数学系11级偏微分方程期末试卷(A卷)

2013/2014 学年第一学期 考试时间 2014.01.06 考试成绩________

院系________ 学号________ 姓名________

一. (10 分) 计算广义导数 $((1+x^2)H(x))'$, 其中 $x \in \mathbb{R}^1$, 其中 $H(x)$ 为 Heaviside 函数.

解: 对任意 $\varphi \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned}\langle((1+x^2)H(x))', \varphi\rangle &= -\langle(1+x^2)H(x), \varphi'(x)\rangle \\ &= -\int_0^{+\infty}(1+x^2)\varphi'(x)dx \\ &= \int_0^{+\infty}2x\varphi(x)dx + (1+x^2)\varphi(x)\Big|_{x=0} \\ &= \langle 2xH(x), \varphi\rangle + \varphi(0) \\ &= \langle 2xH(x)+\delta(x), \varphi\rangle\end{aligned}$$

所以 $((1+x^2)H(x))' = 2xH(x) + \delta(x)$.

二. (10 分) 求方程 $(1-|x|^2)\partial_{x_1}^2 u + (1-2|x|^2)(\partial_{x_2}^2 u + \partial_{x_3}^2 u) = 0$ 的双曲型, 椭圆型与抛物型的区域, 其中 $|x|^2 = x_1^2 + x_2^2 + x_3^2$.

解: 此方程所对应系数矩阵的特征值为 $\lambda_1 = 1-|x|^2$, $\lambda_{2,3} = 1-2|x|^2$,

所以 $|x| < \frac{1}{\sqrt{2}}$ 或者 $|x| > 1$ 时, 方程为椭圆型方程,

$\frac{1}{\sqrt{2}}$ 时, 方程为双曲方程.

三. (12 分) 求解古尔沙问题

$$\begin{cases}\partial_t^2 u - \partial_x^2 = 0, & |x| < t, \quad t > 0, \\ u(x,t) = f(x), & \text{on} \quad t = x, \\ u(x,t) = g(x), & \text{on} \quad t = -x,\end{cases}$$

其中 $f(x) \in C^2([0,+\infty))$, $g(x) \in C^2((-\infty,0])$, 且 $f(0) = g(0)$.

解: 由行波法，可设问题的解 $u(x,t) = F(x+t) + G(x-t)$, 其中 F, G 为二阶连续可微函数.
从而由相应的边值条件可得: $u(x,x) = F(2x)+G(0) = f(x)$, $u(x,-x) = F(0)+G(2x) = g(x)$.
因此有 $F(0)+G(0) = f(0)$, $F(x) = f(\frac{1}{2}x) - G(0)$, $G(x) = g(\frac{1}{2})x) - F(0)$.
所以 $u(x,t) = f(\frac{1}{2}(x+t)) - G(0) + g(\frac{1}{2}(x-t)) - F(0) = f(\frac{1}{2}(x+t)) + g(\frac{1}{2}(x-t)) - f(0)$.

四.（12 分）设 $\Omega \in \mathbb{R}^2$ 是有界光滑区域, $\partial\Omega = \Gamma_1 \cup \Gamma_2$, 且 $\Gamma_1 \cap \Gamma_2 = \emptyset$. 试求下列定解问题所对应的 Green 函数 $G(x;\xi)$ 满足的定解问题, 并写出解 u 的表达式.

解: 若 $G(x;\xi)$ 为对应问题的基本解, 由 Green 公式知,

$$\begin{aligned}
u(\xi) &= \int_{\Gamma_1} \left(u\frac{\partial G(x;\xi)}{\partial \mathbf{n}} - G(x;\xi)\frac{\partial u}{\partial \mathbf{n}} \right) dS + \int_{\Gamma_2} \left(u\frac{\partial G(x;\xi)}{\partial \mathbf{n}} - G(x;\xi)\frac{\partial u}{\partial \mathbf{n}} \right) dS \\
&= \int_{\Gamma_1} \left(u\frac{\partial (x;\xi)}{\partial \mathbf{n}} + G(x;\xi)(u - g_1(x)) \right) dS + \int_{\Gamma_2} \left(g_2\frac{\partial G(x;\xi)}{\partial \mathbf{n}} - G(x;\xi)\frac{\partial u}{\partial \mathbf{n}} \right) dS \\
&= \int_{\Gamma_1} \left(u\big(\frac{\partial G(x;\xi)}{\partial \mathbf{n}} + G(x;\xi)\big) - G(x;\xi)g_1(x) \right) dS + \int_{\Gamma_2} \left(g_2\frac{\partial G(x;\xi)}{\partial \mathbf{n}} - G(x;\xi)\frac{\partial u}{\partial \mathbf{n}} \right) dS.
\end{aligned}$$

