Introduction to Classical Shadows

Jadwiga Wilkens

@Quantum at the Dunes, Natal, Brazil, February 23, 2026

Overview

1. Lecture

1. Overview
2. Basic Notation
3. Pauli measurements
4. Plain State Tomography
5. Measurement Channel
6. Linear inversion

2. Lecture

1. Multi Qubit Measurement Channel
2. Vector t-designs
3. Linear Inversion
4. Observables
5. Classical Shadow Protocol

3. Lecture

1. Complexity Bounds
2. Single Qubit Variances
3. Multi Qubit Variances
4. Sample complexity for local Observables

Reminder: Challenge

Do all the calculations and protocol coding for estimating quadratic functions in rho.

Reward

Invitation to visit our group at JKU with travel expenses to and from Linz being covered.

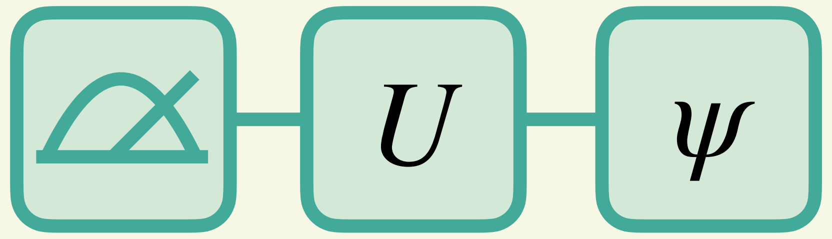
Deadline: In two weeks.
Send your solution, a short motivational letter and your CV to jadwiga.wilkens@jku.at.

Collection of material + Q&A

https://pad.fridaysforfuture.is/p/natal26-intro-classical-shadows

Classical Shadows

Single qubit



$U \in \mathbb{C}(2)$

$\mathscr{M}(\rho) = \mathscr{D}_{1/3}(\rho)$

$\mathscr{M}^{-1} = \mathscr{D}_3$

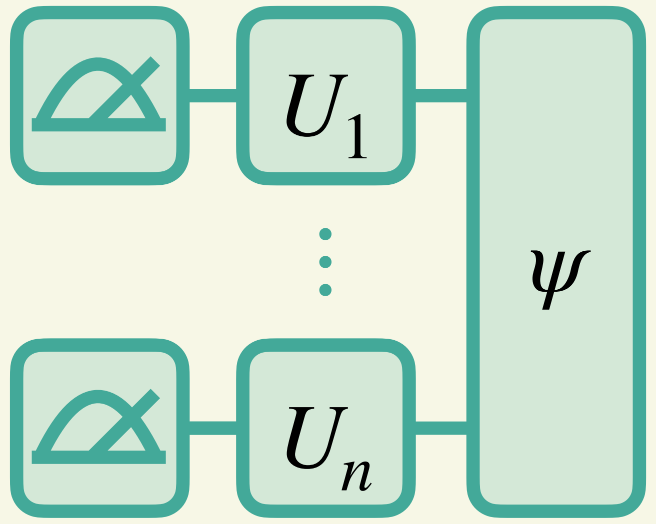
Single Snapshot

Variance

Sample complexity

Multi-qubit local Cliffords

$U_k \in \mathbb{C}(2)$



Measurement Channel

Inverted Measurement Channel

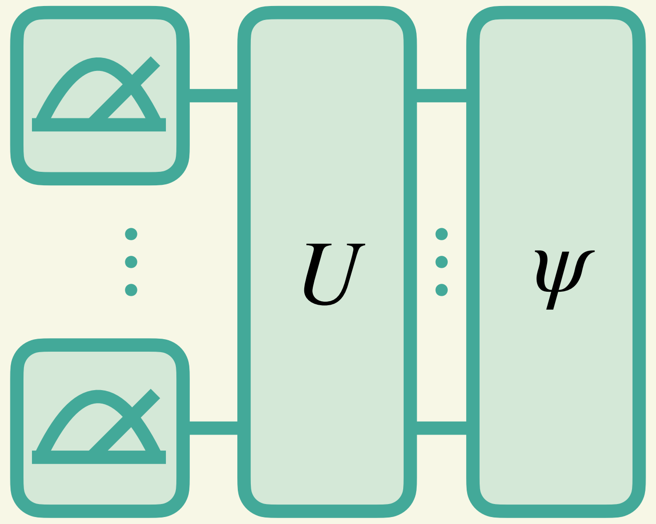
Single Snapshot

Variance

Sample complexity

Multi-qubit global Cliffords

$U \in \mathbb{C}(2^n)$



Measurement Channel

Inverted Measurement Channel

Single Snapshot

Variance

Sample complexity

Recap: Measurement Channel

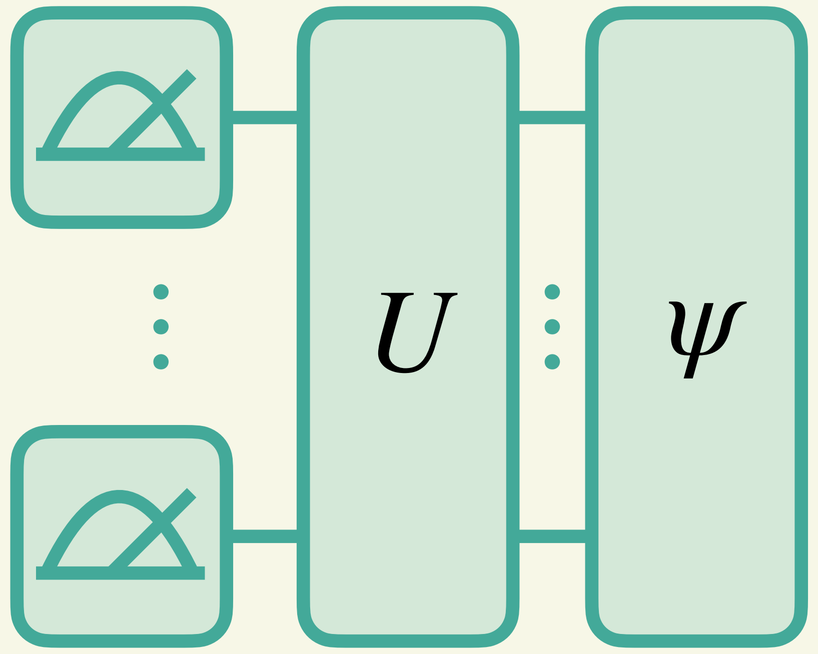
The measurement channel is the sum over all possible post-measurement states, weighted by their probability of appearing.

$$\mathscr{M}_Z(\rho) = \langle 0|\rho|0\rangle |0\rangle\langle 0| + \langle 1|\rho|1\rangle |1\rangle\langle 1| = \begin{pmatrix} \rho_{00} & 0 \\ 0 & \rho_{11} \end{pmatrix}$$

$$\mathscr{M}(\rho) = \frac{1}{3} \sum_{\substack{U \in \mathbb{U} \\ b \in \{0,1\}}} \mathrm{tr}(U^\dagger |b\rangle\langle b| U\rho) U^\dagger |b\rangle\langle b| U$$

$$\mathscr{M}(\rho) = \mathscr{D}_{1/3}(\rho) = \frac{1}{3} \begin{pmatrix} \rho_{00}+1 & \rho_{01} \\ \rho_{10} & \rho_{11}+1 \end{pmatrix}$$

