偏微分方程期中考试试卷答案（2012级）

1. 讨论 Busemann 方程 $\sum_{i,j=1}^{n}(x_i x_j-\delta_{ij})\partial_{ij}u=f$ 的类型.

解：记矩阵 $A=(a_{ij})_{n\times n}$，其中 $a_{ij}=x_i x_j-\delta_{ij}$. 则 A 的特征方程为

$$|\lambda I_{n\times n}-A|=\begin{vmatrix} \lambda+1-x_1^2 & -x_1x_2 & \cdots & -x_1x_n \\ -x_2x_1 & \lambda+1-x_2^2 & & \\ \vdots & & \ddots & \\ -x_nx_1 & & & \lambda+1-x_n^2 \end{vmatrix}$$

$$\overset{\text{若 } x_1\neq 0}{\underset{\text{第}i\text{行}-\text{第}1\text{行}\times\frac{x_i}{x_1}}{=\!=}}\begin{vmatrix} \lambda+1-x_1^2 & -x_1x_2 & \cdots & -x_1x_n \\ -\frac{(\lambda+1)x_2}{x_1} & \lambda+1 & & 0 \\ \vdots & & \ddots & \\ -\frac{(\lambda+1)x_n}{x_1} & 0 & & \lambda+1 \end{vmatrix}$$

$$\overset{\text{第}1\text{列}+\sum_{j=2}^{n}\text{第}j\text{列}\times\frac{x_j}{x_1}}{=\!=}\begin{vmatrix} \lambda+1-|x|^2 & -x_1x_2 & \cdots & -x_1x_n \\ 0 & \lambda+1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \lambda+1 \end{vmatrix}$$

$$=(\lambda+1)^{n-1}(\lambda+1-|x|^2)$$

$\therefore \lambda_1=\cdots=\lambda_{n-1}=-1,\quad \lambda_n=|x|^2-1$

当 $x_1=0$ 时，可得上述同样结论（可设某 $x_k\neq 0$，否则 $x=0$）

从而当 $|x|<1$ 时，$\lambda_n<0$，方程为椭圆型；

$|x|=1$ 时，$\lambda_n=0$，方程为抛物型；

$|x|>1$ 时，$\lambda_n>0$，方程为双曲型。 ※

2. 求解问题：

$$\begin{cases} \partial_t u+(1+u^2)\partial_x u=0, & (x,t)\in\mathbb{R}\times\mathbb{R}^+, \\ u(x,0)=u_0(x), & x\in\mathbb{R} \end{cases}$$

其中 $u_0(x)\in C_0^\infty(\mathbb{R})$. 并求最大的 T^*, 使得 $u\in C^\infty(\mathbb{R}\times[0,T^*))$.

解：设特征线为 $x=X_\alpha(t)$ 满足：

$$X_\alpha'(t)=u^2(X_\alpha(t),t)+1, \quad X_\alpha(0)=\alpha$$

令 $U(t)=u(X_\alpha(t),t)$, 则

$$dU(t)/dt=0, \quad U(0)=u_0(\alpha),$$

从而 $U(t)=u_0(\alpha)$

故 $X_\alpha'(t)=1+u_0^2(\alpha)$, $X_\alpha(0)=\alpha$,

则：$X_\alpha(t)=\alpha+(1+u_0^2(\alpha))t$

从而 $u(x,t)=u_0(\alpha(x,t))$, 其中 $\alpha=\alpha(x,t)$ 由方程

$x=\alpha+(1+u_0^2(\alpha))t$ 给出.

