2010/12/11

第三章

1 (1) $\hat f(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}|x|e^{-i\lambda x}dx$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^a x e^{-i\lambda x}dx + \int_{-a}^0 -x e^{-i\lambda x}dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^a x\, 2\cos\lambda x\, dx$$

$$= \frac{2}{\sqrt{2\pi}}\left[\frac{1}{\lambda}x\sin\lambda x\Big|_0^a - \frac{1}{\lambda}\int_0^a \sin\lambda x\,dx\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[\frac{1}{\lambda}a\sin a\lambda + \frac{1}{\lambda^2}(\cos\lambda a - 1)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{\lambda}a\sin\lambda a + \frac{1}{\lambda^2}\cos\lambda a - \frac{1}{\lambda^2}\right]$$

(2) $\hat f(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\left(1-\frac{|x|}{a}\right)e^{i\lambda x}dx$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{i\lambda}(e^{i\lambda a}-e^{-i\lambda a}) - \frac{1}{|a|}2\left(\frac{1}{\lambda}\cos\lambda a + \frac{1}{\lambda^2}\cos\lambda a - \frac{1}{\lambda^2}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{\lambda}\sin\lambda a - \frac{2}{|a|}\left(\frac{1}{\lambda}\cos\lambda a + \frac{1}{\lambda^2}\cos\lambda a - \frac{1}{\lambda^2}\right)\right]$$

(3) $\hat f(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\sin\lambda_0 x\, e^{-i\lambda x}dx$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{\lambda_0}(-\cos\lambda_0 x)e^{-i\lambda x}\Big|_{-a}^{a} - \frac{1}{\lambda_0}\int_{-a}^{a}(-\cos\lambda_0 x)e^{-i\lambda x}(-i\lambda)dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{\lambda_0}\cos\lambda_0 a\, 2\sin\lambda a - \frac{i\lambda}{\lambda_0^2}\left[\sin\lambda_0 x\cdot e^{-i\lambda x}\Big|_{-a}^{a} - \int_{-a}^{a}\sin\lambda_0 x\cdot e^{-i\lambda x}\cdot(-i\lambda)dx\right]\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{\lambda_0}\cos\lambda_0 a\cdot 2\sin\lambda a - \frac{i\lambda}{\lambda_0^2}(\sin\lambda_0 a\cdot 2\cos\lambda a)\right] + \frac{1}{\sqrt{2\pi}}\frac{\lambda^2}{\lambda_0^2}\int_{-a}^{a}\sin\lambda_0 x\, e^{-i\lambda x}dx]$$

$$\therefore \hat f(\lambda) = \frac{1}{\sqrt{2\pi}}\left(1+\frac{\lambda^2}{\lambda_0}\right)^{-1}\left[\frac{2i}{\lambda_0}\cos\lambda_0 a\sin\lambda a - \frac{2i\lambda}{\lambda_0^2}\sin\lambda_0 a\cos\lambda a\right]$$

(4) $\hat f(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-a|x|}\,e^{-i\lambda x}dx$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^{+\infty}e^{-ax}e^{-i\lambda x}dx + \int_{-\infty}^{0}e^{ax}e^{-i\lambda x}dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{+\infty}e^{-ax}\,2\cos\lambda x\,dx$$

$$= \frac{2}{\sqrt{2\pi}}\left[-\frac{1}{a}\int_0^{+\infty}\cos\lambda x\,de^{-ax}\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[-\frac{1}{a}e^{-ax}\cos\lambda x\Big|_0^{a} - \frac{\lambda}{a}\int_0^{+\infty}e^{-ax}\sin\lambda x\,dx\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[\frac{1}{a} + \frac{\lambda^2}{a^2}(-1)\int_0^{+\infty}\cos\lambda x\,e^{-ax}dx\right]$$

$$\therefore \hat f(\lambda) = \frac{a}{a^2+\lambda^2}\sqrt{\frac{2}{\pi}}$$

2.(1) $f(x) = x^2 U(x)$, $U = \begin{cases}1 & |x|<a\\ 0 & |x|\geq a\end{cases}$

则 $\hat f(\lambda) = -\frac{d^2\hat U(\lambda,t)}{d\lambda^2}$

$$\hat U = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}e^{-i\lambda x}dx = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{-i\lambda}\right)e^{-i\lambda x}\Big|_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{-i\lambda}\left[e^{-i\lambda a}-e^{i\lambda a}\right] = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{i\lambda}\cdot 2i\sin\lambda a$$

$$= \frac{2}{\sqrt{2\pi}}\,\frac{\sin\lambda a}{\lambda}$$

$$\hat f(\lambda) = -\frac{2}{\sqrt{2\pi}}\,\frac{\lambda a\cos\lambda a - \sin\lambda a}{\lambda^2}$$

(2) $f(x) = x e^{-a|x|} \quad (a>0)$

$$\hat{f}(\lambda) = i \frac{d\hat{v}}{d\lambda} \qquad v = e^{-a|x|} \quad \hat{v} = \frac{a}{a^2+\lambda^2}$$

$$\hat{f}(\lambda) = i(-1)\frac{a\,2\lambda}{(a^2+\lambda^2)^2} = i\frac{-2a\lambda}{(a^2+\lambda^2)^2}$$

(3) $$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{\mu x} e^{-i\lambda x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{(\mu-i\lambda)x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{\mu-i\lambda}\left[e^{a}e^{-i\lambda a} - e^{-a}e^{i\lambda a}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{\mu-i\lambda}\left[e^{a}(\cos\lambda a - i\sin\lambda a) - e^{-a}(\cos\lambda a + i\sin\lambda a)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{\mu-i\lambda}\left[\cos\lambda a\cdot 2\sinh a - i\sin\lambda a\cdot 2\cosh a\right]$$

(4) $$\sin(\lambda_0 x) = \frac{1}{2i}\left(e^{i\lambda_0 x} - e^{-i\lambda_0 x}\right)$$

故 $$f(x) = \frac{1}{2i}\left[e^{i\lambda_0 x}e^{-a|x|} - e^{-i\lambda_0 x}e^{-a|x|}\right]$$

对 $\lambda_0 \in \mathbb{R}$ 易证 $\widehat{e^{i\lambda_0 x} f(x)} = \hat{f}(\lambda-\lambda_0)$

$$\hat{f}(x) = \frac{1}{2i}\left[\sqrt{\frac{2}{\pi}}\frac{a}{a^2+(\lambda-\lambda_0)^2} - \sqrt{\frac{2}{\pi}}\frac{a}{a^2+(\lambda+\lambda_0)^2}\right]$$

(5) $$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-L}^{L} e^{i\lambda_0 x}e^{-i\lambda x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{i(\lambda_0-\lambda)} e^{i(\lambda_0-\lambda)x}\Big|_{-L}^{L}$$

$$= \sqrt{\frac{2}{\pi}}\frac{\sin(\lambda_0-\lambda)L}{\lambda_0-\lambda} \quad (\lambda\neq\lambda_0)$$

若 $\lambda=\lambda_0$ 则 $\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-L}^{L} dx = \sqrt{\frac{2}{\pi}}L$

(6) $$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ax^2+ibx+c}\cdot e^{-i\lambda x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-a(x-\frac{ib}{2a})^2 + c + a(\frac{ib}{2a})^2}\cdot e^{-i\lambda x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{c-\frac{b^2}{4a}}\int_{-\infty}^{+\infty} e^{-a(x-\frac{ib}{2a})^2} e^{-i\lambda x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{c-\frac{b^2}{4a}} e^{-i\lambda\frac{ib}{2a}}\int_{-\infty}^{+\infty} e^{-ax^2}e^{-i\lambda x}\,dx$$

$$= e^{c-\frac{b^2}{4a}} e^{\frac{b\lambda}{2a}} \frac{1}{\sqrt{2a}} e^{-\lambda^2/(4a)}$$