Global Clifford Measurement Channel



$$\mathcal{M}(\rho) = \frac{1}{|\mathbb{C}(2^n)|} \sum_{\substack{U \in \mathbb{C}(2^n) \\ b \in \{0,1\}^n}} \mathrm{tr}(U^\dagger |b\rangle\langle b| U\rho) U^\dagger |b\rangle\langle b| U$$

$$\mathcal{M} = ???$$

Vector t-design

A finite set $\mathbb{V} \subset \mathbb{C}^d$ forms a vector t-design if

$$\frac{1}{|\mathbb{V}|} \sum_{v \in \mathbb{V}} \left(|v\rangle\langle v| \right)^{\otimes t} = \binom{d+t-1}{t}^{-1} \frac{1}{t!} \sum_{\pi \in S_t} \mathrm{SWAP}_\pi$$

where S_t is the symmetric group acting on t tensor factors.

Intuition t-design

Example: Scalars

Uniform distribution

$$X \sim [0,1]$$

$$\mathbb{E}[X] = \frac{1}{2}$$

$$\mathbb{E}[X^2] = \frac{1}{3}$$

$$\mathbb{E}[X^3] = \ldots$$

$$Y = \begin{cases} 0 \text{ with prob } 1/2 \\ 1 \text{ with prob } 1/2 \end{cases}$$

$$\mathbb{E}[Y] = \frac{1}{2}$$

$$\mathbb{E}[Y^2] = \frac{1}{2}$$

$$Z = \begin{cases} \frac{1}{2} + \frac{1}{\sqrt{12}} \text{ with prob } 1/2 \\ \frac{1}{2} - \frac{1}{\sqrt{12}} \text{ with prob } 1/2 \end{cases}$$

$$\mathbb{E}[Z] = \frac{1}{2}$$

$$\mathbb{E}[Z^2] = \frac{1}{3}$$

$$\mathbb{E}[Z^3] = \ldots$$

Vector t-design

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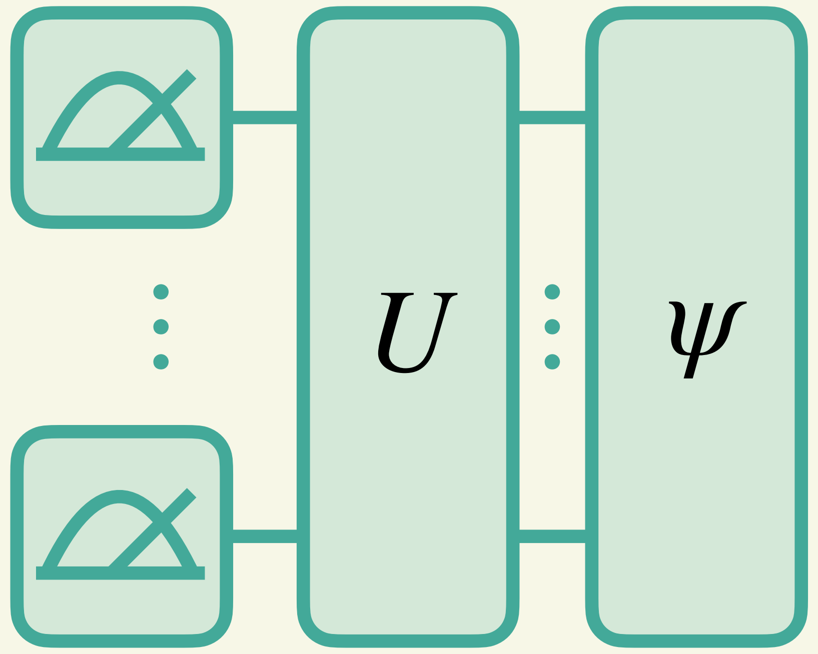
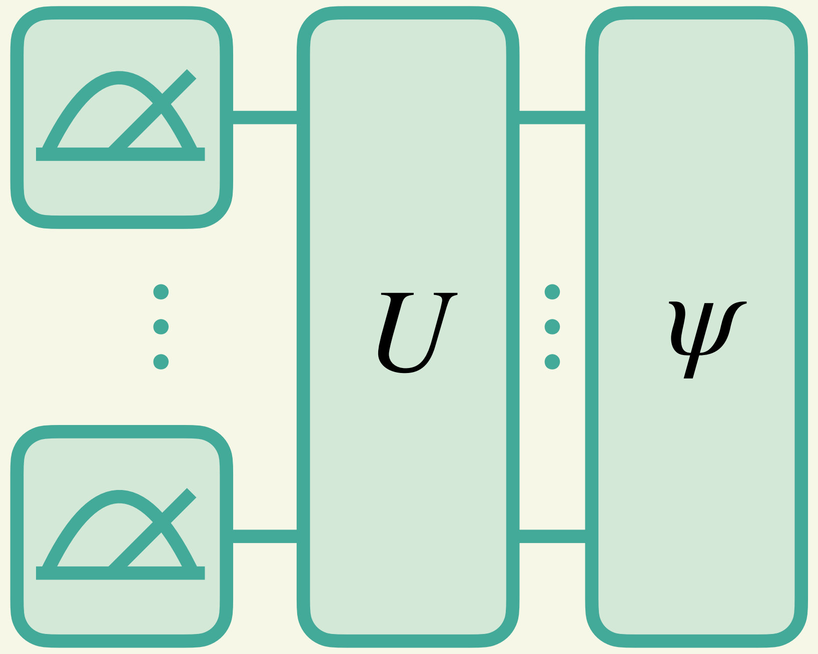
Global Clifford Measurement Channel



$$\mathcal{M}(\rho) = \frac{1}{|\mathbb{C}(2^n)|} \sum_{\substack{U \in \mathbb{C}(2^n) \\ b \in \{0,1\}^n}} \mathrm{tr}(U^\dagger |b\rangle\langle b| U\rho) U^\dagger |b\rangle\langle b| U$$

$$\mathcal{M} = ???$$

Global Clifford Measurement Channel

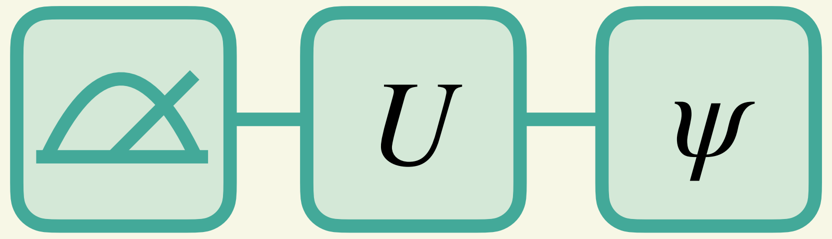


$$\mathscr{M}(\rho) = \frac{1}{|\mathbb{C}(2^n)|} \sum_{\substack{U \in \mathbb{C}(2^n) \\ b \in \{0,1\}^n}} \mathrm{tr}(U^\dagger |b\rangle\langle b| U\rho) U^\dagger |b\rangle\langle b| U$$