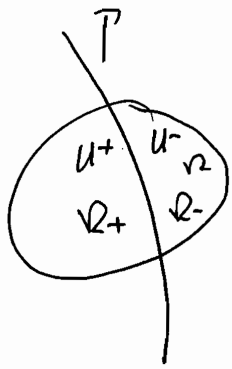
此时，由隐函数存在定理，

$$1+2u_0(\alpha)u_0'(\alpha)t\neq 0$$

故 $$T^*=\frac{1}{-2\min\limits_{x\in\mathbb{R}} u_0(x)u_0'(x)}$$

$\Rightarrow u\in C^\infty(\mathbb{R}\times[0,T^*))$. ※

3. 设u为方程 $\partial_t u+\sum_{i=1}^{n}\partial_i F_i(u)=0$, $(x,t)\in\mathbb{R}^n\times\mathbb{R}^+$ 的分片光滑解，其中曲面 Γ: $m(x,t)=0$ 为其间断面，试推导u为上述方程弱解所需要的 Rankine-Hugoniot 条件。



解：任取 $\mathbb{R}^n\times\mathbb{R}^+$ 一区域 Ω，如图，则

$\forall \varphi\in C_0^\infty(\Omega)$,

$$0=\langle \partial_t u+\sum_{i=1}^{n}\partial_i F_i(u),\varphi\rangle$$

$$=-\iint_\Omega\left(u\partial_t\varphi+\sum_{i=1}^{n}F_i(u)\partial_i\varphi\right)dxdt$$

$$+\int_\Gamma\left[(u^+-u^-)\varphi n_t+\sum_{i=1}^{n}(F_i(u^+)-F_i(u^-))\varphi n_i\right]ds\triangleq I_1+I_2$$

其中 $n=(n_1,n_2,\cdots,n_n,n_t)$ 为 Γ 在 Ω_+ 侧所对应的单位外法向

由于u为方程弱解，故 $I_2=0$.

由Ω及φ的任意性，

$$[u]n_t+\sum_{i=1}^{n}[F_i(u)]n_i=0 \quad \text{on } \Gamma$$

即：$$[u]\partial_t m+\sum_{i=1}^{n}[F_i(u)]\partial_i m=0 \quad \text{on } \Gamma.$$

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4. 设φ,ψ为光滑函数，且$\varphi(0)=\psi(0)$，求解古尔沙问题：

$$\begin{cases} \partial_t^2 u-\partial_x^2 u=1, & t\in\mathbb{R}^+,\ |x|<t, \\ u|_{t=-x}=\varphi(x), & u|_{t=x}=\psi(x) \end{cases}$$

解. 记 $V(x,t)=\partial_t u-\partial_x u$，则：

$$\begin{cases} \partial_t V+\partial_x V=1 \\ V|_{t=-x}=-(\partial_x u-\partial_t u)|_{t=-x}=-\frac{d}{dx}u(x,-x)=-\varphi'(x) \end{cases}$$

则 $V(x,t)=\frac{x+t}{2}-\varphi'(\frac{x-t}{2})$.

从而：

$$\begin{cases} \partial_t u-\partial_x u=V(x,t)=\frac{x+t}{2}-\varphi'(\frac{x-t}{2}) \\ u|_{t=x}=\psi(x) \end{cases}$$

故 $u(x,t)=\varphi(\frac{x-t}{2})+\psi(\frac{x+t}{2})+\frac{t^2-x^2}{4}$ (过程略) ※

5. 设$\varphi(x),\psi(x)$为光滑函数，求解例问题：

$$\begin{cases} \partial_t^2 u-\partial_x^2 u=0, \\ u|_{t=\frac{1}{2}x}=\varphi(x), & \partial_t u|_{t=\frac{1}{2}x}=\psi(x) \end{cases}$$

解：令 $u(x,t)=F(x+t)+G(x-t)$,

则:

$$\begin{cases} F\left(\frac{3x}{2}\right) + G\left(\frac{x}{2}\right) = \varphi(x) \\ F'\left(\frac{3x}{2}\right) - G'\left(\frac{x}{2}\right) = \psi(x) \end{cases}$$

$$\Rightarrow \begin{cases} \frac{3}{2} F'\left(\frac{3x}{2}\right) + \frac{1}{2} G'\left(\frac{x}{2}\right) = \varphi'(x) \\ F'\left(\frac{3x}{2}\right) - G'\left(\frac{x}{2}\right) = \psi(x) \end{cases}$$

故.

