第一章 答案

(a) 线段 z_1z_2 的中垂线

(b). 圆心在原点，半径为1的单位圆周

(c). $x=3$.

(d) $x>c$ ($x\geq c$) 的右半平面.

(e). 设 $a=a_1+ia_2$ $b=b_1+ib_2$ $z=x+iy$

$(a_1+ia_2)(x+iy)+b_1+ib_2$

$=(a_1x-a_2y+b_1)+i(a_1y+a_2x+b_2)$

$Re(az+b)=a_1x-a_2y+b_1>0$

直线 $a_1x-a_2y+b_1=0$ 所分的半个平面

(f). $z=x+iy$.

$\sqrt{x^2+y^2}=x+1$

$x^2+y^2=x^2+2x+1$

$y^2=2x+1$ 抛物线

(g). 直线 $y=C$

$(z,w)=z\overline{w}$

$=(x_1+iy_1)(x_2-iy_2)$

$=x_1x_2+y_1y_2+i(y_1x_2-x_1y_2)$

$\therefore \frac{1}{2}[(z,w)+(w,z)]=x_1x_2+y_1y_2$

$=\langle z,w\rangle$

$z=\sqrt[n]{s}\,e^{i\frac{\varphi+2k\pi}{n}}$ $(k\in\mathbb{Z})$.

根据指数函数的周期性(e^z 以 $2\pi i$ 为周期)

共有 n 个不同的根

注意 "$\succ$" 不是 "$>$"

果 $i\succ 0$, 由(iii). $i\cdot i\succ 0\cdot i$ 即 $-1\succ 0$ 则 $-i\succ 0$

但由(ii) $i-i\succ 0-i$ 即 $0\succ -i$, 矛盾.

果 $i=0$. 则 $\forall z\in\mathbb{C}$ $z=0$. 矛盾

果 $i\prec 0$, 则由(ii). $i-i\prec 0-i$ 即 $-i\succ 0$.

再由(iii) $-i\cdot(-i)\succ 0\cdot(-i)$ 则 $-1\succ 0$. 则 $(-1)(-i)\succ 0\cdot(-i)$

即 $i\succ 0$. 与前提 $i\prec 0$ 矛盾.

至此. 对于 i 与 0, 在"$\succ$"关系下无法满足(i)

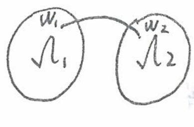
所以在复数内无法定义全序.

5. 设 Ω 为一开集. 证明: Ω 为道路连通 $\Longleftrightarrow$ Ω 为连通.

"$\Longrightarrow$" 假设 Ω 是道路连通的. 以下为反证.

假设 Ω 不是连通的. 则可以找到两个非空不交的开集,

使得 $\Omega=\Omega_1\cup\Omega_2$



取 $w_1\in\Omega_1$, $w_2\in\Omega_2$, 令 γ 表示 Ω 中连通 w_1,w_2 的曲

设 $z:[0,1]\to\Omega$ 是这条曲线的参数化映射

并且满足 $z(0)=w_1$, $z(1)=w_2$. 定义

$t^*=\sup\limits_{0\leq t\leq 1}\{t: z(s)\in\Omega_1,\ \forall\, 0\leq s<t\}$.

若 $z(t^*)\in\Omega_1$, 则 $\exists\, z(t^*)$ 的邻域 $U(z(t^*))\subset\Omega_1$.

选取 $z_0\in\mathring{U}(z(t^*))\cap\gamma$. 对应 $z(t_0)=z_0$.

使得 $t^*<t_0$. 则对 t_0 也满足 $z(s)\in\Omega_1$, $\forall\, 0\leq s<t_0$

这与 t^* 的定义矛盾.

若 $z(t^*)\in\Omega_2$. 则 $\exists\, z(t^*)$ 的邻域 $V(z(t^*))\subset\Omega_2$.

选取 $\tilde{z}_0\in\mathring{V}(z(t^*))\cap\gamma$. 对应 $z(\tilde{t}_0)=\tilde{z}_0$

使得 $\tilde{t}_0<t^*$ 则 $\tilde{z}_0=z(\tilde{t}_0)\in\Omega_2$ 与 t^* 定义矛盾.