(7) $f(x) = f(-x) \quad (f(x))^{\vee}(\lambda) = \hat{f}(-\lambda) = (f(-x))^{\wedge}(\lambda) = \hat{f}(\lambda)$

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{1}{a^2+x^2}e^{-i\lambda x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{1}{a^2+x^2}e^{i\lambda x}\,dx = \left(\frac{1}{a^2+x^2}\right)^{\vee}$$

$$\left(\sqrt{\frac{\pi}{2}}\frac{1}{a}e^{-a|x|}\right)^{\wedge} = \frac{1}{a^2+\lambda^2}$$

$$\sqrt{\frac{\pi}{2}}a^{-1}e^{-a|x|} = \left(\frac{1}{a^2+\lambda^2}\right)^{\vee}$$

$$\hat{f}(\lambda) = a e^{-a|\lambda|}$$

(8) $$\hat{f}(\lambda) = i\frac{d}{d\lambda}\left(\frac{1}{a^2+x^2}\right)^{\wedge} = -ia^2 e^{-a|\lambda|}, \quad \lambda>0$$

(9) $$f(x) = \frac{1}{(a^2+x^2)^2} = \frac{1}{a^2}\frac{a^2+x^2-x^2}{(a^2+x^2)^2} = \frac{1}{a^2}\left[\frac{1}{a^2+x^2} - x^2\frac{1}{a^2+x^2}\right]$$

$$\hat{f}(\lambda) = \frac{1}{a^2}\left[a e^{-a|\lambda|} + \frac{d^2}{d\lambda^2}\left(a e^{-a|\lambda|}\right)\right]$$

3. (1) $$(f(\lambda))^{\vee} = \frac{1}{a\sqrt{2t}} e^{-\frac{x^2}{4a^2 t}}$$

(2) $f(\lambda) = e^{[-a^2(\lambda - \frac{ib}{2a^2})^2 + c + a^2\cdot\frac{-b^2}{4a^4}]t}$

$$= e^{(c-\frac{b^2}{4a^2})t}\, e^{-a^2(\lambda-\frac{ib}{2a^2})^2 t}$$

平移 $(f(\lambda-\xi))^\vee = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(\lambda-\xi)e^{i\lambda x}d\lambda$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(\lambda')e^{i(\lambda'+\xi)x}d\lambda'$$

$$\therefore (f(\lambda))^\vee = e^{(c-\frac{b^2}{4a^2})t}\, e^{i(\frac{ib}{2a^2})x}\, (e^{-a^2\lambda^2 t})^\vee$$

$$= e^{(c-\frac{b^2}{4a^2})t}\, e^{-\frac{b}{2a^2}x}\cdot\frac{1}{a\sqrt{2t}}\, e^{-x^2/(4a^2t)}$$

(3) $(f(\lambda))^\vee = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-|\lambda|y}e^{i\lambda x}d\lambda$

$$=\frac{1}{\sqrt{2\pi}}\left(\int_0^{+\infty} e^{-\lambda y}e^{i\lambda x}d\lambda + \int_{-\infty}^0 e^{\lambda y}e^{i\lambda x}d\lambda\right)$$

$$=\frac{1}{\sqrt{2\pi}}\left(\frac{1}{-y+ix}(-1)+\frac{1}{y+ix}\right)$$

$$=\frac{1}{\sqrt{2\pi}}\frac{-y-ix+(-y)+ix}{(-y+ix)(y+ix)}$$

$$=\frac{1}{\sqrt{2\pi}}\frac{-2y}{-y^2-x^2}$$

$$=\sqrt{\frac{2}{\pi}}\frac{y}{y^2+x^2}$$

4. (1) 对 $u(x,t)$ 关于 x 作 Fourier 变换 记 $\hat u(\lambda,t) = (u(x,t))^\wedge$

则有 $\frac{d\hat u(\lambda,t)}{dt} + a^2\lambda^2\hat u - ib\lambda\hat u - c\hat u = \hat f(\lambda,t)$

故 $\begin{cases} \frac{d\hat u(\lambda,t)}{dt} + (a^2\lambda^2 - ib\lambda - c)\hat u = \hat f \\ \hat u|_{t=0} = \hat\varphi(\lambda) \end{cases}$

$$\Longrightarrow \hat u(\lambda,t) = e^{(-a^2\lambda^2+ib\lambda+c)t}\hat\varphi(\lambda) + \int_0^t e^{(-a\lambda^2+ib\lambda+c)(t-\tau)}\hat f(\lambda,\tau)d\tau$$

$$\therefore u(x,t) = e^{(c-\frac{b^2}{4a^2})t}\frac{1}{a\sqrt{2t}}\int_{-\infty}^{+\infty} e^{-(x-\xi)^2/(4a^2t)}\cdot e^{-\frac{b}{2a^2}(x-\xi)}\varphi(\xi)d\xi$$

$$+\int_0^t \frac{1}{a\sqrt{t-\tau}}\, e^{(c-\frac{b^2}{4a^2})(t-\tau)}\int_{-\infty}^{+\infty} e^{-(x-\xi)^2/4a^2(t-\tau)}\cdot e^{-\frac{b}{2a^2}(x-\xi)}\cdot f(\xi,\tau)\,d\xi\, d\tau$$

(2) $(u(x,y))^\wedge = \hat u(\lambda,y)$ 对方程关于 x 作 Fourier 变换，得

$\begin{cases} (i\lambda)^2\hat u(\lambda,y)+\frac{\partial^2\hat u}{\partial y^2}=0 \\ \hat u|_{y=0}=\hat\varphi(\lambda) \end{cases} \Longrightarrow \frac{d^2\hat u}{dy^2}-\lambda^2\hat u(\lambda,y)=0$

$\Longrightarrow \hat u(\lambda,y) = Ce^{-|\lambda|y}$ 由初始条件 $\Rightarrow \hat u(\lambda,y)=\hat\varphi(\lambda)e^{-|\lambda|y}$

$$u(x,y) = (e^{-|\lambda|y})^\vee * \varphi$$

$$=\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{y}{y^2+(x-\xi)^2}\varphi(\xi)d\xi$$

5.(1) $\forall \varphi \in \mathcal{D}(\mathbb{R})$ 有

$$\langle \varphi(x)\delta(x), \psi(x)\rangle = \langle \delta(x), \varphi(x)\psi(x)\rangle$$
$$= \varphi(0)\psi(0) = \varphi(0)\langle \delta, \psi\rangle = \langle \varphi(0)\delta, \psi\rangle$$
$$\therefore \varphi(x)\delta(x) = \varphi(0)$$

(2) $\forall \psi \in \mathcal{D}(\mathbb{R})$ 有

$$\langle \varphi(x)\delta'(x), \psi\rangle = \langle \delta', \varphi(x)\psi\rangle = -\langle \delta, (\varphi\psi)'\rangle$$
$$= -\langle \delta, \varphi\psi' + \varphi'\psi\rangle = -(\varphi(0)\psi'(0) + \varphi'(0)\psi(0))$$
$$= \varphi(0)\langle \delta', \psi\rangle - \varphi'(0)\langle \delta, \psi\rangle$$
$$= \langle \varphi(0)\delta' - \varphi'(0)\delta, \psi\rangle$$
$$\therefore \varphi(x)\delta'(x) = \varphi(0)\delta' - \varphi'(0)\delta$$

(3) $\langle x\delta^{(m)}(x), \varphi\rangle = \langle \delta^{(m)}(x), x\varphi\rangle = (-1)^m\langle \delta, (x\varphi)^{(m)}\rangle$

$$= (-1)^m\langle \delta, x\cdot\varphi^{(m)}(x) + m\varphi^{(m-1)}(x)\rangle$$
$$= (-1)^m m\cdot\varphi^{(m-1)}(0)$$
$$= (-1)^m m\langle \delta, \varphi^{(m-1)}\rangle$$
$$= -m\langle \delta^{(m-1)}, \varphi\rangle$$
$$\therefore x\delta^{(m)}(x) = -m\delta^{(m-1)}$$