$$\mathscr{M} = \mathscr{D}_{1/(2^n+1)}$$

Classical Shadows

Single qubit



$U \in \mathbb{C}(2)$

$\mathcal{M}(\rho) = \mathcal{D}_{1/3}(\rho)$

$\mathcal{M}^{-1} = \mathcal{D}_3$

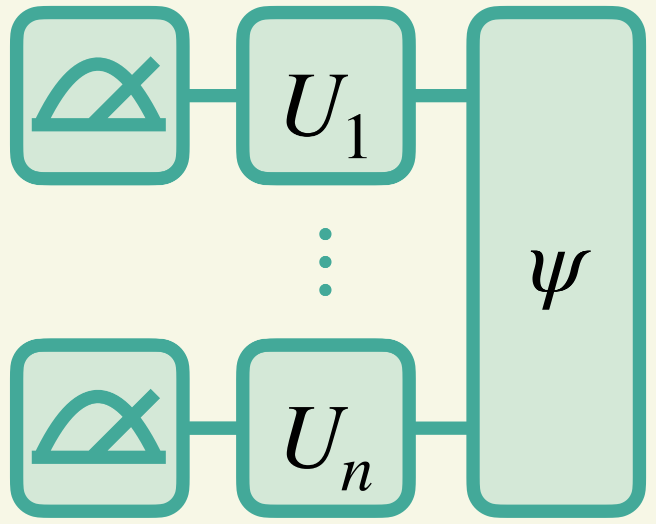
Single Snapshot

Variance

Sample complexity

Multi-qubit local Cliffords

$U_k \in \mathbb{C}(2)$



Measurement Channel

Inverted Measurement Channel

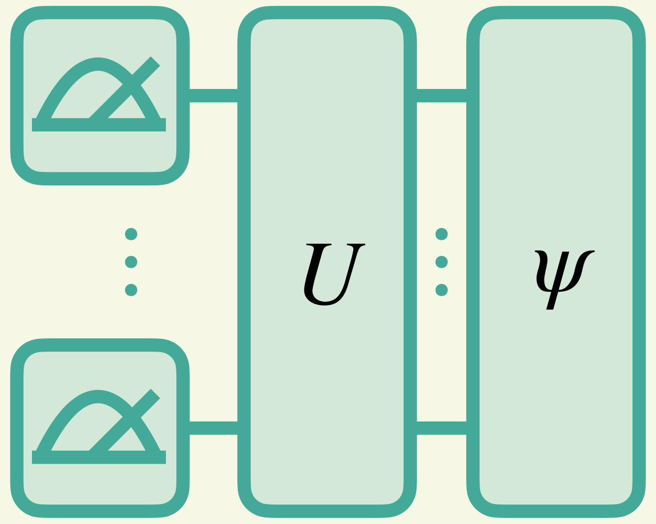
Single Snapshot

Variance

Sample complexity

Multi-qubit global Cliffords

$U \in \mathbb{C}(2^n)$



Measurement Channel

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Single Snapshot

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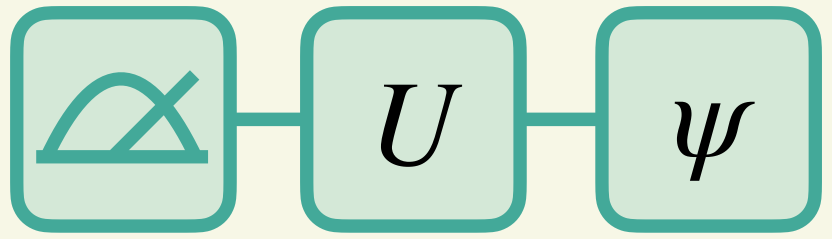
Classical Shadows

Single qubit



$U \in \mathbb{C}(2)$

$\mathcal{M}(\rho) = \mathcal{D}_{1/3}(\rho)$

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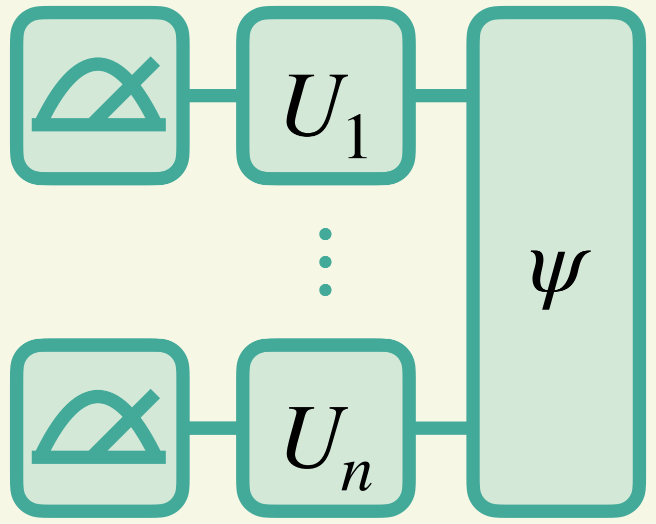

Measurement Channel

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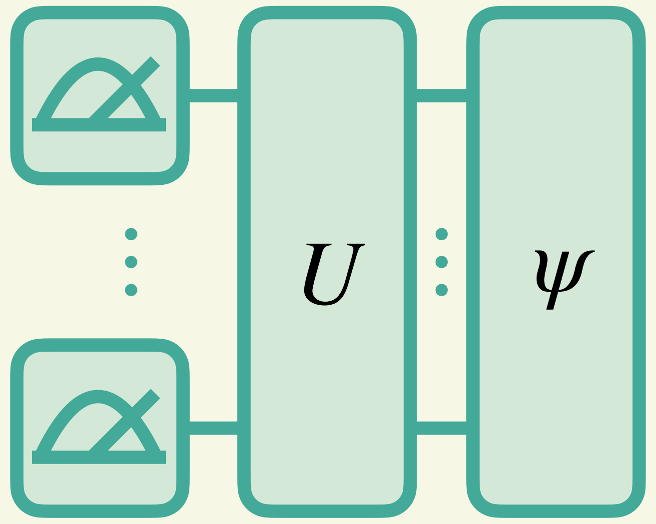