所以 $z(t^*)\notin\Omega$. 但 $z(t^*)$ 在 γ 上 所以 $z(t^*)\in\Omega$.

导出矛盾. 所以 Ω 是连通的.

"$\Longleftarrow$" 假设 Ω 是连通的. 任取 $w\in\Omega$.

Ω_1 表示 Ω 中所有能与 w 道路连通的点的集合.

Ω_2 表示 Ω 中所有不能与 w 道路连通的点的集合.

显然 $w\in\Omega_1$。若 $v\in\Omega_1$, 则 $v\in\Omega$,

则 $\exists\, U(v)\subset\Omega$. 因为 w 与 v 连通. 所以 w 与 $U(v)$

中的任意点连通. 即 $U(v)\subset\Omega_1$. 故 Ω_1 为开集.

类似易证 Ω_2 为开集. 且 $\Omega_1\cap\Omega_2=\phi$

所以 Ω_2 为空集 (否则与 Ω 连通矛盾).

所以 Ω 为道路连通.

(a). $\forall w \in C_z$. 则 $w \in \Omega$.

则 $\exists U(w) \subset \Omega$. 因为存在连接 $z w$ 的曲线 $(\subset \Omega)$

则存在连接 z 与 $U(w)$ 内任意点的曲线 $(\subset \Omega)$

$\therefore U(w) \subset C_z$

$\therefore C_z$ 为开集

$\forall w_1, w_2 \in C_z$, $\exists \gamma_1 (\subset \Omega)$ 连接 $z w_1$

$\exists \gamma_2 (\subset \Omega)$ 连接 $z w_2$.

$\therefore \gamma = \gamma_1 \cup \gamma_2 (\subset \Omega)$ 连接 $w_1 w_2$

$\therefore C_z$ 为道路连通

$\therefore$ 由5题结论 C_z 是连通的.

$v \in C_z$ 是一个等价关系

i) ii) 显然.

iii) 若 $w \in C_z$, 则 $\exists \gamma_1 (\subset \Omega)$ 连接 $z w$

$z \in C_\xi$, 则 $\exists \gamma_2 (\subset \Omega)$ 连接 ξz.

则 $\gamma = \gamma_1 \cup \gamma_2 (\subset \Omega)$ 连接 $w \xi$.

所以 $w \in C_\xi$.

). 设 $\Omega = \bigcup_i C_{z_i}$, $C_{z_i} \cap C_{z_j} = \phi \, (i \neq j)$

因为 C_{z_i} 为 $\mathbb{C}$ 中的一个开集.

所以其中一定含有一个点实部、虚部均为有理数

因为 Q^2 的基数为 $\aleph_0$. (即 Q^2 是可数的)

所以 Q^2 的任何子集也是可数的.

所以 Ω 只有可数多个不同的连通部分.

). 假设 K 是一个紧集. 则 K 是一个有界闭集.

则存在一个以原点为圆心的圆盘 D, 使得 $K \subset D$.

显然 D^c 为一个连通集

设 $\Omega = \bigcup_{i \in I} C_i$, C_i 为 Ω 互不相交的连通部分.

因为 $\Omega \supset D^c$. 所以必然存在某个 $C_j \, (j \in I)$

使得 $C_j \supset D^c$. 显然 C_j 即为唯一的一个无界连通部分.

$\forall C_i \, (i \neq j, i \in I)$, $C_i \subset D$ 即 C_i 有界.

7. (无需假设 z 为实数).

(a). $\left|\frac{w-z}{1-\bar{w}z}\right| < 1 \Leftrightarrow \left|\frac{w-z}{1-\bar{w}z}\right|^2 < 1$.

因为 $\left|\frac{w-z}{1-\bar{w}z}\right|^2 = \left(\frac{w-z}{1-\bar{w}z}\right)\left(\frac{\bar{w}-\bar{z}}{1-w\bar{z}}\right)$

$$= \frac{|w|^2 - z\bar{w} - w\bar{z} + |z|^2}{1 - \bar{w}z - w\bar{z} + |w|^2|z|^2}$$

所以只要证 $|w|^2 + |z|^2 \leq 1 + |w|^2|z|^2$

只要证 $(|w|^2-1)(|z|^2-1) \geq 0$.