从而 $G(x;\xi)$ 满足

$$\begin{cases}
-\Delta G(x;\xi) = \delta(x-\xi), \quad x \in \Omega, \\
\frac{\partial G(x;\xi)}{\partial \mathbf{n}} + G(x;\xi) = 0, \quad \text{on} \quad \Gamma_1, \\
G(x;\xi) = 0 \quad \text{on} \quad \Gamma_2.
\end{cases}$$

五.（12 分）记 $Q = \{(x,t) : 0 < x < l, t > 0\}$, 用变量分离法求解问题

$$\begin{cases}
\partial_t u - \partial_x^2 u = f(x,t), \quad \text{in} \quad Q, \\
u(x,0) = \varphi(x), \quad 0 < x < l, \\
u(0,t) = 0, \quad \partial_x u(l,t) = 0.
\end{cases}$$

解: 设其次方程的解具有形式 $X(x)T(t)$, 从而 $T'(t)X(x) = T(t)X''(x) = 0$.

从而令 $X''(x) + \lambda X(x) = 0$.

所以 $X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \quad X'(l) = 0$.

故 $\lambda_k = \left(\frac{k\pi - \frac{1}{2}\pi}{l}\right)^2, X_k(x) = sin(\sqrt{\lambda_k}x), k = 1, 2, \cdots$.

设 $\varphi(x) = \sum\limits_{k=1}^{\infty} \varphi_k sin(\sqrt{\lambda_k}x), f(x,t) = \sum\limits_{k=1}^{\infty} f_k(t) sin(\sqrt{\lambda_k}x)$,

其中 $\varphi_k = \frac{2}{l}\int_0^l \varphi(x) sin(\sqrt{\lambda_k}x)dx, \quad f_k(t) = \frac{2}{l}\int_0^l f(x,t) sin(\sqrt{\lambda_k}x)dx$.

所以$T_k'(t) + \lambda_k T_k(t) = f_k(t), \quad T_k(0) = \varphi_k$.

所以 $T_k(t) = e^{-\lambda_k t}\left(\varphi_k + \int_0^t e^{\lambda_k s} f_k(s)ds\right)$.

从而原方程的解为 $u(x,t) = \sum\limits_{k=1}^{\infty} T_k(t)X_k(x)$.

六. (12 分) 记 $B_1(0)$ 为 $\mathbb{R}^2$ 中单位球, $Q=\{(x,t):x\in B_1(0),0<t<T\}$, $\mathcal{L}=\partial_t-(1-|x|^2)\partial_{x_1}^2-\partial_{x_2}^2$. 设二阶连续可微函数 $u(x,t)$ 为下列问题

$$\begin{cases} \mathcal{L}u=f(x,t), \quad \text{in} \quad Q, \\ u(x,0)=\varphi(x),\ x\in B_1(0), \\ u(x,t)=\phi(x,t), \quad (x,t)\in\partial B_1(0)\times[0,T), \end{cases}$$

的解, 证明:

(1). $\mathcal{L}$ 在 Q 上有弱极值原理成立, 即若在 Q 上满足 $\mathcal{L}v\geq 0$, 有

$$\inf_{(x,t)\in Q} v(x,t)\geq \inf{(x,t)\in \text{Q的抛物边界}} v^-(x,t).$$

(2). 存在正常数 C, 使得

$$\sup_{(x,t)\in Q}|u(x,t)|\leq C\left(\sup_{(x,t)\in Q}|f(x,t)|+\sup_{x\in B_1(0)}|\phi(x)|+\sup_{(x,t)\in \text{Q的抛物边界}}|\phi(x,t)|\right).$$

解: (1). 记 $w(x,t)=v(x,t)+\epsilon t$, $\mathcal{L}w=\sqsubseteq+\epsilon=f(x,t)+\epsilon>0$,

若 w 在 Q 内某点 (x_0,t_0) 处取到负极小值, 则

$\partial_t w(x_0,t_0)\leq 0, \quad -(1-|x|^2)\partial_{x_1}^2-\partial_{x_2}^2 w(x_0,t_0)\leq 0$,

从而 $\mathcal{L}w(x_0,t_0)\leq 0$, 与已知条件矛盾.

