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**Analysis of Black Hole Thermodynamics**

Wilkie Hoare WW4

*All information that follows, including more detailed results, can be found on my GitHub:* [github.com/wilkiehoare/BlackHoles](http://www.github.com/wilkiehoare/BlackHoles)

**Method:**

As the universe ages, the universe will expand and cool, and eventually radiate away its mass and evaporate. In this investigation, I take an analytical look at the effects of this temperature change on Black Holes, comparing attributes of Black Holes as their solar masses change. I programmed a mathematical model in C++ to compare with known attributes about other stars (including Sirius, Eta Carinae and R136a1), all details of which can be found in my GitHub. I take a look at five key attributes, all of which are their own functions in the code, returning a floating-point number:

1. Schwarzschild Radius,

2. Schwarzschild Density,

3. Hawking Radiation,

4. Hawking Temperature,

5. Time taken for the black hole to reach thermal equilibrium.

**Theory:**

A black hole is a region of space with a gravitational field intense enough such that nothing can escape – its mass must also exceed the Chandrasekhar limit (about 1.4 solar masses). In this investigation, I assume that the Black Holes are not rotating and uncharged, that is and . The solution taken for Einstein’s field equations differ if we cannot make these assumptions. In reality, Black Holes often do have , but observed astronomical objects rarely have .

We can relate the fundamental laws of thermodynamics to Black Holes:

The zeroth law states that a simple, non-rotating Black Hole has uniform gravity at its event horizon. An event horizon can be thought of as the boundary surrounding a black hole beyond which nothing can escape (light, radiation etc). It is at thermal equilibrium at the event horizon.

The first law states that the mass, rotation, and charge (3 defining attributes) of a Black Hole are related to its entropy. We relate this in a simplified manner to internal energy at A level:

The entropy of a black hole, according to Hawking, can in fact be directly related to its area:

The second law states that the entropy of a Black Hole system cannot decrease. This means that when two Black Holes merge, the surface area of the event horizon must be greater than both Black Holes.

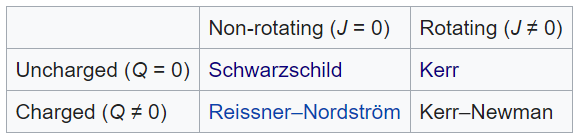
The third law states that ‘extreme’ Black Holes (maximum possible rotation and charge – which would take the Kerr-Newman model) would have minimum entropy, meaning that such a case is never possible.

**Derivations:**

Several important equations are used to calculate the five attributes.

Schwarzschild Radius:

Firstly, the Schwarzschild radius equation drops out of the Schwarzschild metric, the solution to the Einstein Field Equations for an uncharged, non-rotating black hole. Out of interest, all cases are listed in the table below:



There are two routes of derivation: the first, more rigorous route uses tensor maths beyond me at this stage, and an easier route would be via Lagrangian mechanics.

We must make some assumptions before we start:

* The spacetime is *spherically symmetric* – it is invariant under rotations.
* The spacetime is *static* – all metric components are independent of , and the spacetime is unchanged under a time-reversal .
* The solution is a *vacuum solution* – it satisfies , the energy-momentum tensor.
* In fact, Birkhoff’s theorem tells us that if our first and third conditions are satisfied, it must be true that the spacetime is also static.

Begin with the general metric with coefficients to be found:

Recall the arc length integral:

Recall the Euler-Lagrange equation:

I’ll use dot notation for derivatives, so it isn’t as messy. Apply (3) to (2) so we have equations for ():

In a circular orbit, , so:

Recall Kepler’s third law of motion:

In a circular orbit, , and the point mass is negligible compared to , so:

Integrating:

If we set then and so :

When the point mass is temporarily stationary, and . (1) becomes and (3) becomes . is the acceleration of gravity, , so:

From and the Schwarzschild radius drops out as:

Schwarzschild Density:

Begin with the well-known equation:

Inside the event horizon, the volume (sphere) can be given by:

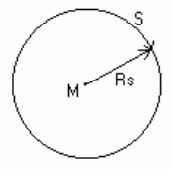
Substitute (15) into (14):

Substitute (13) into (16):

The practical result for when we have a measured radius (e.g., stars) is given by:

Hawking Temperature:

Again, there are two routes through this derivation – via the Schwarzschild solution or by thinking of the surface horizon as a circle; the latter is easier to follow.



Consider the surface horizon of a black hole a circle which perimeter is equal to . A fluctuating field which can radiate, will be considered as a transverse field and we suppose that it has equal probability to assume positive or negative values. A test particle of mass coupled to this field, integrated over a small interval of the circle’s arc will give on average null contribution for the energy, due to the fluctuating character of the field. Thus, we need to consider the second moment of this ‘elastic’ energy, and we write:

We divided by to consider the possibility of producing a particle pair from the vacuum.

We can use a constant to clean this up:

This is treated as a spring constant of a harmonic oscillator. The angular frequency is given by:

The radial field at the surface horizon is given by:

We propose that the vacuum fluctuations are great enough such that (set (23) equal to (24)):

Relating constants:

Hawking Radiation:

‘Hawking Radiation’ refers to the proposal of Hawking that during pair production occurring just outside the event horizon, a black hole slowly loses mass or evaporates as particles are radiated away. In this process, anti-particles with negative energy fall into the black hole actually causing the mass to decrease.