显然 当 $|w|<1$ 且 $|z|<1$ 时, 上述不等式取 "$>$"

当且仅当 $|w|=1$ 或者 $|z|=1$ 时, 上述不等式取 "$=$".

(b). 从 (a) 的结论易知 (i) 的前半论断成立.

$$\lim_{h\to 0} \frac{\frac{w-(z+h)}{1-\bar{w}(z+h)} - \frac{w-z}{1-\bar{w}z}}{h}$$

$$= \lim_{h\to 0} \frac{(w-(z+h))(1-\bar{w}z) - (w-z)(1-\bar{w}(z+h))}{h(1-\bar{w}(z+h))(1-\bar{w}z)}$$

$$= \lim_{h\to 0} \frac{\cancel{w} - (z+h) - |w|^2 z + \cancel{\bar{w}z(z+h)} - \cancel{w} + z + |w|^2(z+h) - \cancel{\bar{w}z(z+h)}}{h(1-\bar{w}(z+h))(1-\bar{w}z)}$$

$$= \lim_{h\to 0} \frac{(|w|^2-1)h}{h(1-\bar{w}(z+h))(1-\bar{w}z)}$$

$$= \frac{|w|^2-1}{(1-\bar{w}z)^2}$$

所以 F 为全纯映射.

ii) $F(0) = \frac{w-0}{1-\bar{w}\cdot 0} = w$.

$F(w) = \frac{w-w}{1-\bar{w}w} = 0$.

iii) 由 (a) 中等式成立的条件易知

若 $|z|=1$, 则 $|F(z)|=1$.

iv). 单射: 若 $F(z_1) = F(z_2)$

则 $\frac{w-z_1}{1-\bar{w}z_1} = \frac{w-z_2}{1-\bar{w}z_2}$

$(w-z_1)(1-\bar{w}z_2) = (w-z_2)(1-\bar{w}z_1)$

$(|w|^2-1)z_1 = (|w|^2-1)z_2$

因为 $|w|<1$, 所以 $z_1 = z_2$

满射: $\forall v \in \mathbb{D}$. 若 $\frac{w-z}{1-\bar{w}z} = v$

则 $w - z = v - \bar{w}vz$

则 $z = \frac{w-v}{1-\bar{w}v} \in \mathbb{D}$. 即 $F(z) = v$.

$\therefore \frac{\partial h}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)(g\circ f)$.

设 $f(z)=u(x,y)+iv(x,y)$ $g(w)=g_1(w_1,w_2)+ig_2(w_1,w_2)$

(其中 $z=x+iy$, $w=w_1+iw_2$)

则 $g\circ f(z) = g(u+iv) = g_1(u,v)+ig_2(u,v)$.

$$\frac{\partial h}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y}\right)(g_1(u,v)+ig_2(u,v))$$

$$= \frac{1}{2}\left[\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + i\left(-\frac{\partial g_1}{\partial y} + \frac{\partial g_2}{\partial x}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\partial g_1}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial g_1}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial g_2}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial g_2}{\partial v}\cdot\frac{\partial v}{\partial y} + i\left(-\frac{\partial g_1}{\partial u}\cdot\frac{\partial u}{\partial y} - \frac{\partial g_1}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial g_2}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial g_2}{\partial v}\cdot\frac{\partial v}{\partial x}\right)\right]$$

$$= \frac{\partial g_1}{\partial u}\cdot\frac{\partial u}{\partial x} - \frac{\partial g_1}{\partial v}\cdot\frac{\partial u}{\partial y} + i\left(-\frac{\partial g_1}{\partial u}\frac{\partial u}{\partial y} - \frac{\partial g_1}{\partial v}\cdot\frac{\partial u}{\partial x}\right).$$

$$= \frac{\partial g_1}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{1}{i}\frac{\partial u}{\partial y}\right) + \frac{1}{i}\frac{\partial g_1}{\partial v}\left(\frac{\partial u}{\partial x}+\frac{1}{i}\frac{\partial u}{\partial y}\right)$$

$$= \left(\frac{\partial g_1}{\partial u}+\frac{1}{i}\frac{\partial g_1}{\partial v}\right)\left(\frac{\partial u}{\partial x}+\frac{1}{i}\frac{\partial u}{\partial y}\right)$$

