Introduction to Classical Shadows

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@Quantum at the Dunes, Natal, Brazil, February 23, 2026

JOHANNES KEPLER UNIVERSITÄT LINZ

Quantum Information & Computation @ Kepler


JKU
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QUANTUM AT
THE DUNES

Challenge

Do all the calculations and protocol coding for estimating quadratic functions in rho.

Reward

Invitation to visit our group at JKU with travel expenses to and from Linz being covered.

Deadline: In two weeks. Send us your solution, a short motivational letter and your CV.

Active poll

0

Join at

slido.com

#2361 874

What is your primary background?

Theoretical physics

0%

Experimental physics

0%

Computer science

0%

Mathematics

0%

Engineering

0%

Other

Overview

1. Lecture

1. Overview
2. Basic Notation
3. Pauli measurements
4. Plain State Tomography
5. Measurement Channel
6. Linear inversion

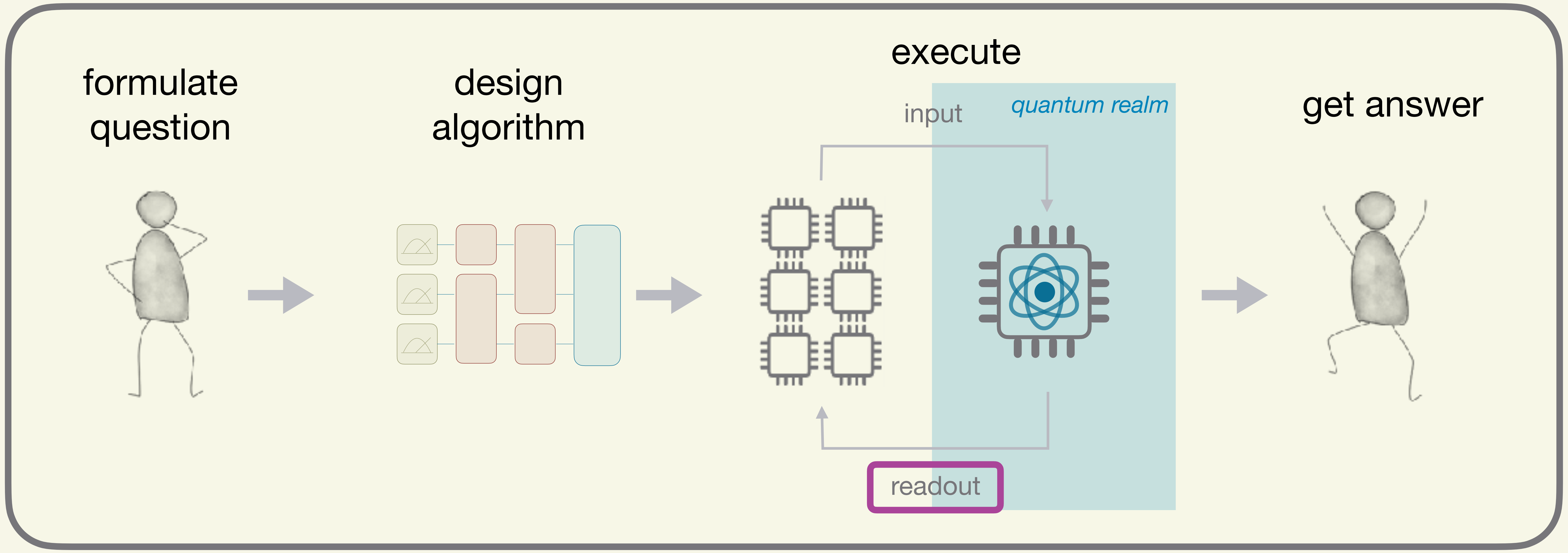
2. Lecture

1. Multi Qubit Measurement Channel
2. Vector t-designs
3. Linear Inversion
4. Observables
5. Classical Shadow Protocol

3. Lecture

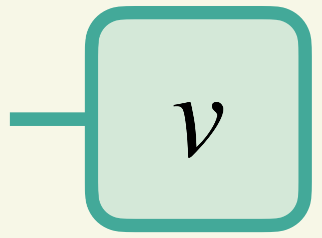
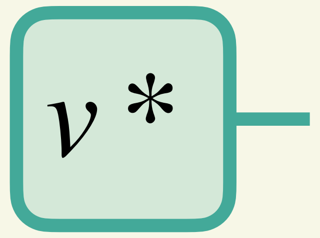
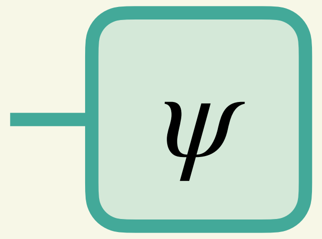
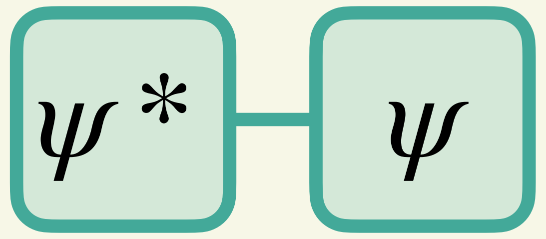
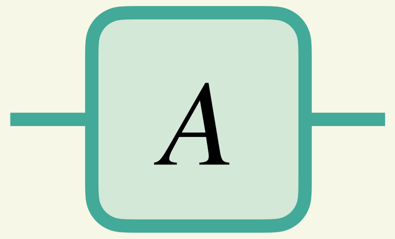
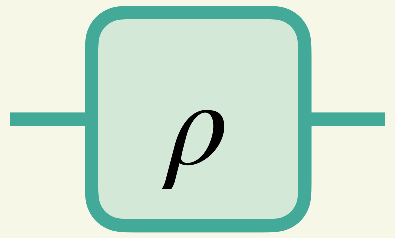
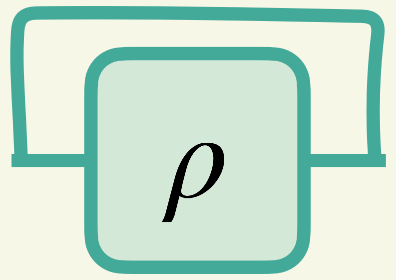
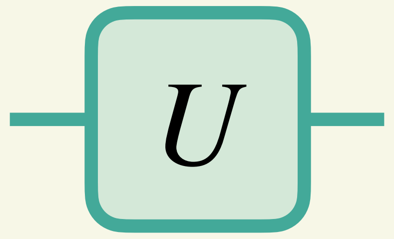
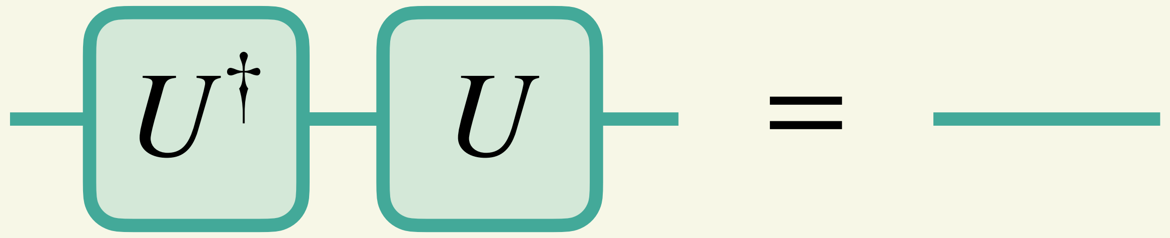
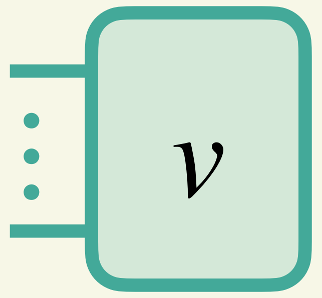
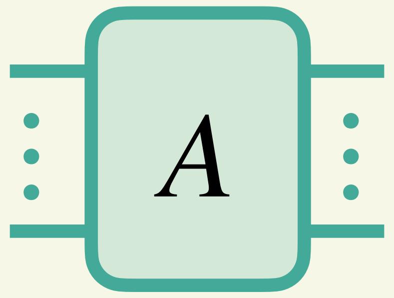
1. Complexity Bounds
2. Single Qubit Variances
3. Multi Qubit Variances
4. Sample complexity for local Observables

Overview



question/algorithm	answer
Finding ground states	$\mathrm{tr}\left(\rho H\right)$
Characterizing spin-spin correlations	$\mathrm{tr}\left(\rho(Z\otimes Z\ldots)\right)$
Evaluate fidelity to target state	$\mathrm{tr}\left(\rho\sigma\right)$
Certify entanglement/ Reny entropy	$\log\mathrm{tr}\left(\rho^2\right)$
Factoring large integers (Shor's algorithm)	Single shot measurement after QFT
Sampling from quantum state (Quantum machine learning)	Samples themselves
⋮	⋮