所以 $\inf\limits_{(x,t)\in Q} w(x,t)\geq \inf\limits_{(x,t)\in \text{Q的抛物边界}} w^-(x,t)$,

即 $\inf\limits_{(x,t)\in Q} v(x,t)+\epsilon T\geq \inf\limits_{(x,t)\in \text{Q的抛物边界}} v^-(x,t)-\epsilon T$,

让 $\epsilon\to 0^+$, 从而有 $\inf\limits_{(x,t)\in Q} v(x,t)\geq \inf{(x,t)\in \text{Q的抛物边界}} v^-(x,t)$.

(2). 记 $v(x,t)=Ft\pm u(x,t)$,

其中 $F=\sup\limits_{(x,t)\in Q}|f(x,t)|+\sup\limits_{x\in B_1(0)}|\varphi(x)|+\sup\limits_{(x,t)\in \text{Q的抛物边界}}|\phi(x,t)|$.

从而 $\mathcal{L}v(x,t)\geq 0$, $v\Big|_{\text{Q的抛物边界}}\geq 0$,

由 (1), $v(x,t)\geq 0$, 从而 $|u(x,t)|\leq TF$.

七. (12 分)记 $B_1(0)=\{x\in\mathbb{R}^2:|x|<1\}$, 若 $u\in C^2(B_1(0))\cap C^1(\overline{B_1(0)})$ 为定解问题

$$\begin{cases} -\Delta u+6u=f(x), \quad \text{in} \quad B_1(0), \\ \frac{\partial u}{\partial \mathbf{n}}=\phi(x), \quad \text{on} \quad \partial B_1(0), \end{cases}$$

的解, 其中 $\mathbf{n}$ 为 $\partial B_1(0)$ 的单位外法向, 证明:

(1). 对任意已知函数 $\psi(x)\in C^2(\overline{B_1(0)})$, 求 $v(x)=u(x)e^{\psi(x)}$ 所满足的定解问题;

(2). 存在正常数 C, 使得 $\sup\limits_{x\in B_1(0)}|u(x)|\leq C\left(\sup\limits_{x\in B_1(0)}|f(x)|+\sup\limits_{x\in\partial B_1(0)}|g(x)|\right)$.

解: (1). 记 $u(x)=v(x)e^{-\psi(x)}$, 从而

$\partial_{x_i}u=\left(\partial_{x_i}ve-\partial_{x_i}\psi v\right)e^{-\psi(x)}$,

$\partial_{x_i}^2 u=\left(\partial_{x_i}^2 v-2\partial_{x_i}\psi\partial_{x_i}v+(\partial_{x_i}\psi)^2v-\partial_{x_i}^2\psi v\right)e^{-\psi(x)}$,

从而有:

$$-\Delta u+6u=e^{-\psi(x)}\left(-\Delta v+2\sum_{i=1}^2\partial_{x_i}\psi\partial_{x_i}v+(6+\Delta\psi-(\partial_{x_i}\psi)^2)v\right)=f(x),$$

所以 v 满足下列边值问题,

$$\begin{cases} -\Delta v+2\sum\limits_{i=1}^2\partial_{x_i}\psi\partial_{x_i}v+(6+\Delta\psi-(\partial_{x_i}\psi)^2)v=f(x)e^{\psi(x)}, \quad \text{in} \quad B_1(0), \\ \frac{\partial v}{\partial \mathbf{n}}-(1+\frac{\partial\psi}{\partial\mathbf{n}})v=\phi(x)e^{\psi(x)}, \quad \text{on} \quad \partial B_1(0). \end{cases}$$

(2). 记 $c(x)=6+\Delta\psi-\sum\limits_{i=1}^2(\partial_{x_i}\psi)^2$, $\alpha(x)=-(1+\frac{\partial\psi}{\partial\mathbf{n}}$.