$$= 2\frac{\partial g_1}{\partial w}\cdot 2\frac{\partial u}{\partial \bar{z}} = \frac{\partial g}{\partial w}\cdot\frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{w}}\cdot\frac{\partial \bar{f}}{\partial \bar{z}} \quad \left(\frac{\partial g}{\partial \bar{w}}=0\right)$$

$$\frac{\partial h}{\partial \bar{z}} = 0 = \frac{\partial g}{\partial w}\cdot\frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{w}}\cdot\frac{\partial \bar{f}}{\partial \bar{z}}$$

$$\left(\frac{\partial f}{\partial \bar{z}}=0,\ \frac{\partial g}{\partial \bar{w}}=0\right).$$

由柯西-黎曼方程 $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$.

因为 $x=r\cos\theta$, $y=r\sin\theta$.

所以 $\frac{\partial x}{\partial r}=\cos\theta$, $\frac{\partial y}{\partial r}=\sin\theta$. $\frac{\partial x}{\partial \theta}=-r\sin\theta$, $\frac{\partial y}{\partial \theta}=r\cos\theta$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cdot\cos\theta + \frac{\partial u}{\partial y}\cdot\sin\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta} = -r\frac{\partial v}{\partial x}\sin\theta + r\frac{\partial v}{\partial y}\cdot\cos\theta$$

$$= r\frac{\partial u}{\partial y}\sin\theta + r\frac{\partial u}{\partial x}\cos\theta$$

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}\cdot r\cos\theta$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r} = -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\cdot\sin\theta$$

$$\therefore \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

对数函数 $\log z = \log r + i\theta$

$u(r,\theta)=\log r$, $v(r,\theta)=\theta$

$\frac{\partial u}{\partial r}=\frac{1}{r}$. $\frac{\partial u}{\partial \theta}=0$, $\frac{\partial v}{\partial r}=0$, $\frac{\partial v}{\partial \theta}=1$

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{1}{r}\frac{\partial u}{\partial \theta}=-\frac{\partial v}{\partial r}$$

所以对数函数在指定区域全纯.

10. 设 $z=x+iy$, $f(z)=u(x,y)+iv(x,y)$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x}-\frac{1}{i}\frac{\partial}{\partial y}\right)(u+iv)$$

$$= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+i\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]$$

$$4\frac{\partial}{\partial \bar{z}}\left(\frac{\partial f}{\partial z}\right) = \left(\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y}\right)\left[\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}+i\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]$$

$$= \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial x\partial y} + \frac{1}{i}\left(\frac{\partial^2 u}{\partial y\partial x} - \frac{\partial^2 v}{\partial y^2}\right)$$

$$+ i\left(\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial x^2}\right) + \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y\partial x}\right)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + i\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u+iv) = \Delta f$$

同理可证 $4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = \Delta$.

11. 因为 f 全纯.

所以 $\frac{\partial f}{\partial \bar{z}}=0$.

所以 $\Delta f = \left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+i\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$

$= 0$.

所以 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$

所以 u, v 为调和函数.

12. $f(x+iy)=\sqrt{|x||y|}$, $h=h_1+ih_2$

$$\lim_{h\to0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to0}\frac{\sqrt{|h_1||h_2|}-0}{h}$$

因为$h_2=h_1>0$ 与 $h_2=h_1<0$ 上的极限不相等. 所以f在(0,0)不可导.

$u(x,y)=\sqrt{|x||y|}$, $v(x,y)=0$

$$\left.\frac{\partial u}{\partial x}\right|_{(0,0)}=\lim_{x\to0}\frac{\sqrt{|x|\cdot0}-\sqrt{0\cdot0}}{x}=0$$

$$\left.\frac{\partial u}{\partial y}\right|_{(0,0)}=\lim_{y\to0}\frac{\sqrt{0\cdot|y|}-\sqrt{0\cdot0}}{y}=0$$

$$\left.\frac{\partial v}{\partial x}\right|_{(0,0)}=\left.\frac{\partial v}{\partial y}\right|_{(0,0)}=0$$

显然 $\left.\frac{\partial u}{\partial x}\right|_{(0,0)}=\left.\frac{\partial v}{\partial y}\right|_{(0,0)}$ $\left.\frac{\partial u}{\partial y}\right|_{(0,0)}=-\left.\frac{\partial v}{\partial x}\right|_{(0,0)}$

(a). 因为 $u=Re(f)$ 为常数

所以 $\frac{\partial u}{\partial x}\equiv\frac{\partial u}{\partial y}\equiv0$.

