Introduction to Classical Shadows

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@Quantum at the Dunes, Natal, Brazil, February 25, 2026

Overview

1. Lecture

1. Overview
2. Basic Notation
3. Pauli measurements
4. Plain State Tomography
5. Measurement Channel
6. Linear inversion

2. Lecture

1. Multi Qubit Measurement Channel
2. Vector t-designs
3. Linear Inversion
4. Observables
5. Classical Shadow Protocol

3. Lecture

1. Complexity Bounds
2. Single Qubit Variances
3. Multi Qubit Variances
4. Sample complexity for local Observables

Reminder: Challenge

Do all the calculations and protocol coding for estimating quadratic functions in rho.

Reward

Invitation to visit our group at JKU with travel expenses to and from Linz being covered.

Deadline: In two weeks.
Send your solution, a short motivational letter and your CV to jadwiga.wilkens@jku.at.

Collection of material, Q&A

https://pad.fridaysforfuture.is/p/natal26-intro-classical-shadows

Classical Shadow Protocol

Single-shot estimator

$$\hat{\rho}$$

Multi-shot estimator

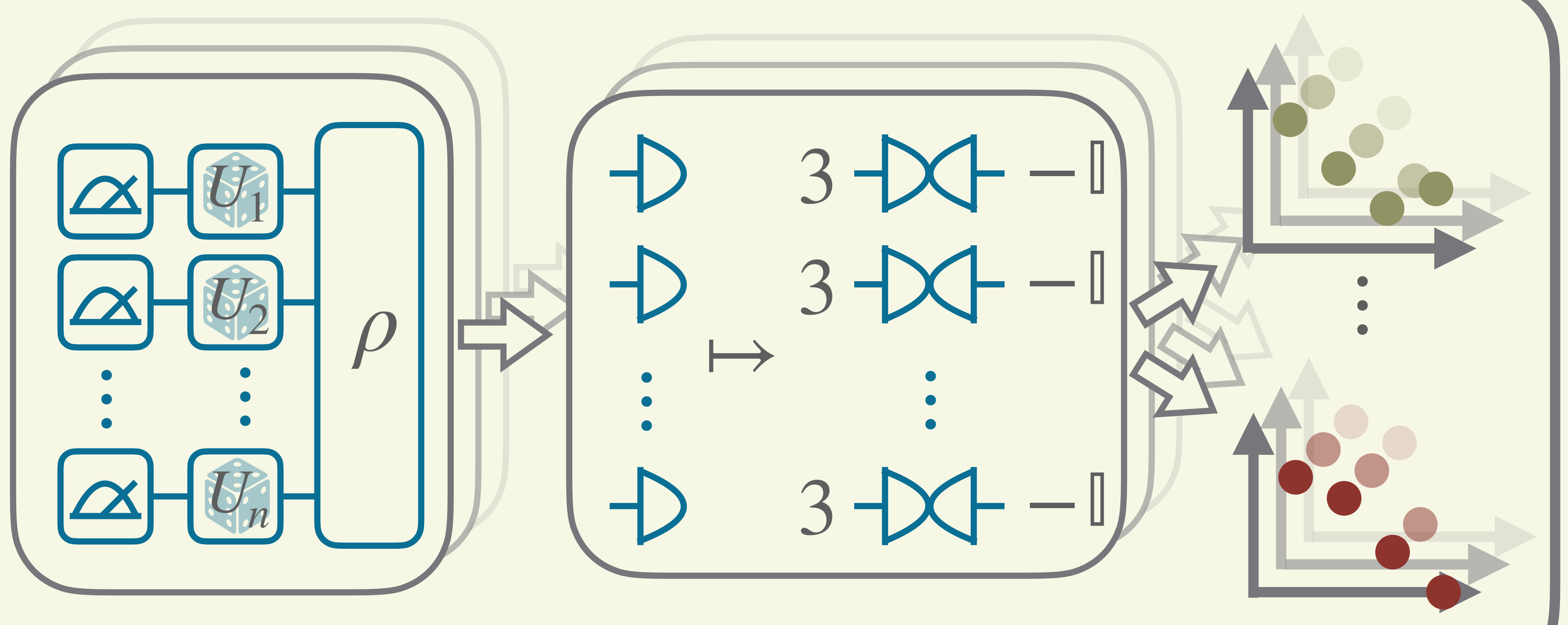
$$\bar{\rho} = \frac{1}{N} \sum_{t=1}^{N} \hat{\rho}_t$$

Expected value

$$\rho$$

$$(U, b) \rightarrow \hat{\rho} = \mathscr{M}^{-1}(U^\dagger | b \rangle\langle b | U)$$

$\frac{1}{T} \sum_{t=1}^{N} \hat{\rho}_t$ for $T \rightarrow \infty$: $\mathbb{E}(\hat{\rho}) = \rho$.