Basic Notations

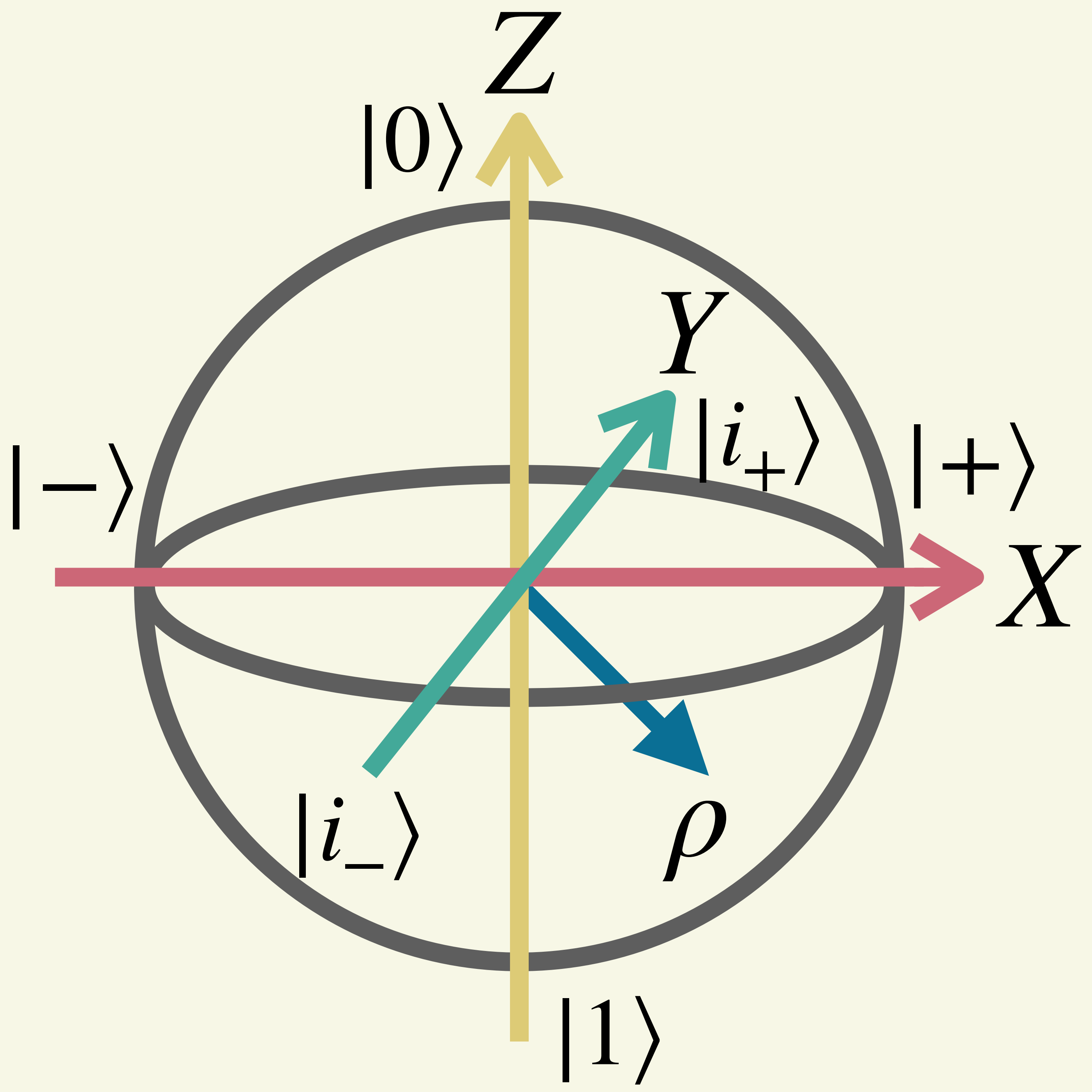
	Algebraic expression	Lives in	Diagram	Condition	Diagram Examples
Vector	$\lvert v\rangle$	$\in \mathscr{H}$			
Dual Vector	$\langle v \rvert$	$\in \mathscr{H}^*$			
Pure states	$\lvert \psi\rangle$	$\in \mathscr{H}$		$\langle \psi \vert \psi \rangle = 1$	 $= 1$
Operators	A	$\in L\left(\mathscr{H}\right)$			
Density Operator	ρ	$\in L\left(\mathscr{H}\right)$		$\rho = \rho^\dagger,\ \rho \geq 0,$ $\mathrm{tr}(\rho) = 1$	 $= 1$
Gates	U	$\in L\left(\mathscr{H}\right)$		$UU^\dagger = \mathbb{I}$	
Composite System: Vector	$\lvert v\rangle$	$\in \mathscr{H}^{\otimes n}$			
Composite System: operator	A	$\in L(\mathscr{H}^{\otimes n})$			

Blochsphere

Pauli Operators

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$



Single qubit

$$\mathbb{P} = \{X, Y, Z, \mathbb{1}\}$$

$$\rho = \frac{1}{2} \sum_{P \in \mathbb{P}} \mathrm{tr}(\rho P) P$$

Multi qubit

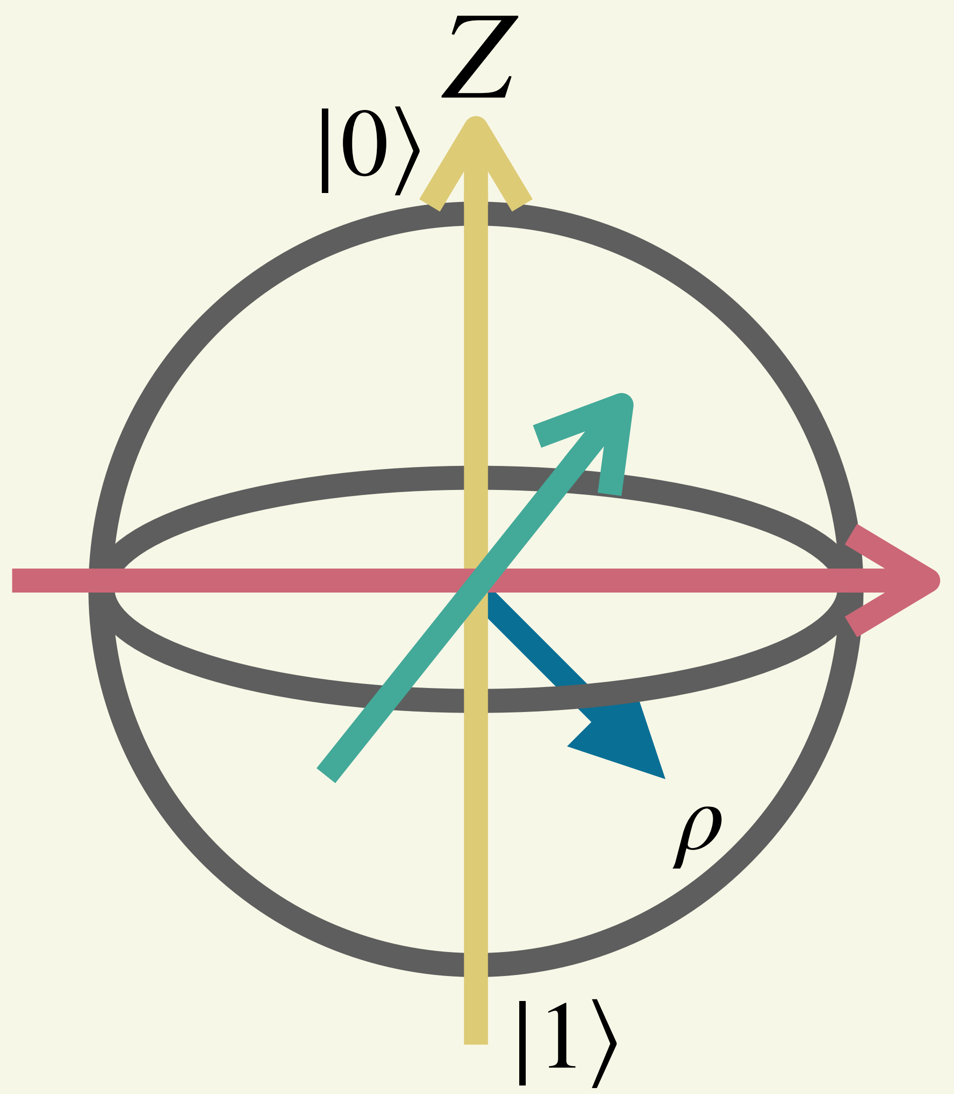
$$\mathbb{P}^n = \{X, Y, Z, \mathbb{1}\}^{\otimes n}$$

$$P = P_1 \otimes P_2 \otimes \cdots \otimes P_n$$

for $P_k \in \mathbb{P}$

$$\rho = \frac{1}{2^n} \sum_{P \in \mathbb{P}^n} \mathrm{tr}(\rho P) P$$

Measurements



$$\Pr_\rho\left[b=0\right] = \mathrm{tr}\left[\rho\,|0\rangle\langle 0|\right] = \langle 0|\rho|0\rangle$$

$$\Pr_\rho\left[b=1\right] = \mathrm{tr}\left[\rho\,|1\rangle\langle 1|\right] = \langle 1|\rho|1\rangle$$

$$\mathrm{tr}(\rho Z) = \langle 0|\rho|0\rangle - \langle 1|\rho|1\rangle = 2\langle 0|\rho|0\rangle - 1$$