(4) $\forall \varphi \in \mathcal{D}(\mathbb{R})$，有

$$\langle x^m\delta^{(m)}(x), \varphi\rangle = \langle \delta^{(m)}, x^m\varphi(x)\rangle$$
$$= \langle \delta, (x^m\varphi(x))^{(m)}\rangle$$
$$= (-1)^{(m)}\langle \delta, C_m^0 x^m\varphi^{(m)} + \cdots + C_m^m m!\varphi(x)\rangle$$
$$= (-1)^{(m)} m!\varphi(0) = (-1)^m m!\langle \delta, \varphi\rangle$$
$$\therefore x^m\delta^{(m)}(x) = (-1)^m m!\delta(x)$$

(5) $\forall \varphi \in \mathcal{D}(\mathbb{R})$，有

$$\langle (H(x)\rho(x))', \varphi\rangle = -\langle H(x)\rho(x), \varphi'\rangle$$
$$= -\int_0^{+\infty}\rho(x)\varphi'(x)dx$$
$$= \rho(0)\varphi(0) + \int_{-\infty}^{+\infty}H(x)\rho'(x)\varphi(x)dx$$
$$= \rho(0)\langle \delta, \varphi\rangle + \langle H\rho', \varphi\rangle$$
$$= \langle \rho(0)\delta + H\rho', \varphi\rangle$$
$$\therefore (H(x)\rho(x))' = \delta(x)\rho(0) + H(x)\rho'(x)$$

6.(1) $\forall \varphi \in \mathcal{D}(\mathbb{R})$，若 $m \geq 2$

$$\langle |x|^{(m)}, \varphi\rangle = (-1)^m\langle |x|, \varphi^{(m)}\rangle$$
$$= (-1)^m\int_{-\infty}^{+\infty}|x|\varphi^{(m)}dx$$
$$= (-1)^m\left[\int_0^{+\infty}x\varphi^{(m)}(x)dx + \int_{-\infty}^0(-x)\varphi^{(m)}(x)dx\right]$$
$$= (-1)^m[2\varphi^{(m-2)}(0)]$$
$$= 2\cdot(-1)^m\langle \delta, \varphi^{(m-2)}\rangle$$
$$= 2\langle \delta^{(m-2)}, \varphi\rangle$$
$$\therefore (|x|)^{(m)} = 2\delta^{(m-2)}$$

若 $m = 1$，则

$$\langle |x|', \varphi\rangle = -\langle |x|, \varphi'\rangle = -\left(\int_0^{+\infty}x\varphi'(x)dx + \int_{-\infty}^0 -x\varphi'(x)dx\right)$$

$$= -\left[-\int_0^{+\infty}\varphi\,dx + \int_{-\infty}^{0}\varphi\,dx\right]$$

$$= \int_{-\infty}^{+\infty} H(x)\varphi(x)\,dx - \int_{-\infty}^{+\infty} H(-x)\varphi(x)\,dx$$

$$= \langle H(x) - H(-x), \varphi\rangle$$

$$\therefore (|x|)' = H(x) - H(-x)$$

(2) $\langle (H(x)\sin x)', \varphi\rangle = -\langle H(x)\sin x, \varphi'\rangle$

$$= -\int_0^{+\infty}\sin x\cdot\varphi(x)\,dx$$

$$= -\sin x\,\varphi(x)\Big|_0^{+\infty} + \int_0^{+\infty}\cos x\,\varphi(x)\,dx$$

$$= \int_{-\infty}^{+\infty} H(x)\cos x\,\varphi(x)\,dx$$

$$\therefore (H(x)\sin x)' = H(x)\cos x$$

(3) $\langle (H(x)e^{ax})'', \varphi\rangle = \langle H(x)e^{ax}, \varphi''\rangle$

$$= \int_0^{+\infty} e^{ax}\varphi''(x)\,dx$$

$$= e^{ax}\cdot\varphi'(x)\Big|_0^{+\infty} - a\int_0^{+\infty} e^{ax}\varphi'(x)\,dx$$

$$= \varphi'(0) - a\left(e^{ax}\varphi\Big|_0^{+\infty} - a\int_0^{+\infty} e^{ax}\varphi(x)\,dx\right)$$

$$= \varphi'(0) + a\varphi(0) + a^2\int_{-\infty}^{+\infty} H(x)e^{ax}\varphi(x)\,dx$$

$$= -\langle\delta', \varphi\rangle + a\langle\delta, \varphi\rangle + a^2\langle H(x)e^{ax}, \varphi\rangle$$

$$\therefore (H(x)e^{ax})'' = -\delta' + a\delta + a^2 H(x)e^{ax}$$

7. (1) $\langle f', \varphi\rangle = -\langle f, \varphi'\rangle = -\int_0^{+\infty}\sin x\,\varphi'(x)\,dx$

$$= -\sin x\cdot\varphi\Big|_0^{+\infty} + \int_0^{+\infty}\cos x\cdot\varphi(x)\,dx$$

$$= \langle H(x)\cos x, \varphi\rangle$$

$$\therefore f' = H(x)\cos x$$

(2) $\langle f', \varphi\rangle = -\langle f, \varphi'\rangle = -\int_0^{+\infty}\cos x\,\varphi'(x)\,dx$

$$= -\varphi'(0) - \int_{-\infty}^{+\infty} H(x)\sin x\,\varphi(x)\,dx$$

$$= \langle\delta', \varphi\rangle - \langle H(x)\sin x, \varphi\rangle$$

$$\therefore f' = \delta' - H(x)\sin x$$

(3) $\langle f', \varphi\rangle = -\langle f, \varphi'\rangle = -\int_{-1}^{1} x^2\varphi'(x)\,dx$

$$= -x^2\varphi\Big|_{-1}^{1} + \int_{-1}^{1} 2x\cdot\varphi(x)\,dx$$

$$= -\varphi(1) + \varphi(-1) + \int_{-1}^{1} 2x\,\varphi(x)\,dx$$

$$= -\langle\delta(x-1), \varphi\rangle + \langle\delta(x+1), \varphi\rangle + \int_{-1}^{1} 2x\,\varphi(x)\,dx$$

记 $g(x) = \begin{cases} 2x, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

则 $f' = -\delta(x-1) + \delta(x+1) + g(x)$

8. (1) 令 $z=\dfrac{x}{2a\sqrt{t}}$

$$\frac{\partial\Phi}{\partial t}-a^2\frac{\partial^2\Phi}{\partial x^2}=\Phi'(z)\frac{1}{2a}\left(-\frac{x}{2\sqrt{t}\cdot t}\right)-a^2\left(\Phi'(z)\frac{1}{2a\sqrt{t}}\right)_x$$

$$a^2\left(\Phi'(z)\frac{1}{2a\sqrt{t}}\right)_x=a^2\frac{1}{2a\sqrt{t}}\left(\Phi'(z)\cdot\left(-2\frac{x}{2a\sqrt{t}}\right)\frac{1}{2a\sqrt{t}}\right)$$

$$=\Phi'(z)\cdot\frac{1}{2t}\cdot\frac{-x}{2a\sqrt{t}}$$

$$\therefore\ \frac{\partial\Phi}{\partial t}-a^2\frac{\partial^2\Phi}{\partial x^2}=0$$

令 $V=u-U_0$ 则

$$\begin{cases}V_t-a^2V_{xx}=0\\ V(x,0)=u(x,0)-U_0=-U_0\\ V|_{x=0}=0\end{cases}$$