$\mathcal{M}(\rho) = \mathcal{D}_{1/(2^n+1)}(\rho)$

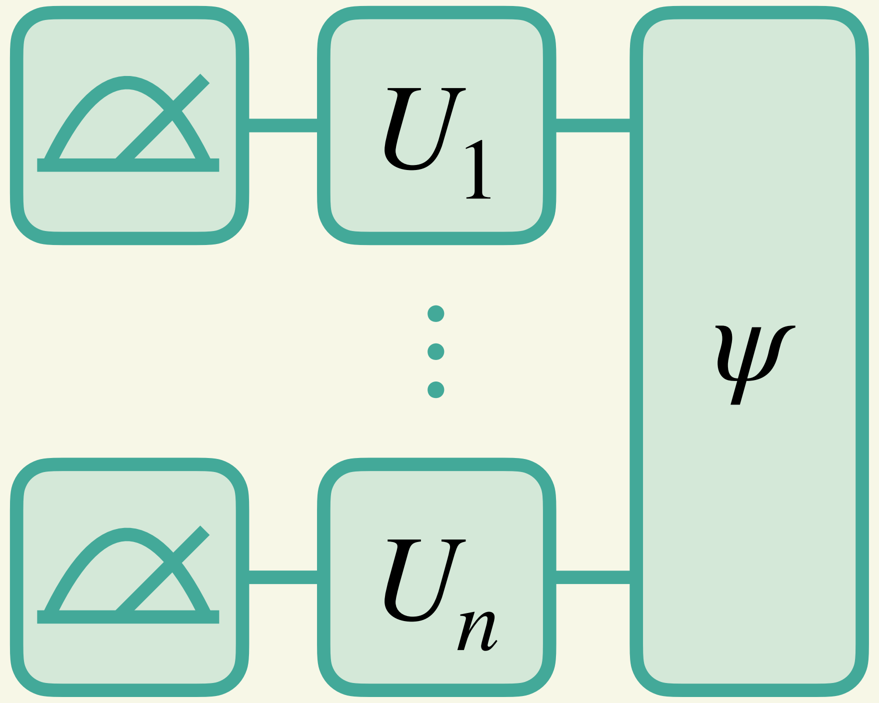
$\mathcal{M}^{-1} = \mathcal{D}_{2^n+1}$

Single Snapshot

Variance

Sample complexity

Local n-qubit Clifford Measurement Channel

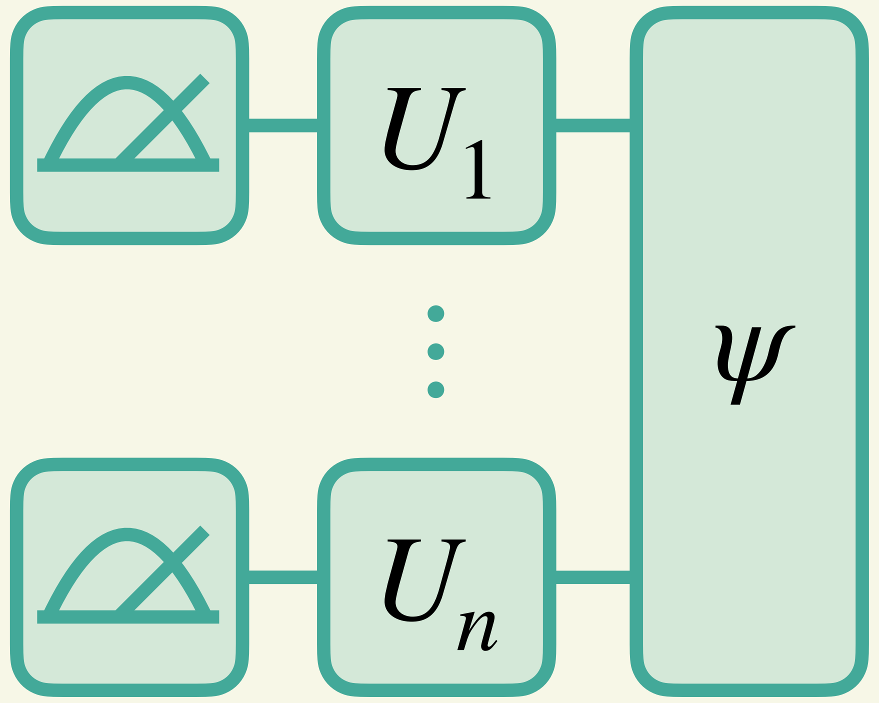


$U_k \in \{\mathbb{I}, H, HS^\dagger\}$

$$\mathcal{M}(\rho) = \frac{1}{3^n} \sum_{\substack{U \in \mathbb{U}^n \\ b \in \{0,1\}^n}} \mathrm{tr}(U^\dagger |b\rangle\langle b| U\rho) U^\dagger |b\rangle\langle b| U$$

$$\mathcal{M} = ???$$

Local n-qubit Clifford Measurement Channel



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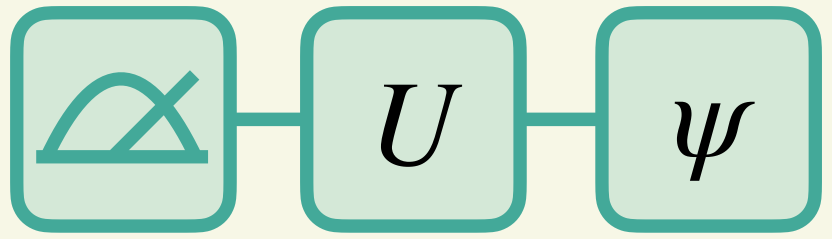
$$\mathcal{M}(\rho) = \frac{1}{3^n} \sum_{\substack{U \in \mathbb{U}^n \\ b \in \{0,1\}^n}} \mathrm{tr}(U^\dagger |b\rangle\langle b| U\rho) U^\dagger |b\rangle\langle b| U$$

