CALIBRATION OF A LARGE DISTRIBUTED LOW FREQUENCY RADIO ASTRONOMICAL ARRAY (LOFAR)

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ABSTRACT

Several low frequency radio astronomy arrays are currently under development. For example the LOFAR array (with 13,000 dipole elements) will operate in the frequency range of 20MHz-240Mhz. At these frequencies the effect of the ionosphere cannot be ignored. Due to the fact that the size of the array is larger than the size of the irregularities in the ionosphere the calibration problem is direction dependent and time varying. The most general form of this problem does not have a unique solution given a single sample covariance estimate. In this paper we explore several constraints derived from the physics of the problem which make the problem solvable.

1. INTRODUCTION

This paper considers a sensor array which must be calibrated using existing signals of opportunity as reference sources. It is assumed that these sources have known position and are relatively intense. This problem is often referred to as "self calibration" [1]. This technique has been widely used for higher frequency synthesis arrays, several improved algorithms have been proposed, and the estimation statistic are well understood [2]. We consider the particularly difficult low frequency case where calibration gain parameters are not only a function of variations in instrument electronics, but of refraction through an unknown randomly structured propagation medium. We will further assume that this refractive field is not uniform across elements of the array, or over the differing directions pointing to the self calibration sources. Such a problem can arise in a variety of applications, including sonar direction finding through turbulent water, and radio astronomy at low frequencies where the signal interacts strongly with the ionosphere.

Several groups are currently working on low frequency radio astronomy arrays. The largest of these, called LOFAR, is being developed by ASTRON in The Netherlands. This system will consist of 13,000 dipole antennas grouped into 72 stations. It will operate in the 10MHz-240MHz frequency range. The maximum planned station-to-station distance is 100 km. At station level a beamformer combines the signals of the dipoles. At central level a station can be considered as a single directional antenna. Cross correlations between stations are computed periodically over many hours to exploit earth rotation as it repositions the array relative to the desired space objects. Further correlation data processing consists of self calibration and interferometric synthesis imaging. The end product is a data cube consisting of an image or intensity map of the sky per frequency channel.

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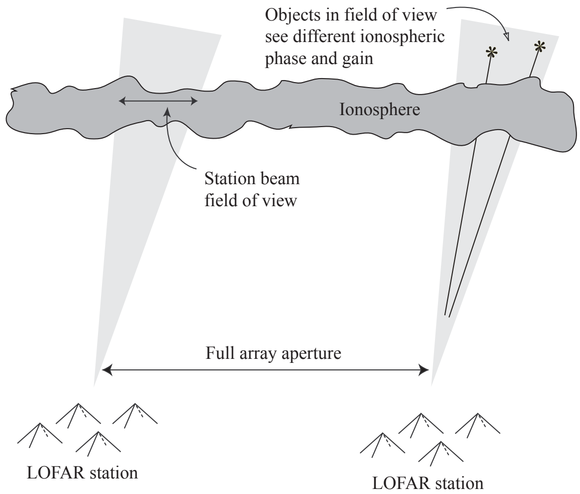

Figure 1: The problem of LOFAR calibration through ionospheric refraction. Unknown complex gains through the ionosphere are different for each source at each station. (after C. Lonsdale)

The LOFAR calibration problem will apparently require extensions to existing methods of self calibration used in synthesis array radio imaging. This is due to several factors, including the very large number of antennas and the random phase and gain perturbations caused by refractive propagation through the ionosphere. These perturbations vary not only with antenna position, but also with signal source direction.

The intent of this paper is to evaluate at a fundamental level the mathematical structure of the LOFAR calibration problem. This is done with an eye toward determining what constraints and outside information are required to yield calibration solutions. We will initially assume no noise, contaminating sources, or modeling error for the known sources (sky model) so that basic questions regarding parameter identifiability and uniqueness in the estimation problem can be established. By studying several different scenarios related to properties of the refractive medium and array configuration we hope to establish which conditions can yield suitable calibration solutions. Specific algorithmic approaches to find these