$$\begin{cases} F'\left(\frac{3x}{2}\right) = \frac{1}{2}\varphi'(x) + \frac{1}{4}\psi(x) \Rightarrow F'(x) = \frac{1}{2}\varphi'\left(\frac{2}{3}x\right) + \frac{1}{4}\psi\left(\frac{2}{3}x\right) \\ G'\left(\frac{x}{2}\right) = \frac{1}{2}\varphi'(x) - \frac{3}{4}\psi(x) \Rightarrow G'(x) = \frac{1}{2}\varphi'(2x) - \frac{3}{4}\psi(2x) \end{cases}$$

从而

$$\begin{cases} F(x) = F(0) + \frac{3}{4}\left[\varphi\left(\frac{2}{3}x\right) - \varphi(0)\right] + \frac{3}{8}\int_0^{\frac{2}{3}x}\psi(t)\,dt \\ G(x) = G(0) + \frac{1}{4}\left[\varphi(2x) - \varphi(0)\right] - \frac{3}{8}\int_0^{2x}\psi(t)\,dt \end{cases}$$

其中 $F(0) + G(0) = \varphi(0)$.

从而

$$u(x,t) = \frac{3}{4}\varphi\left(\frac{2}{3}(x+t)\right) + \frac{1}{4}\varphi\left(2(x-t)\right)$$

$$+ \frac{3}{8}\int_0^{\frac{2}{3}(x+t)}\psi(t)\,dt - \frac{3}{8}\int_0^{2(x-t)}\psi(t)\,dt \quad \#$$

6. 设Ω为$\mathbb{R}^n$中有界单连通区域，$\partial\Omega=\Gamma_1\cup\Gamma_2$且$\Gamma_1\cap\Gamma_2=\phi$. 推导变分问题 $J(u)=\min\limits_{v\in M}J(v)$ 所对应的微分方程问题，其中$M=C^2(\Omega)\cap C(\bar{\Omega})$

以及 $J(v)=\frac{1}{2}\int_\Omega(|\nabla v|^2+v^2)dx+\frac{1}{2}\int_{\Gamma_1}\alpha(x)v^2ds-\int_\Omega fv\,dx-\int_{\Gamma_2}gv\,ds\ (\alpha>0)$.

解: $\forall\varepsilon\in\mathbb{R}$, $\forall v\in M$, 则: $j(\varepsilon)\triangleq J(u+\varepsilon v)$

由题意知: $j'(0)=0$.

此时:

$$j(\varepsilon)=\frac{1}{2}\int_\Omega\left(|\nabla u+\varepsilon\nabla v|^2+(u+\varepsilon v)^2\right)dx+\frac{1}{2}\int_{\Gamma_1}\alpha(x)(u+\varepsilon v)^2ds$$

$$-\int_\Omega f(u+\varepsilon v)\,dx-\int_{\Gamma_2}g(u+\varepsilon v)\,ds$$

$$j'(\varepsilon)=\int_\Omega\left[(\nabla u+\varepsilon\nabla v)\cdot\nabla v+(u+\varepsilon v)v\right]dx+\int_{\Gamma_1}\alpha(x)(u+\varepsilon v)v\,ds$$

$$-\int_\Omega fv\,dx-\int_{\Gamma_2}gv\,ds$$

$$j'(0)=\int_\Omega[\nabla u\nabla v+uv]\,dx+\int_{\Gamma_1}\alpha(x)uv\,ds-\int_\Omega fv\,dx-\int_{\Gamma_2}gv\,ds$$

$$=\int_\Omega(-\Delta u\cdot v+uv-fv)dx+\int_{\partial\Omega}\frac{\partial u}{\partial n}v\,ds+\int_{\Gamma_1}\alpha(x)uv\,ds$$

$$-\int_{\Gamma_2}gv\,ds$$

由v的任意性:

$$\begin{cases}-\Delta u+u=f, & x\in\Omega\\ \frac{\partial u}{\partial n}+\alpha(x)u=0, & x\in\Gamma_1\\ \frac{\partial u}{\partial n}=g, & x\in\Gamma_2\end{cases}$$