Single-shot estimator

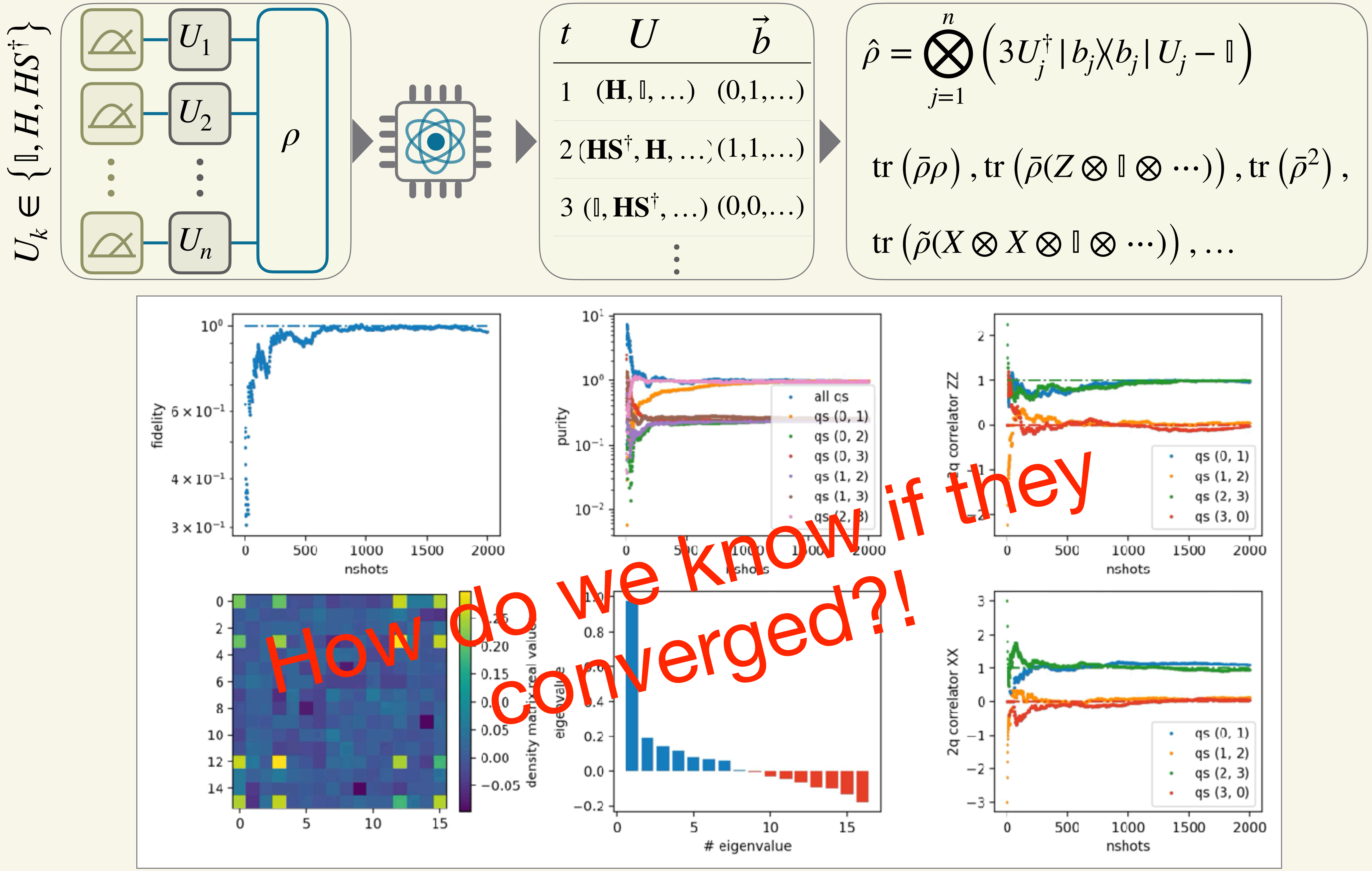
$$\hat{o} = \mathrm{tr}(O\hat{\rho})$$

Multi-shot estimator

$$\bar{o} = \frac{1}{N} \sum_{t=1}^{N} \hat{o}_t$$

Expected value

$$o = \mathrm{tr}(O\rho)$$



$U_k \in \{\mathbb{1}, H, HS^\dagger\}$
U_1
U_2
U_n
ρ
t U $\vec{b}$
1 $(\mathbf{H}, \mathbb{1}, \ldots)$ $(0,1,\ldots)$
2 $(\mathbf{HS}^\dagger, \mathbf{H}, \ldots)$ $(1,1,\ldots)$
3 $(\mathbb{1}, \mathbf{HS}^\dagger, \ldots)$ $(0,0,\ldots)$
$\hat{\rho} = \bigotimes_{j=1}^{n} \left(3U_j^\dagger |b_j\rangle\langle b_j| U_j - \mathbb{1}\right)$
$\mathrm{tr}\left(\bar{\rho}\rho\right), \mathrm{tr}\left(\bar{\rho}(Z \otimes \mathbb{1} \otimes \cdots)\right), \mathrm{tr}\left(\bar{\rho}^2\right),$
$\mathrm{tr}\left(\tilde{\rho}(X \otimes X \otimes \mathbb{1} \otimes \cdots)\right), \ldots$
fidelity
nshots
purity
all cs
qs (0, 1)
qs (0, 2)
qs (0, 3)
qs (1, 2)
qs (1, 3)
qs (2, 3)
2q correlator ZZ
qs (3, 0)
density matrix real value
eigenvalue
eigenvalue
2q correlator XX
How do we know if they converged?!

Complexity Bounds

Complexity Bounds

Absolute deviation $\epsilon < 0$

$$\Pr\left[\left|\frac{1}{N}\sum_{t=1}^{N}\hat{o}_t - \mathrm{tr}(O\rho)\right| \geq \epsilon\right] \leq f(N) \text{ with } f(N) = ??$$

Complexity Bounds

Example: Biased Coin

Single toss: $\hat{x}$

$$\Pr[\hat{x} = \text{tails}] = p$$

