第二章习题

31. 利用特征线法求解以下拟线性方程的Cauchy问题.

$$\begin{cases} \partial_t u = \dfrac{1}{1-x-u}(1+\partial_x u), & x\in\mathbb{R},\ t>0 \\ u(0,x)=0, & x\in\mathbb{R} \end{cases}$$

解: 设特征线 $x=X(t)$ 满足

$$X'(t)=\frac{1}{X(t)+u(X(t),t)-1},\quad X(0)=\alpha.\quad V(t)=u(X(t),t)+X(t)$$

则
$$\begin{cases} \dfrac{d}{dt}V(t)=0 \\ V(0)=u(\alpha,0)+X(0)=\alpha \end{cases}$$

$$\Rightarrow V(t)=V(0)=\alpha$$

$$\therefore X'(t)=\frac{1}{\alpha-1}\quad X(0)=\alpha \Rightarrow X(t)=\alpha+\frac{t}{\alpha-1}$$

$$\therefore u(X(t),t)+\alpha+\frac{t}{\alpha-1}=\alpha \Rightarrow u(X(t),t)=\frac{t}{\alpha-1}.$$

由 $x=\alpha+\dfrac{t}{\alpha-1} \Rightarrow \alpha=\alpha(x,t)$

$$\therefore u(x,t)=\frac{t}{\alpha(x,t)-1}$$

第三章习题.

9. 用分离变量法求解下列混合问题

$$(1)\begin{cases} u_t-u_{xx}=u, & 0<x<l,\ t>0 \\ u|_{t=0}=1, & 0\le x\le l \\ u|_{x=0}=0,\quad u|_{x=l}=0, & t>0 \end{cases}$$

解：令 $u(x,t)=T(t)X(x)$，则

$$T'(t)X(x)-T(t)X''(x)=T(t)X(x)$$

$$\Rightarrow \frac{T'(t)}{T(t)}=\frac{X''(x)}{X(x)}+1$$

令 $\frac{X''(x)}{X(x)}+1=-\lambda$

$\therefore X''(x)+(1+\lambda)X(x)=0,\quad X(0)=X(l)=0$

$\therefore \lambda>-1,\quad X(x)=C_1\sin(\sqrt{1+\lambda}\,x)+C_2\cos(\sqrt{1+\lambda}\,x)$

$X(0)=0\Rightarrow C_2=0,\quad X(l)=C_1\sin(\sqrt{1+\lambda}\,l)=0$

$\therefore \sqrt{1+\lambda_k}\,l=k\pi,\ k=1,2,\cdots,\Rightarrow \lambda_k=\left(\frac{k\pi}{l}\right)^2-1$

$\therefore u(x,t)=\sum\limits_{k=1}^{\infty}T_k(t)X_k(x)\quad \left(X_k(x)=\sin(\sqrt{1+\lambda_k}\,x)\right)$

$$\therefore \begin{cases} T_k'(t)+\lambda_k T_k(t)=0 \\ T_k(0)=\int_0^l \sin(\sqrt{1+\lambda_k}\,x)dx\Big/\int_0^l \sin^2(\sqrt{1+\lambda_k}\,x)dx \end{cases}$$

$$=\frac{2}{k\pi}(1-\cos(k\pi))=\frac{2}{k\pi}(1-(-1)^k)$$

$\therefore T_k(t)=\frac{2}{k\pi}(1-(-1)^k)e^{-\lambda_k t}$

$\therefore u(x,t)=\sum\limits_{k=1}^{\infty}\frac{4}{(2k-1)\pi}e^{-\lambda_k t}\cdot\sin(\sqrt{1+\lambda_k}\,x)$

$\lambda_k=\left(\frac{k\pi}{l}\right)^2-1$

(2) $\begin{cases} u_t = a^2 u_{xx} & 0<x<\pi,\ t>0 \\ u|_{t=0} = \sin x & 0\le x\le \pi \\ u_x|_{x=0} = u_x|_{x=\pi} = 0, & t>0. \end{cases}$

解：令 $u(x,t) = T(t)X(x)$

$$T'(t)X(x) = a^2 T(t)X''(x) \Rightarrow \frac{T'(t)}{T(t)} = a^2\frac{X''(x)}{X(x)}$$

$\therefore X''(x) + \lambda X(x) = 0,\quad X'(0) = X'(\pi) = 0$

$\therefore \lambda>0,\quad X(x) = C_1\sin(\sqrt{\lambda}x) + C_2\cos(\sqrt{\lambda}x)$

$X'(0)=0 \Rightarrow C_1 = 0,\quad X'(\pi) = -C_2\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) = 0$