进一步：

$$j''(\varepsilon)=\int_\Omega[|\nabla v|^2+v^2]\,dx+\int_{\Gamma_1}\alpha(x)v^2\,ds>0 \quad (v\neq 0)$$

故 $\varepsilon=0$ 为 $j(\varepsilon)$ 的极小值点，从而所对应微分问题：

$$\begin{cases}-\Delta u+u=f, & x\in\Omega\\ \frac{\partial u}{\partial n}+\alpha(x)u=0, & x\in\Gamma_1\\ \frac{\partial u}{\partial n}=g, & x\in\Gamma_2\end{cases}$$

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7. 设 $B_1(0)$ 为 $\mathbb{R}^2$ 中的单位圆盘，若 $u\in C^2$ 为下列问题的解：

$$\begin{cases}\partial_t^2u-\Delta u=f(x,t), & (x,t)\in B_1(0)\times[0,T],\\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x), & x\in B_1(0),\\ \partial_n u+\alpha(x)u=0, & (x,t)\in\partial B_1(0)\times[0,T]\quad(\alpha(x)>0)\end{cases}$$

证明：存在依赖于 T 的常数 $C(T)$，对于 $0<t<T$，有：

$$\int_{B_1(0)}(|\partial_t u|^2+|\nabla u|^2)(x,t)\,dx$$

$$\leq C(T)\left(\int_{B_1(0)}((\nabla\varphi)^2+\psi^2)\,dx+\int_{B_1(0)\times[0,T]}f^2\,dx\,dt+\int_{\partial B_1(0)}\alpha\varphi^2\,dx\right)$$

证明：方程两边乘 $\partial_t u$，在 $B_1(0)\times[0,\tau]$ 上积分，得：$(0<\tau\leq T)$

$$\int_{B_1(0)\times[0,\tau]}\partial_t u f\,dx\,dt=\int_{B_1(0)\times[0,\tau]}[(\partial_t^2u-\Delta u)\partial_t u]\,dx\,dt$$

$$= \int_{B_1(0)\times[0,\tau]} \left[\partial_t\left(\frac{1}{2}|\partial_t u|^2+\frac{1}{2}|\nabla u|^2\right) - \sum_{i=1}^{2}\partial_i(\partial_i u\,\partial_t u)\right] dx\,dt$$

$$= \int_{B_1(0)} \left(\frac{1}{2}|\partial_t u|^2+\frac{1}{2}|\nabla u|^2\right)(x,\tau)\,dx - \int_{B_1(0)} \left(\frac{1}{2}|\nabla\varphi|^2+\frac{1}{2}\psi^2\right)(x)\,dx$$

$$-\int_{\partial B_1(0)\times[0,\tau]} \partial_n u\,\partial_t u\,ds$$

$$= \int_{B_1(0)} \left(\frac{1}{2}|\partial_t u|^2+\frac{1}{2}|\nabla u|^2\right)(x,\tau)\,dx - \int_{B_1(0)} \left(\frac{1}{2}|\nabla\varphi|^2+\frac{1}{2}\psi^2\right)(x)\,dx$$

$$+\int_{\partial B_1(0)\times[0,\tau]} \frac{1}{2}\partial_t\left[\alpha(x)\,u^2\right] ds$$

$$= \int_{B_1(0)} \left(\frac{1}{2}|\partial_t u|^2+\frac{1}{2}|\nabla u|^2\right)(x,\tau)\,dx + \int_{\partial B_1(0)} \frac{1}{2}\alpha(x)\,u^2(x,\tau)\,ds$$

$$-\int_{B_1(0)} \left(\frac{1}{2}|\nabla\varphi|^2+\frac{1}{2}\psi^2\right)(x)\,dx - \int_{\partial B_1(0)} \alpha(x)\,\varphi^2(x)\,ds$$

$$\Rightarrow \int_{B_1(0)} \left(|\partial_t u|^2+|\nabla u|^2\right)(x,\tau)\,dx \le \int_{B_1(0)} \left(|\nabla\varphi|^2+\psi^2\right)(x)\,dx + \int_{B_1(0)\times[0,\tau]} f^2\,dx\,dt$$

$$+\int_{\partial B_1(0)} \alpha(x)\,\varphi^2(x)\,ds \triangleq F(\tau)$$

$$+\int_{B_1(0)\times[0,\tau]} u^2\,dx\,dt$$

故记 $G(\tau)=\int_{B_1(0)\times[0,\tau]} \left(|\partial_t u|^2+|\nabla u|^2\right)(x,t)\,dx\,dt$，则

