分) 设$f(x)$ 是区间$[a,b]$ 上的一个充分光滑的函数, 在区间$[a,b]$ 上给定一组基点:

$$a = x_1 < x_2 < \cdots < x_n < x_{n+1} = b.$$

表示函数

$$S(x) = \begin{cases} S_1(x)\ , & x \in [x_1, x_2]\ ; \\ \cdots\cdots\cdots & \\ S_i(x)\ , & x \in [x_i, x_{i+1}]\ ; \\ \cdots\cdots\cdots & \\ S_n(x)\ , & x \in [x_n, x_{n+1}] \end{cases}$$

续可微的, $S_i(x), i = 1, \cdots, n$都是不高于三次的多项式, 且$S(x_i) = f(x_i), i = 1, \cdots, n+$
$f(x)$ 在区间两端点的导数$f'(a)$ 和$f'(b)$ 为已知. 记$m_i = S'(x_i), f_i = f(x_i)$, 试导出关
$_3, \cdots, m_n$ 的方程组.

x_i x_{i+1}

$m_i = S'(x_i)$, $m_{i+1} = S'(x_{i+1})$

$f_i = S(x_i)$, $f_{i+1} = S(x_{i+1})$ $\qquad h_i = x_{i+1} - x_i$

由Hermite插值得

$$S_i(x) = \frac{1}{h_i^3}[h_i + 2(x - x_i)](x - x_{i+1})^2 f(x_i) + \frac{1}{h_i^3}[h_i - 2(x - x_{i+1})](x - x_i)^2 f(x_{i+1}) + \frac{1}{h_i^2}(x - x_i)(x - x_{i+1})^2 m_i + \frac{1}{h_i^2}(x - x_{i+1})(x - x_i)^2 m_{i+1}$$

$$S_i''(x) = \frac{6}{h_i^3}[h_i + 2(x - x_{i+1})]f(x_i) + \frac{6}{h_i^3}[h_i - 2(x - x_i)]f(x_{i+1}) + \frac{1}{h_i^2}[6(x - x_{i+1}) + 2h_i]m_i + \frac{1}{h_i^2}[6(x - x_i) - 2h_i]m_{i+1}$$

由 $S_i''(x_i^+) = S_{i-1}''(x_i^-)$ 得 $\quad \frac{1}{h_{i-1}}m_{i-1} + \left(\frac{2}{h_{i-1}} + \frac{2}{h_i}\right)m_i + \frac{1}{h_i}m_{i+1} = 3\left[\frac{1}{h_{i-1}}f[x_{i-1},x_i] + \frac{1}{h_i}f[x_i,x_{i+1}]\right]$

$$\frac{h_i}{h_i + h_{i-1}} m_{i-1} + 2m_i + \frac{h_{i-1}}{h_i + h_{i-1}} m_{i+1} = 3\left[\frac{h_i}{h_i + h_{i-1}} \frac{f(x_i) - f(x_{i-1})}{h_{i-1}} + \frac{h_{i-1}}{h_i + h_{i-1}} \frac{f(x_{i+1}) - f(x_i)}{h_i}\right]$$

结合边界条件 $S'(x_0) = f'(a)$, $S'(x_{n+1}) = f'(b)$ 得

$$\begin{bmatrix} 2 & \mu_2 & & \\ \lambda_3 & 2 & \mu_3 & \\ & \ddots & \ddots & \mu_{n-1} \\ & & \lambda_n & 2 \end{bmatrix} \begin{bmatrix} m_2 \\ \vdots \\ \vdots \\ m_n \end{bmatrix} = \begin{bmatrix} c_2' \\ c_3 \\ \vdots \\ c_n' \end{bmatrix}$$

其中 $\lambda_i = \frac{h_i}{h_i + h_{i-1}}$, $\mu_i = \frac{h_{i-1}}{h_i + h_{i-1}}$

$c_i = 3(\lambda_i f[x_{i-1}, x_i] + \mu_i f[x_i, x_{i+1}])$ $\quad i = 2, \cdots n$

$c_2' = c_2 - \lambda_2 f'(a)$, $\quad c_n' = c_n - \mu_n f'(b)$.