令 $V=C\cdot\Phi\left(\dfrac{x}{2a\sqrt{t}}\right)$ 则 V满足方程 且 $V|_{x=0}=0$

由 $V|_{t=0}=-U_0$ 得

$$C\frac{2}{\sqrt{\pi}}\int_0^{+\infty}e^{-\xi^2}d\xi=C=-U_0$$

$$\therefore\ u(x,t)=-U_0\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2a\sqrt{t}}}e^{-\xi^2}d\xi+U_0$$

(2) 偶延拓 $\bar{u}(x,0)=\begin{cases}U_0 & 0\le x\le 1\\ 0 & x>1\\ U_0 & -1\le x<0\\ 0 & x<-1\end{cases}$

则 $\begin{cases}\bar{u}_t-a^2\bar{u}_{xx}=0\\ \dfrac{\partial\bar{u}}{\partial x}\Big|_{x=0}=0\\ \bar{u}|_{t=0}=\text{given}\end{cases}$

令 $\bar{u}=C_1\left[\Phi\left(\frac{x}{2a\sqrt{t}}\right)+\Phi\left(\frac{-x}{2a\sqrt{t}}\right)\right]+C_2\left[\Phi\left(\frac{x-1}{2a\sqrt{t}}\right)+\Phi\left(\frac{-x-1}{2a\sqrt{t}}\right)\right]$

满足方程和边界条件，再由初始条件知

$x\in[0,1]\Rightarrow$

$$C_1\frac{2}{\sqrt{\pi}}\left(\int_0^{+\infty}e^{-\xi^2}d\xi+\int_0^{-\infty}e^{-\xi^2}d\xi\right)+C_2\frac{2}{\sqrt{\pi}}\left(\int_0^{-\infty}e^{-\xi^2}d\xi+\int_0^{-\infty}e^{-\xi^2}d\xi\right)=U_0$$

$x\in(1,+\infty)\Rightarrow$

$$C_1\frac{2}{\sqrt{\pi}}\left(\int_0^{+\infty}e^{-\xi^2}d\xi+\int_0^{-\infty}e^{-\xi^2}d\xi\right)+C_2\frac{2}{\sqrt{\pi}}\left(\int_0^{+\infty}e^{-\xi^2}d\xi+\int_0^{-\infty}e^{-\xi^2}d\xi\right)=0$$

$$\Longrightarrow\ \begin{matrix}C_1\cdot 0+C_2(-2)=U_0\\ C_1\cdot 0+C_2\cdot 0=0\end{matrix}\qquad\Rightarrow\ C_2=-\frac{U_0}{2}$$

解为 $u=-\dfrac{U_0}{2}\left[\Phi\left(\frac{x-1}{2a\sqrt{t}}\right)+\Phi\left(\frac{-x-1}{2a\sqrt{t}}\right)\right]+C\left[\Phi\left(\frac{x}{2a\sqrt{t}}\right)+\Phi\left(\frac{-x}{2a\sqrt{t}}\right)\right]$

(解不唯一)

(3)

(4)

9. (1) 令 $u = X(x)T(t)$ 代入方程

$$X(x)T'(t) - X''(x)T(t) - X(x)T(t) = 0$$

$$\therefore \frac{X''(x)}{X(x)} = -\lambda = \frac{T'(t) - T(t)}{T(t)}$$

由边界条件 $\Rightarrow X_n(x) = \sin\frac{n\pi}{l}x \qquad \lambda_n = \left(\frac{n\pi}{l}\right)^2$

代入 $T'(t) + (\lambda - 1)T(t) = 0 \Rightarrow T_n(t) = C_n e^{-a^2\left[\left(\frac{n\pi}{l}\right)^2 - 1\right]t}$

$$u = \sum_{n=1}^{\infty} C_n e^{-a^2\left[\left(\frac{n\pi}{l}\right)^2 - 1\right]t} \sin\frac{n\pi}{l}x$$

由初始条件得 $\sum C_n \sin\frac{n\pi}{l}x = 1 \qquad C_n = 1$

$$\therefore u = \sum_{n=1}^{\infty} e^{-\left[\left(\frac{n\pi}{l}\right)^2 - 1\right]} \sin\frac{n\pi}{l}x$$

(2) $u = \sum_{n=1}^{\infty} C_n e^{-a^2\left(\frac{n\pi}{\pi}\right)^2 t} \cdot \cos\frac{n\pi}{\pi}x$

$$= \sum_{n=0}^{\infty} C_n e^{-a^2n^2t} \cos nx$$

由初始条件 得 $C_n = \dfrac{\int_0^{\pi} \sin x \cos nx \, dx}{\int_0^{\pi} \cos^2 nx \, dx} = \begin{cases} 0 & n \geq 1 \\ \frac{2}{\pi} & n = 0 \end{cases}$

(3) $u(x,t) = \sum_{n=0}^{\infty} C_n e^{-a^2\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)^2 t} \cos\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)x$

由 $\sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)x = x^2(1-x)$

$$C_n = \frac{\int_0^l x^2(1-x)\cos\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)x \, dx}{\int_0^l \cos\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)x \, dx}$$

(4) 令 $V = u - Atx/l$. 则有

$$\begin{cases} V_t - a^2 V_{xx} = -Ax/l \\ V|_{x=0} = 0 \\ V|_{x=l} = u|_{x=0} - At = 0 \\ V|_{t=0} = u|_{t=0} = 0 \end{cases}$$

$$V(x,t) = \int_0^t \sum_{n=1}^{\infty} \left(f_n(\tau) e^{-a^2\left(\frac{n\pi}{l}\right)^2(t-\tau)} d\tau \sin\frac{n\pi}{l}x\right)$$

$$= \sum_{n=1}^{\infty} \frac{2}{l}\int_0^l (-Ax/l)\sin\frac{n\pi}{l}x \, dx \int_0^t e^{-a^2\left(\frac{n\pi}{l}\right)^2(t-\tau)} d\tau \cdot \sin\frac{n\pi}{l}x$$

(5) 令 $V = u - x$. 则有

$$\begin{cases} V_t - a^2 V_{xx} = x(l-x) = f(x) \\ V|_{t=0} = \sin\frac{n\pi}{l}x - x = \varphi(x) \\ V|_{x=0} = 0 \\ V_x|_{x=l} = u_x|_{x=l} - 1 = 0 \end{cases}$$

$$V(x,t) = \sum_{n=1}^{\infty} \varphi_n \cdot e^{-a^2\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)^2 t} \sin\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)x +$$

$$\sum_{n=1}^{\infty} \int_0^t f_n e^{-a^2\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)^2(t-\tau)} d\tau \sin\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)x$$

其中 $\varphi_n = \int_0^l \varphi(x)\sin\frac{n\pi + \frac{\pi}{2}}{l}x\,dx \Big/ \int_0^l \sin^2\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)x\,dx$

$$f_n = \int_0^l f(x)\sin\frac{n\pi + \frac{\pi}{2}}{l}x\,dx \Big/ \int_0^l \sin^2\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)x\,dx$$

(6) 令 $V = u - q x^2/2l$ 则有

$$\begin{cases} V_t - a^2 V_{xx} = a^2 q/l = f(x) \\ V|_{t=0} = -q x^2/2l = \varphi(x) \\ V_x|_{x=0} = 0 \\ V_x|_{x=l} = 0 \end{cases}$$

$$V(x,t) = \sum_{n=0}^{\infty} \varphi_n \cdot e^{-(\frac{n\pi}{l})^2 t} \cos\frac{n\pi}{l}x + \sum_{n=0}^{\infty} \int_0^t f_n e^{-(\frac{n\pi}{l})^2(t-\tau)} d\tau \cos\frac{n\pi}{l}x$$

$$\sum \varphi_n \cos\beta_n x = -u_0$$

$$\therefore \varphi_n = \int_0^l -u_0 \cos\beta_n x\, dx \Big/ \int_0^l \cos^2\beta_n x\, dx$$