Post-measurement state

$|0\rangle\langle 0|$

$|1\rangle\langle 1|$

Clifford Operators

Hadamard gate:

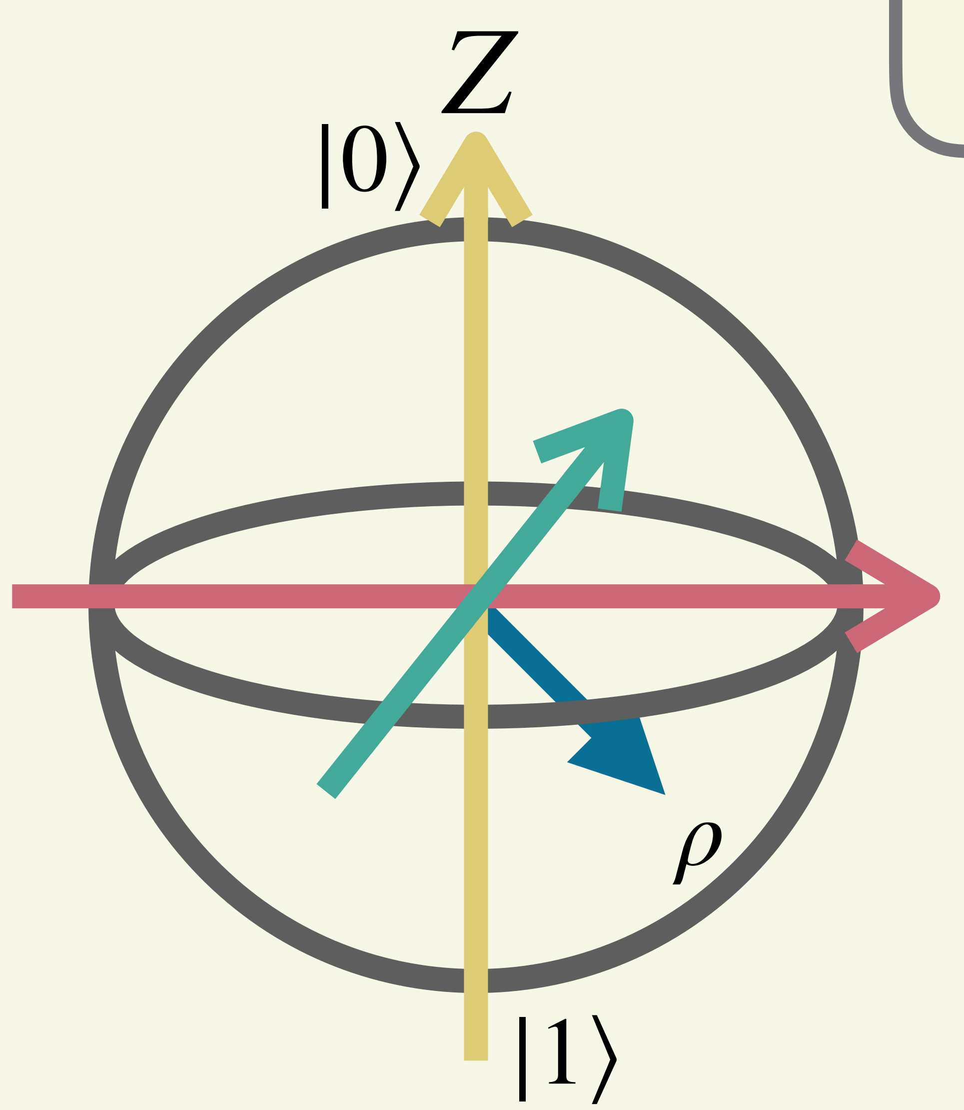
$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Phase gate:

$$S = \sqrt{Z} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}$$

Single qubit: They map Paulis to Paulis

$$\mathbb{C}(2) = \left\{ U \in U(2) \,|\, UPU^\dagger = P' \,\forall\, P, P' \in \mathbb{P} \right\}$$



$$\mathrm{tr}(U\rho U^\dagger Z) = \mathrm{tr}(\rho U^\dagger Z U)$$

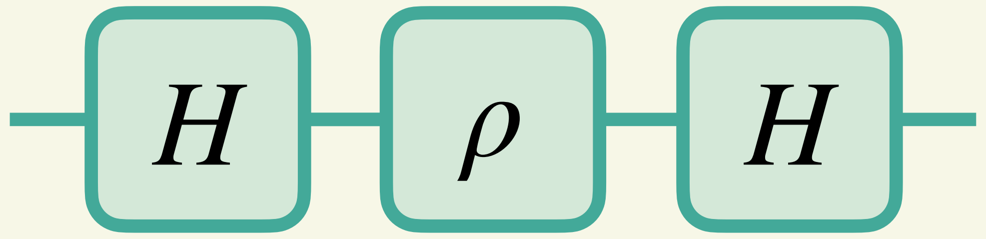
$$\mathrm{tr}(\rho HZH) = \mathrm{tr}(\rho X)$$

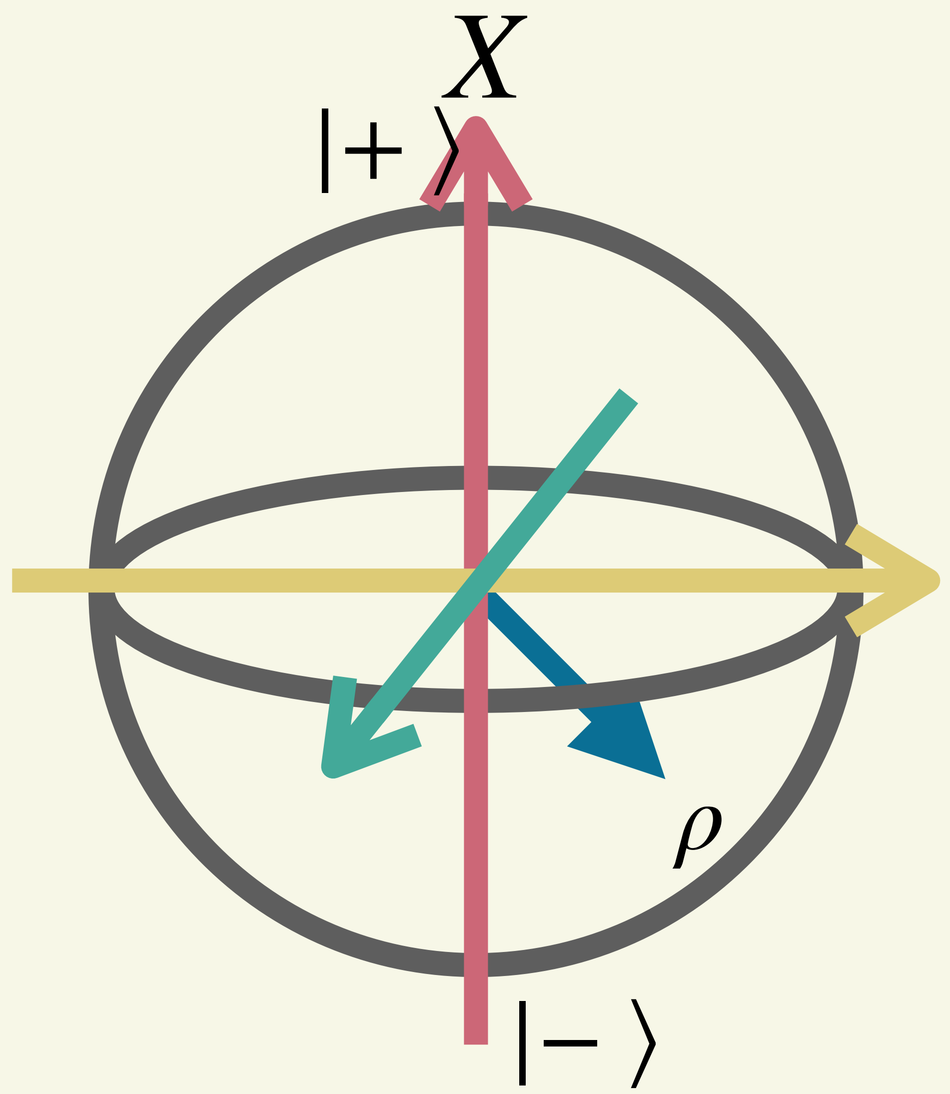
$$\mathrm{tr}(\rho SHZHS^\dagger) = \mathrm{tr}(\rho Y)$$

$UZU^\dagger \simeq Z$	$UZU^\dagger \simeq X$	$UZU^\dagger \simeq Y$
$\mathbb{I}$	H	SH
S	HS	SHS
SS	HSS	$SHSS$
SSS	$HSSS$	$SHSSS$
$HSSH$	SSH	$SSSH$
$HSSHS$	$SSHS$	$SSSHS$
$HSSHSS$	$SSHSS$	$SSSHSS$
$HSSHSSS$	$SSHSSS$	$SSSHSSS$

Global: They map Paulis to Paulis

$$\mathbb{C}(2^n) = \left\{ U \in U(2^n) \,|\, UPU^\dagger = P' \,\forall\, P, P' \in \mathbb{P}^n \right\}$$