$\therefore \sqrt{\lambda_k}\pi = k\pi,\ k=1,2,\cdots \Rightarrow \lambda_k = k^2,\ k=1,2,\cdots$

$\therefore u(x,t) = \sum_{k=1}^{\infty} T_k(t)X_k(x) \quad (X_k(x) = \cos(kx))$

$\therefore T_k'(t) + a^2\lambda_k T_k(t) = 0$

$$T_k(0) = \int_0^{\pi}\sin x\cos(kx)\,dx \Big/ \int_0^{\pi}\cos^2(kx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\sin x\cos(kx)\,dx$$

$$= \begin{cases} \frac{1}{\pi}\left(\frac{1}{k+1}\left(1-\cos[(k+1)\pi]\right) - \frac{1}{k-1}\left(1-\cos[(k-1)\pi]\right)\right), & k>1 \\ 0, & k=1 \end{cases}$$

$$= \begin{cases} \frac{1}{\pi}\frac{-2}{k^2-1}\left(1+(-1)^k\right), & k>1 \\ 0, & k=1 \end{cases}$$

$$\therefore u(x,t) = \sum_{k=1}^{\infty}\frac{-4}{[(2k)^2-1]\pi}e^{-4a^2k^2}\cos(2kx)$$

(3) $\begin{cases} u_t = a^2 u_{xx}, & 0<x<l,\ t>0 \\ u|_{t=0} = x^2(l-x), & 0\le x\le l. \\ u_x|_{x=0} = u|_{x=l}=0, & t>0 \end{cases}$

解：设 $u(x,t) = T(t)X(x)$,

$T'(t)X(x) = a^2 T(t) X''(x).$

由题(2),

$X''(x)+\lambda X(x)=0,\quad X'(0)=X(l)=0$

$\therefore \lambda>0,\quad X(x)=C_1\cos(\sqrt{\lambda}x)+C_2\sin(\sqrt{\lambda}x)$

$\because X'(0)=0 \Rightarrow C_2=0,\quad X(l)=0\Rightarrow \cos(\sqrt{\lambda}l)=0$

$\therefore \sqrt{\lambda_k}\,l=(k-\frac{1}{2})\pi,\ k=1,2,\cdots \Rightarrow \lambda_k=\left(\frac{(k-\frac{1}{2})\pi}{l}\right)^2$

$\therefore u(x,t)=\sum_{k=1}^{\infty} T_k(t)X_k(x).\ (X_k(x)=\cos(\sqrt{\lambda_k}x))$

$T_k'(t)+a^2\lambda_k T_k(t)=0.$

$$T_k(0)=\int_0^l x^2(l-x)\cos(\sqrt{\lambda_k}x)dx \Big/ \int_0^l \cos^2(\sqrt{\lambda_k}x)dx$$

$$=\frac{2}{l}\int_0^l x^2(l-x)\cos(\sqrt{\lambda_k}x)dx\cdots$$

$$\therefore u(x,t)=\sum_{k=1}^{\infty} T_k(0)e^{-a^2\lambda_k t}\cos(\sqrt{\lambda_k}x)$$

(4) $\begin{cases} u_t = a^2 u_{xx} & 0<x<l,\ t>0. \\ u|_{t=0}=0. & 0\le x\le l \\ u|_{x=0}=0,\ u|_{x=l}=At,\ t>0 \end{cases}$

解：令 $V(x,t)=u(x,t)-\frac{A}{l}tx$，则

$$\begin{cases} V_t - a^2 V_{xx} = -\frac{A}{l}x, & 0<x<l,\ t>0. \\ V|_{t=0}=0. \\ V|_{x=0} = V|_{x=l}=0 \end{cases}$$

种问题所对应 S-L 问题：

$$X''(x)+\lambda X(x)=0\ ,\ X(0)=X(l)=0.$$

$\therefore \lambda>0,\ X(x)=C_1\sin(\sqrt{\lambda}x)+C_2\cos(\sqrt{\lambda}x)$

$X(0)=0 \Rightarrow C_2=0,\ X(l)=0 \Rightarrow \sin(\sqrt{\lambda}l)=0$

$\therefore \sqrt{\lambda_k}\,l=k\pi,\ k=1,2,\cdots;\ \Rightarrow \lambda_k=\left(\frac{k\pi}{l}\right)^2$

$\therefore V(x,t)=\sum_{k=1}^{\infty} T_k(t)\,X_k(x),\ \left(X_k(x)=\sin\left(\frac{k\pi}{l}x\right)\right)$

$$\begin{cases} T_k'(t)+a^2\lambda_k T_k(t) = -\frac{A}{l}\int_0^l x\sin\left(\frac{k\pi}{l}x\right)dx \Big/ \int_0^l \sin^2\left(\frac{k\pi}{l}x\right)dx \\ \qquad \triangleq a_k \\ T_k(0)=0 \end{cases}$$