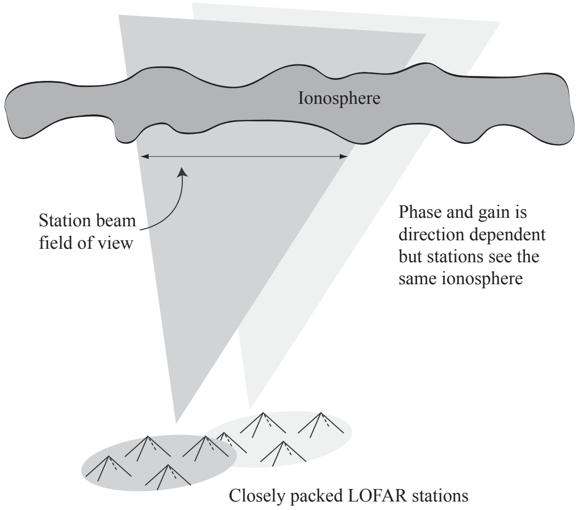

Figure 2: Calibration scenario for closely spaced LOFAR central core stations. Due to beam overlap at ionospheric altitude, each station sees the same direction dependence which is cancelled out in the cross correlation computation. (after C. Lonsdale)

solutions will not be addressed.

Though LOFAR is the motivating application for the analysis, the signal and array models are quite general. Specific geometrical characteristics or design details of LOFAR are represented by abstractions that attempt to capture only the underlying issues which affect the calibration problem.

1.1 Data model

Consider a telescope array with M sensors (stations). For each 1 kHz frequency band the output of the m^{th} station at time sample n is a beamformed signal $x_{m,n}$. For simplicity only scalar (non polarized) propagation and sensing is considered. The array output vector $\mathbf{x}_n$ is obtained by stacking the signals $x_{m,n}$. The Q brightest point sources in the sky are used as calibration references. Because of their relative intensity the remaining astronomical signals can be ignored during calibration, which leads to the observed data model

$$\mathbf{x}_n = \sum_{q=1}^{Q} \mathbf{a}_q s_{q,n} \;=\; \sum_{q=1}^{Q} (\mathbf{g}_q \odot \mathbf{k}_q) s_{q,n}.$$

$\mathbf{a}_q$ is the array response to the q^{th} source, including geometric and ionospheric propagation effects. $s_{q,n}$ is the corresponding Gaussian signal time sequence, with variance σ_q^2. These calibrators have known sky positions and intensities (from tables). Station positions are also known accurately. Thus $\mathbf{a}_q$ can be factored into a known geometric term $\mathbf{k}_q$ and the unknown calibration complex gains $\mathbf{g}_q$ which must be estimated. $\odot$ indicates the Hadamard element–wise matrix product.

Due to non-homogeneity of the refractive ionospheric layer, the effective complex gain at these frequencies for each source as seen by each station can be an independent value. This scenario is illustrated in Figure 1. The mutually independent calibration signals can be stacked in a vector $\mathbf{s}_k = [s_{1,n} \ldots s_{Q,n}]^{\mathrm{T}}$ so the corresponding covariance matrix $\mathbf{\Sigma}_s = \mathrm{E}[\mathbf{s}\mathbf{s}^{\mathrm{H}}] = \mathrm{diag}(\sigma_1^2 \ldots \sigma_Q^2)$ is diagonal and is known from tables.

The signals are correlated and integrated to obtain the sample correlation matrix

$$\hat{\mathbf{R}} = \frac{1}{N} \sum_{n=1}^{N} \mathbf{x}_n \mathbf{x}_n^{\mathrm{H}}.$$

Gathering vectors $\mathbf{a}_q$, $\mathbf{k}_q$ and $\mathbf{g}_q$ into the columns of matrices $\mathbf{A}$, $\mathbf{K}$, and $\mathbf{G}$ respectively leads to expressing the expected value of $\hat{\mathbf{R}}$ as

$$\mathbf{R} = \mathbf{A}\mathbf{\Sigma}_s\mathbf{A}^{\mathrm{H}} \;=\; (\mathbf{G} \odot \mathbf{K})\mathbf{\Sigma}_s(\mathbf{G} \odot \mathbf{K})^{\mathrm{H}}. \qquad (1)$$

In radio astronomical context, entries of $\hat{\mathbf{R}}$ are called visibilities. Each visibility represents the interferometric correlation along the baseline vector between the two corresponding array elements (stations). It is desirable to obtain visibilities from as many unique baseline vectors as possible in order to fill in the frequency domain sample space used for synthesis imaging [1]. Large arrays provide many element pairings and thus a large number of baselines, but earth rotation is also exploited to reposition the baselines relative to sky sources and provide dense frequency domain sampling. Therefore a larger $\hat{\mathbf{R}}$ may be built up from entries computed over a series of time snapshots.