$$\Pr[\hat{x} = \text{heads}] = 1 - p$$

$$\mathbb{E}(\hat{x}) = \mu = p$$

$$\text{Var}(\hat{x}) = p(1 - p)$$

Multi-tosses:

$$\bar{x} = \bar{\mu} = \frac{1}{N} \sum_{t=1}^{N} \hat{x}_t$$

$$\mathbb{E}(\bar{\mu}) = \mu$$

$$\text{Var}(\bar{\mu}) = \frac{p(1 - p)}{N}$$

Single toss: $\hat{x}$

$\Pr[\hat{x} = \text{tails}] = p$

$\Pr[\hat{x} = \text{heads}] = 1 - p$

$\mathbb{E}(\hat{x}) = \mu = p$

Multi-toss: $\bar{x} = \bar{\mu} = \frac{1}{N} \sum_{t=1}^{N} \hat{x}_t$

Complexity Bounds

Single toss: $\hat{x}$

$\Pr[\hat{x} = \text{tails}] = p$

$\Pr[\hat{x} = \text{heads}] = 1 - p$

$\mathbb{E}(\hat{x}) = \mu = p$

Multi-toss: $\bar{x} = \bar{\mu} = \frac{1}{N}\sum_{t=1}^{N} \hat{x}_t$

Complexity Bounds

Markov's inequality

For every constant $a > 0$:

$$\Pr\left[\hat{x} \geq a\right] \leq \frac{\mathbb{E}[\hat{x}]}{a}.$$

Single toss: $\hat{x}$

$\Pr[\hat{x} = \text{tails}] = p$

$\Pr[\hat{x} = \text{heads}] = 1 - p$

$\mathbb{E}(\hat{x}) = \mu = p$

Multi-toss: $\bar{x} = \bar{\mu} = \frac{1}{N}\sum_{t=1}^{N} \hat{x}_t$

Complexity Bounds

Chebychev's inequality

with iid:

$$\mathrm{Var}(\bar{\mu}) = \frac{\mathrm{Var}(\hat{x})}{N}$$

For every constant $\epsilon > 0$:

$$\Pr\left[\left|\bar{\mu} - \mu\right| \geq \epsilon\right] \leq \frac{\mathrm{Var}[\bar{\mu}]}{\epsilon^2}.$$