10. (1) 令 $V = u - \left(\frac{1+hl}{2h+h^2l} - \frac{hx}{2h+h^2l}\right)hu_1 + \left(\frac{1}{2h+h^2l} + \frac{h}{2h+h^2l}x\right)hu_2$

$= u - w$

则 $\begin{cases} V_t - a^2 V_{xx} = 0 \\ V|_{t=0} = \varphi(x) - w \\ \left[-\frac{\partial V}{\partial x} + hV\right]\Big|_{x=0} = 0 \\ \left(\frac{V}{\partial x} + hV\right)\Big|_{x=l} = 0 \end{cases}$

特征值 $\lambda_n = \beta_n^2$ β_n 满足 $\tan\beta l = \frac{2h\beta}{\beta^2 - h^2}$

记正根为 $0 < \beta_1 < \beta_2 < \cdots$ $\lim\limits_{n\to\infty}\beta_n = +\infty$ 则

$$V(x,t) = \sum_{n=1}^{\infty} (\varphi_n - w_n) e^{-a^2\beta_n^2 t}\left(\cos\beta_n x + \frac{h}{\beta_n}\sin\beta_n x\right)$$

易证 $\lim\limits_{t\to+\infty} V(x,t) = 0$

故 $\lim\limits_{t\to+\infty} u = \lim\limits_{t\to+\infty}(V + w) = w = \frac{xh}{2+hl}(u_2 - u_1) + \frac{u_1 + u_2 + hlu_1}{2+hl}$

(2) 令 $V = e^{ht}(u - u_0)$ 则

$$\begin{cases} V_t - a^2 V_{xx} = he^{ht}(u - u_0) + e^{ht}u_t - a^2 u_{xx} e^{ht} = 0 \\ V|_{t=0} = \varphi(x) - u_0 \\ V_x|_{x=0,l} = e^{ht} u|_{x=0,l} = 0 \end{cases}$$

$$\therefore V(x,t)=\sum_{n=0}^{\infty}(\varphi_n-V_{0n})e^{-(\frac{an\pi}{l})^2 t}\cos\frac{n\pi}{l}x$$
$$=(\varphi_0-V_{00})+\sum_{n=1}^{\infty}(\varphi_n-V_{0n})e^{-(\frac{an\pi}{l})^2 t}\cos\frac{n\pi}{l}x$$
$$\longrightarrow \varphi_0-V_{00}\quad (t\to+\infty)$$

$$\lim_{t\to+\infty}u=\lim_{t\to+\infty}(e^{-ht}V+V_0)=V_0$$

11. 记 $Q_T=\{(x,t)\,|\,0<x<l=l_1+l_2,\ 0<t\le T\}$

$$u_t-a_1^2u_{xx}=0\qquad 0<x<l_1,\ t>0$$
$$u_t-a_2^2u_{xx}=0\qquad 0<x<l_1+l_2,\ t>0$$
$$u|_{t=0}=\varphi(x)\qquad 0<x<l_1+l_2$$
$$u|_{x=0,\,l_1+l_2}=0$$

$$\begin{cases}u(l_1-0,t)=u(l_1+0,t)\\ k_1u_x(l_1-0,t)=k_2u_x(l_1+0,t)\end{cases}$$

设 $u(l_1-0,t)=u(l+0,t)=g(t)$ (待定)

求解 $\begin{cases}u_t-a_1^2u_{xx}=0 & 0<x<l_1\\ u|_{t=0}=\varphi(x)\\ u|_{x=0}=0\\ u|_{x=l}=g\end{cases}$

令 $v=u-\frac{x}{l}g$ 则

$$v_t-a^2v_{xx}=-\frac{x}{l}g'(t)+0=-\frac{x}{l}g'(t)=f(x,t)$$
$$v|_{t=0}=-\frac{x}{l}g(0)=G$$
$$v|_{x=0}=0$$
$$v|_{x=l}=0$$

$$V(x,t)=\sum_{n=1}^{\infty}G_n e^{-(\frac{an\pi}{l_1})^2t}\sin\frac{n\pi}{l_1}x+\sum_{n=1}^{\infty}\int_0^t f_n(\tau)e^{-(\frac{an\pi}{l_1})^2(t-\tau)}d\tau\sin\frac{n\pi}{l_1}x$$

$\therefore u=v+\frac{x}{l}g$ in $0<x<l_1$

考虑 $\begin{cases}u_t-a^2u_{xx}=0 & \text{in } l_1<x<l_1+l_2\\ u|_{t=0}=\varphi\\ u|_{x=l_1}=g\\ u|_{x=l_1+l_2}=0\end{cases}$

令 $h(x,t)=u(x,t)-\frac{l_2+l_1-x}{l_2}g(t)$ 则

$$\begin{cases}h_t-a^2h_{xx}=-\frac{l_2+l_1-x}{l_2}g'(t)=\tilde f(x,t) & l_1<x<l_1+l_2\\ h|_{t=0}=\varphi(x)-\frac{l_2+l_1-x}{l_2}g(0)=\Phi(x)\\ h|_{x=l_1}=0\\ h|_{x=l_1+l_2}=0\end{cases}$$

故 $h(x,t)=\sum C_n e^{-(\frac{an\pi}{l_2})^2t}\sin\frac{n\pi}{l_2}(x-l_1)$
$$+\sum\int_0^t\tilde f_n(\tau)e^{-(\frac{an\pi}{l_2})^2(t-\tau)}d\tau\sin\frac{n\pi}{l_2}(x-l_1)$$
$$=\sum\Phi_n e^{-(\frac{an\pi}{l_2})^2t}\sin\frac{n\pi}{l_2}(x-l_1)$$
$$+\sum\int_0^t\tilde f_n(\tau)e^{-(\frac{an\pi}{l_2})^2(t-\tau)}d\tau\sin\frac{n\pi}{l_2}(x-l_1)$$

$\therefore u = h(x,t) + \frac{l_1+l_2-x}{l_2} g(t)$, $l_1 < x < l_1+l_2$

由连接条件得

$V|_{x=l_1} + g\cdot\frac{x}{l} = h|_{x=l_1} + \frac{l_2+l_1-x}{l_2} g$ 自然成立

另外 $k_1\left[\sum_{n=1}^{\infty}\int_0^t f_n(\tau) e^{-\left(\frac{an\pi}{l_1}\right)^2(t-\tau)} d\tau \frac{n\pi}{l_1}(-1)^n\right.$

$\left. + \sum_{n=1}^{\infty} G_n e^{-\left(\frac{an\pi}{l_1}\right)^2 t}\cdot\frac{n\pi}{l_1}(-1)^n + \frac{g}{l_1}\right]$

$= k_2\left[\sum_{n=1}^{\infty}\Phi_n e^{-\left(\frac{an\pi}{l_2}\right)^2 t}\frac{n\pi}{l_2} + \sum_{n=1}^{\infty}\int_0^t f_n(\tau) e^{-\left(\frac{an\pi}{l_2}\right)^2(t-\tau)} d\tau\right.$

$\left.\cdot\frac{n\pi}{l_2} - \frac{g(t)}{l_2}\right]$

找 $g(t)$ 满足的方程和定解条件 ??