$\therefore T_k(t)=e^{-a^2\lambda_k t}\int_0^t a_k e^{a^2\lambda_k s}\,ds$

$$= \frac{a_k}{a^2\lambda_k}\left(1-e^{-a^2\lambda_k t}\right)$$

$\therefore u(x,t)=\sum_{k=1}^{\infty}\frac{a_k}{a^2\lambda_k}\left(1-e^{-a^2\lambda_k t}\right)\sin\left(\frac{k\pi}{l}x\right)$

注：余下 9-10 题所剩题目可按上述过程完成，此处略。

在以下各题中，假设设区域 $Q=\{(x,t)\mid 0<x<l,\ 0<t\le T\}$，$\Gamma$是$Q$的抛物边界.

13、设 $u\in C^{2,1}(\bar{Q})$，$u_t\in C^{2,1}(Q)$ 且满足以下定解问题.

$$\begin{cases} u_t-u_{xx}=f(x,t) & (x,t)\in Q \\ u|_{t=0}=\varphi(x) & 0\le x\le l \\ u|_{x=0}=u|_{x=l}=0, & 0\le t\le T \end{cases}$$

则有以下估计

$$\max_{\bar{Q}}|u_t(x,t)|\le C\left[\|f\|_{C^1(\bar{Q})}+\|\varphi''\|_{C[0,l]}\right]$$

其中C仅依赖于T.

证：令 $v(x,t)=u_t(x,t)$，则

$$\begin{cases} v_t-v_{xx}=\partial_t f(x,t) \\ v|_{t=0}=\varphi''(x)+f(x,0) \\ v|_{x=0}=v|_{x=l}=0 \end{cases}$$

令 $w(x,t)=\|f\|_{C^1(\bar{Q})}t+\|\varphi''\|_{C[0,l]}+\|f\|_{C^1(\bar{Q})}\pm v$

$$\Rightarrow\begin{cases} w_t-w_{xx}\ge 0 \\ w|_{t=0}\ge 0 \\ w|_{x=0}\ge 0,\ w|_{x=l}\ge 0 \end{cases}$$

∴由抛物方程极值原理知，$w(x,t)\ge 0$.

∴ $|v(x,t)|\le C\left[\|f\|_{C^1(\bar{Q})}+\|\varphi''\|_{C[0,l]}\right]$. ※

第四章 习题

1. 设 $u(x)$ 是定解问题 $\begin{cases} -\Delta u + c(x)u = f(x), & x\in\Omega \\ u|_{\partial\Omega}=0 \end{cases}$ 的一个解。

(1) 如果 $c(x)\geq c_0>0$，则有估计 $\max\limits_{\Omega}|u(x)|\leq c_0^{-1}\sup\limits_{\Omega}|f(x)|$.

(2) 如果 $c(x)\geq 0$ 且有界，则 $\max\limits_{\Omega}|u(x)|\leq M\sup\limits_{\Omega}|f(x)|$，其中 M 依赖于 $c(x)$ 的界与 Ω 的直径. (3) 如果 $c(x)<0$，试举反例说明上述最大模估计一般不成立.

证明：(1) 如果 $u(x)$ 在 Ω 内某点 x_0 处达到正极大值，则 $-\Delta u(x_0)\geq 0$.

$\Rightarrow$ $c(x_0)\,u(x_0)\leq f(x_0) \Rightarrow u(x_0)\leq c_0^{-1}f(x_0)\leq c_0^{-1}\sup\limits_{\Omega}|f(x)|$.

同理 $u(x)$ 在 Ω 内某点 x_1 处达到负极小值时，$u(x_1)\geq -c_0^{-1}\sup\limits_{\Omega}|f(x)|$

$\therefore \max\limits_{\Omega}|u(x)|\leq c_0^{-1}\sup\limits_{\Omega}|f(x)|+0 = c_0^{-1}\sup\limits_{\Omega}|f(x)|$.

(2) 设 $|c(x)|\leq \tilde{M}$ $(\forall x\in\bar{\Omega})$，不妨设原点在 Ω 内，令 $w(x)=\sup\limits_{\Omega}|f(x)|\,(d^2-|x|^2)\pm u$，

其中 d 为 Ω 的直径. 从而

$$\begin{cases} -\Delta w(x)+c(x)\,w(x)\geq 0 \\ w(x)|_{\partial\Omega}\geq 0 \end{cases}$$

由极值原理知，$w(x)\geq 0 \Rightarrow |u|\leq M\sup\limits_{\Omega}|f(x)|$ $(M=d^2)$.