Single toss: $\hat{x}$

$\Pr[\hat{x} = \text{tails}] = p$

$\Pr[\hat{x} = \text{heads}] = 1 - p$

$\mathbb{E}(\hat{x}) = \mu = p$

Multi-toss: $\bar{x} = \bar{\mu} = \frac{1}{N} \sum_{t=1}^{N} \hat{x}_t$

Complexity Bounds

Chebychev's inequality

with iid:

$$\mathrm{Var}(\bar{\mu}) = \frac{\mathrm{Var}(\hat{x})}{N}$$

For every constant $\epsilon > 0$:

$$\Pr\left[\left|\bar{\mu} - \mu\right| \geq \epsilon\right] \leq \frac{\mathrm{Var}[\bar{\mu}]}{\epsilon^2}.$$

With confidence $1 - \delta$:

Sample complexity: $N \geq \dfrac{\mathrm{Var}[\hat{x}]}{\epsilon^2 \delta}$

Single toss: $\hat{x}$

$\Pr[\hat{x} = \text{tails}] = p$

$\Pr[\hat{x} = \text{heads}] = 1 - p$

$\mathbb{E}(\hat{x}) = \mu = p$

Multi-toss: $\bar{x} = \bar{\mu} = \frac{1}{N} \sum_{t=1}^{N} \hat{x}_t$

Complexity Bounds

Hoeffding's inequality

For every constant $\epsilon > 0$:

$$\Pr\left[\left|\bar{\mu} - \mu\right| \geq \epsilon\right] \leq 2\exp\left(-\frac{2N\epsilon^2}{c}\right)$$

With confidence $1 - \delta$:

Sample complexity: $N \geq \dfrac{c}{2\epsilon^2} \log \dfrac{2}{\delta}$

** Only for iid*

$a \leq \hat{x} \leq b$

$c = b - a$

Single toss: $\hat{x}$

$\Pr[\hat{x} = \text{tails}] = p$

$\Pr[\hat{x} = \text{heads}] = 1 - p$

$\mathbb{E}(\hat{x}) = \mu = p$

Multi-toss: $\bar{x} = \bar{\mu} = \frac{1}{N}\sum_{t=1}^{N} \hat{x}_t$

Complexity Bounds

Bernstein inequality

* *Only for iid*

$$R \geq |\hat{x} - \mathbb{E}(\hat{x})|$$

$$\sigma^2 \geq \mathrm{Var}(\hat{x})$$

$$C \leq 2R$$

For every constant $\epsilon > 0$:

$$\Pr\left[\left|\bar{\mu} - \mu\right| \geq \epsilon\right] \leq \begin{cases} 2\exp\left(-\dfrac{3N\epsilon^2}{8\sigma^2}\right) & \text{if } \epsilon \leq \dfrac{\sigma^2}{R}, \\ 2\exp\left(-\dfrac{3N\epsilon}{8R}\right) & \text{if } \epsilon \geq \dfrac{\sigma^2}{R}. \end{cases}$$

With confidence $1 - \delta$: $N \geq \dfrac{8\sigma^2}{3\epsilon^2}\log\dfrac{2}{\delta}$

Complexity Bounds

Bound	Uses	Decay	Sharpness
Markov	Mean only	—	Very weak
Chebyshev	Variance	1/N	Weak
Hoeffding	Boundedness + iid	exp(-N)	Strong
Bernstein	Variance + bounded + iid	exp(-N)	Strongest