13. 证明：令 $V = u_t$ 则有

$$\begin{cases} V_t - V_{xx} = f_t(x,t) \\ V|_{x=0,l} = 0 \\ V|_{t=0} = u_t|_{t=0} = (f+u_{xx})|_{t=0} = f(x,0) + \varphi''(x) \end{cases}$$

由最大模估计，知

$\max_{\overline{Q}} |V(x,t)| \le C\left[\|f_t\|_{C(\overline{Q})} + \|f(x,0)\|_{C[0,l]} + \|\varphi''\|_{C[0,l]}\right]$

即 $\max_{\overline{Q}} |u_t(x,t)| \le C\left[\|f\|_{C^1(\overline{Q})} + \|\varphi''\|_{C[0,l]}\right]$

14 (1) 因为 $\varphi\in C^1[0,l]$ $\forall x\in[0,l]$

有 $\varphi(x) = \varphi(x)-\varphi(0) = \varphi'(\xi)x \le \|\varphi'\|_{C[0,l]}\cdot x$

记 $V(x,t) = \|\varphi'\|_{C[0,l]}\cdot x$

$u_t - u_{xx} = 0 \le V_t - V_{xx} = 0$

$u|_{t=0} = \varphi(x) \le V(x,0) = V(x)$

$u|_{x=0} = 0 \le V|_{x=0}$

$u|_{x=l} = 0 \le V|_{x=l}$

由比较原理 $\Rightarrow u\le V$ in $\overline{Q}$

$\frac{u(x,t)-u(0,t)}{x} \le \frac{V(x,t)-V(0,t)}{x}$, $x>0$

令 $x\to 0^+$ 得 $\frac{\partial u(0,t)}{\partial x} \le \lim_{x\to 0^+}\|\varphi'\|_{C[0,l]} = \|\varphi'\|_{C[0,l]}$

同法 由 $\varphi(x) \ge -\|\varphi'\|_C\cdot x$

$\frac{u(x,t)-u(0,t)}{x} \ge \frac{V(x,t)-\|\varphi'\|_C}{x} = -\|\varphi'\|_{C[0,l]}$

令 $x\to 0^+$，得 $\frac{\partial u(0,t)}{\partial x} \ge -\|\varphi'\|_{C[0,l]}$

$\Rightarrow \left\|\frac{\partial u}{\partial x}(0,t)\right\| \le \|\varphi'\|_{C[0,l]} \le \|\varphi\|_{C^1[0,l]}$

再由 $\varphi(x)-\varphi(l) = \varphi'(\xi)(l-x) \Rightarrow$

$-\|\varphi'\|_C(l-x) \le \varphi(x) \le \|\varphi'\|_C\cdot(l-x)$

类似证明 $-\|\varphi'\|_C(l-x) \le u(x,t) \le \|\varphi'\|_C(l-x)$

$\frac{-\|\varphi'\|_C\cdot(l-x)}{l-x} \le \frac{u(x,t)-u(l,t)}{l-x} \le \frac{\|\varphi'\|_C\cdot(l-x)-0}{l-x}$

$-\|\varphi'\|_C \le -\frac{\partial u(l,t)}{\partial x} \le \|\varphi'\|_C$

$\left|\frac{\partial u}{\partial x}(l,t)\right| \le \|\varphi'\|_C \le \|\varphi\|_{C^1[0,l]}$

(2) 令 $v = u_x$, 则

$$\begin{cases} v_t - v_{xx} = 0 \\ v|_{t=0} = \varphi'(x) \\ v|_{x=0} = \frac{\partial u}{\partial x}(0,t) \\ v|_{x=l} = \frac{\partial u}{\partial x}(l,t) \end{cases}$$

由最大模估计得

$$\max_{\bar Q} |v| \le C \max\left\{ \|\varphi'\|_{C[0,l]}, \left\|\frac{\partial u(0,t)}{\partial x}\right\|_{C[0,T]}, \left\|\frac{\partial u(l,T)}{\partial x}\right\|_{C[0,T]} \right\}$$
$$\le \tilde C$$

$\tilde C$ 和 $\|\varphi\|_{C^1}$, C 有关

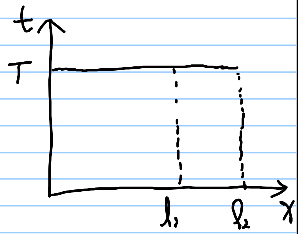
15. 令 $v = \frac{\partial u}{\partial x}$, 则有

$$\begin{cases} Lv = f_x(x,t) \\ v|_{t=0} = \varphi'(x) \\ v|_{x=0} = u_x|_{x=0} = \alpha u(0,t) + g_1(t) \\ v|_{x=l} = g_2(t) - \beta u(l,t) \end{cases}$$

由最大模估计, 得

$$\max_{\bar Q} |v(x,t)| \le C\Big(\|f_x\|_{C(\bar Q)} + \max\big\{ \|\varphi'\|_{C[0,l]}, \|g_1\|_{C[0,T]} + \alpha\|u(0,t)\|_{C[0,T]}, \|g_2\|_{C[0,T]} + \beta\|u(l,t)\|_{C[0,T]} \big\}\Big)$$
$$\le C\big[\|f_x\|_{C(\bar Q)} + \|\varphi'\|_{C[0,l]} + \|g_1\|_{C[0,T]} + \|g_2\|_{C[0,T]} + \alpha(\|f\|_{C(\bar Q)} + \|\varphi\|_{C[0,l]} + \|g_1\|_{C[0,T]} + \|g_2\|_{C[0,T]}) +$$
$$\beta(\|f\|_{C(\bar Q)} + \|\varphi\|_{C[0,l]} + \|g_1\|_{C[0,T]} + \|g_2\|_{C[0,T]})\big]$$
$$= C(\alpha,\beta)\big[\|f\|_{C^1(\bar Q)} + \|\varphi\|_{C^1[0,l]} + \|g_1\|_{C[0,T]} + \|g_2\|_{C[0,T]} \big]$$

16. 因为
$$\begin{cases} Lu_{l_2} = \frac{\partial u_{l_2}}{\partial t} - \frac{\partial^2 u_{l_2}}{\partial x^2} = 0 \text{ in } Q^{l_2} \\ u_{l_2}|_{t=0} = 0 \\ u_{l_2}(0,t) = g_1(t) \\ u_{l_2}(l_2,t) = 0 \end{cases}$$



$g_1(t) \ge 0 \Longrightarrow u_{l_2}(x,t) \ge 0$ in $\bar Q^{l_2}$

$\Longrightarrow u_{l_2}(l_1,t) \ge 0$

在 Q^{l_1} 上 $Lu_{l_1} = Lu_{l_2} = 0$

$u_{l_1}|_{t=0} = u_{l_2}|_{t=0} = 0$

$u_{l_1}|_{x=0} = u_{l_2}|_{x=0} = g_1(t)$

$u_{l_1}|_{x=l_1} = 0 \le u_{l_2}|_{x=l_1}$

$\Longrightarrow u_{l_1}(x,t) \le u_{l_2}(x,t) \quad \forall (x,t) \in Q^{l_1}$

17. (1) 反证: 设 $u \ge 0$ 不成立, 即存在 $(x_0,t_0) \in Q$.

s.t. $u(x_0,t_0) < 0$ 故 $\min_{\bar Q} u < 0$

由极值原理: $\min_{\bar Q} u = \min_{\Gamma} u$ 设 $(x^*,t^*) \in \Gamma$

s.t. $u(x^*,t^*) = \min_{\bar Q} u < 0$

由条件 $(x^*,t^*) \in \Gamma \cap \{x=0\}$ 由极值必要条件, 得

$\frac{\partial u}{\partial x}\Big|_{(x^*,t^*)} \ge 0$

$h(u_0 - u|_{(x^*,t^*)}) > 0$

$$\Longrightarrow \left[\frac{\partial u}{\partial x}+h(u_0-u)\right]\Big|_{(x^*,t^*)}>0$$ 与条件矛盾

故 $u\geq 0$ in $\overline{Q}$

令 $v=u_0-u$ 则有

$$v_t-v_{xx}=0$$
$$v|_{t=0}=u_0$$
$$\left[\frac{\partial v}{\partial x}+h(v_0-v)\right]\Big|_{x=0}=h(u_0-u_0)=0$$
$$v|_{x=l}=u_0$$