(3) 反例：$u(x,y)=\sin x\sin y$，$c(x)=2$，$f(x)=0$，$\Omega=[0,2\pi]^2$.

则 $\begin{cases} -\Delta u(x,y)+2u(x,y)=0 \\ u|_{\partial\Omega}=0 \end{cases}$

但 $u(x,y)$ 在 Ω 内部取到最大值 1 和最小值 -1.

2. 设 $u(x)$ 是定解问题 $\begin{cases} -\Delta u + c(x)u = f(x), & x\in\Omega, \\ \frac{\partial u}{\partial n} + \alpha(x)u|_{\Gamma_1} = \varphi_1, & u|_{\Gamma_2} = \varphi_2 \end{cases}$ 的解，其中 $\Gamma_1\cup\Gamma_2 = \partial\Omega$，

$\Gamma_1\cap\Gamma_2 = \phi$，$\Gamma_2 \neq \phi$. 如果 $c(x)\geq 0$，$\alpha(x)\geq\alpha_0>0$，则有估计：

$$\max_{\Omega}|u(x)| \leq C\left(\sup_{\Omega}|f| + \sup_{\Gamma_1}|\varphi_1| + \sup_{\Gamma_2}|\varphi_2|\right)$$

其中常数 C 依赖于 α_0 与 Ω 直径.

证明：不妨设原点在 Ω 内，记 $F = \sup_{\Omega}|f| + \sup_{\Gamma_1}|\varphi_1| + \sup_{\Gamma_2}|\varphi_2|$，令

$$w(x) := F\left(\frac{4+\alpha_0+d^2}{\alpha_0} + d^2 - x_1^2\right) \pm u$$

则：$-\Delta w(x) + c(x)w \geq 2F \pm f(x) \geq 0$，$x\in\Omega$

$$\begin{cases} \frac{\partial w}{\partial n} + \alpha(x)w \geq (4+d^2)F - 2Fn_1x_1 \pm \varphi_1 \\ \qquad \geq (4+d^2)F - 2Fd \pm \varphi_1 \\ \qquad \geq (4+d^2)F - F(1+d^2) \pm \varphi_1 \\ \qquad \geq 0 \quad x\in\Gamma_1 \\ w|_{\Gamma_2} \geq F \pm \varphi_2 \geq 0 \end{cases}$$

由极值原理，$w\geq 0 \Rightarrow |u| \leq CF$. #

3. 试用辅助函数 $w(x) = |x|^{-\alpha} - r^{-\alpha}$ 证明边界点引理（其中常数 $\alpha>0$ 待定，r 是 S 的半径）.

证明：设 u 在 S 的边界点某点 x_0 达到严格最大值，记 $S^* = \{x \mid \frac{r}{2} < |x| < r\}$.

令 $\qquad V(x) = u(x) - u(x^0) + \varepsilon w(x)$,

在 $|x| = \frac{r}{2}$ 上，$u(x) - u(x^0) < 0 \Rightarrow$ 存在 $M>0$, s.t. $u(x) - u(x^0) < -M < 0$

$\therefore$ 存在 $\varepsilon>0$，使得 $V(x)$ 在 $|x| = \frac{r}{2}$ 上小于0.

在 $|x|=r$ 上，$V(x)$ 在 x^0 处取最大值0.

$$LV = -\Delta V + c(x)V$$

$$= Lu - c(x)u(x^0) + \varepsilon c(x)(|x|^{-\alpha} - r^{-\alpha}) - \varepsilon\left(\alpha(\alpha+1)|x|^{-\alpha-2} + \alpha(1-n)|x|^{-\alpha-2}\right)$$

$$\le Lu - \varepsilon\left(\alpha^2 + 2\alpha - 2n + c(x)|x|^2\right)|x|^{-\alpha-2}$$

$$\le -\varepsilon\left(\alpha^2 + 2\alpha - 2n + c(x)|x|^2\right)|x|^{-\alpha-2}$$

$\therefore$ 当 $c(x)$ 有界，存在充分大 α，使得 $LV<0$

$\therefore$ 由极值原理知，V 在 x^0 处取非负最大值.

$$\therefore \frac{\partial V}{\partial n}(x^0) \geq 0 \Rightarrow \left.\frac{\partial u}{\partial n}\right|_{x^0} \geq -\left.\frac{\partial w}{\partial n}\right|_{x^0} = \varepsilon\alpha r^{-\alpha-1} > 0 \quad \#$$

4. 考虑一般二阶椭圆型方程

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x)\,\partial_{ij}u + \sum_{i=1}^{n} b_i(x)\,\partial_i u + c(x)u = 0$$

其中矩阵 $(a_{ij}(x))$ 是正定，即存在正常数 $\alpha>0$，使得：

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2, \quad \forall x\in\Omega,\ \xi\in\mathbb{R}^n.$$

证明：当 $c(x)\geq 0$ 时弱极值原理成立.