再由C-R方程. $\frac{\partial v}{\partial x}\equiv\frac{\partial v}{\partial y}\equiv0$

所以 u, v均为常数

所以 f 为常数

b) 证明同(a).

c). ∵ $|f|=C$ ∴ $|f|^2=C$.

∴ $f\cdot\bar{f}=C$

∴ $\frac{\partial(f\cdot\bar{f})}{\partial z}\equiv0$

∴ $\frac{\partial(u^2+v^2)}{\partial z}\equiv0$ 即$\left(\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y}\right)(u^2+v^2)\equiv0$

∴ $$\begin{cases}2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}=0\\ 2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}=0\end{cases}$$

C-R方程:

$$\begin{cases}u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}=0 & ①\\ v\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}=0 & ②\end{cases}$$

①×u + ②×v 得.

$$(u^2+v^2)\frac{\partial u}{\partial x}\equiv0$$

$u^2+v^2\equiv0$. 则 $f\equiv0$. 原命题得证

则 $\frac{\partial u}{\partial x}\equiv0$. 同理可得. $\frac{\partial u}{\partial y}\equiv0$

∴ u为常数.

再由C-R方程可得 v为常数

∴ f为常数.

14. $$\sum_{n=M}^{N}a_nb_n=\sum_{n=M}^{N}a_n(B_n-B_{n-1})$$

$=a_NB_N-a_NB_{N-1}+a_{N-1}B_{N-1}-a_{N-1}B_{N-2}$

$+\cdots+a_{M+1}(B_{M+1}-B_M)+a_M(B_M-B_{M-1})$

$=a_NB_N-(a_N-a_{N-1})B_{N-1}-(a_{N-1}-a_{N-2})B_{N-2}$

$-\cdots-(a_{M+1}-a_M)B_M-a_MB_{M-1}$

$=a_NB_N-a_MB_{M-1}-\sum_{n=M}^{N-1}(a_{n+1}-a_n)B_n$

15. ∵ $\sum_{n=1}^{\infty}a_n$收敛

∴ $\forall\varepsilon>0$, $\exists N$, $\forall n>N$ 及 $p\in\mathbb{N}^+$, 均有

$\left|\sum_{k=n}^{n+p}a_k\right|<\varepsilon$.

∴ $\left|\sum_{k=n}^{n+p}a_kx^k\right|=\left|x^{n+p}A_{n+p}-x^nA_{n-1}-\sum_{k=n}^{n+p-1}(x^{k+1}-x^k)A_k\right|$

$\le|A_{n+p}|x^{n+p}+|A_{n-1}|x^n+\sum_{k=n}^{n+p-1}|x^{k+1}-x^k|\cdot|A_k|$

$\le\varepsilon(x^{n+p}+x^n+x^n-x^{n+1}+x^{n+1}-x^{n+2}+\cdots+x^{n+p-1}-x^{n+p})$

$=2x^n\varepsilon\le2\varepsilon$ $(x\le1)$

∴ $\sum_{n=1}^{\infty}a_kx^k$ 在 $x\in[0,1]$ 上一致收敛.

∴ $\lim_{x\to1}\sum_{n=1}^{\infty}a_kx^k=\lim_{x\to1}S(x)=S(1)=\sum_{n=1}^{\infty}a_k$

(在计算A_{n+p}时, 虽然 $A_{n+p}=\underbrace{a_1+a_2+\cdots+a_{n-1}}+a_n+\cdots+a_{n+p}$, 但是由于前n项完全相同. 所以不妨假设均为0. 则 $|A_{n+p}|=\left|\sum_{k=n}^{n+p}a_k\right|<\varepsilon$).