1.2 Calibration Problem

The calibration strategy for LOFAR is to estimate the unknown gains in the direction of a few (i.e. Q) calibration sources. This amounts to finding a $\mathbf{G}$ such that the measured data $\hat{\mathbf{R}}$ match the model in (1) given known values for $\mathbf{K}$ and $\mathbf{\Sigma}_s$.

Several calibration sources are required in the imaging field of view (FOV), for example over a solid angle of six degrees. The ionosphere is varying over this scale, and sufficient calibration sampling is needed to parameterize a smoothly varying phase sheet model. It is also necessary to calibrate on the most intense sources outside the FOV in the station beam sidelobes. Energy from these source can then be accurately removed from $\hat{\mathbf{R}}$ to achieve the best possible imaging dynamic range.

The single snapshot (i.e. one $\hat{\mathbf{R}}$ realization) calibration problem is not uniquely defined since by substitution one can verify that for any solution $\mathbf{G}$ there is a whole set for which the model gives the same $\mathbf{R}$. To illustrate, note that (1) can be written as

$$\mathbf{R} = (\mathbf{G} \odot \mathbf{K})\mathbf{\Sigma}_s^{\frac{1}{2}}\mathbf{U}\mathbf{U}^{\mathrm{H}}\mathbf{\Sigma}_s^{\frac{1}{2}}(\mathbf{G} \odot \mathbf{K})^{\mathrm{H}}, \qquad (2)$$

where $\mathbf{U}$ is any unitary matrix such that $\mathbf{U}\mathbf{U}^{\mathrm{H}} = \mathbf{I}$. For any given $\mathbf{U}$ the corresponding ambiguous calibration solution is

$$\tilde{\mathbf{G}}(\mathbf{U}) = ((\mathbf{G} \odot \mathbf{K})\mathbf{\Sigma}_s^{\frac{1}{2}}\mathbf{U}\mathbf{\Sigma}_s^{-\frac{1}{2}}) \odot \mathbf{K}^{\odot -1}, \qquad (3)$$

where $^{\odot -1}$ indicates element-wise matrix inverse.

To find a unique solution more constraints on $\mathbf{G}$ are needed. These constraints must be based on the physics of the problem. We have identified and analyzed the following possibilities, which do in fact result in unambiguous calibration solutions.

1. Calibrated subarray
2. Deterministic frequency dependence
3. Known absolute gain, unknown phase.
4. Diversity of $\mathbf{K}$ over frequency and time

In the next sections we will discuss the first two of these.

2. CALIBRATED SUBARRAY

We now assume the array includes a subarray of M_c stations with known calibration. This could occur when the compact central core of LOFAR stations are operating in "regime 3" described in [3] as shown in Figure 2. The station beam fields of view are much larger than, and the total subarray aperture is much less than, the ionospheric irregularity scale. Though each source undergoes a different phase delay through the irregular ionosphere, the closely packed antennas all see this same bulk phase, which is lost when computing the visibilities (correlations). Thus the subarray sees a coherent scene without direction dependence. The rest of the array operates in "regime 4" of Figure 1 with direction dependent gains. In this case calibration can be first estimated for the subarray using existing techniques. The question we address is whether, and under which conditions, the known calibration of a subarray can be used to resolve the type of ambiguity described above for the rest of the array.

The array is partitioned by re-indexing antenna elements so that the first M_c elements have corresponding known calibration vectors, $\mathbf{g}_q^c = [g_{1,q}, \cdots g_{M_c,q}]^t$, $1 \le q \le Q$. These gains may be constant with respect to q (i.e. source direction independent calibration) or they may depend on q (direction or source varying calibration), but it is assumed that they are known. The unknown calibration terms are $\mathbf{g}_q^r = [g_{(M_c+1),q}, \cdots g_M]^t$, $1 \le q \le Q$. Superscript 'c' indicates the core subarray with known calibration and 'r' denotes the remainder of the full array, which in this scenario requires calibration for direction dependent gains. Using the notation of the previous section we have

$$\begin{aligned} \mathbf{A} &= \mathbf{G} \odot \mathbf{K}, \\ &= \begin{bmatrix} \mathbf{G}^c \odot \mathbf{K}^c \\ \mathbf{G}^r \odot \mathbf{K}^r \end{bmatrix} = \begin{bmatrix} \mathbf{A}^c \\ \mathbf{A}^r \end{bmatrix} \end{aligned} \tag{4}$$

where $\mathbf{A}^c$ and $\mathbf{K}^r$ are known, and

$$\mathbf{G}^c = \left[\mathbf{g}_1^c \cdots \mathbf{g}_Q^c\right], \ \ \mathbf{G}^r = \left[\mathbf{g}_1^r \cdots \mathbf{g}_Q^r\right]$$