$$\mathcal{M} = \mathcal{D}_{1/3}^{\otimes 3}$$

Classical Shadows

Single qubit



$U \in \mathbb{C}(2)$

$\mathcal{M}(\rho) = \mathcal{D}_{1/3}(\rho)$

$\mathcal{M}^{-1} = \mathcal{D}_3$

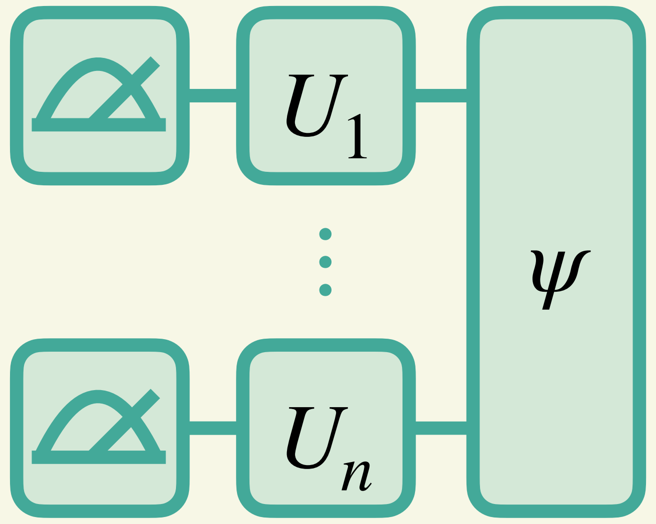
Single Snapshot

Variance

Sample complexity

Multi-qubit local Cliffords

$U_k \in \mathbb{C}(2)$



Measurement Channel

Inverted Measurement Channel

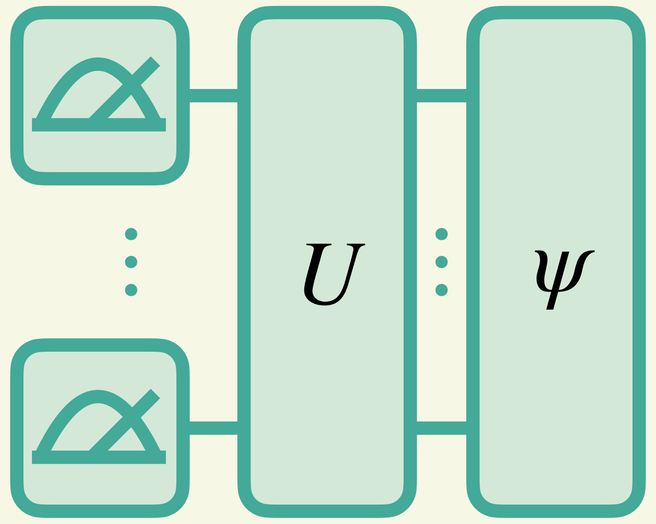
Single Snapshot

Variance

Sample complexity

Multi-qubit global Cliffords

$U \in \mathbb{C}(2^n)$



$\mathcal{M}(\rho) = \mathcal{D}_{1/(2^n+1)}(\rho)$

$\mathcal{M}^{-1} = \mathcal{D}_{2^n+1}$

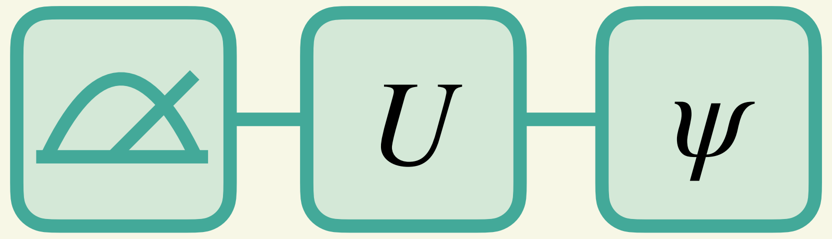
Single Snapshot

Variance

Sample complexity

Classical Shadows

Single qubit



$U \in \mathbb{C}(2)$

$$\mathcal{M}(\rho) = \mathcal{D}_{1/3}(\rho)$$

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Single Snapshot

Variance

Sample complexity

Multi-qubit local Cliffords

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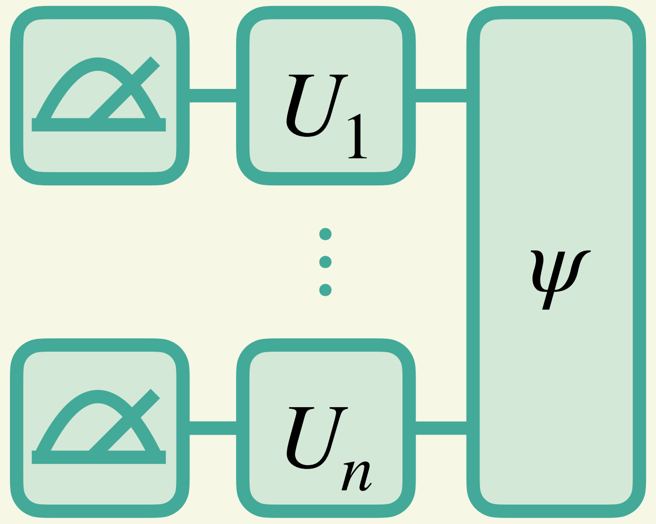


$$\mathcal{M}(\rho) = \mathcal{D}_{1/3}^{\otimes n}(\rho)$$

$$\mathcal{M}^{-1} = \mathcal{D}_3^{\otimes n}$$

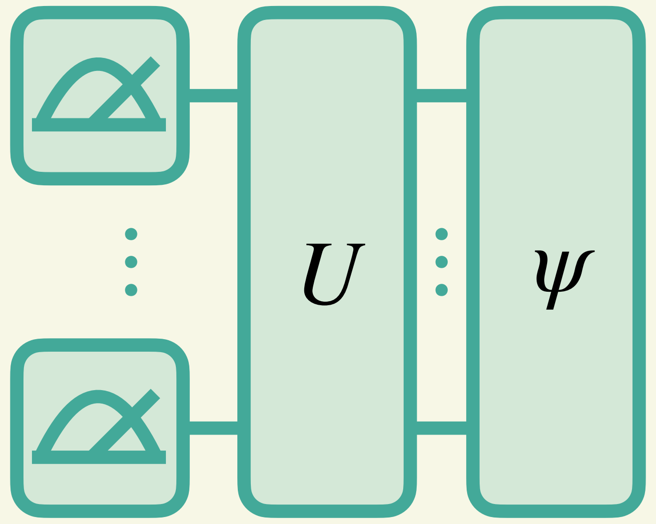
Single Snapshot

Variance

Sample complexity

Multi-qubit global Cliffords

$U \in \mathbb{C}(2^n)$



$$\mathcal{M}(\rho) = \mathcal{D}_{1/(2^n+1)}(\rho)$$

$$\mathcal{M}^{-1} = \mathcal{D}_{2^n+1}$$

Single Snapshot

Variance

Sample complexity

Snapshots

$$(U, b) \rightarrow \hat{\rho} = \mathcal{M}^{-1}(U^\dagger |b\rangle\langle b| U)$$

Single qubit

$$\mathcal{M}^{-1} = \mathcal{D}_3$$

$$\hat{\rho} = 3U^\dagger |b\rangle\langle b| U - \mathbb{I}$$

Multi-qubit local Cliffords

$$\mathcal{M}^{-1} = \mathcal{D}_3^{\otimes n}$$

$$\hat{\rho} = \bigotimes_{j=1}^{n} \left(3U_j^\dagger |b_j\rangle\langle b_j| U_j - \mathbb{I} \right)$$

Multi-qubit global Cliffords

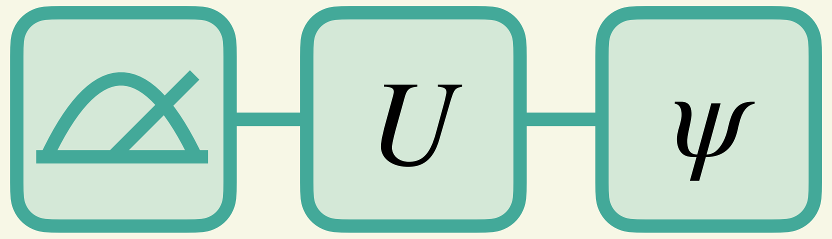
$$\mathcal{M}^{-1} = \mathcal{D}_{2^n+1}$$

$$\hat{\rho} = (2^n + 1)U^\dagger |b\rangle\langle b| U - \mathbb{I}$$

$$\mathcal{D}_p(\rho) = p\rho + (1-p)\mathrm{tr}(\rho)\frac{\mathbb{I}^{\otimes n}}{2^n}$$