Complexity Bounds

Union Bound

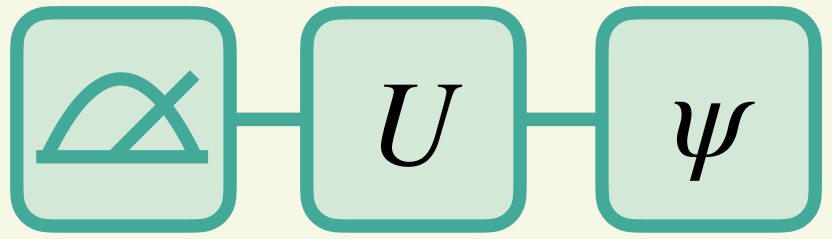
For every constant $\epsilon > 0$ and M many properties to estimate:

$$\Pr\left[\bigcup_{i=1}^{M} \left|\bar{\mu}_i - \mu_i\right| \geq \epsilon\right] \leq \sum_{i=1}^{M} \Pr\left[\left|\bar{\mu}_i - \mu_i\right| \geq \epsilon\right] \leq M \max_{0 \leq i \leq M} \Pr\left[\left|\bar{\mu}_i - \mu_i\right| \geq \epsilon\right]$$

With Bernstein: $N \leq \frac{8\sigma_{\max}}{3\epsilon^2} \log \frac{2}{\delta} \log M$

Classical Shadows

Single qubit



$U \in \mathbb{C}(2)$

$$\mathcal{M}(\rho) = \mathcal{D}_{1/3}(\rho)$$

$$\mathcal{M}^{-1} = \mathcal{D}_3$$

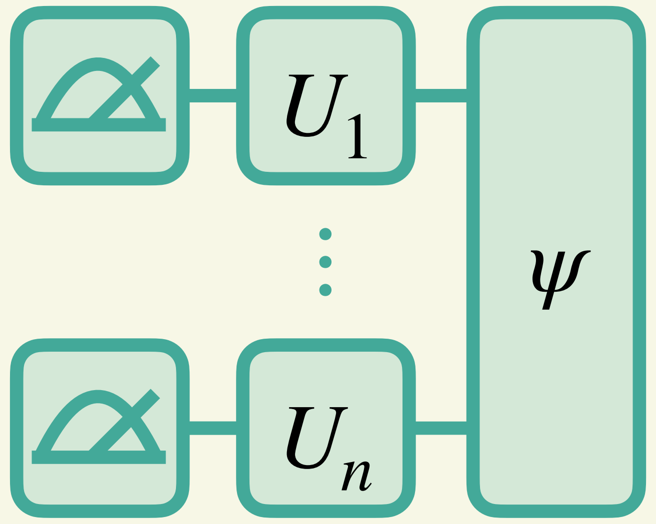
$$\hat{\rho} = 3U^\dagger |b\rangle\langle b| U - \mathbb{1}$$

Variance

Sample complexity

Multi-qubit local Cliffords

$U_k \in \mathbb{C}(2)$



$$\mathcal{M}(\rho) = \mathcal{D}_{1/3}^{\otimes n}(\rho)$$

$$\mathcal{M}^{-1} = \mathcal{D}_3^{\otimes n}$$

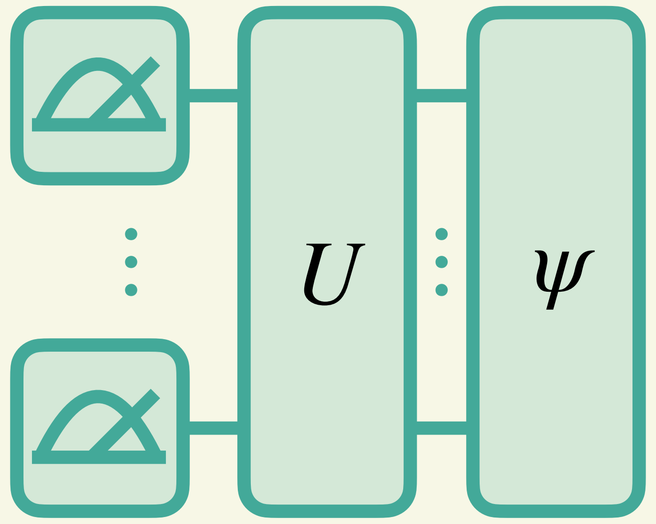
$$\hat{\rho} = \bigotimes_{j=1}^{n} \left(3U_j^\dagger |b_j\rangle\langle b_j| U_j - \mathbb{1} \right)$$

Variance

Sample complexity

Multi-qubit global Cliffords

$U \in \mathbb{C}(2^n)$



$$\mathcal{M}(\rho) = \mathcal{D}_{1/(2^n+1)}(\rho)$$

$$\mathcal{M}^{-1} = \mathcal{D}_{2^n+1}$$

$$\hat{\rho} = (2^n + 1)U^\dagger |b\rangle\langle b| U - \mathbb{1}^{\otimes n}$$