If distinct matrices, $\mathbf{G}^r$, can be found which lead to the same visibilities in $\mathbf{R}$, then the problem is ambiguous and calibration is not possible. As in (2) we use an internal unitary matrix term, $\mathbf{U}$, to form a new array response matrix, $\tilde{\mathbf{A}}$, which yields unchanged visibilities such that $\mathbf{R} = \tilde{\mathbf{A}}\mathbf{\Sigma}_s\tilde{\mathbf{A}}^{\mathrm{H}}$. Such ambiguous solutions can be constructed as

$$\begin{aligned} \tilde{\mathbf{A}} &= \mathbf{A}\mathbf{\Sigma}_s^{\frac{1}{2}}\mathbf{U}\mathbf{\Sigma}_s^{-\frac{1}{2}}, \\ &= \begin{bmatrix} \mathbf{A}^c\mathbf{\Sigma}_s^{\frac{1}{2}}\mathbf{U}\mathbf{\Sigma}_s^{-\frac{1}{2}} \\ \mathbf{A}^r\mathbf{\Sigma}_s^{\frac{1}{2}}\mathbf{U}\mathbf{\Sigma}_s^{-\frac{1}{2}} \end{bmatrix}. \end{aligned} \tag{5}$$

But, since $\mathbf{A}^c$ is known, $\mathbf{U}$ is constrained such that

$$\mathbf{A}^c\mathbf{\Sigma}_s^{\frac{1}{2}}\mathbf{U}\mathbf{\Sigma}_s^{-\frac{1}{2}} = \mathbf{A}^c. \tag{6}$$

There are two possible cases.

Case I, More calibration sources than subarray elements.

For this situation $\mathbf{A}^c$ is a short-fat matrix, i.e. $M_c < Q$. We can first rewrite (6) as

$$\begin{aligned} \mathbf{C}\mathbf{U} &= \mathbf{C}, \text{ where} \\ \mathbf{C} &= \mathbf{A}^c\mathbf{\Sigma}_s^{\frac{1}{2}}. \end{aligned} \tag{7}$$

Assuming the rank of $\mathbf{A}^c$ is equal to the number of rows, and since $\mathbf{\Sigma}_s$ is full rank, we can write the SVD of $\mathbf{C}$ as

$$\begin{aligned} \mathbf{C} &= \mathbf{W}\mathbf{S}\mathbf{Z}^{\mathrm{H}}, \\ &= \mathbf{W}\begin{bmatrix} s_1 & & & & & \\ & \ddots & & & & \\ & & s_{M_c} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}\begin{bmatrix} \mathbf{Z}^c & \mathbf{Z}^r \end{bmatrix}^{\mathrm{H}}. \end{aligned}$$

where $\mathbf{W}$ and $\mathbf{Z}$ are unitary, $\mathbf{Z}^c$ is the partition of $\mathbf{Z}$ (the first M_c columns) containing the signal subspace, and $\mathbf{Z}^r$ is the corresponding nullspace. We require that $(\mathbf{Z}^c)^{\mathrm{H}}\mathbf{U} = (\mathbf{Z}^c)^{\mathrm{H}}$ to satisfy (7), but since $\mathbf{Z}^r$ is in the nullspace of $\mathbf{C}$, we do not care what value $(\mathbf{Z}^r)^{\mathrm{H}}\mathbf{U}$ takes on. Further, if an ambiguous calibration exists then

$$\tilde{\mathbf{A}}^r = \mathbf{A}^r\mathbf{\Sigma}_s^{\frac{1}{2}}\mathbf{U}\mathbf{\Sigma}_s^{-\frac{1}{2}} \neq \mathbf{A}^r, \tag{8}$$

which implies we must require $(\mathbf{Z}^r)^{\mathrm{H}}\mathbf{U} \neq (\mathbf{Z}^r)^{\mathrm{H}}$, since the projection of $\mathbf{U}$ onto the nullspace of $\mathbf{C}$ is the only unconstrained component which can contribute to an ambiguous calibration solution.