$\mathbb{U} = \{\mathbb{I}, H, HS^\dagger\}$

Subset of $\mathbb{C}(2)$

Measurements

$$\Pr_\rho[b=0] = \mathrm{tr}\left[\rho\,|0\rangle\langle 0|\right] = \langle 0|\rho|0\rangle$$

$$\Pr_\rho[b=1] = \mathrm{tr}\left[\rho\,|1\rangle\langle 1|\right] = \langle 1|\rho|1\rangle$$

$$\mathrm{tr}(\rho Z) = \langle 0|\rho|0\rangle - \langle 1|\rho|1\rangle = 2\langle 0|\rho|0\rangle - 1$$

$$\Pr_{H\rho H}[b=0] = \mathrm{tr}\left[H\rho H\,|0\rangle\langle 0|\right] = \langle +|\rho|+\rangle$$

$$\Pr_{H\rho H}[b=1] = \mathrm{tr}\left[H\rho H\,|1\rangle\langle 1|\right] = \langle -|\rho|-\rangle$$

$$\mathrm{tr}(\rho X) = 2\langle +|\rho|+\rangle - 1$$

Post-measurement state

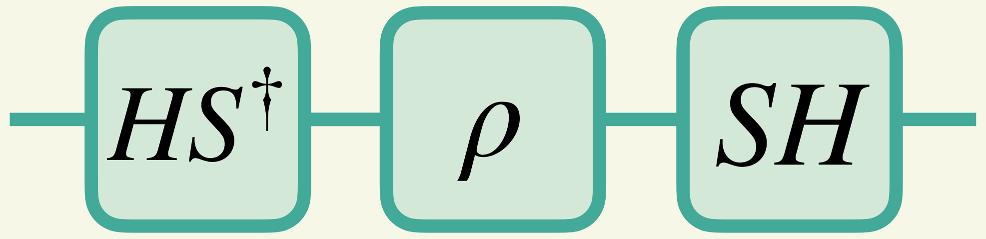
$|0\rangle\langle 0|$

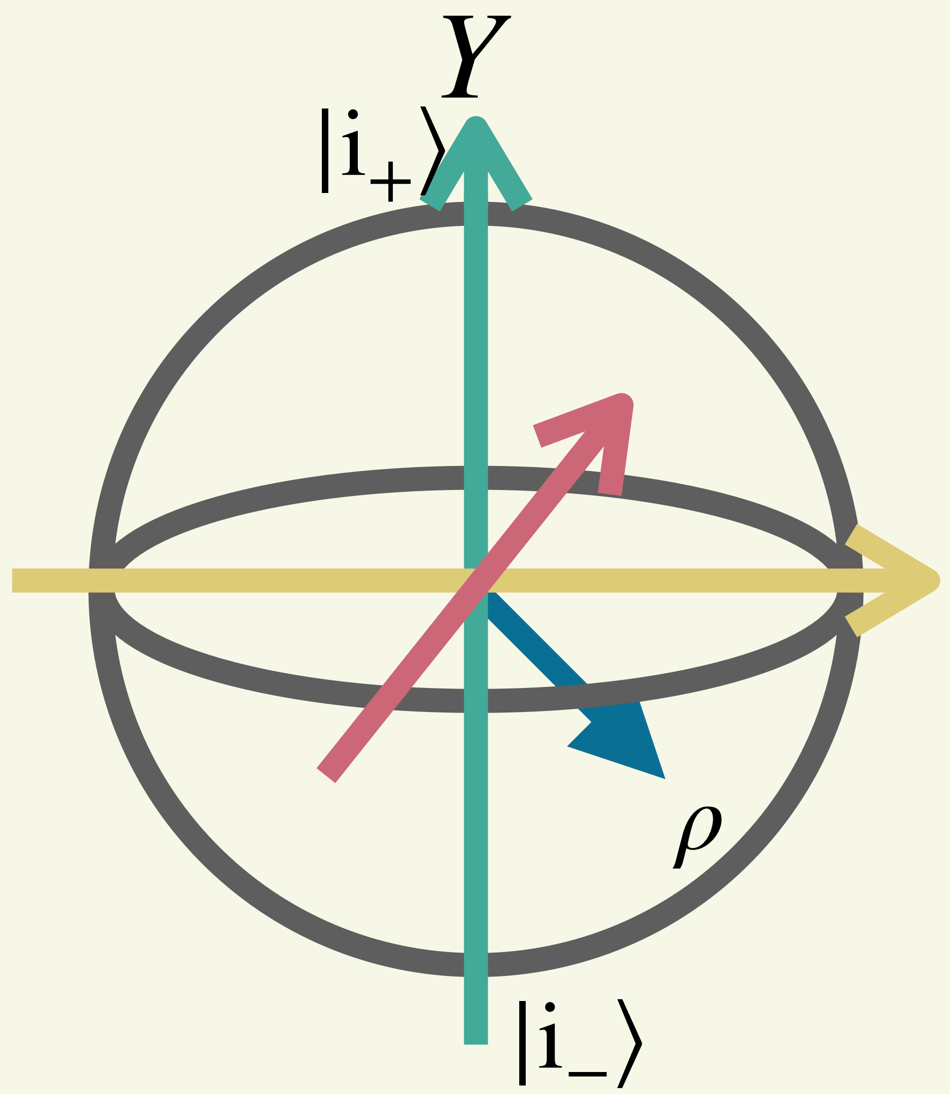
$|1\rangle\langle 1|$

$H|0\rangle\langle 0|H$

$H|1\rangle\langle 1|H$

Measurements





$\mathbb{U} = \{\mathbb{I}, H, HS^\dagger\}$
Subset of $\mathbb{C}(2)$

$$\Pr_\rho[b=0] = \mathrm{tr}\left[\rho|0\rangle\langle 0|\right] = \langle 0|\rho|0\rangle$$
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$$\Pr_{H\rho H}[b=0] = \mathrm{tr}\left[H\rho H|0\rangle\langle 0|\right] = \langle +|\rho|+\rangle$$
$$\Pr_{H\rho H}[b=1] = \mathrm{tr}\left[H\rho H|1\rangle\langle 1|\right] = \langle -|\rho|-\rangle$$
$$\mathrm{tr}(\rho X) = 2\langle +|\rho|+\rangle - 1$$

$$\Pr_{HS^\dagger\rho SH}[b=0] = \mathrm{tr}\left[HS^\dagger\rho SH|0\rangle\langle 0|\right] = \langle i_+|\rho|i_+\rangle$$
$$\Pr_{HS^\dagger\rho SH}[b=1] = \mathrm{tr}\left[HS^\dagger\rho SH|1\rangle\langle 1|\right] = \langle i_-|\rho|i_-\rangle$$
$$\mathrm{tr}(\rho Y) = 2\langle i_+|\rho|i_+\rangle - 1$$

Post-measurement state

$|0\rangle\langle 0|$
$|1\rangle\langle 1|$

$H|0\rangle\langle 0|H$
$H|1\rangle\langle 1|H$

$SH|0\rangle\langle 0|HS^\dagger$
$SH|1\rangle\langle 1|HS^\dagger$

Quantum State Tomography

Single qubit

Data acquisition

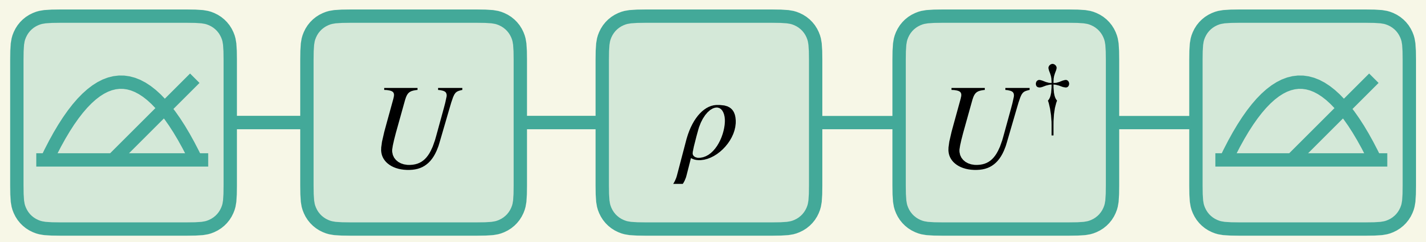
1. For $U \in \mathbb{U}$:

