRETE MATHEMATICS

DATE: NOVEMBER, 2010 TIME: 2 HOURS

INSTRUCTIONS: Answer Question **ONE** and any other **TWO** questions.

QUESTION ONE

a) Write the converse, inverse and the contra positive of the following sentence."If the Sun shines brightly today , then it will set early" (4mks)

b) Define the terms: Tautology, Logical equivalence and a propositional function. (3mks)

c) State three methods of proving theorems (3Mks)

d) Explain the meaning of the term lattice (2mks)

e) How many ways are there to select 5 players from a 10-member tennis team to make a trip to a match at another school? (3mks)

f) Differentiate between the cardinality of a set and the Cartesian product of sets(4mks)

g) What is a partially ordered set (2mks)

h) Simplify the following Boolean expressions (5mks)

a. $\overline{(1+0)} + \overline{(1.0)}$

b. $\overline{(1+1)}+\overline{(0)}+1.1$

i) Draw the venn diagram showing the intersection between two sets A and B. (2 mks)

QUESTION TWO

Use a relevant method of proving theorems to:

a) Prove that the following formula for the sum of a finite number of terms of a geometric progression is given by,

$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots \ldots \ldots \ldots \ldots \ldots ar^n = \frac{ar^{n+1} - a}{r-1}$

(7mks)

b) Prove that $\sqrt{2}$ is irrational.

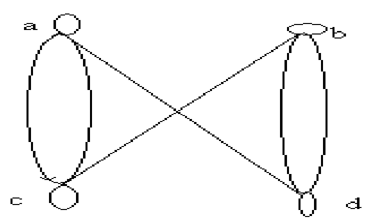
(7mks)

c) Prove the theorem "the integer n is odd if and only if $n^2$is odd (6mks)

QUESTION THREE

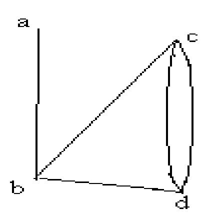
a) Represent each of the following graphs using an adjacency matrix

i)



(4mks)

ii)



(4mks)

b) Draw the graphs with the following adjacency matrices

i) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$ (6mks)

ii) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$ (6mks)

QUESTION FOUR

a) What is the Cartesian product $AXBXC$ where $A = \{0,1\}$,$B = \{1,2\}$ and $C = \{a, b, c\}$ (6mks)

b) Find the power set of the set and $\{0,1,2\}$

(4mks)

c) Let A, B and C are sets. Show that $A \cup (B \cap C$=$(\bar{C} \cup \bar{B}) \cap \bar{A}$ using identities

(4mks)

d) Suppose that f is defined recursively by;

f (0)=3

f (n+1)=2f(n)+3.

Find f (1), f(2), f(3) and f(4)

(4mks)

e) What is a Venn diagram?

(2mks)

QUESTION FIVE

a) Show that the propositions $\neg p \vee q$ and $p \rightarrow q$ are logically equivalent

(6mks)

b) Let R(x,y,z) denote the statement $z = x^2 + y^2$.Find the truth values of R(0,0,1),R(1,1,2) and R(2,3,13).

(6mks)

c) Draw the truth tables for negation and the implication of a proposition p and q

(4mks)

d) Explain the meaning of the following terms.

i) Open sentence

ii) Logical Reasoning

(4mks)