$G'(\tau) = \int_{B_1(0)} \left(|\partial_t u|^2+|\nabla u|^2\right)(x,\tau)\,dx$. 且.

$$G'(\tau) \le F(\tau) + G(\tau)$$

由Gronwall不等式：

$$G(\tau) \le \left[G(0) + \int_0^\tau e^{-s} F(s) ds\right] e^\tau \le C(\tau) F(\tau)$$

从而

$$G'(\tau) \le C(\tau) F(\tau)$$

即：命题结论。 ※

8. 设 $\varphi(x)$, $\psi(x)$ 为光滑函数，对于下面问题：

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, & 0<x<l,\ t>0 \\ \partial_x u(0,t) = \partial_x u(l,t) = 0, & \\ u(x,0) = \varphi(x),\ \partial_t u(x,0) = \psi(x), & 0<x<l \end{cases}$$

(1) 用分离变量法求解方程；

(2) 写出问题直到3阶的相容性条件；

(3) 在上述条件下，证明分离变量的解二次连续可微。

解：(1) 问题对应S-L问题为：

$$X''(x) + \lambda X(x) = 0, \quad X'(0) = X'(l) = 0$$

从而

$$\lambda_0 = 0, \quad X_0(x) = 1$$

$$\lambda_k = \left(\frac{k\pi}{l}\right)^2, \quad X_k(x) = \cos\left(\frac{k\pi}{l}x\right), \ k = 1, 2, \cdots$$

故令：

$$u(x,t) = \sum_{k=0}^{\infty} T_k(t) X_k(x),$$

$$\varphi(x) = \sum_{k=0}^{\infty} \varphi_k X_k(x), \quad \varphi_k = \int_0^l \varphi(x) X_k(x) dx \Big/ \int_0^l X_k^2(x) dx$$

$$\psi(x) = \sum_{k=0}^{\infty} \psi_k X_k(x), \quad \psi_k = \int_0^l \psi(x) X_k(x) dx \Big/ \int_0^l X_k^2(x) dx$$

从而

$$\begin{cases} T_k''(t) + \lambda_k T_k(t) = 0 \\ T_k(0) = \varphi_k, \quad T_k'(0) = \psi_k \end{cases}$$

$$\Rightarrow T_k(t) = \varphi_k \cos(\sqrt{\lambda_k} t) + \frac{\psi_k}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} t), \quad T_0(t) = \varphi_0 + \psi_0 t$$

则

$$u(x,t) = \sum_{k=1}^{\infty} \left[\varphi_k \cos(\sqrt{\lambda_k} t) + \frac{\psi_k}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} t)\right] X_k(x)$$

$$+ \varphi_0 + \psi_0 t$$

(2) 0阶相容性条件：无

1阶相容性条件：$\varphi'(0) = \varphi'(l) = 0$

2阶相容性条件：$\psi'(0) = \psi'(l) = 0$

3阶相容性条件：$\varphi'''(0) = \varphi'''(l) = 0$

(3) 证明.

$$\varphi_k = \int_0^l \varphi(x) X_k(x)\,dx \Big/ \int_0^l X_k^2(x)\,dx$$

$$\Rightarrow \varphi_0 = \frac{1}{l}\int_0^l \varphi(x)\,dx$$

$$\varphi_k = \frac{2}{l}\int_0^l \varphi(x)\cos\left(\frac{k\pi}{l}x\right)dx$$

$$= -\frac{2}{l}\left(\frac{l}{k\pi}\right)\varphi(x)\sin\left(\frac{k\pi}{l}x\right)\Big|_0^l$$