    1. Prepare ρ

    2. Apply rotation $U\rho U^\dagger$

    3. Perform computational measurement

    4. Repeat N times



Post-Processing

1. Compute $\widetilde{\mathrm{tr}(\rho Z)}$, $\widetilde{\mathrm{tr}(\rho X)}$, $\widetilde{\mathrm{tr}(\rho Y)}$

2. Reconstruct $\tilde{\rho} = \frac{1}{2} \sum_{P \in \mathbb{P}} \widetilde{\mathrm{tr}(\rho P)} P$

3. Project $\tilde{\rho}$ onto set of valid quantum states

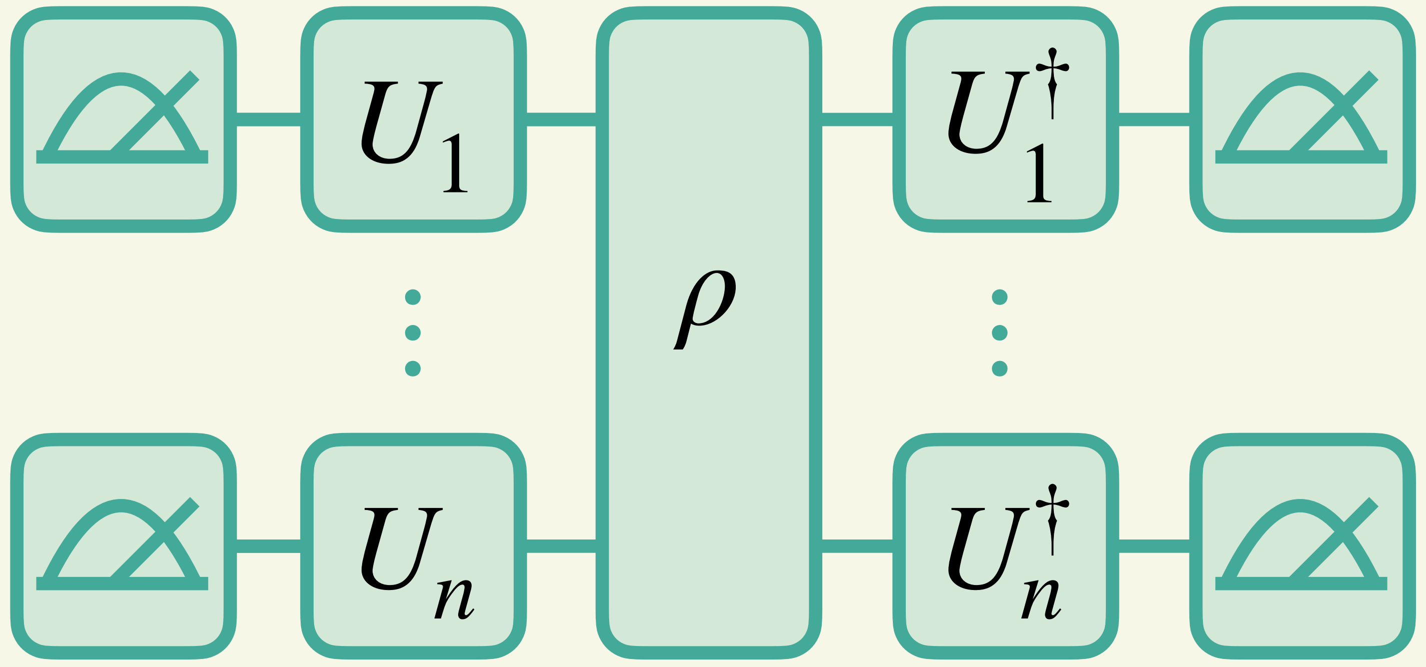
Estimate $\mathrm{tr}(O\tilde{\rho}), \ldots$

Quantum State Tomography

Multi qubit

Data acquisition

1. For $U \in \mathbb{U}^n$:
    1. Prepare ρ
    2. Apply rotation $U\rho U^\dagger$
    3. Perform computational measurement
    4. Repeat N times



Post-Processing

1. Compute $\widetilde{\mathrm{tr}(\rho Z)}$, $\widetilde{\mathrm{tr}(\rho X)}$, $\widetilde{\mathrm{tr}(\rho Y)}$
2. Reconstruct $\tilde{\rho} = \frac{1}{2^n} \sum_{P \in \mathbb{P}^n} \widetilde{\mathrm{tr}(\rho P)} P$
3. Project $\tilde{\rho}$ onto set of valid quantum states

Estimate $\mathrm{tr}(O\tilde{\rho}), \ldots$

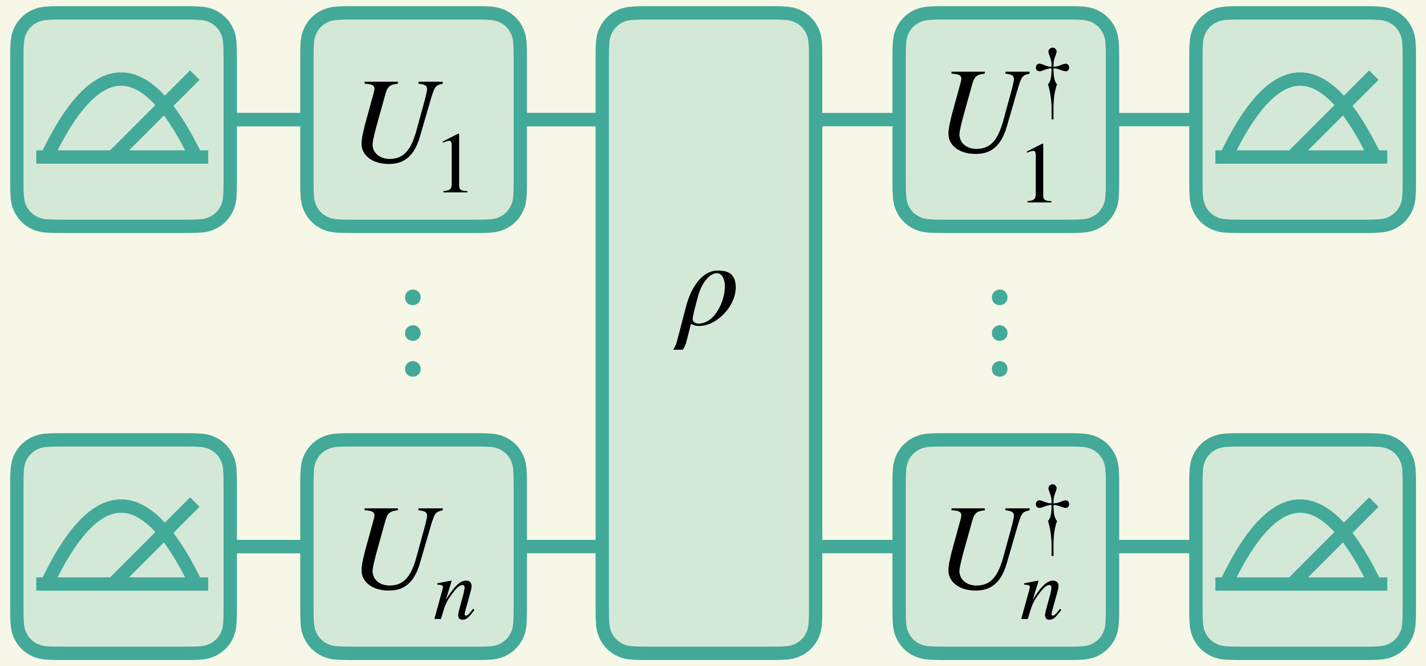
Quantum State Tomography

Multi qubit

Data acquisition

1. For $U \in \mathbb{U}^n$:
   1. Prepare ρ
   2. Apply rotation $U\rho U^\dagger$
   3. Perform computational measurement
   4. Repeat N times

3^n Pauli words



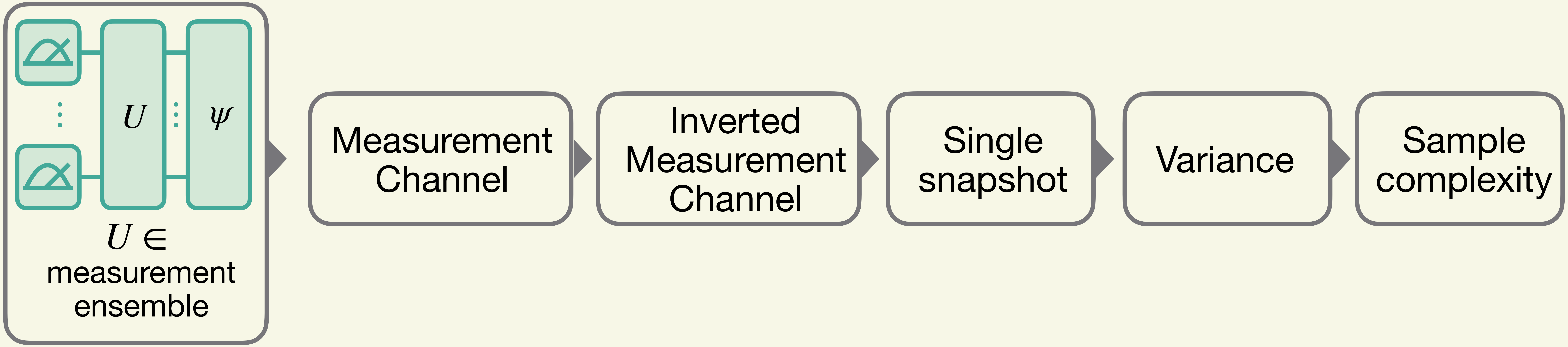
Post-Processing

1. Compute $\widetilde{\mathrm{tr}(\rho Z)}, \widetilde{\mathrm{tr}(\rho X)}, \widetilde{\mathrm{tr}(\rho Y)}$
2. Reconstruct $\tilde{\rho} = \frac{1}{2^n} \sum_{P \in \mathbb{P}^n} \widetilde{\mathrm{tr}(\rho P)} P$
3. Project $\tilde{\rho}$ onto set of valid quantum states