(a) $\lim\limits_{n\to\infty}\sqrt[n]{a_n}=\lim\limits_{n\to\infty}(\log n)^{\frac{2}{n}}$

$$=\lim_{n\to\infty}e^{\frac{2}{n}\log(\log n)}$$

$$\lim_{x\to+\infty}\frac{2\log(\log x)}{x}=\lim_{x\to+\infty}2\cdot\frac{1}{\log x}\cdot\frac{1}{x}=0$$

$$\therefore\lim_{n\to\infty}\sqrt[n]{a_n}=e^{\lim\limits_{n\to\infty}\frac{2\log(\log n)}{n}}=1$$

$\therefore R=1.$

b). $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\frac{(n+1)!}{n!}=\lim\limits_{n\to\infty}(n+1)=\infty$

$\therefore R=0.$ （用17题结论）.

). $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\frac{(n+1)^2}{4^{n+1}+3(n+1)}\cdot\frac{4^n+3n}{n^2}$

$$=\lim_{n\to\infty}\frac{n^2+2n+1}{n^2}\cdot\frac{4^n+3n}{4^{n+1}+3n+3}$$

$$=\lim_{n\to\infty}\frac{1+\frac{3n}{4^n}}{4+\frac{3n+3}{4^n}}=\frac{1}{4}$$

$\therefore R=4$

). $\because n!\sim cn^{n+\frac12}e^{-n}\quad(c>0)$

$$\therefore\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{(n!)^{\frac3n}}{((3n)!)^{\frac1n}}$$

$$=\lim_{n\to\infty}\frac{c^{\frac3n}n^{(n+\frac12)\frac3n}(e^{-n})^{\frac3n}}{c^{\frac1n}(3n)^{(3n+\frac12)\frac1n}(e^{-3n})^{\frac1n}}$$

$$=\lim_{n\to\infty}\frac{n^{(3+\frac{3}{2n})}e^{-3}}{(3n)^{(3+\frac{1}{2n})}e^{-3}}$$

$$=\lim_{n\to\infty}\frac{1}{3^{3+\frac{1}{2n}}}\cdot n^{3+\frac{3}{2n}-3-\frac{1}{2n}}$$

$$=\frac{1}{27}$$

$\therefore R=27$

). 若 $\alpha=-k$, k 取 $0,1,2,\cdots$

或 $\beta=-k$, k 取 $0,1,2,\cdots$

则显然 $R=\infty$.

否则 $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\frac{|\alpha(\alpha+1)\cdots(\alpha+n)\beta(\beta+1)\cdots(\beta+n)|}{(n+1)!\,|\gamma(\gamma+1)\cdots(\gamma+n)|}\cdot\frac{n!\,|\gamma(\gamma+1)\cdots(\gamma+n-1)|}{|\alpha(\alpha+1)\cdots(\alpha+n-1)|\,|\beta(\beta+1)\cdots(\beta+n-1)|}$

$$=\lim_{n\to\infty}\frac{1}{n+1}\frac{|\alpha+n|\,|\beta+n|}{|\gamma+n|}=\lim_{n\to\infty}\frac{1}{n+1}\frac{\sqrt{(\alpha_1+n)^2+\alpha_2^2}\sqrt{(\beta_1+n)^2+\beta_2^2}}{\sqrt{(\gamma_1+n)^2+\gamma_2^2}}=1.\quad\therefore R=1$$

(f). $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\frac{|(-1)^{n+1}|}{(n+1)!\,(n+1+r)!}\cdot\frac{4^{n}}{4^{n+1}}\cdot\frac{n!\,(n+r)!}{|(-1)^n|}$

$$=\lim_{n\to\infty}\frac14\,\frac{1}{n+1}\cdot\frac{1}{n+1+r}=0.$$

$\therefore R=+\infty$

17. $\because\lim\limits_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=L$

$\therefore\forall\varepsilon>0,\ \exists N$. 当 $n\geq N$ 时. $|a_{n+1}|\leq(L+\varepsilon)|a_n|$

$$\leq(L+\varepsilon)^2|a_{n-1}|$$

$$\vdots$$

$$\leq(L+\varepsilon)^{n+1-N}|a_N|$$

不妨写成 $|a_n|\leq(L+\varepsilon)^{n-N}|a_N|\quad\forall n>N.$

同理可得 $(L-\varepsilon)^{n-N}|a_N|\leq|a_n|\quad\forall n>N.$

$$(L-\varepsilon)^{\frac{n-N}{n}}|a_N|^{\frac1n}\leq|a_n|^{\frac1n}\leq(L+\varepsilon)^{\frac{n-N}{n}}|a_N|^{\frac1n}$$

$$\because\lim_{n\to\infty}(L+\varepsilon)^{1-\frac Nn}|a_N|^{\frac1n}=L+\varepsilon$$

$$\lim_{n\to\infty}(L-\varepsilon)^{1-\frac Nn}|a_N|^{\frac1n}=L-\varepsilon$$

$$\therefore L-\varepsilon\leq\lim_{n\to\infty}|a_n|^{\frac1n}\leq L+\varepsilon$$

