a) The Lagrangian of the system is defined as:

$$L = T - U + \lambda(3x_1 + x_2 - c)$$
$$L = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{k}{2}\left(x_1^2 + (x_2 - x_1)^2 + x_2^2\right) + \lambda(3x_1 + x_2 - c)$$
$$L = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) - k\left(x_1^2 - x_1x_2 + x_2^2\right) + \lambda(3x_1 + x_2 - c)$$

$b) The\ equations\ of\ motion\ can\ then\ be\ derived:$

$$\frac{d}{dt}\frac{dL}{d\dot{x}_i} - \frac{dL}{dx_i} = 0$$

$$\frac{d}{dt}\frac{dL}{d\dot{x}_i} = m\ddot{x}_i \quad for\ i = 1,2$$

$$\frac{dL}{dx_1} = k(-2x_1 + x_2) + 3\lambda$$

$$\frac{dL}{dx_2} = k(-2x_2 + x_1) + \lambda$$

$$\frac{dL}{d\lambda} = 3x_1 + x_2 - c = 0 \quad (*)$$

$$\Rightarrow m\ddot{x}_1 - k(-2x_1 + x_2) - 3\lambda = 0$$
$$m\ddot{x}_2 - k(-2x_2 + x_1) - \lambda = 0$$

$c)$

$Eliminate\ \lambda:$

$$\Rightarrow m\ddot{x}_1 - k(-2x_1 + x_2) - 3m\ddot{x}_2 + 3k(-2x_2 + x_1)$$
$$= m(\ddot{x}_1 - 3\ddot{x}_2) + k(5x_1 - 7x_2) = 0$$

$Use\ (*)\ to\ eliminate\ x_2:$

$$x_2 = c - 3x_1$$
$$\Rightarrow m(\ddot{x}_1 - 3\ddot{x}_2) + k(5x_1 - 7x_2) =$$
$$= m(\ddot{x}_1 - 9\ddot{x}_1) + k(5x_1 - 7(c - 3x_1))$$

$$= -8m\ddot{x}_1 + k(26x_1 - 7c) = 0$$

$$\Rightarrow x_1 = \frac{8m\ddot{x}_1 + 7kc}{26k}$$

We can now find the equlibirium positions under the constraint.

$$\text{Equilibrium} \Rightarrow \ddot{x}_i = 0$$

$$\Rightarrow x_1 = \frac{7c}{26} \quad \Rightarrow \quad x_2 = c\left(1 - \frac{21}{26}\right) = \frac{5c}{26}$$

d)

To find the eigenfrequencies we first define:

$$\ddot{x}_1 = \frac{k(26x_1 - 7c)}{8} = \frac{k(5x_1 - 7x_2)}{8}$$

$$\ddot{x}_2 = \frac{-3k(26x_1 - 7c)}{8} = \frac{-3k(5x_1 - 7x_2)}{8}$$

$$\ddot{X}(t) = K\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \qquad \text{where } K = \frac{k}{8}\begin{pmatrix} 5 & -7 \\ 15 & -21 \end{pmatrix}$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} * exp(i\omega t) = X(t)$$

Thus we have:

$$det\left(K + Iw^2\right) = 0 = \begin{vmatrix} 5 + w^2 & -7 \\ 15 & -21 + w^2 \end{vmatrix}$$

$$\Rightarrow (5 + w^2)(-21 + w^2) + 105 = 0$$

$$\Rightarrow w^4 - 16w^2 - 105 + 105 = 0$$

$$\Rightarrow w^2\left(w^2 - 16\right) = 0$$

$$\Rightarrow w^2 = \pm 4$$

$$\Rightarrow \omega_1 = \omega_2 = 2$$