证：令 $w(x) = u + \varepsilon e^{\sum_{i=1}^{n} a_i x_i}$

$$\Rightarrow Lw = Lu + \varepsilon\left(-\sum_{i,j=1}^{n} a_{ij}(x)a_i a_j + \sum_{i=1}^{n} b_i(x)a_i + c(x)\right)e^{\sum_{i=1}^{n} a_i x_i}$$

$$\le \varepsilon\left(-\alpha|a|^2 + |b|\cdot|a| + c(x)\right)e^{\sum_{i=1}^{n} a_i x_i}$$

从而可选适当 $a_i\ (i=1,2,\cdots n)$，使得 $|a|^2 = \sum_{i=1}^{n} a_i^2$ 充分大，

$$\Rightarrow Lw < 0$$

假设在 Ω 内某点 x^0，w 达到非负最大值，由于 $(a_{ij}(x^0))$ 正定，则存在正交矩阵 B，使得 $B^{-1}(a_{ij}(x^0))B = \begin{pmatrix} \lambda_1(x^0) & & \\ & \ddots & \\ & & \lambda_n(x^0) \end{pmatrix}$，$\lambda_i(x^0) \geq \alpha \ (i=1,2,\cdots,n)$

令 $y = x^0 + B(x - x^0)$

$$\Rightarrow \partial_{x_i x_j} u = \partial_{y_l y_k} u \cdot b_{li} b_{kj}$$

$$\Rightarrow -\sum_{i,j=1}^{n} a_{ij}(x^0)\partial_{x_i x_j} u(x^0) = -\sum_{i,j=1}^{n}\sum_{k,l=1}^{n} a_{ij}(x^0) b_{li} b_{kj} \partial_{y_l y_k} u(x^0)$$

$$= -\sum_{i=1}^{n} \lambda_i(x^0)\partial_{y_i}^2 u(x^0) \geq 0$$

$$\sum_{i=1}^{n} b_i(x^0)\partial_{x_i} u(x^0) = 0, \quad c(x^0)u(x^0) \geq 0$$

$$\Rightarrow Lw(x^0) \geq 0 \text{ 与 } Lw < 0 \text{ 矛盾}$$

$$\therefore \sup_{\Omega} w(x) \leq \sup_{\partial\Omega} w^+(x)$$

$$\Rightarrow \sup_{\Omega} u(x) \leq \sup_{\partial\Omega} u^+(x) + \varepsilon \sup_{\partial\Omega} e^{\sum_{i=1}^{n} a_i x_i}$$

由 Ω 有界，令 $\varepsilon \to 0^+$，$\sup_{\Omega} u(x) \leq \sup_{\partial\Omega} u^+(x)$.

6. 设 $u(x)$ 是边值问题 $\begin{cases} -\Delta u + u^3 = 0, & x \in \Omega \\ \frac{\partial u}{\partial n} + \alpha(x) u \big|_{\partial\Omega} = \varphi \end{cases}$ 属于 $C^2(\Omega) \cap C^1(\bar{\Omega})$ 的解，其中

$\alpha(x) \geq \alpha_0 > 0$，则：$\max_{\bar{\Omega}} |u(x)| \leq \frac{1}{\alpha_0} \max_{\partial\Omega} |\varphi(x)|$

证明：令 $L = -\Delta + u^2$，则 $Lu = -\Delta u + u^3 = 0$

$\therefore$ 由极值原理知，$\max_{\bar{\Omega}} |u(x)| \leq \max_{\partial\Omega} |u(x)|$.

若 u 在 $\partial\Omega$ 上某点 x^0 处取非负最大值，则 $\frac{\partial u}{\partial \nu}(x^0)\geq 0$.

$$\therefore \alpha(x^0)u(x^0)\leq\varphi(x^0)\Rightarrow u(x^0)\leq\frac{1}{\alpha_0}\varphi(x^0)\leq\frac{1}{\alpha_0}\max_{\partial\Omega}|\varphi(x)|.$$

同理若 u 在 $\partial\Omega$ 上某点 x_1 处取非正最小值，有：

$$u(x_1)\geq-\frac{1}{\alpha_0}\max_{\partial\Omega}|\varphi(x)|$$

$$\therefore\ \max_{\partial\Omega}|u|\leq\frac{1}{\alpha_0}\max_{\partial\Omega}|\varphi(x)|.\quad\therefore\ \max_{\Omega}|u(x)|\leq\frac{1}{\alpha_0}\max_{\partial\Omega}|\varphi(x)|.$$