由 ε 的任意性 知 $\lim\limits_{n\to\infty}|a_n|^{\frac1n}=L$

18. 假设 $f(z)=\sum\limits_{k=0}^{\infty} a_k z^k$, $|z|<R$

则由Th2.6. $f'(z)=\sum\limits_{k=1}^{\infty} k a_k z^{k-1}$, $|z|<R$.

$$f''(z)=\sum_{k=2}^{\infty} k(k-1)a_k z^{k-2},\ |z|<R$$
$$\vdots$$
$$f^{(n)}(z)=\sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)a_k z^{k-n},\ |z|<R.$$
$$\vdots$$

令 $z_0\in|z|<R$. 则 $f(z_0), f'(z_0), \cdots f^{(n)}(z_0)\cdots$ 均存在.

$$f(z)=\sum_{k=0}^{\infty} a_k z^k=\sum_{k=0}^{\infty} a_k(z_0+z-z_0)^k$$
$$=\sum_{k=0}^{\infty} a_k\left[z_0^k+C_k^1 z_0^{k-1}(z-z_0)+\cdots+C_k^{k-1} z_0(z-z_0)^{k-1}+(z-z_0)^k\right]$$
$$=\sum_{k=0}^{\infty} a_k z_0^k+\left[\sum_{k=1}^{\infty} a_k C_k^1 z_0^{k-1}\right](z-z_0)+\left[\sum_{k=2}^{\infty} a_k C_k^2 z_0^{k-2}\right](z-z_0)^2$$
$$+\cdots+\left[\sum_{k=n}^{\infty} a_k C_k^n z_0^{k-n}\right](z-z_0)^n+\cdots$$
$$=f(z_0)+f'(z_0)(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\cdots+\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n+\cdots$$
$$=\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

19. (a). $\lim\limits_{n\to\infty}\sqrt[n]{a_n}=\lim\limits_{n\to\infty}\sqrt[n]{n}=1$. $\therefore R=1$.

若 $|z|=1$. 则 $\lim\limits_{n\to\infty} n|z|^n=\lim\limits_{n\to\infty} n=\infty\neq 0$.

$\therefore \sum n z^n$ 发散.

(b). 因为 $\left|\frac{z^n}{n^2}\right|=\frac{1}{n^2}$, $\forall |z|=1$.

而 $\sum\frac{1}{n^2}$ 收敛. 所以 $\sum\frac{z^n}{n^2}$ 绝对收敛($\forall|z|=1$)

(c). 若 $z=1$. 则 $\sum\frac{z^n}{n}=\sum\frac{1}{n}$ 发散

若 $|z|=1$ 且 $z\neq1$. 令 $S_n=\sum\limits_{k=1}^{n} z^k$

$$\sum_{k=1}^{n}\frac{z^k}{k}=\frac{1}{n}S_n-\sum_{k=1}^{n-1}\left(\frac{1}{k+1}-\frac{1}{k}\right)S_k$$
$$=\frac{1}{n}\frac{z(1-z^n)}{1-z}+\sum_{k=1}^{n-1}\frac{1}{k(k+1)}\frac{z(1-z^k)}{1-z}$$
$$=\frac{z}{1-z}\left[\frac{1-z^n}{n}+\sum_{k=1}^{n-1}\frac{1}{k(k+1)}-\sum_{k=1}^{n-1}\frac{z^k}{k(k+1)}\right]$$
$$=\frac{z}{1-z}\left[\frac{1-z^n}{n}+\left(1-\frac{1}{n}\right)-\sum_{k=1}^{n-1}\frac{z^k}{k^2+k}\right]$$
$$=\frac{z}{1-z}\left[1-\frac{z^n}{n}-\sum_{k=1}^{n-1}\frac{z^k}{k^2+k}\right]$$