Classical Shadows

Single qubit



$U \in \mathbb{C}(2)$

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Single Snapshot

Variance

Sample complexity

Multi-qubit local Cliffords

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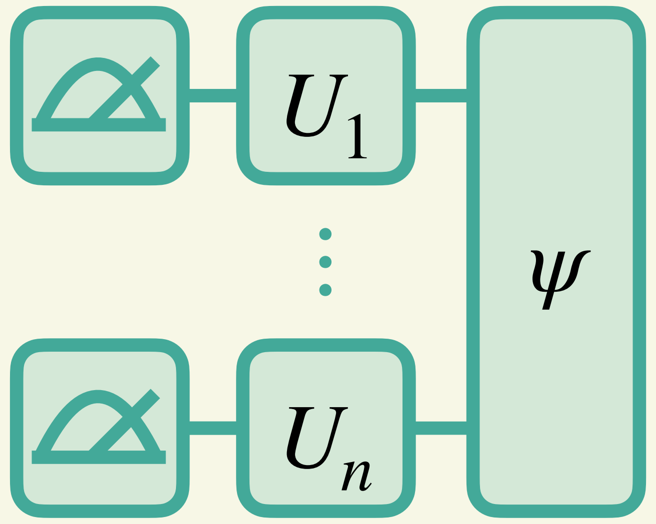


$$\mathcal{M}(\rho) = \mathcal{D}_{1/3}^{\otimes n}(\rho)$$

$$\mathcal{M}^{-1} = \mathcal{D}_3^{\otimes n}$$

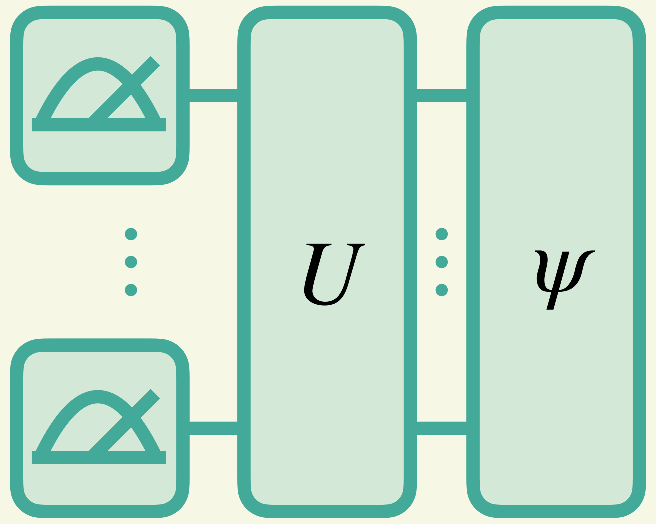
Single Snapshot

Variance

Sample complexity

Multi-qubit global Cliffords

$U \in \mathbb{C}(2^n)$



$$\mathcal{M}(\rho) = \mathcal{D}_{1/(2^n+1)}(\rho)$$

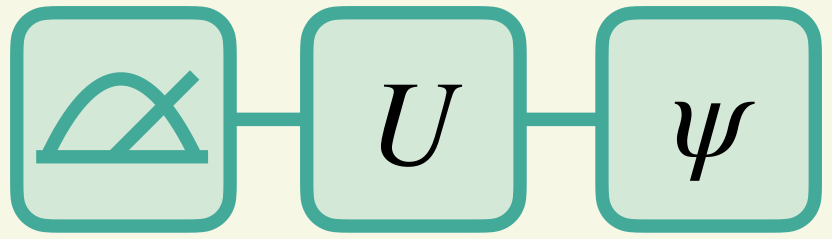
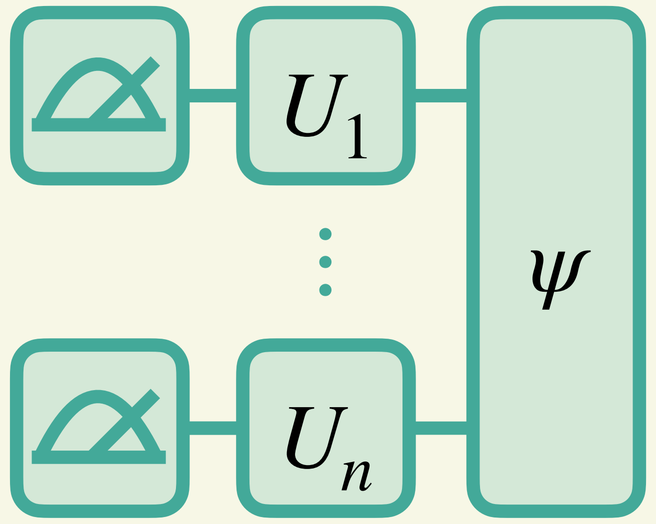
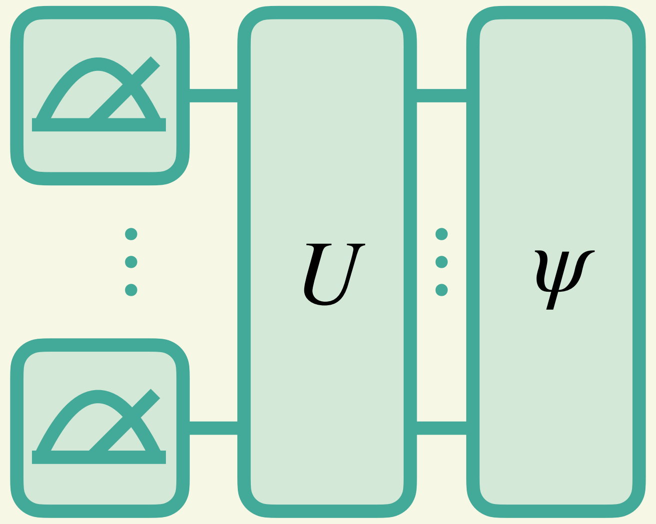
$$\mathcal{M}^{-1} = \mathcal{D}_{2^n+1}$$

Single Snapshot

Variance

Sample complexity

Classical Shadows

Single qubit	*Multi-qubit local Cliffords*	*Multi-qubit global Cliffords*
 $U \in \mathbb{C}(2)$	$U_k \in \mathbb{C}(2)$ 	$U \in \mathbb{C}(2^n)$ 
$\mathcal{M}(\rho) = \mathcal{D}_{1/3}(\rho)$	$\mathcal{M}(\rho) = \mathcal{D}_{1/3}^{\otimes n}(\rho)$	$\mathcal{M}(\rho) = \mathcal{D}_{1/(2^n+1)}(\rho)$
$\mathcal{M}^{-1} = \mathcal{D}_3$	$\mathcal{M}^{-1} = \mathcal{D}_3^{\otimes n}$	$\mathcal{M}^{-1} = \mathcal{D}_{2^n+1}$
$\hat{\rho} = 3U^\dagger \lvert b \rangle\langle b \rvert U - \mathbb{1}$	$\hat{\rho} = \bigotimes_{j=1}^{n} \left(3U_j^\dagger \lvert b_j \rangle\langle b_j \rvert U_j - \mathbb{1} \right)$	$\hat{\rho} = (2^n + 1)U^\dagger \lvert b \rangle\langle b \rvert U - \mathbb{1}^{\otimes n}$
Variance	Variance	Variance
Sample complexity	Sample complexity	Sample complexity