Begin with Stefan-Boltzmann’s Law:

*The total electromagnetic radiation energy emitted per unit time by a black body is given by:*

Assume a sphere:

Substitute (30) into (29) (and we’ll let the radiation be called ):

Substitute (28) into (31):

Substitute (13) into (32):

Thermal Equilibrium:

* Find the rate of mass loss due to Hawking radiation.
* Find the rate of mass gain due to CBR.
* Equate them at equilibrium and solve for time.

By definition:

We know:

Differentiate (36) and substitute (34):

The temperature of CBR in terms of time is given by:

The black hole absorbs CBR:

Flux also is equivalent to:

Multiplying by gives:

In another formulation:

Substitute (38) into (42):

Simplifying:

We know (the radiation constant):

Substitute (46) into (44):

Differentiate (47):

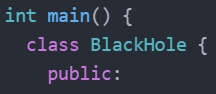
Equating (37) and (48):

Substitute (30) and (33):

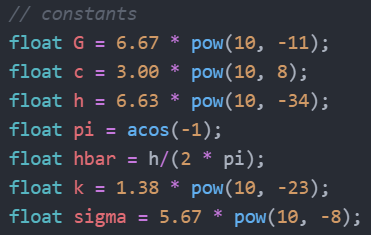
Substitute (28) into (52):

Square root:

**Code:**

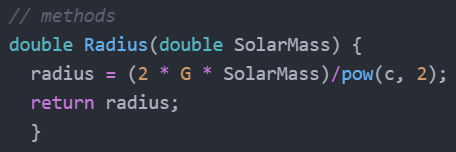


Within the main body of the code, I set up the model ‘template’ called BlackHole as a class. I can then make instances of these later, ‘generating’ black holes with different solar masses.

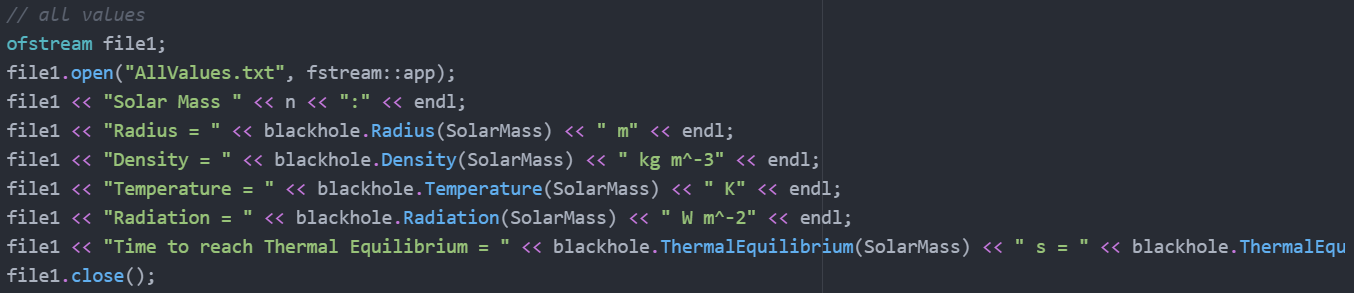


Defining all physical constants used in calculations. Note that it doesn’t recognise/have a built-in variable , so instead I get the precise value from in radians. I use the function that comes with the library to do any exponentiation for me:

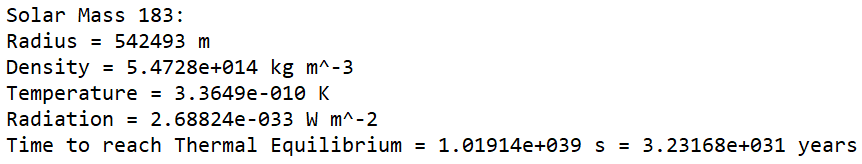
* N m2 kg-2
* m s-1
* J s
* m2 kg s-1
* J K-1
* W m-2 K-4



An example ‘method’ in the class, in which the code will find the solar mass of the black hole and calculate its corresponding Schwarzschild radius, using (13).



This is the largest file generated by the code, entitled ‘AllValues’ and saved as a text file. This writes to the file all the data sequentially for the given black hole with appropriate units. Example from this file:



**Graphs:**

This graph can be modelled parametrically and gives .

This graph can be modelled parametrically and gives . As temperature increases with greater solar mass, Hawking radiation eventually varies linearly at approximately K.

**Conclusion:**

The larger the Black Hole, the longer it would take to reach equilibrium with the temperature of the universe because it is cooler than smaller Black Holes. Once it does reach equilibrium, a larger Black Hole radiates less than a smaller one, so it would take longer for the larger Black Hole to radiate all its mass away. According to my results, my hypothesis is correct. However, it would take a long time for a large solar mass to reach equilibrium with the temperature of the universe and would take an even longer time for it to radiate its mass away.

**Error & Improvement:**

In an ideal world, there would be more recorded results and models to discover, compared to my highly mathematical model, with which I had very little data to compare (probably due to the theoretical nature of this area of physics). Exploring the more realistic situation of the Kerr solution to Einstein’s Field Equations for a rotating black hole would also draw some interesting results. The Schwarzschild metric is a slight extension of the derivation for its radius:

One can then obtain the Kerr metric through one of three routes: the Newman-Janis algorithm, which changes the coordinates to complex coordinates, the Ernst equation, or by ellipsoid coordinate transformation:

The very nature of the coordinate systems used, and routes of derivation show how theoretical this area of physics is.

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