8. 记 B^+ 为二维半圆 $\{(x,y)\mid x^2+y^2<1,\ y>0\}$，设 u 是定解问题

$$\begin{cases}-\partial_x^2u-y\,\partial_y^2u+c(x,y)u=f(x,y), & (x,y)\in B^+,\\ u|_{\partial B^+}=\varphi\end{cases}$$

属于 $C^2(B^+)\cap C(\bar{B}^+)$ 的解

(1) 如果 $c(x,y)\geq c_0>0$，则

$$\max_{\bar{B}^+}|u(x,y)|\leq\frac{1}{c_0}\sup_{B^+}|f(x,y)|+\max_{\partial B^+}|\varphi(x,y)|$$

(2) 如果 $c(x,y)\geq 0$ 且有界，则

$$\max_{\bar{B}^+}|u(x,y)|\leq M\left(\sup_{B^+}|f(x,y)|+\max_{\partial B^+}|\varphi(x,y)|\right)$$

其中 M 依赖于 $c(x,y)$ 的界.

证：提示 (1) 的证明同问题 (1).

(2) 的证明构造如下辅助函数.

$$w(x,y)=\max_{\partial B^+}|\varphi(x)|+\sup_{B^+}|f(x,y)|\left(1-\tfrac{1}{2}|x|^2\right)$$

11. 考虑一维椭圆问题

$$\begin{cases} -\dfrac{d^2u}{dx^2}+g(x)u=f(x), \quad 0<x<1 \\ u(0)=u(1)=0 \end{cases}$$

(1) 设 $\sup\limits_{[0,1]}|u(x)|\le M_0$，$g(x)$，$f(x)$ 在 $[0,1]$ 上有界，则有

$|u'(0)|$，$|u'(1)|\le C_1$. 其中 C_1 只依赖于 $g(x)$ $f(x)$ 的界与 M_0.

(2) 进一步假设 $g(x)\in C^1[0,1]$，$f(x)\in C^1[0,1]$，$u\in C[0,1]\cap C^3(0,1)$，则

$|u'(x)|\le C_2$.，其中 C_2 只依赖于 M_0，$\|g\|_{C^1[0,1]}$，$\|f\|_{C^1[0,1]}$.

证明：(1) 令 $w(x)=Mx(1-x)\pm u$，其中 M 待定.

$$\begin{cases} -\dfrac{d^2w}{dx^2}=2M\pm(f(x)-g(x)u) \\ \qquad\ge 2M-\sup\limits_{x\in[0,1]}|f(x)|-\sup\limits_{x\in[0,1]}|g(x)|M_0. \\ w(0)=w(1)\ge 0 \end{cases}$$

取 $M=\sup\limits_{x\in[0,1]}|f(x)|+\sup\limits_{x\in[0,1]}|g(x)|M_0$.

$\Rightarrow w(x)\ge 0$

$\Rightarrow |u(x)|\le Mx(1-x)$

$\Rightarrow |u'(0)|+|u'(1)|\le 2M$.

(2) (方法一) 令 $v=\dfrac{du}{dx}$. $\Rightarrow -\dfrac{d^2v}{dx^2}+g(x)v=f(x)-g(x)u$

利用极值原理证. $\max\limits_{x\in[0,1]}|v|\le C\left(|u'(0)|+|u'(1)|+\max\limits_{x\in[0,1]}|f(x)|+\max\limits_{x\in[0,1]}|g(x)|M_0\right)$

(方法二)

$u(x)=\int_0^x \dfrac{d^2u}{dx^2}(s)\,ds+u'(0)\Rightarrow |u'(x)|\le\int_0^1\left(\sup\limits_{x\in[0,1]}|f(x)|+\sup\limits_{x\in[0,1]}|g(x)|M_0\right)+M$

$\le 2M$.

12. 设 $u,v\in C^2(\Omega)\cap C(\bar\Omega)$，且是以下方程组

$$\begin{cases} -\Delta u+2u-v=f(x), & x\in\Omega \\ -\Delta v+2v-u=g(x), & x\in\Omega \end{cases}$$

满足边界条件 $u|_{\partial\Omega}=v|_{\partial\Omega}=0$.