$2^n \times 2^n$ complex values

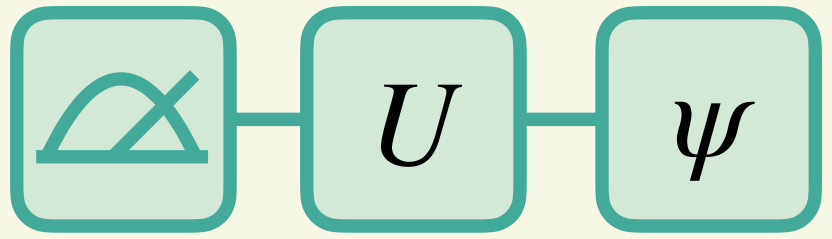
Estimate $\mathrm{tr}(O\tilde{\rho}), \ldots$

Outlook: Classical Shadows



Classical Shadows

Single qubit



$U \in \mathbb{C}(2)$

Measurement Channel

Inverted Measurement Channel

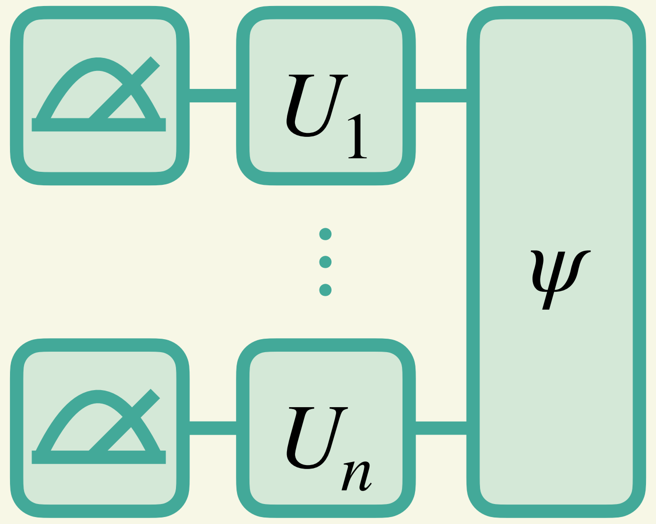
Single Snapshot

Variance

Sample complexity

Multi-qubit local Cliffords

$U_k \in \mathbb{C}(2)$



Measurement Channel

Inverted Measurement Channel

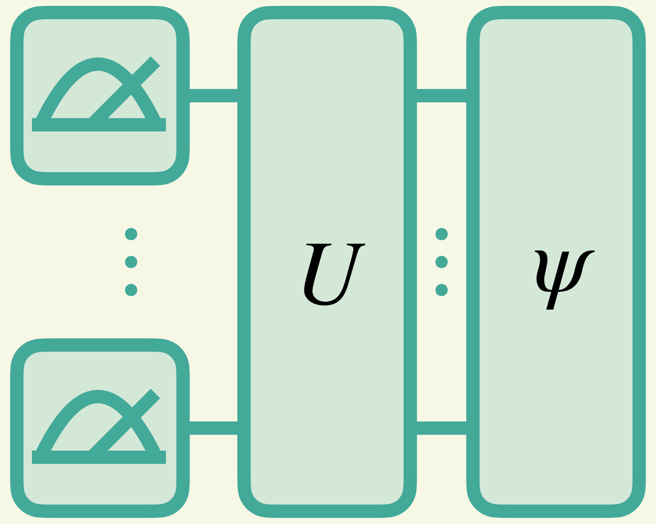
Single Snapshot

Variance

Sample complexity

Multi-qubit global Cliffords

$U \in \mathbb{C}(2^n)$



Measurement Channel

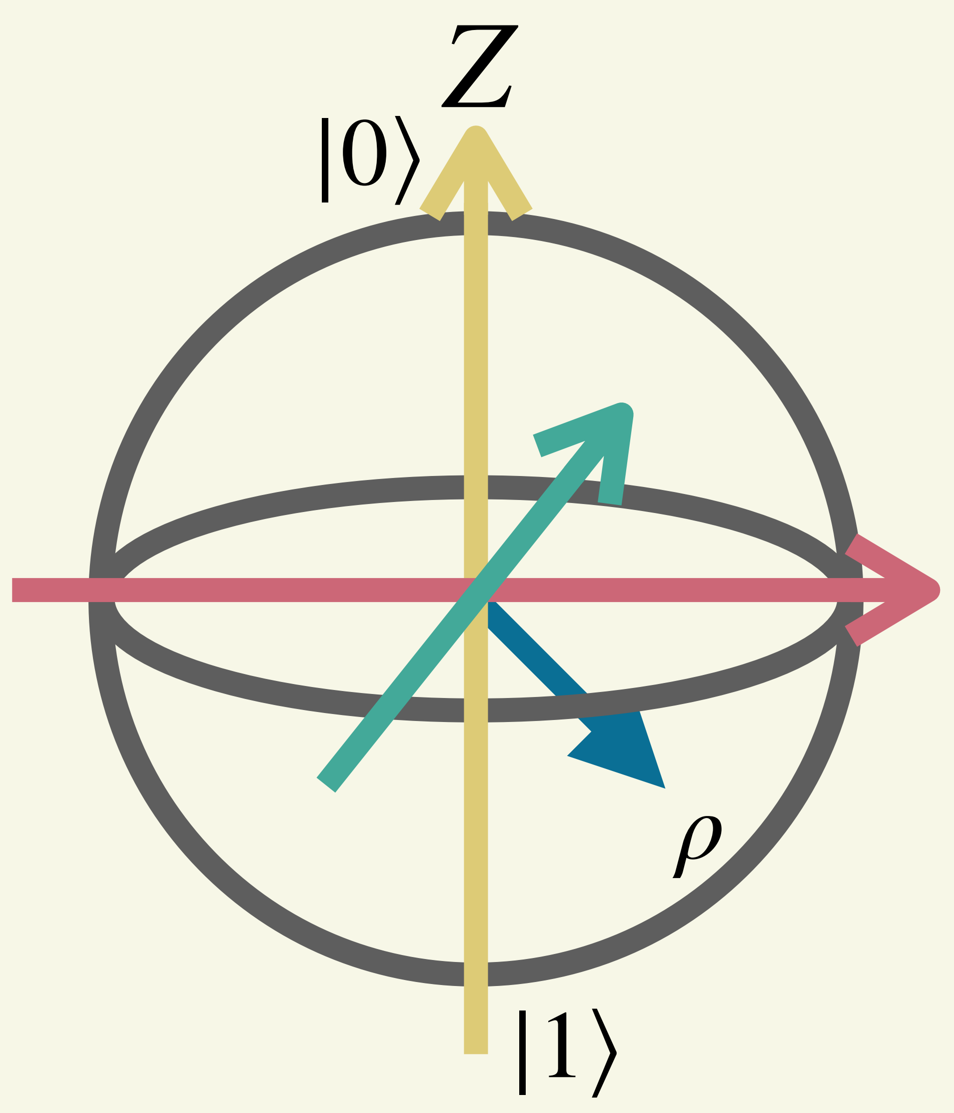
Inverted Measurement Channel

Single Snapshot

Variance

Sample complexity

Measurements



$\mathbb{U} = \{\mathbb{I}, H, HS^\dagger\}$

Subset of $\mathbb{C}$

$$\Pr_\rho\left[b=0\right] = \mathrm{tr}\left[\rho\,|0\rangle\!\langle 0|\right] = \langle 0|\rho|0\rangle$$

$$\Pr_\rho\left[b=1\right] = \mathrm{tr}\left[\rho\,|1\rangle\!\langle 1|\right] = \langle 1|\rho|1\rangle$$

$$\mathscr{M}_Z(\rho) = \langle 0|\rho|0\rangle|0\rangle\!\langle 0| + \langle 1|\rho|1\rangle|1\rangle\!\langle 1| = \begin{pmatrix}\rho_{00} & 0\\ 0 & \rho_{11}\end{pmatrix}$$

$$\Pr_{H\rho H}\left[b=0\right] = \mathrm{tr}\left[H\rho H\,|0\rangle\!\langle 0|\right] = \langle +|\rho|+\rangle$$

$$\Pr_{H\rho H}\left[b=1\right] = \mathrm{tr}\left[H\rho H\,|1\rangle\!\langle 1|\right] = \langle -|\rho|-\rangle$$

$$\mathscr{M}_X(\rho) = \langle +|\rho|+\rangle|+\rangle\!\langle +| + \langle -|\rho|-\rangle|-\rangle\!\langle -|$$

$$\Pr_{HS^\dagger\rho SH}\left[b=0\right] = \mathrm{tr}\left[HS^\dagger\rho SH\,|0\rangle\!\langle 0|\right] = \langle \mathrm{i}_+|\rho|\mathrm{i}_+\rangle$$

$$\Pr_{HS^\dagger\rho SH}\left[b=1\right] = \mathrm{tr}\left[HS^\dagger\rho SH\,|1\rangle\!\langle 1|\right] = \langle \mathrm{i}_-|\rho|\mathrm{i}_-\rangle$$

$$\mathscr{M}_Y(\rho) = \langle \mathrm{i}_+|\rho|\mathrm{i}_+\rangle|\mathrm{i}_+\rangle\!\langle \mathrm{i}_+| + \langle \mathrm{i}_-|\rho|\mathrm{i}_-\rangle|\mathrm{i}_-\rangle\!\langle \mathrm{i}_-|$$