Classical Shadow Protocol

State Estimation

Single-shot estimator

$$\hat{\rho}$$

Multi-shot estimator

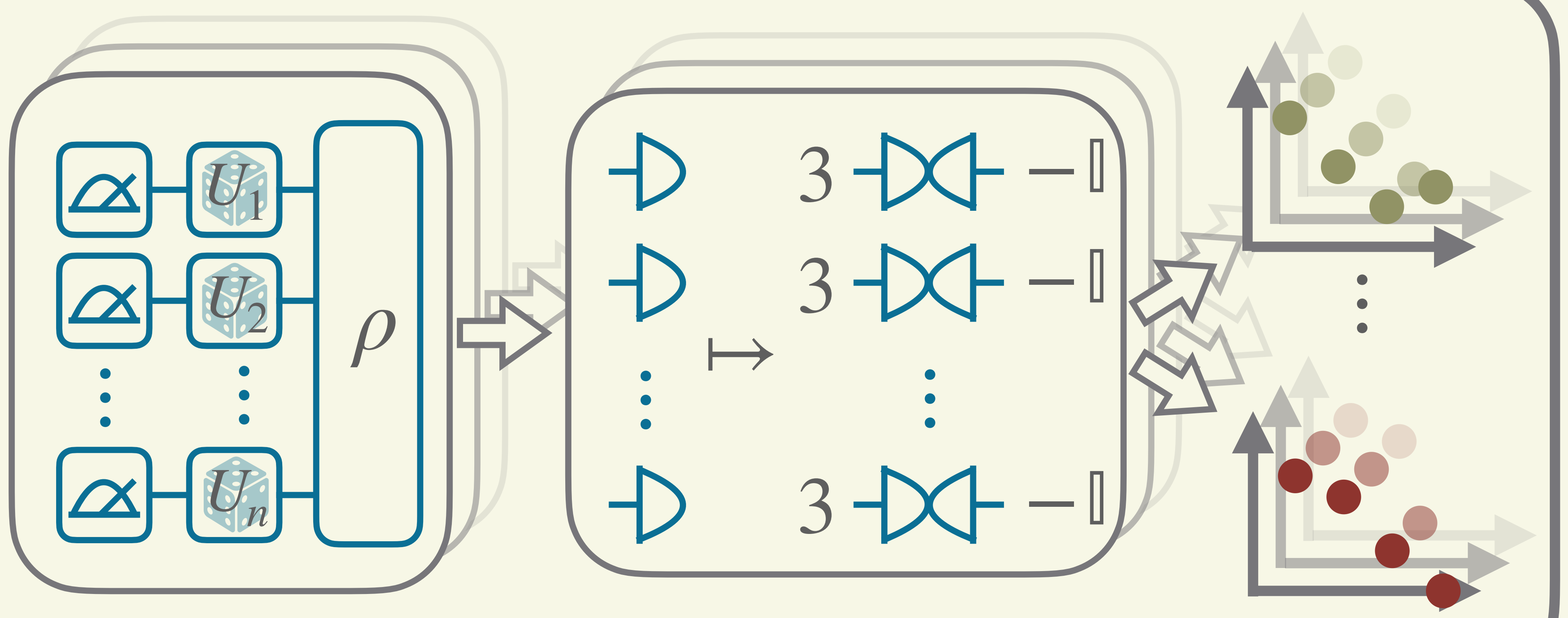
$$\bar{\rho} = \frac{1}{N}\sum_{t=1}^{N}\hat{\rho}_t$$

Expected value

$$\rho$$

$$(U, b) \to \hat{\rho} = \mathscr{M}^{-1}(U^\dagger\,|\,b\rangle\langle b\,|\,U)$$

$\frac{1}{N}\sum_{t=1}^{N}\hat{\rho}_t$ for $T \to \infty$: $\mathbb{E}(\hat{\rho}) = \rho$.



Observable Estimation

Single-shot estimator

$$\hat{o} = \mathrm{tr}(O\hat{\rho})$$

Multi-shot estimator

$$\bar{o} = \frac{1}{N}\sum_{t=1}^{N}\hat{o}_t$$

Expected value

$$o = \mathrm{tr}(O\rho)$$

Observables

Observable	$O \in L(\mathscr{H})$ with $O = O^\dagger$
Linear function in ρ	$\mathrm{tr}(O\rho) = \langle O \rangle_\rho$

Traceless observable	set $O := O - \frac{\mathrm{tr}(O)}{2^n}\mathbb{I}^{\otimes n}$ such that $\mathrm{tr}(O) = 0$
k-local observable	$O = O_K \otimes \mathbb{I}_{\neg K}$ with $\lvert K \rvert = k$

M many observables	$\{O_1, O_2, \ldots, O_M\}$

Observables

Example: Pauli Observables

Single shot estimator

$$\mathrm{tr}(\hat{\rho} Z) = ??$$

Observables

Example: Pauli Observables

Single shot estimator

$$\mathrm{tr}(\hat{\rho}Z) = 3(-1)^b \delta_{U,\mathbb{I}}$$

→ No need to construct matrix

$$\mathrm{tr}(\hat{\rho}Z) = 3(-1)^b \delta_{U,\mathbb{I}}$$

$$\mathrm{tr}(\hat{\rho}X) = 3(-1)^b \delta_{U,H}$$

$$\mathrm{tr}(\hat{\rho}Y) = 3(-1)^b \delta_{U,HS^\dagger}$$

Observables

Example: 2q Pauli Correlations

$$\mathrm{tr}(\hat{\rho} Z \otimes Z) = 9(-1)^{b_1+b_2}\delta_{(U_1,U_2),(\mathbb{I},\mathbb{I})}$$

$$\mathrm{tr}(\hat{\rho} X \otimes X) = 9(-1)^{b_1+b_2}\delta_{(U_1,U_2),(H,H)}$$

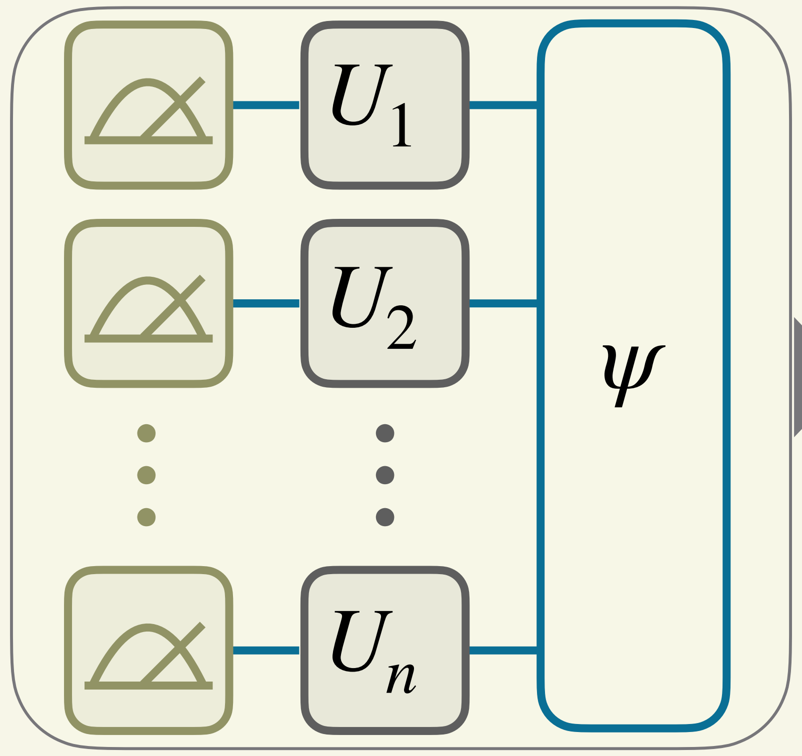
$$\mathrm{tr}(\hat{\rho} Y \otimes Y) = 9(-1)^{b_1+b_2}\delta_{(U_1,U_2),(HS^\dagger,HS^\dagger)}$$