As an example of how to construct a suitable $\mathbf{U}$, let

$$\mathbf{U}(\mathbf{T}) = \mathbf{Z}^c(\mathbf{Z}^c)^{\mathrm{H}} + \mathbf{Z}^r\mathbf{T}(\mathbf{Z}^r)^{\mathrm{H}}, \tag{9}$$

where $\mathbf{T}$ is any $(Q-M_c)\times(Q-M_c)$ unitary matrix $\neq \mathbf{I}$. It is easily verified that

$$\begin{aligned} (\mathbf{Z}^c)^{\mathrm{H}}\mathbf{U}(\mathbf{T}) &= (\mathbf{Z}^c)^{\mathrm{H}}, \\ (\mathbf{Z}^r)^{\mathrm{H}}\mathbf{U}(\mathbf{T}) &\neq (\mathbf{Z}^r)^{\mathrm{H}}, \text{ and} \\ \mathbf{U}(\mathbf{T})\mathbf{U}(\mathbf{T})^{\mathrm{H}} &= \mathbf{I}, \end{aligned} \tag{10}$$

which satisfy all the requirements to find a unitary $\mathbf{U}$ which does not rotate the known calibration, $\mathbf{A}^c$, but leads to a distinct solution, $\tilde{\mathbf{G}}^r \neq \mathbf{G}^r$. Substituting (9) and (8) into (5) and exploiting the orthogonality of the subspaces yields

$$\tilde{\mathbf{A}}(\mathbf{T}) = \begin{bmatrix} \mathbf{A}^c \\ \tilde{\mathbf{G}}^r(\mathbf{T}) \odot \mathbf{K}^r \end{bmatrix}, \tag{11}$$

where

$$\tilde{\mathbf{G}}^r(\mathbf{T}) = \left((\mathbf{G}^r \odot \mathbf{K}^r)\mathbf{\Sigma}_s^{\frac{1}{2}}\mathbf{U}(\mathbf{T})\mathbf{\Sigma}_s^{-\frac{1}{2}}\right) \odot (\mathbf{K}^r)^{\odot -1} \tag{12}$$

is a calibration solution for elements outside the known subarray which is different from $\mathbf{G}^r$ but consistent with observed visibilities $\hat{\mathbf{R}}$. Since $\tilde{\mathbf{G}}^r(\mathbf{T})$ is continuously variable with respect to $\mathbf{T}$, we conclude that when $M_c < Q$, the known subarray calibration does not remove the essential ambiguities in solving for calibration, $\mathbf{G}^r$ of the remaining antennas (or LOFAR stations).

Case II, Fewer sources than subarray elements.

In this situation $\mathbf{A}^c$ is a square or tall matrix, i.e. $M_c \geq Q$. Thus $M_c \times Q$ matrix $\mathbf{A}^c$ (assuming linearly independent columns) and $\mathbf{C}$ defined in (7) both have a rank of Q. Thus $\mathbf{CU} = \mathbf{C}$ is only satisfied when $Q \times Q$ matrix $\mathbf{U} = \mathbf{I}$. Therefore there is no ambiguity.

We conclude that the solution for antenna and source dependent calibration gains with a known, calibrated subarray is unique only when the number of antennas in this subarray is equal to or greater than the number of calibration sources. Any attempt to find a solution when these conditions are not satisfied will not succeed without additional constraints on $\mathbf{G}$. Suitable constraints cannot be expressed in the measurement equation or sky model.

3. DETERMINISTIC FREQUENCY DEPENDENCE

Consider an electromagnetic wave propagating through an ionic cloud along a ray path $s_{m,q}$ from the q^{th} source to the m^{th} antenna. The total propagation time is approximated by [4]

$$t_{m,q} \approx \frac{\delta_{m,q}}{c} + \frac{2\pi\,\epsilon^2}{c\,\mu\,\omega^2}\int_0^{\delta_{m,q}} n(s_{m,q})\,ds_{m,q}, \quad (13)$$

where δ is the total distance, c is the speed of light, ω is the signal radian frequency, $n(s)$ is the local density along s expressed as the number of ions per cm^3, ϵ is the ion charge, and μ is its mass. For narrowband observation (13) can be expressed as a wavelength dependent phase delay by substituting $\omega = \frac{2\pi c}{\lambda}$,

$$\begin{aligned} \bar{\phi}_{m,q}(\lambda) &= \frac{2\pi c}{\lambda}\, t_{m,q} \\ &= \frac{2\pi\,\delta_{m,q}}{\lambda} + \lambda\left(\frac{\epsilon^2}{\mu\,c^2}\int n(s_{m,q})\,ds_{m,q}\right) \quad (14) \\ &= \angle k_{m,q} + \lambda\,(\psi_{m,q}). \quad (15) \end{aligned}$$

In (15) we have recognized that the first term in (14) corresponds to the bulk geometric delay phase expressed by elements of $\mathbf{K} = [k_{m,q}]$. $\psi_{m,q} = \frac{\epsilon^2}{\mu\,c^2}\int n(s_{m,q})\,ds_{m,q}$ contains all ionospherically induced unknown calibration terms. These depend on both source direction and sensor position. Note that the corresponding phase term is directly proportional to λ. Thus with this deterministic relationship between λ and the ionospherically induced phase shift, we can use visibilities from several different frequency bands to estimate the common parameter set $\Psi = [\psi_{m,q}]$ in $\mathbf{G}$.