$$+\frac{2}{l}\left(\frac{l}{k\pi}\right)\int_0^l \varphi'(x)\sin\left(\frac{k\pi}{l}x\right)dx$$

$$= \frac{2}{l}\left(\frac{l}{k\pi}\right)^2 \varphi'(x)\cos\left(\frac{k\pi}{l}x\right)\Big|_0^l \quad (\varphi'(0)=\varphi'(l)=0$$

$$-\frac{2}{l}\left(\frac{l}{k\pi}\right)^2\int_0^l \varphi''(x)\cos\left(\frac{k\pi}{l}x\right)dx$$

$$= \frac{2}{l}\left(\frac{l}{k\pi}\right)^3 \varphi''(x)\sin\left(\frac{k\pi}{l}x\right)\Big|_0^l$$

$$-\frac{2}{l}\left(\frac{l}{k\pi}\right)^3\int_0^l \varphi'''(x)\sin\left(\frac{k\pi}{l}x\right)dx$$

$$= -\frac{2}{l}\left(\frac{l}{k\pi}\right)^4 \varphi'''(x)\cos\left(\frac{k\pi}{l}x\right)\Big|_0^l \quad (\varphi'''(0)=\varphi'''(l)=0)$$

$$+\frac{2}{l}\left(\frac{l}{k\pi}\right)^4\int_0^l \varphi^{(4)}(x)\cos\left(\frac{k\pi}{l}x\right)dx$$

$$= \frac{2}{l}\left(\frac{l}{k\pi}\right)^4\int_0^l \varphi^{(4)}(x)\cos\left(-\frac{k\pi}{l}x\right)dx \quad (k\geq 1)$$

$$\Rightarrow |\varphi_k| \leq \max_{x\in[0,l]}|\varphi^{(4)}(x)|\cdot\frac{2l^3}{\pi^4}\cdot\frac{1}{k^4} \quad (k\geq 1)$$

$$\varphi_k = \int_0^l \varphi(x) X_k(x)\,dx \Big/ \int_0^l X_k^2(x)\,dx$$

$$\Rightarrow \varphi_0 = \frac{1}{l}\int_0^l \varphi(x)\,dx$$

$$\varphi_k = \frac{2}{l}\int_0^l \varphi(x)\cos\left(\frac{k\lambda}{l}x\right)dx$$

$$= -\frac{2}{l}\varphi(x)\sin\left(\frac{k\lambda}{l}x\right)\cdot\frac{l}{k\lambda}\Big|_0^l$$

$$+\frac{2}{l}\left(\frac{l}{k\lambda}\right)\int_0^l \varphi'(x)\sin\left(\frac{k\lambda}{l}x\right)dx$$

$$= \frac{2}{l}\left(\frac{l}{k\lambda}\right)^2\varphi'(x)\cos\left(\frac{k\lambda}{l}x\right)\Big|_0^l \quad (\varphi'(0)=\varphi'(l)=0)$$

$$-\frac{2}{l}\left(\frac{l}{k\lambda}\right)^2\int_0^l \varphi''(x)\cos\left(\frac{k\lambda}{l}x\right)dx$$

$$= \underline{\frac{2}{l}\left(\frac{l}{k\lambda}\right)^3\varphi''(x)\sin\left(\frac{k\lambda}{l}x\right)\Big|_0^l} = 0$$

$$-\frac{2}{l}\left(\frac{l}{k\lambda}\right)^3\int_0^l \varphi'''(x)\sin\left(\frac{k\lambda}{l}x\right)dx$$

$$\Rightarrow |\varphi_k| \le \max_{x\in[0,l]}|\varphi^{(3)}(x)|\cdot\frac{2l^2}{\lambda^3}\cdot\frac{1}{k^3} \quad (k\ge 1)$$

从而:

$$|u(x,t)|+|\partial_t u|+|\partial_x u|+|\partial_t^2 u|+|\partial_{tx} u|+|\partial_x^2 u| \le C\sum_{k=1}^{\infty}\frac{1}{k^2}+C_0$$

$$\le \tilde{C}.$$

即: $u(x,t)$ 为二次连续可微函数.

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