要建立对 v 的最大模估计, 我们需要找合适的 ψ, 使得 $c(x)>0$, $\alpha(x)\Big|_{|x|=1}>0$.

从而选 $\psi(x)=-\frac{3}{4}$, 有 $(x)>\frac{3}{4}$, $\alpha(x)\Big|_{|x|=1}=\frac{3}{2}-1=\frac{1}{2}>0$.

所以有标准的极值原理知,

$$\sup_{x\in B_1(0)}|v(x)|\leq C\left(\sup_{x\in B_1(0)}|f(x)|+\sup_{x\in\partial B_1(0)}|g(x)|\right),$$

最终,

$$\sup_{x\in B_1(0)}|u(x)|\leq C\left(\sup_{x\in B_1(0)}|f(x)|+\sup_{x\in\partial B_1(0)}|g(x)|\right).$$

八. (20 分) 记 $Q=\{(x,t):x\in\Omega,0<t<T\}$, 其中 Ω 为 $\mathbb{R}^n$ 中有界开区域. 若 $u\in C^2(\overline{Q})$ 满足下列定解问题

$$\begin{cases} \partial_t^2 u-\Delta u=f(x,t), \quad \text{in} \quad Q, \\ u(x,0)=\varphi(x), \quad \partial_t u(x,0)=\psi(x), \quad x\in\Omega, \\ \frac{\partial u}{\partial \mathbf{n}}+u=0, \quad \text{on} \quad \partial\Omega\times[0,T), \end{cases}$$

其中 $\mathbf{n}$ 为 $\partial\Omega$ 的单位外法向, 则存在依赖于 T 的常数 $C(T)$, 使得:

$$\begin{aligned} &\int_\Omega \left((\partial_t u)^2+|\nabla u|^2\right)(x,t)dx+\int_{\partial\Omega}u^2(x,t)dx \\ \le\ & C(T)\left(\int_0^t\int_\Omega f^2(x,s)dxds+\int_\Omega(|\nabla\varphi|^2+|\psi|^2)(x)dx+\int_{\partial\Omega}\varphi^2(x)dS\right). \end{aligned}$$

解: 方程两边乘以 $\partial_t u$, 在 $\Omega\times[0,t]$ 上分部积分, 有

$$\begin{aligned} &\frac{1}{2}\int_\Omega \left((\partial_t u)^2+|\nabla u|^2\right)(x,t)dx-2\int_0^t\int_{\partial\Omega}\partial u\frac{\partial u}{\partial \mathbf{n}}(x,s)dSds \\ =\ & \int_\Omega \left((\psi)^2+|\nabla\varphi|^2\right)(x)dx+2\int_0^t\int_\Omega(f\partial_t u)(x,s)dxds, \end{aligned}$$

带入相应的边值条件, 有

$$\begin{aligned} &\frac{1}{2}\int_\Omega \left((\partial_t u)^2+|\nabla u|^2\right)(x,t)dx-2\int_0^t\int_{\partial\Omega}(\partial_t uu)(x,s)dSds \\ =\ & \int_\Omega \left((\psi)^2+|\nabla\varphi|^2\right)(x)dx+2\int_0^t\int_\Omega(f\partial_t u)(x,s)dxds, \end{aligned}$$

所以,

$$\begin{aligned} &\int_\Omega \left((\partial_t u)^2+|\nabla u|^2\right)(x,t)dx+\int_{\partial\Omega}u^2(x,t)dS \\ =\ & \int_\Omega \left(\psi^2+|\nabla\varphi|^2\right)(x)dx+\int_{\partial\Omega}\varphi^2dS+\int_0^t\int_\Omega f^2(x,s)dxds+\int_0^t\int_\Omega(\partial_t u)^2(x,s)dxds. \end{aligned}$$

从而由 Granwall 不等式, 得

$$\begin{aligned} &\int_\Omega \left((\partial_t u)^2+|\nabla u|^2\right)(x,t)dx+\int_{\partial\Omega}u^2(x,t)dx \\ \le\ & C(T)\left(\int_0^t\int_\Omega f^2(x,s)dxds+\int_\Omega(|\nabla\varphi|^2+|\psi|^2)(x)dx+\int_{\partial\Omega}\varphi^2(x)dS\right). \end{aligned}$$