Let $\{\omega_k \mid \forall\, 1 \leq k \leq L\}$ be a set of distinct frequencies where visibilities (covariances) $\hat{\mathbf{R}}_k$ are observed. For each ω_k there is an independent unitary rotation matrix $\mathbf{U}_k$ associated with calibration ambiguity, so

$$\mathbf{R}_k = (\mathbf{G}_k \odot \check{\mathbf{K}}_k)\mathbf{U}_k\mathbf{U}_k^{\text{H}}(\mathbf{G}_k \odot \check{\mathbf{K}}_k)^{\text{H}},$$

where $\check{\mathbf{K}}_k = \mathbf{K}_k\mathbf{\Sigma}_s^{\frac{1}{2}}$. The question is whether (15) can be exploited to constrain all the $\mathbf{U}_k$ and produce a unique solution. To this end we propose the following model

$$\mathbf{G}_k = \mathbf{D}_k(\mathbf{G}_0 \odot \exp(j\lambda_k\Psi)). \quad (16)$$

$\mathbf{D}_k$ is diagonal and represents frequency dependent complex gains that do not follow (15) but which are common across all sources. This could for example arise from direction independent electronic instrumentation calibration terms. Elements of Ψ represent deterministic frequency dependent phase for each source direction as seen by each array element, and $\mathbf{G}_0$ is the corresponding gain magnitude.

Set $k = 1$ and arbitrarily select some unitary $\mathbf{U}_1$. There must exist some $\tilde{\mathbf{D}}_1 \neq \mathbf{D}_1$, $\tilde{\mathbf{G}}_0 \neq \mathbf{G}_0$ and $\tilde{\Psi} \neq \Psi$ such that

$$\begin{aligned} (\mathbf{G}_1 \odot \check{\mathbf{K}}_1)\mathbf{U}_1 &= \tilde{\mathbf{G}}_1 \odot \check{\mathbf{K}}_1 \quad \text{for} \\ \tilde{\mathbf{G}}_1 &= ((\mathbf{G}_1 \odot \check{\mathbf{K}}_1)\mathbf{U}_1) \odot \check{\mathbf{K}}_1^{\odot\,-1} \quad (17) \\ &= \tilde{\mathbf{D}}_1(\tilde{\mathbf{G}}_0 \odot \exp(j\lambda_1\tilde{\Psi})). \quad (18) \end{aligned}$$

For a non trivial $\mathbf{U}_1$, (17) and (18) confine $\tilde{\mathbf{G}}_0$ and $\tilde{\Psi}$ and $\tilde{\mathbf{D}}_1$ to small solution spaces. But (16) requires that given our choice for $\mathbf{U}_1$, all other $\mathbf{U}_k$ and $\tilde{\mathbf{D}}_k$ $k \neq 1$ must then be chosen to satisfy

$$((\mathbf{G}_k \odot \check{\mathbf{K}}_k)\mathbf{U}_k) \odot \check{\mathbf{K}}_k^{\odot\,-1} = \tilde{\mathbf{D}}_k(\tilde{\mathbf{G}}_0 \odot \exp[j\lambda_k\tilde{\Psi}]). \quad (19)$$

Since $\tilde{\mathbf{D}}_k$ is diagonal and $\tilde{\mathbf{G}}_0$ and $\tilde{\Psi}$ are confined to small solutions spaces by (17) and (18), no non trivial choice of $\mathbf{U}_k$ can be found to produce equality. In other words, an arbitrary choice for $\mathbf{U}_1$ forces a violation of (16) for all other frequency bands $\omega_l \neq \omega_k$. This makes the calibration solution unique.

We conclude that if the assumed phase delay dependence on λ is an accurate model, using even a few frequency bands in the calibration can eliminate the ambiguity due to direction dependent gains. The structure of equation (16) can be explicitly included in any selfcal optimization search to improve performance and eliminate ambiguous (incorrect) calibration estimates.

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