Break 5 min

Coding time

https://github.com/wilkensJ/natal26-intro-classical-shadows

$U_k \in \{\mathbb{1}, H, HS^\dagger\}$



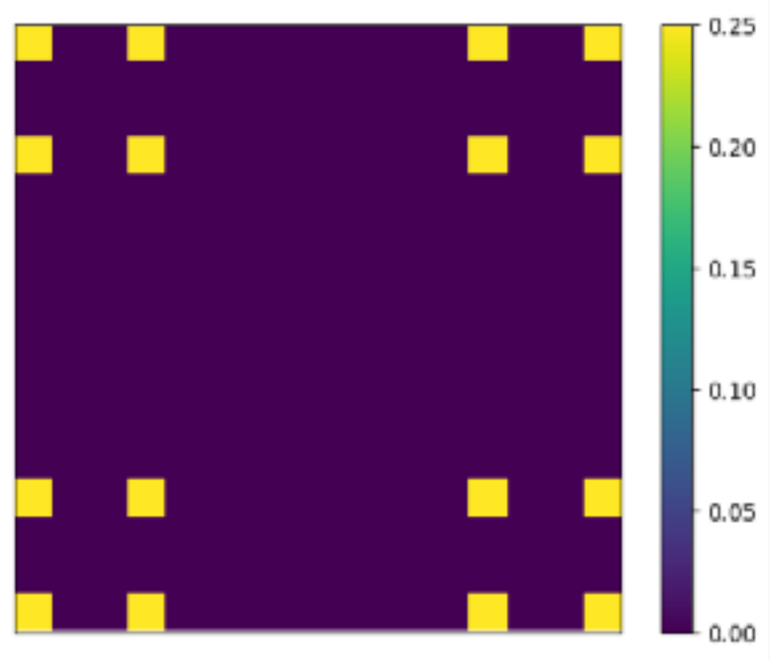
t	U	$\vec{b}$
1	$(\mathbf{H}, \mathbb{1}, \ldots)$	$(0,1,\ldots)$
2	$(\mathbf{HS}^\dagger, \mathbf{H}, \ldots)$	$(1,1,\ldots)$
3	$(\mathbb{1}, \mathbf{HS}^\dagger, \ldots)$	$(0,0,\ldots)$
⋮		

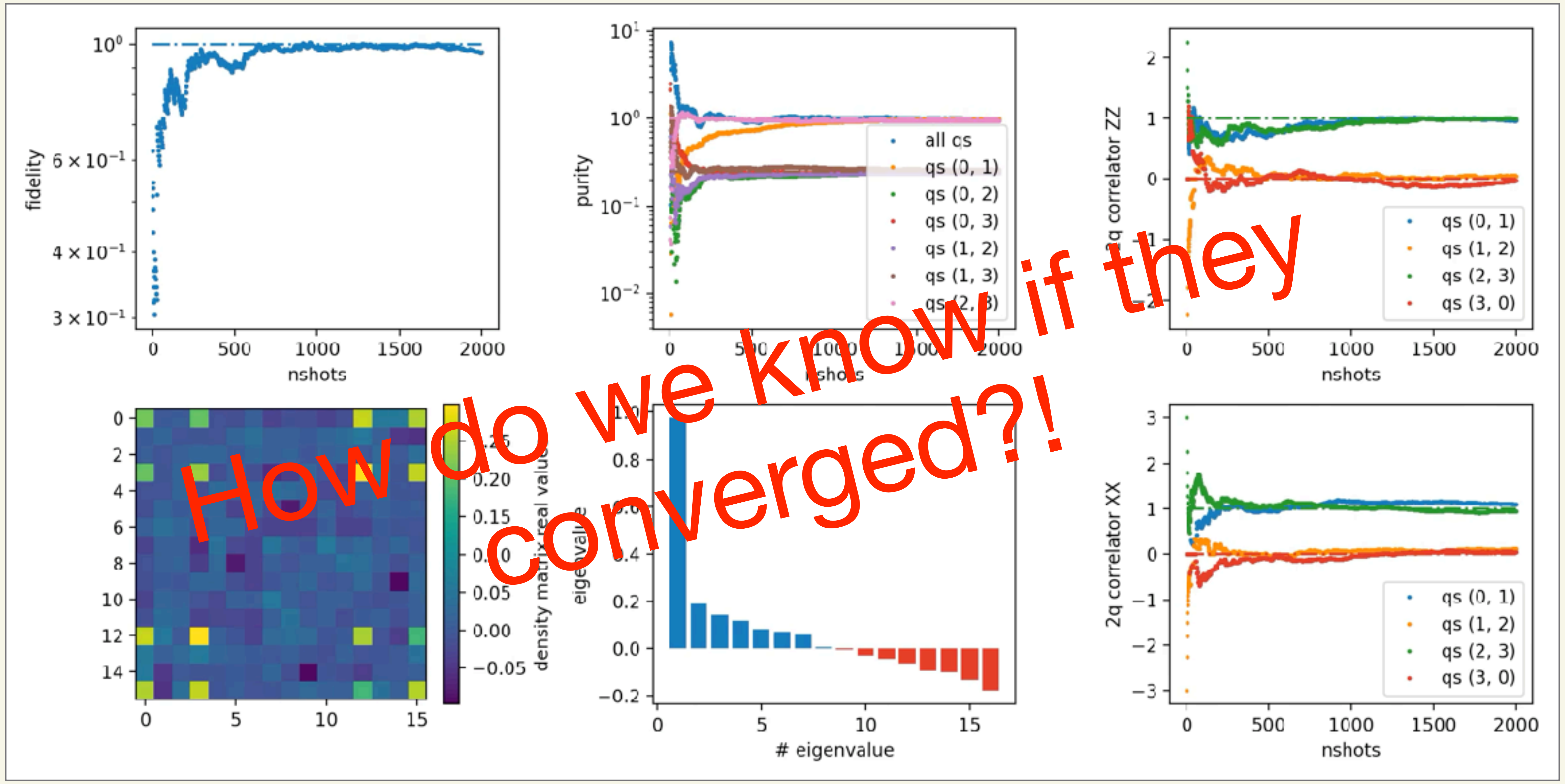
$$\hat{\rho} = \bigotimes_{j=1}^{n} \left(3U_j^\dagger \,|b_j\rangle\langle b_j|\, U_j - \mathbb{1}\right)$$

$$\mathrm{tr}\left(\bar{\rho}\rho\right), \mathrm{tr}\left(\bar{\rho}(Z \otimes \mathbb{1} \otimes \cdots)\right), \mathrm{tr}\left(\bar{\rho}^2\right),$$

$$\mathrm{tr}\left(\tilde{\rho}(X \otimes X \otimes \mathbb{1} \otimes \cdots)\right), \ldots$$

$$|\psi\rangle = |\mathrm{bell}\rangle \otimes |\mathrm{bell}\rangle$$





How do we know if they converged?!

Thank you for your attention!