Variance

Sample complexity

Recipe: Bernstein

1. Calculate upper bound for variance σ^2
2. Calculate upper bound for magnitude R
3. Calculate σ^2/R to bound ϵ
4. State sample complexity $N \geq \dfrac{8\sigma^2}{3\epsilon^2} \log \dfrac{2}{\delta}$

Sample complexity Single Qubit

Observable estimation

1. Calculate upper bound for variance σ^2
2. Calculate upper bound for magnitude R
3. Calculate σ^2/R to bound ϵ
4. State sample complexity $N \geq \dfrac{8\sigma^2}{3\epsilon^2} \log \dfrac{2}{\delta}$

Measurement Channel

$$\mathcal{M}(\rho) = \frac{1}{|\mathbb{C}(2^n)|} \sum_{\substack{U \in \mathbb{C}(2^n) \\ b \in \{0,1\}^n}} \mathrm{tr}(U^\dagger | b \rangle\langle b | U\rho) U^\dagger | b \rangle\langle b | U$$

The measurement channel is self adjoint:

$$\mathrm{tr}(\mathcal{M}^{-1}(A)B) = \mathrm{tr}(A\mathcal{M}^{-1}(B))$$

Sample complexity Single Qubit

Observable estimation

1. Calculate upper bound for variance σ^2
2. Calculate upper bound for magnitude R
3. Calculate σ^2/R to bound ϵ
4. State sample complexity $N \geq \frac{8\sigma^2}{3\epsilon^2} \log \frac{2}{\delta}$

Sample complexity Single Qubit

Observable estimation

1. $\sigma^2 = 3\mathrm{tr}(O^2)$
2. $R = 4\mathrm{tr}(O^2)$
3. $\epsilon \leq \sigma^2/R = 3/4$
4. $N \geq \dfrac{8\mathrm{tr}(O^2)}{\epsilon^2} \log \dfrac{2}{\delta}$

Sample complexity Single Qubit

Observable estimation

For 1 Observable and
$\sigma^2 = 3\mathrm{tr}(O^2)$ and $R = 4\mathrm{tr}(O^2)$,
and therefore an absolute deviation $\epsilon \in \left(0, \frac{3}{4}\right)$

For $N \geq \frac{8\mathrm{tr}(O^2)}{\epsilon^2} \log \frac{2}{\delta}$

Sample complexity Single Qubit

Observable estimation

For M many Observables and
$\sigma^2 = 3\mathrm{tr}(O^2)$ and $R = 4\mathrm{tr}(O^2)$,
and therefore an absolute deviation $\epsilon \in \left(0, \frac{3}{4}\right)$:

For $N \geq \frac{8\mathrm{tr}(O_{\max}^2)}{\epsilon^2} \log \frac{2}{\delta} \log M$

Sample complexity Mutli Qubit (Global)

Observable estimation

With $\sigma^2 = 3\mathrm{tr}(O^2)$ and $R = (2^n + 1)\mathrm{tr}(O^2)$,

and therefore an absolute deviation $\epsilon \in \left(0, \frac{3}{(2^n+1)}\right)$:

Exponential in n?!

For $N \geq \frac{8\mathrm{tr}(O^2)}{\epsilon^2} \log \frac{2}{\delta}$

Sample complexity Mutli Qubit (Global)

Observable estimation: Median of Means

With K independent sample means,
each of them using T samples:

$$K \geq 2 \log \frac{2M}{\delta} \text{ and } T \geq \frac{34}{\epsilon^2} 3\mathrm{tr}(O_{\max}^2)$$

$$N \geq \mathcal{O}\left(\frac{\log M}{\epsilon^2}\mathrm{tr}(O_{\max}^2)\right)$$

Sample complexity Multi Qubit (local)

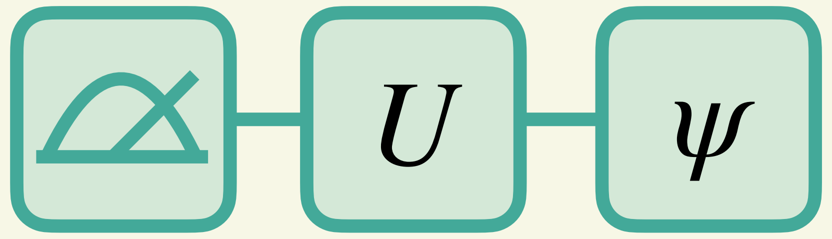
k-local Observable estimation

$$\sigma^2 = 4^k \mathrm{tr}(O^2)$$

$$N \leq \frac{8}{3} \frac{4^k \mathrm{tr}(O^2)}{\epsilon^2} \log \frac{2}{\delta}$$