同前证明有 $v\geq 0$ in $\overline{Q}$ $\Rightarrow u_0\geq u$ in $\overline{Q}$

(2) 设 $h_1<h_2$ u_{h_1}, u_{h_2} 是相应的解，则有 $v=u_{h_1}-u_{h_2}$ 满足

$$v_t-v_{xx}=0 \quad \text{in } Q$$
$$v|_{t=0}=0$$
$$v|_{t=l}=0$$

$$\frac{\partial v}{\partial x}+(h_2-h_1)(u_0-u_{h_2})-h_1v$$
$$=\frac{\partial u_{h_2}}{\partial x}-\frac{\partial u_{h_1}}{\partial x}+(h_2-h_1)(u_0-u_{h_2})-h_1(u_{h_2}-u_{h_1})$$
$$=\frac{\partial u_{h_2}}{\partial x}-h_2u_{h_2}+h_2u_0-\left[\frac{\partial u_{h_1}}{\partial x}-h_1u_{h_1}+h_1u_0\right]=0$$

故 $\frac{\partial v}{\partial x}+h_1\left[\frac{h_2-h_1}{h_1}(u_0-u_{h_2})-v\right]=0$

$h_1>0$ $\frac{h_2-h_1}{h_1}>0$ $u_0-u_{h_2}>0$ 故 $v\geq 0$

即 $u_{h_2}\geq u_{h_1}$ u_h 关于 h 单调递增

18. 设 u 在 $\overline{Q}$ 上存在非负最大值，即 $\max\limits_{\overline{Q}} u\geq 0$，设 $(x_0,t_0)\in\overline{Q}$

s.t $u(x_0,t_0)=\max\limits_{\overline{Q}} u\geq 0$

分两种情况

i) $Lu<0$ 则证明 $(x_0,t_0)\in\Gamma$

反证 若 $(x_0,t_0)\notin\Gamma$，则 $(x_0,t_0)\in Q$ 则有

$$\frac{\partial u}{\partial x}\Big|_{(x_0,t_0)}=0$$
$$\frac{\partial^2 u}{\partial x^2}\Big|_{(x_0,t_0)}\leq 0$$
$$\frac{\partial u}{\partial t}\Big|_{(x_0,t_0)}\geq 0$$

故 $Lu|_{(x_0,t_0)}=\left(\frac{\partial u}{\partial t}-a^2u_{xx}+c(x,t)u\right)\Big|_{(x_0,t_0)}\geq 0$ 与 $Lu<0$ 矛盾

因此 $(x_0,t_0)\in\Gamma$，即 $\max\limits_{\overline{Q}} u=\max\limits_{\Gamma} u$

ii) 若 $Lu\leq 0$. 作辅助函数 $v(x,t)=u+\varepsilon e^{-Mt}$, $\varepsilon>0$

$M=\sup\limits_{Q} c+1$

则有 $Lv=Lu+\varepsilon(-Me^{-Mt}+ce^{-Mt})$

$$=Lu+\varepsilon(-M+c)e^{-Mt}\leq -\varepsilon e^{-Mt}<0$$

$\max\limits_{\overline{Q}} v\geq\max\limits_{\overline{Q}} u\geq 0$ 由(i)得 $\max\limits_{\overline{Q}} v=\max\limits_{\Gamma} v$

另外有

$$\max_{\Gamma} u\leq\max_{\overline{Q}} u\leq\max_{\overline{Q}} v=\max_{\Gamma} v\leq\max_{\Gamma} u+\varepsilon$$

令 $\varepsilon\to 0$，得 $\max\limits_{\overline{Q}} u=\max\limits_{\Gamma} u$

推论：若 $Lu\leq 0$ 则 $u|_{\overline{Q}}\leq 0$ $u|_{\Gamma}\leq 0$

证明：若 $\exists\, p(x_0,t_0)\in\overline{Q}$, s.t $u|_p>0$ 则 $\max\limits_{\overline{Q}} u>0$

故有 $\max\limits_{\overline{Q}} u=\max\limits_{\Gamma} u\le 0$ 与上面矛盾.

推论. 若 $Lu\ge 0$, 则 $u|_{\overline{Q}}\ge 0$. $u|_\Gamma\ge 0$

19. 令 $u=e^{Mt}v$

因为 $Lu=Me^{Mt}v+e^{Mt}v_t-a^2e^{Mt}v_{xx}+ce^{Mt}v$

$=e^{Mt}[v_t-a^2v_{xx}+(M+c)v]\le 0$

故 $v_t-a^2v_{xx}+(M+c)v\le 0$

取 $M=c_0$ 则 $M+c\ge c_0-c_0=0$

由 $u|_\Gamma\le 0\Rightarrow v|_\Gamma\le 0$ 由上题推论得

$v|_{\overline{Q}}\le 0\Rightarrow u|_{\overline{Q}}\le 0$

20. 证明: $Lu=u_t-u_{xx}+(u-a)u=0\quad (x,t)\in Q$

$u|_\Gamma\ge 0\qquad u-a\ge\min(u-a)$

由上题推论 知 $u|_{\overline{Q}}\ge 0$

记 $\Phi=\max\limits_{[0,l]}\varphi(x)$, $v=\Phi$ 则 $u|_\Gamma\le v|_\Gamma$

$c=\max\limits_{\overline{Q}}|a(x,t)|$ 令 $u=e^{ct}w$. 则

$Lw=w_t-w_{xx}+(c-a(x,t))w$

$=-ce^{-ct}u+e^{-ct}u_t-e^{-ct}w_{xx}+(c-a(x,t))e^{-ct}u$

$=e^{-ct}(u_t-u_{xx}-a(x,t)u)$

$=-e^{-ct}u\le 0$

$$\begin{cases} Lv=v_t-v_{xx}+(c-a(x,t))v=(c-a(x,t))\Phi\ge 0\ge Lw\\ v|_\Gamma\ge u|_\Gamma\ge w|_\Gamma\end{cases}$$

18题推论 $v\ge w$ in $\overline{Q}$ $\quad v\ge e^{-ct}u\Rightarrow$

$u\le e^{ct}v\le e^{cT}\max\limits_{[0,l]}\varphi(x)$

另法：设 $v=M\Phi$, $M>1$ 为常数. 则有 $v|_\Gamma\ge u|_\Gamma$ 且

$L(u-v)=v_t-u_{xx}+(u-v-a)(u-v)$

$=v_t-u_{xx}+(u-a)u-vu-uv-v^2+av$

$=-2uv-v^2+av=-v(v-a+2u)$

$\le -v^2+av=-v(v-a)\quad(u\ge 0,\ v\ge 0)$

若 $\Phi=0$ 则 $L(u\cdot v)=0$

若 $\Phi\ne 0$ 则 取 $M=(\Phi+\max|a(x,t)|)/\Phi$.

则有 $L(u\cdot v)\le 0$

故对 Lu 用19题 得 $u\le v$ in $\overline{Q}$ 即

$u\le M\cdot\max\limits_{[0,l]}\varphi(x)$

21. 证明: $\begin{cases} Lu=u_t-a^2u_{xx}=0 \\ u|_{t=0}=0 \\ u|_{x=0}=0\end{cases}$ 第一边值 的有界解只有零解

记 $Q_{LT}=\{(x,t)\mid 0<x<L,\ 0<t\le T\}$ u有界 故 $\exists\, k>0$.