的解，试证明：

$$\max\{\sup_\Omega|u(x)|,\ \sup_\Omega|v(x)|\}\le\max\{\sup_\Omega|f(x)|,\ \sup_\Omega|g(x)|\}$$

证明：令 $w=u+v$，$\bar w=u-v$

$$\Rightarrow \begin{cases} -\Delta w+w=f(x)+g(x), & x\in\Omega \\ w|_{\partial\Omega}=0 \end{cases} \qquad \begin{cases} -\Delta\bar w+3\bar w=f(x)-g(x), & x\in\Omega \\ \bar w|_{\partial\Omega}=0 \end{cases}$$

由极值原理：$|w|\le \sup_{x\in\Omega}|f(x)+g(x)|$

$|\bar w|\le \sup_{x\in\Omega}|f(x)-g(x)|$

$$\Rightarrow \max\{\sup_\Omega|u(x)|,\ \sup_\Omega|v(x)|\}\le\max\{\sup_{x\in\Omega}|f(x)+g(x)|,\ \sup_{x\in\Omega}|f(x)-g(x)|\}$$

$\boxed{???}$

正确的方法：

由极值原理知：

$$\sup_\Omega|u(x)|\le\frac12\sup_\Omega|v(x)|+\frac12\sup_\Omega|f(x)|$$

$$\sup_\Omega|v(x)|\le\frac12\sup_\Omega|u(x)|+\frac12\sup_\Omega|g(x)|.$$

$$\Rightarrow \sup_\Omega|u(x)|\le\frac13\sup_\Omega|g(x)|+\frac23\sup_\Omega|f(x)|\le\max\{\sup_\Omega|f(x)|,\ \sup_\Omega|g(x)|\}$$

同理，$\sup_\Omega|v(x)|\le\max\{\sup_\Omega|f(x)|,\ \sup_\Omega|g(x)|\}$

13. 设 $c(x)\geq c_0>0$, $\alpha(x)\geq 0$, $u\in C^2(\Omega)\cap C(\bar\Omega)$ 是问题

$$\begin{cases}-\Delta u+c(x)u=f(x), & x\in\Omega\\ \left[\frac{\partial u}{\partial n}+\alpha(x)u\right]\Big|_{\partial\Omega}=0\end{cases}$$

的解，则有估计：

$$\int_\Omega|\nabla u(x)|^2dx+\frac{c_0}{2}\int_\Omega|u(x)|^2dx+\int_{\partial\Omega}\alpha(x)u^2(x)dl\leq M\int_\Omega|f(x)|^2dx.$$

其中 M 也只依赖于 c_0.

证明：
$$f(x)\cdot u=-\Delta u\cdot u+c(x)u^2$$
$$=-\sum_{i=1}^n\partial_{x_i}(u\partial_{x_i}u)+|\nabla u|^2+c(x)u^2$$
$$\Rightarrow\int_\Omega|\nabla u|^2dx+\int_\Omega c(x)u^2dx-\int_{\partial\Omega}u\frac{\partial u}{\partial n}dl=\int_\Omega f(x)u\,dx$$
$$\Rightarrow\int_\Omega|\nabla u|^2dx+c_0\int_\Omega u^2dx+\int_{\partial\Omega}\alpha u^2dl\leq\frac{c_0}{2}\int_\Omega u^2dx+\frac{1}{2c_0}\int_\Omega f^2dx$$
$$\Rightarrow\int_\Omega|\nabla u|^2dx+\frac{c_0}{2}\int_\Omega u^2dx+\int_{\partial\Omega}\alpha u^2dl\leq\frac{1}{2c_0}\int_\Omega|f|^2dx.$$

14. 考虑问解问题 $\begin{cases}-\Delta u+\sum_{i=1}^n b_i(x)\partial_{x_i}u+c(x)u=f(x), & x\in\Omega.\\ u|_{\partial\Omega}=0.\end{cases}$

如果 $c(x)-\frac{1}{4}\sum_{i=1}^n b_i^2(x)>0$，试用能量估计证明上述边值问题解的唯一性.

证明：此时，令 $f(x)=0$.
$$-\Delta u\cdot u+\sum_{i=1}^n b_i(x)\partial_{x_i}u\cdot u+c(x)u^2=0$$
$$\Rightarrow-\sum_{i=1}^n\partial_i(\partial_i u\cdot u)+|\nabla u|^2+c(x)u^2=-\sum_{i=1}^n b_i(x)\partial_{x_i}u\cdot u$$

$$\therefore \int_\Omega |\nabla u|^2 dx + \int_\Omega c(x)|u|^2 dx \le \int_\Omega \sum_{i=1}^n |b_i(x)\partial_{x_i}u \cdot u| dx$$

$$\le \int_\Omega |\nabla u|^2 dx + \frac{1}{4}\int_\Omega u^2 \sum_{i=1}^n |b_i(x)|^2 dx$$

$$\Rightarrow \int_\Omega \left(c(x) - \frac{1}{4}\sum_{i=1}^n b_i^2(x)\right) u^2(x) dx \le 0$$

由于 $c(x) - \frac{1}{4}\sum_{i=1}^n b_i^2(x) > 0 \Rightarrow u \equiv 0$.

