ESE 351 Spring 2024 Homework #4
Assigned Thursday, February 15
Due Monday, February 26
Material to be covered in **Quiz 2** (with CT Fourier Series) in class **Thurs., Feb. 29** (Lecture 14)

Finish reading Chapter 3 in Oppenheim and Willsky. The '*Basic Problems with Answers*' are all good review problems. Emphasize content from lecture in your reading of the text.

Solve the problems below. Acknowledgement to course AI Brian Sun for some of the problems.

Written problems to turn in (20 points each) (Partial credit will be given. If you cannot solve a problem completely, describe your approach and the difficulty you are having.)

For each problem, show all supporting work, e.g., consider the following items:

- A brief problem statement (rephrased in your own words when possible),
- your approach, including key expressions and variables, sketches when appropriate,
- the results of your solution and interpretation when possible.
- Include justification of any steps used to produce your results (results without justification will not receive full credit).

1. Given the following periodic signals, determine their fundamental frequency ω_0 and Fourier coefficients a_k. Where possible, report a_k in magnitude and phase form. Use Matlab to verify your solutions and include stem plots of $x[n]$ and a_k (note that a_k is generally complex-valued, so plot the real and imaginary parts, or the magnitude and phase, whichever helps interpret the result more clearly). Recall that **fft**() and **ifft**() compute the forward and inverse DTFS based on one period ($[0,1,\dots,N-1]$) of both $x[n]$ and a_k. Recall also that the convention in Matlab is to scale by $1/N$ when computing $x[n]$ from a_k whereas we used the opposite convention in lectures, following the course text. Optional: as mentioned in class, **fftshift**() can facilitate visualization of symmetry in a_k or $x[n]$. For submission, plot vs. $[0,1,\dots,N-1]$ (for n and k).
    a. $x[n] = 3 + \sin\left(\frac{4\pi}{5}n + \frac{\pi}{10}\right) + \cos(2\pi n) + (-1)^n$.
    b. $x[n] = \sum_{k=-\infty}^{\infty} \delta[n-5k] - 2\delta[n-2-5k]$.
    c. $x[n] = 1 - \sin\left(\frac{\pi n}{2}\right)$, when the period is assumed to be (i) $N = 4$ and (ii) $N = 16$.
    d. $x[n] = \sin\left(\frac{7\pi}{2}n\right) + e^{j\frac{\pi}{4}n}$
2. Given the Fourier series coefficients a_k below, determine the signal $x[n]$. Each function is periodic with period $N = 8$. As in problem 1, verify your results using Matlab and include stem plots.
    a. The only nonzero coefficients a_k for $k \in [-4,3]$ are $a_0 = 2, a_1 = a_{-1} = 3, a_3 = a_{-3}^* = j, a_{-4} = -1$. Express $x[n]$ in the form $x[n] = \sum_{k=0}^{\infty} A_k \cos(k\omega_0 N + \phi_k)$.
    b. $a_k = \cos\left(\frac{\pi k}{4}\right)$
    c. $a_k = \begin{cases} 1, & -2 \le k \le 2 \\ 0, & \text{else} \end{cases}$ (a DT square wave in frequency)

3. Solve the following problems (recall properties of the Fourier series).
   a. Let $x[n] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$, find a general expression for the Fourier series coefficients of each of these functions:
      i. $x[n] - x[n-5]$
      ii. $(-1)^n x[n-4]$
   b. Consider the DT square wave $x[n] = \begin{cases} 1, & -N_1 \le n \le N_1 \\ 0, & N_1 < n < N - N_1 \end{cases}$, periodic with period N. For each of the following functions, use Matlab to plot the Fourier series coefficients a_k (stem plot of $|a_k|, \angle a_k$) and provide a brief interpretation of the results, including any symmetry in a_k.
      i. $x[n]$, with $N_1 = 3$, and $N = 16, 32$ and 64.
      ii. $x[n]$, with $N = 32$, and $N_1 = 2, 6$ and 10.
      iii. For $N_1 = 3, N = 16$.
         1. $x[n-3]$
         2. $x[n] - \frac{1}{2}$
         3. $\cos(\pi n)\, x[n]$
   c. Recall that convolution in time is equivalent to multiplication in frequency. For periodic DT functions, that convolution is periodic (also termed circular). Consider the convolution of two rectangular pulses, $x[n] = rect[n] * rect[n]$, where (as in Homework #2 Matlab),

$$rect[n] = \begin{cases} 1, \ 0 \le n \le N \\ 0, \ else \end{cases}$$

      i. For $N = 10$, compute the convolution result $x[n]$ as the inverse fft of the product of the FS coefficients, e.g., $(a_k)^2$. You will need to select a length of the vector representing $rect[n]$ for the FS/fft and ifft calculations. Show these results using fft's of length 10, 15 and 20, and interpret them, including comparison to the intended (linear) convolution.
      ii. For finite-length DT functions, performing convolution in the frequency domain can be computationally efficient, but care must be taken to ensure the proper result. To generate the convolution of two finite-length DT functions with lengths M and N, what is the longest possible length of their linear convolution (i.e., what length fft must be used to generate the linear convolution result via frequency-domain multiplication)?
4. Consider a DT LTI system with frequency response

$$H(e^{j\omega}) = \begin{cases} 1, \ |\omega| \le \dfrac{\pi}{3} \\ j, \ \dfrac{\pi}{3} \le \omega \le \dfrac{2\pi}{3} \\ -j, -\dfrac{2\pi}{3} \le \omega \le -\dfrac{\pi}{3} \\ 0, \dfrac{2\pi}{3} \le |\omega| \le \pi \end{cases}$$

   Find the system output $y[n]$ when input $x[n]$ is:

a. $x[n] = 1 - \sin\left(\frac{\pi n}{2}\right)$
b. $x[n] = \sin\left(\frac{9\pi}{4}n\right) + \cos\left(\frac{3\pi}{4}n\right) + (-1)^n$

5. Solve the following problems (3.42 a,b,c and 3.54 in the course text)
   a. (Symmetry properties for real-valued $x(t)$) Let $x(t)$ be a real-valued signal with fundamental period T and Fourier series coefficients a_k.
      i. Show that $a_k = a_{-k}^*$ and a_0 must be real.
      ii. Show that if $x(t)$ is even, then its Fourier series coefficients must be real and even.
      iii. Show that if $x(t)$ is odd, then its Fourier series coefficients are imaginary and odd and $a_0 = 0$.
   b. (Orthogonality relation for DT complex exponentials) Consider the function

$$a[k] = \sum_{n=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)kn}.$$ (cont'd next page)

      i. Show that $a[k] = N$ for $k = 0, \pm N, \pm 2N, \ldots$
      ii. Show that $a[k] = 0$ whenever k is not an integer multiple of N. Hint: use the formula for the finite sum of a geometric series.
      iii. Repeat parts (i) and (ii) if $a[k] = \sum_{n=<N>} e^{j\left(\frac{2\pi}{N}\right)kn}$.

There is no extended Matlab assignment for this homework. Use that time instead to work on the first case study.