$\sup\limits_{Q}|u|\le k\quad Q=\{(x,t)\mid x\in(0,+\infty),\ t>0\}$

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作辅助函数

$$W(x,t)=\frac{k}{L^2}(2a^2t+x^2)\pm u$$

则有 $LW=0$

$$W|_{t=0}=\frac{k}{L^2}x^2\pm 0\geq 0$$

$$W|_{x=0}=\frac{k}{L^2}(2a^2t)\pm 0\geq 0$$

$$W|_{x=L}=\frac{k}{L^2}(2a^2t+L^2)\pm u\geq 0$$

$W\geq 0$ in $\overline{Q}$ 即 $|u|\leq \frac{k}{L^2}(2a^2t+x_0^2)$

对 $\forall (x_0,t_0)\in Q$ 有 L 充分大，T 充分大 s.t.

$(x_0,t_0)\in Q_{TL}$ 则有 $|u(x,t)|\leq \frac{k}{L^2}(2a^2t_0+x_0^2)\to 0$

$L\to\infty$

22. 23 同书上证明.

故 $u(x,t)\equiv 0 \quad \forall (x,t)\in Q$ 解唯一

若考虑第二，三边值问题，则证明问题

$$\begin{cases} Lu=u_t-a^2u_{xx}=0 \\ u|_{t=0}=0 \\ (u_x+\alpha u)|_{x=0}=0, \quad \alpha>0 \end{cases}$$

的有界解为零

取 $W(x,t)=\frac{k}{L^2}(2a^2t+x^2)\pm u$ 则有

$$\begin{cases} LW=0 \\ W|_{t=0}\geq 0 \\ W|_{x=L}\geq 0 \\ (-W_x+\alpha W)|_{x=0}=\alpha\cdot\frac{k}{L^2}(2a^2t)\pm 0\geq 0 \end{cases}$$

第四章

比较定理 $Lu=-\Delta u+c(x)u$，$c(x)\geq 0$ 则有

$$\begin{cases} Lu|_\Omega\leq 0 \\ u|_{\partial\Omega}\leq 0 \end{cases} \Longrightarrow u|_\Omega\leq 0$$

$$\begin{cases} Lu|_\Omega\geq 0 \\ u|_{\partial\Omega}\geq 0 \end{cases} \Longrightarrow u|_\Omega\geq 0$$

1. (1) 作辅助函数 $W(x)=\frac{F}{c_0}\pm u$，则

$$\begin{cases} LW=c(x)\cdot\frac{F}{c_0}\pm f(x)\geq F+f\geq 0 \\ W|_{\partial\Omega}=\frac{F}{c_0}\geq 0 \end{cases}$$

$$\Rightarrow W \geq 0 \text{ in } \Omega \Rightarrow |u| \leq \frac{F}{C_0} = C^{-1} \sup_{\Omega} |f(x)|$$

(2) 作辅助函数 $W(x) = \frac{F}{2n}(d^2 - |x|^2) \pm u$

$$d = \sup_{x,y\in\Omega} |x-y|$$

$$LW = -\frac{F}{2n}(-2n) + c(x)\frac{F}{2n}(d^2-|x|^2) \pm f$$

$$= F \pm f + c(x)\frac{F}{2n}(d^2-|x|^2) \geq 0$$

$$W|_{\partial\Omega} = \frac{F}{2n}(d^2-|x|^2)\Big|_{\partial\Omega} \pm 0 \geq 0$$

$$\Rightarrow W \geq 0 \text{ in } \Omega \Rightarrow |u| \leq \frac{F}{2n}(d^2-|x|^2)$$

$$\leq \sup_{x\in\Omega}\left(\frac{d^2-|x|^2}{2n}\right)F \leq \frac{d^2}{2n}F$$

2. 记 $\Phi_1 = \sup_{\Gamma_1}|\varphi_1|$, $\Phi_2 = \sup_{\Gamma_2}|\varphi_2|$, $F = \sup_{\Omega}|f|$

$$W(x) = \frac{\Phi_1}{\alpha_0} + \Phi \quad \text{——}$$

F

3. $v(x) = |x|^{-a} - r^{-a} = |x|^{2\cdot(-\frac{a}{2})} - r^{-a}$

$$Lv = -\Delta v + c\cdot v$$

$$= -\left[\left(\frac{a^2}{4}+\frac{a}{2}\right)\cdot 4|x|^{-a-2} - an|x|^{-a-2}\right] + cv$$

$$= -[a^2+2a-an]|x|^{-a-2} + cv < 0$$

a 充分大. $x \in S^*$

$v|_{|x|=r} = 0$, $w|_{|x|=\frac{r}{2}} < 0$ (1(3)右上).

$$-\frac{\partial v}{\partial \nu}\Big|_{x=x^0} = -\frac{\partial v}{\partial n}\cos(\nu, n)\Big|_{x^0}$$

$$= a|x_0|^{-a-1}\cos(\nu, n) = a r^{-a-1}\cos(\nu, n)$$

$$> 0$$

4. 证明: 记 $Lu = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$

先证若 $Lu = f < 0$ 则 $\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+$

设 u 在 $\bar{\Omega}$ 上为负, 则 $\sup_{\Omega} u \leq 0 = \sup_{\partial\Omega} u^+$

若 $\max_{\bar{\Omega}} u \geq 0$, 设 $u(x^0) = \max_{\bar{\Omega}} \overset{\geq 0}{u}$ 则征 $x^0 \in \partial\Omega$

否则 $x^0 \in \Omega$ 有 $\frac{\partial u}{\partial x_i}\Big|_{x^0} = 0$

$$J = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)\Big|_{x^0} \leq 0 \qquad J = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & \frac{\partial^2 u}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & \cdots & \cdots & \frac{\partial^2 u}{\partial x_n^2} \end{pmatrix}\Bigg|_{x^0}$$

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}\Big|_{x^0} = tr(AJ) \le 0,$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$ 由条件 A 正定.

J 半负定.

$\Rightarrow AJ$ 半负定 ?

$\therefore tr(AJ) \le 0$.

$\left(-\sum_{i,j=1} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_i \frac{\partial u}{\partial x_i} + cu\right)\Big|_{x^0} \ge 0$ 与条件矛盾.

若 $f \le 0$, 作 $w = u(x) + \varepsilon e^{ax_1}$, $Lw = Lu + \varepsilon[-a_{11}a^2 e^{ax_1} + b_1 a e^{ax_1} + c(x) e^{ax_1}] \le \varepsilon e^{ax_1}(-a_{11}a^2 + b_1 a + c) < 0$ (a充分大)

同理上.

5. 由 $\lim_{|x|\to\infty} u(x) = l$,

$\forall \varepsilon > 0$, $\exists a > 0$, 当 $|x| > a$ 时, 有

$$|u(x) - l| < \varepsilon$$

取 a 充分大, s.t. $B_{a+1} = \{x \mid |x| < a+1\}$ 包含 Ω_0. 记 $\Omega' = \Omega \cap B_{a+1}$, 在 Ω' 上用最大模估计

$$\max_{\Omega'} |u(x)| \le \max\left\{\max_{\partial B_{a+1}} |u|, \max_{\partial\Omega} |u|\right\}$$

$$\le \max\left\{|l| + \varepsilon, \max_{\partial\Omega} |u|\right\}$$

$$\Longrightarrow \sup_{\Omega} |u(x)| \le \max\left\{|l| + \varepsilon, \max_{\partial\Omega} |u|\right\}$$

令 $\varepsilon \to 0$ 则有 $\sup_{\Omega} |u(x)| \le \max\left\{|l|, \max_{\partial\Omega} |\varphi(x)|\right\}$

6. 因为 $-\Delta u + u^2 \cdot u = 0$

$$\left(\frac{\partial u}{\partial n} + \alpha(x) u\right)\Big|_{\partial\Omega} = \varphi$$

记 $c(x) = u^2(x) \ge 0$, 又 $c(x)$ 有上界, 由Th2.6 得

$$\max_{\Omega} |u(x)| \le \frac{\Phi}{\alpha_0} = \frac{1}{\alpha_0} \cdot \max_{\partial\Omega} |\varphi(x)|$$