例1. 设 $f(x) \in C^\infty(\overline{B_1(0)})$, $g(x) \in C^\infty(\overline{B_1(0)})$, 求解

$$\begin{cases} -\Delta u = f(x), & x \in B_1(0) \subset \mathbb{R}^2, \\ u|_{\partial\Omega} = g(x), & x \in \partial B_1(0) \end{cases}$$

解: 记 $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $-\Delta u = \left(-\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\theta^2\right)u$

首先找齐次方程 $-\Delta u = 0$ 非零解.

设 $u(r,\theta) = R(r)\Theta(\theta)$, 则:

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

$$\Rightarrow \frac{R''(r) + \frac{1}{r}R'(r)}{R(r)} \cdot r^2 = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda.$$

$$\therefore \Theta''(\theta) + \lambda\Theta(\theta) = 0 \qquad \Theta(\theta + 2\pi) = \Theta(\theta)$$

$$\therefore \lambda_n = n^2, \ n = 1, 2, \cdots, \quad \Theta_n^1(\theta) = \cos(n\theta), \ \Theta_n^2(\theta) = \sin(n\theta)$$

$$\lambda_0 = 0, \quad \Theta_0(\theta) = 1.$$

$\therefore$ 对于非齐次方程. 令

$$u(r,\theta) = u_0(r)\Theta_0(\theta) + \sum_{n=1}^\infty \left(u_n^1(r)\cos(n\theta) + u_n^2(r)\sin(n\theta)\right)$$

$$f(x) = f_0(r)\Theta_0(\theta) + \sum_{n=1}^\infty \left(f_n^1(r)\cos(n\theta) + f_n^2(r)\sin(n\theta)\right)$$

$$g(x)|_{\partial B_1(0)} = g_0\Theta_0(\theta) + \sum_{n=1}^\infty \left(g_n^1\cos(n\theta) + g_n^2\sin(n\theta)\right)$$

$f_0(r)=\frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)\,d\theta \qquad f_n^1(r)=\frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\cos(n\theta)\,d\theta \quad f_n^2(r)=\frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\sin(n\theta)\,d\theta$

$g_0=\frac{1}{2\pi}\int_0^{2\pi} g(\theta)\,d\theta,\quad g_n^1=\frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos(n\theta)\,d\theta,\ g_n^2=\frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin(n\theta)\,dx.$

$$\therefore \begin{cases} -(u_0(r))''-\frac{1}{r}(u_0(r))'=f_0(r) \\ u_0(1)=g_0 \end{cases}$$

$$\begin{cases} -(u_n^i(r))''-\frac{1}{r}(u_n^i(r))' \ \overset{+\frac{n^2}{r^2}u_n^i(r)}{} = f_n^i(r),\ i=1,2, \\ u_n^i(1)=g_n^i,\ i=1,2 \end{cases}$$

例2. 在例1.的假设下，求解下列问题的有界光滑解.

$$\begin{cases} -\Delta u=f(x),\quad x\in B_1(0). \\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega}=g(x),\quad x\in\partial B_1(0) \end{cases}$$

解：同例1过程.

$-(u_0(r))''-\frac{1}{r}(u_0(r))'=f_0(r) \qquad u_0'(1)=g_0.$

$$\begin{cases} -(u_n^i(r))''-\frac{1}{r}(u_n^i(r))'+\frac{n^2}{r^2}u_n^i(r)=f_n^i(r),\ i=1,2 \\ (u_n^i)'(1)=g_n^i,\ i=1,2 \end{cases}$$

$\Rightarrow\ (r u_0'(r))'=-rf_0(r)$

$\Rightarrow\ r u_0'(r)=\int_1^r -sf_0(s)\,ds+g_0$

$\Rightarrow\ u_0'(r)=\frac{1}{r}\left(g_0-\int_1^r s f_0(s)\,ds\right)$

如 u 为有界光滑解，则：

$$\lim_{r\to 0} \frac{1}{r}\left(g_0+\int_r^1 s f_0(s)ds\right) \text{ 有界}$$

$$\Leftrightarrow \lim_{r\to 0}\left(g_0+\int_r^1 s f_0(s)ds\right)=0$$

$$\Leftrightarrow \int_0^{2\pi} g(\theta)d\theta+\int_0^1\int_0^{2\pi} s f(s,\theta)d\theta ds=0$$

$$\Leftrightarrow \int_{\partial B_1(0)} g(x)dl+\int_{B_1(0)} f(x)dx=0.$$