Post-measurement state

$|0\rangle\!\langle 0|$

$|1\rangle\!\langle 1|$

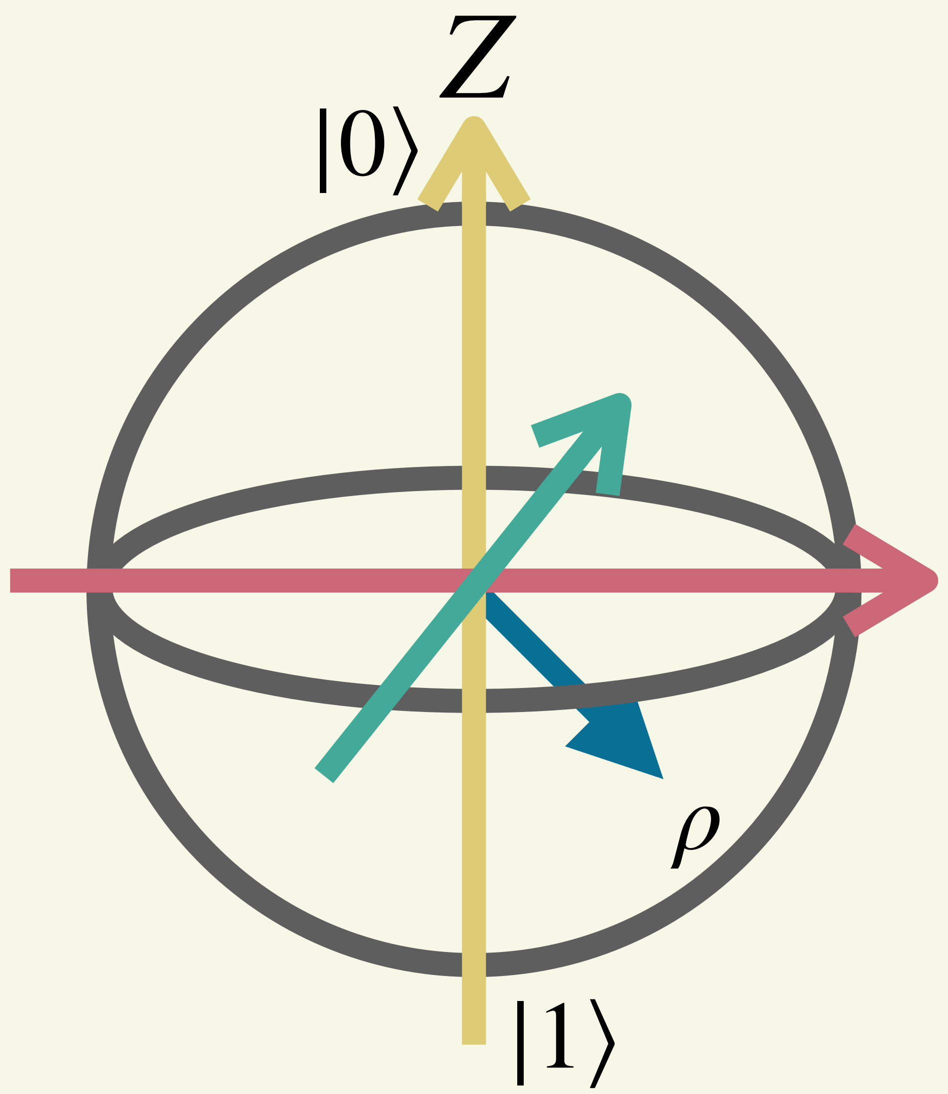
$H|0\rangle\!\langle 0|H$

$H|1\rangle\!\langle 1|H$

$SH|0\rangle\!\langle 0|HS^\dagger$

$SH|1\rangle\!\langle 1|HS^\dagger$

Single Qubit Measurement Channel



$$\mathbb{U} = \{\mathbb{I}, H, HS^\dagger\}$$

Subset of $\mathbb{C}$

$$\mathscr{M}(\rho) = \frac{1}{3}\left(\mathscr{M}_Z(\rho) + \mathscr{M}_X(\rho) + \mathscr{M}_Y(\rho)\right)$$

$$\mathscr{M}(\rho) = \frac{1}{3}\sum_{\substack{U \in \mathbb{U} \\ b \in \{0,1\}}} \mathrm{tr}(U^\dagger |b\rangle\langle b| U\rho) U^\dagger |b\rangle\langle b| U$$

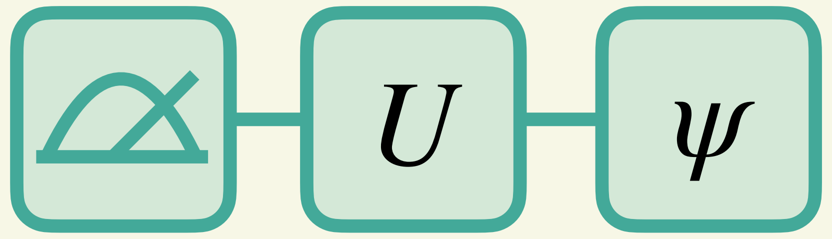
$$\mathscr{M}(\rho) = \mathscr{D}_{1/3}(\rho)$$

For depolarizing parameter $0 \leq p \leq 1$

$$\mathscr{D}_p(\rho) = p\rho + (1-p)\mathrm{tr}(\rho)\frac{\mathbb{I}^{\otimes n}}{2^n}$$

Classical Shadows

Single qubit



$U \in \mathbb{C}(2)$

$\mathscr{M}(\rho) = \mathscr{D}_{1/3}(\rho)$

Inverted Measurement Channel

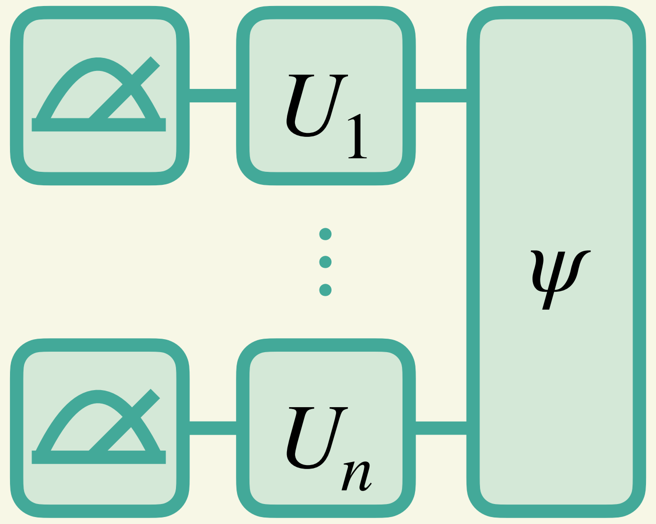
Single Snapshot

Variance

Sample complexity

Multi-qubit local Cliffords

$U_k \in \mathbb{C}(2)$



Measurement Channel

Inverted Measurement Channel

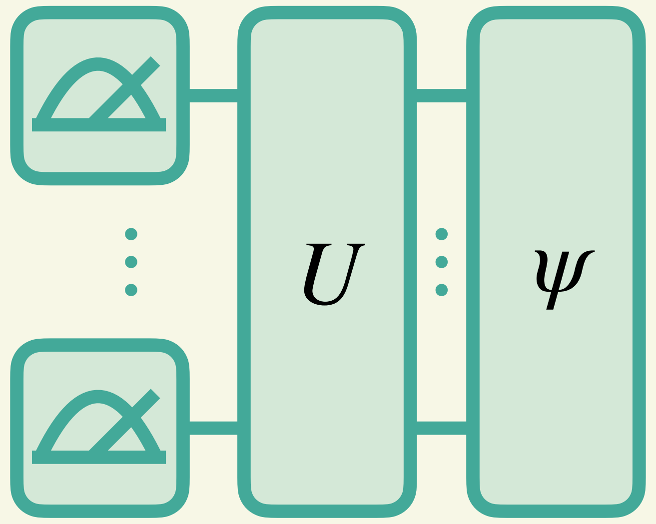
Single Snapshot

Variance

Sample complexity

Multi-qubit global Cliffords

$U \in \mathbb{C}(2^n)$



Measurement Channel

Inverted Measurement Channel

Single Snapshot

Variance

Sample complexity

Linear Inversion

$$\mathscr{M}^{-1} \circ \mathscr{M}(\rho) = \rho$$

$$(\mathscr{D}_p)^{-1} \circ \mathscr{D}_p(\rho) = \rho$$

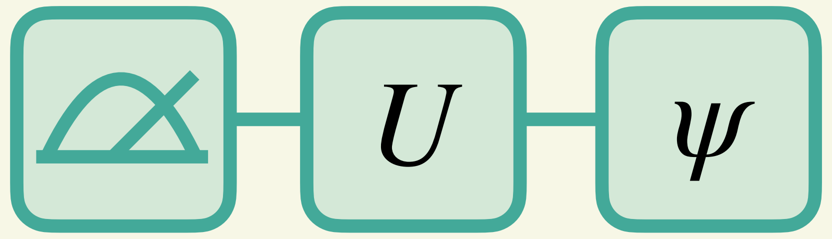
$$\mathscr{D}_{p'} \circ \mathscr{D}_p(\rho) = \rho$$