Classical Shadows

Single qubit



$U \in \mathbb{C}(2)$

$$\mathcal{M}(\rho) = \mathcal{D}_{1/3}(\rho)$$

$$\mathcal{M}^{-1} = \mathcal{D}_3$$

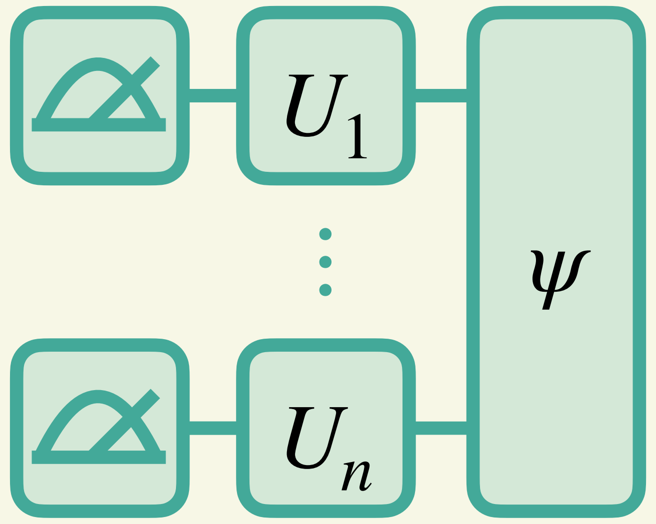
$$\hat{\rho} = 3U^\dagger |b\rangle\langle b| U - \mathbb{I}$$

$$\mathrm{Var}(\hat{o}) \leq 3\mathrm{tr}(O^2)$$

$$N \geq \frac{8\mathrm{tr}(O_{\max}^2)}{\epsilon^2} \log\frac{2}{\delta} \log M$$

Multi-qubit local Cliffords

$U_k \in \mathbb{C}(2)$



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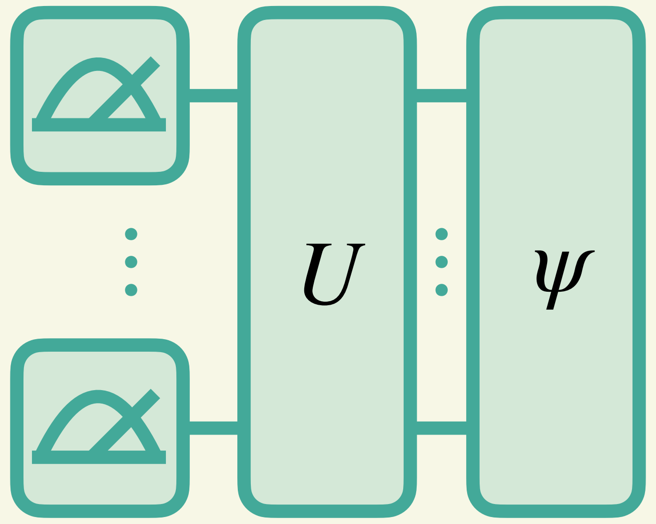
$$\mathrm{Var}(\hat{o}) \leq 4^k \mathrm{tr}(O^2)$$

$$N \leq \frac{8}{3} \frac{4^k \mathrm{tr}(O_{\max}^2)}{\epsilon^2} \log\frac{2}{\delta} \log M$$

Multi-qubit global Cliffords

$U \in \mathbb{C}(2^n)$



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$$\mathcal{M}^{-1} = \mathcal{D}_{2^n+1}$$

$$\hat{\rho} = (2^n + 1)U^\dagger |b\rangle\langle b| U - \mathbb{I}^{\otimes n}$$

$$\mathrm{Var}(\hat{o}) \leq 3\mathrm{tr}(O^2)$$

$$N \geq \mathcal{O}\left(\frac{\log M}{\epsilon^2} \mathrm{tr}(O_{\max}^2)\right)$$

Thank you for your attention!