$$p' = \frac{1}{p}$$

$$\mathscr{D}_p^{-1}(\rho) = \mathscr{D}_{p^{-1}}(\rho)$$

Classical Shadows

Single qubit



$U \in \mathbb{C}(2)$

$\mathscr{M}(\rho) = \mathscr{D}_{1/3}(\rho)$

Inverted Measurement Channel

Single Snapshot

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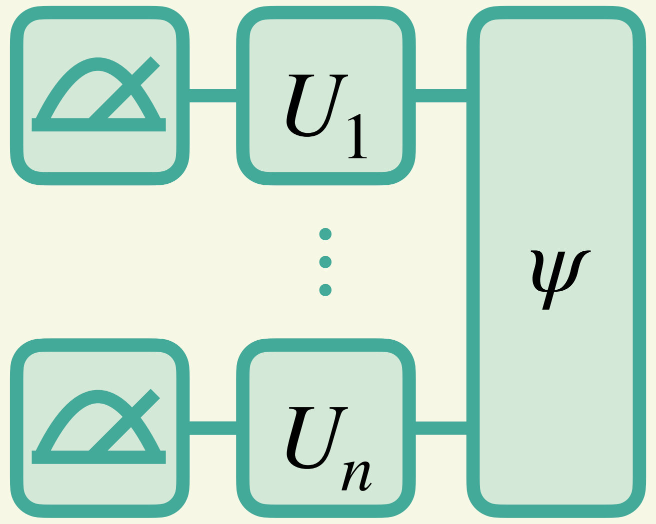

Measurement Channel

Inverted Measurement Channel

Single Snapshot

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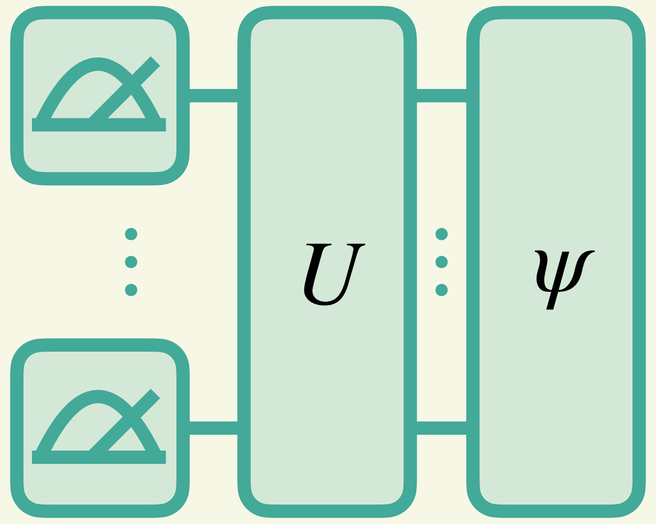
Multi-qubit global Cliffords

$U \in \mathbb{C}(2^n)$



Measurement Channel

Inverted Measurement Channel

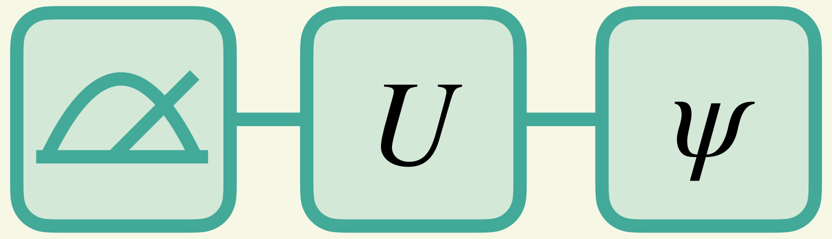
Single Snapshot

Variance

Sample complexity

Classical Shadows

Single qubit



$U \in \mathbb{C}(2)$

$\mathscr{M}(\rho) = \mathscr{D}_{1/3}(\rho)$

$\mathscr{M}^{-1} = \mathscr{D}_3$

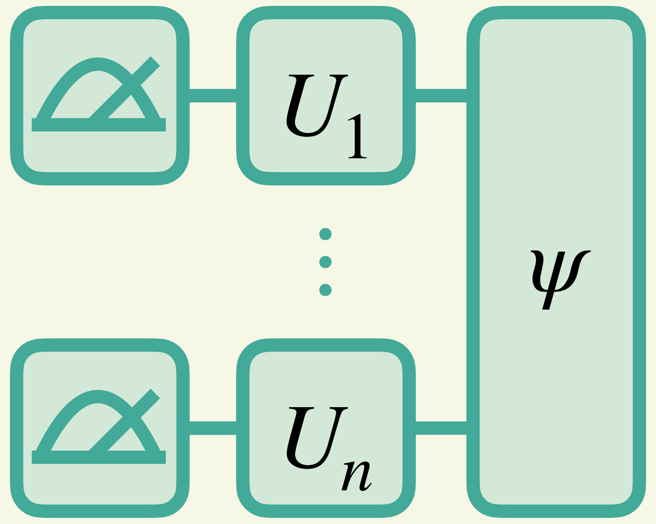
Single Snapshot

Variance

Sample complexity

Multi-qubit local Cliffords

$U_k \in \mathbb{C}(2)$



Measurement Channel

Inverted Measurement Channel

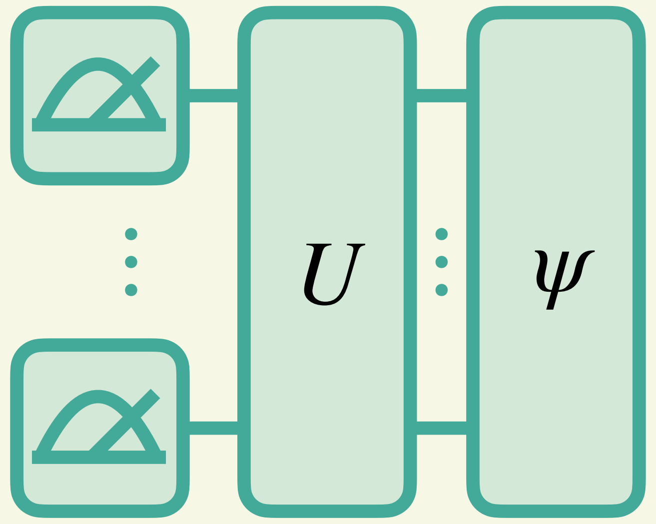
Single Snapshot

Variance

Sample complexity

Multi-qubit global Cliffords

$U \in \mathbb{C}(2^n)$



Measurement Channel

Inverted Measurement Channel

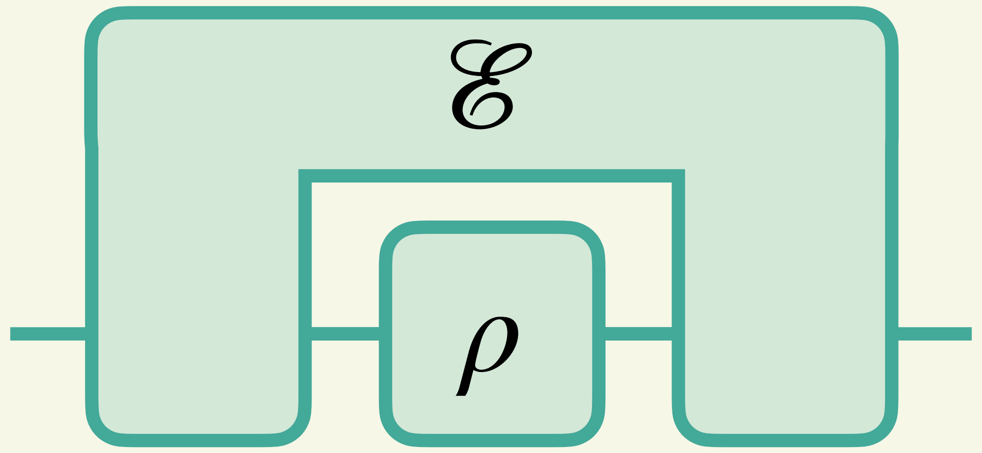
Single Snapshot

Variance

Sample complexity

Bonus: Channel representation

Channel $\mathscr{E}(\rho) = \rho'$



Pauli-Liouville Channel Representation:

$$(\mathscr{E})_{i,j} = \frac{1}{2}\mathrm{tr}(P_i \mathscr{E}(P_j)) \text{ with } P_i, P_j \in \mathbb{P}$$

Active poll

0

What was the hardest concept to understand?

Pauli basis decomposition
0%

Projective measurements
0%

Measurement channel
0%

State tomography protocol
0%

SWAP trick in proofs
0%

Concept of quantum channels
0%

Linear inversion

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Thank you for your attention!