$\because \sum\limits_{k=1}^{\infty}\frac{z^k}{k^2+k}$ 收敛 $(|z|=1)$ $\therefore$ 不妨设 $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n-1}\frac{z^k}{k^2+k}=S$

$$\therefore \lim_{n\to\infty}\sum_{k=1}^{n}\frac{z^k}{k}$$
$$=\frac{z}{1-z}\lim_{n\to\infty}\left[1-\frac{z^n}{n}-\sum_{k=1}^{n-1}\frac{z^k}{k^2+k}\right]$$
$$=\frac{z}{1-z}\left(1-\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{z^k}{k^2+k}\right)$$
$$=\frac{z}{1-z}(1-S).$$

$\therefore \sum\limits_{k=1}^{\infty}\frac{z^k}{k}$ 收敛 ($|z|=1$ 且 $z\neq1$)

20. 若 $\frac{1}{(1-z)^m}=\sum\limits_{k=0}^{\infty} a_k z^k$

则 $a_k=\dfrac{\left(\frac{1}{(1-z)^m}\right)^{(k)}}{k!}$

令 $f(z)=\frac{1}{(1-z)^m}$

$$\text{则 } f'(z)=\frac{-m(1-z)^{m-1}(-1)}{(1-z)^{2m}}=\frac{m}{(1-z)^{m+1}}$$
$$f''(z)=m\,\frac{-(m+1)(1-z)^m(-1)}{(1-z)^{2(m+1)}}=\frac{m(m+1)}{(1-z)^{m+2}}$$
$$\vdots$$
$$f^{(n)}(z)=\frac{m(m+1)\cdots(m+n-1)}{(1-z)^{m+n}}$$
$$\vdots$$
$$\therefore f^{(n)}(0)=\begin{cases} m(m+1)\cdots(m+n-1) & n\geq1\\ 1, & n=0\end{cases}$$
$$\therefore \frac{1}{(1-z)^m}=1+mz+\frac{m(m+1)}{2!}z^2+\cdots+\frac{m(m+1)\cdots(m+n-1)}{n!}z^n+\cdots$$
$$\therefore a_n=\frac{m(m+1)\cdots(m+n-1)}{n!}=\frac{(m+n-1)!}{n!\,(m-1)!}=\frac{(n+1)\cdots(n+m-1)}{(m-1)!}\sim\frac{n^{m-1}}{(m-1)!}\quad n\to\infty.$$

21. $\sum_{k=0}^{n} \frac{z^{2^k}}{1-z^{2^{k+1}}} = \sum_{k=0}^{n} \frac{1+z^{2^k}-1}{(1-z^{2^k})(1+z^{2^k})}$

$= \sum_{k=0}^{n} \left(\frac{1}{1-z^{2^k}} - \frac{1}{1-z^{2^{k+1}}} \right)$

$= \frac{1}{1-z} - \frac{1}{1-z^{2^{n+1}}}$

$\therefore$ 左边 $= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{z^{2^k}}{1-z^{2^{k+1}}}$

$= \lim_{n\to\infty} \left(\frac{1}{1-z} - \frac{1}{1-z^{2^{n+1}}} \right)$

$= \frac{1}{1-z} - 1 = \frac{+z}{1-z} \doteq \frac{z}{1-z} \quad (|z|<1).$

$\frac{z}{1-z} - \sum_{k=0}^{n} \frac{2^k z^{2^k}}{1+z^{2^k}} = \frac{z}{1-z} - \frac{z}{1+z} - \frac{2z^2}{1+z^2} - \cdots - \frac{2^n z^{2^n}}{1+z^{2^n}}$

$= \frac{2z^2}{1-z^2} - \frac{2z^2}{1+z^2} - \cdots - \frac{2^n z^{2^n}}{1+z^{2^n}}$

$\cdots\cdots$

$= \frac{2^{n+1} z^{2^{n+1}}}{1-z^{2^{n+1}}}$

$\therefore \sum_{k=0}^{n} \frac{2^k z^{2^k}}{1+z^{2^k}} = \frac{z}{1-z} - \frac{2^{n+1} z^{2^{n+1}}}{1-z^{2^{n+1}}}$

$\therefore$ 左边 $= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{2^k z^{2^k}}{1+z^{2^k}} = \lim_{n\to\infty} \frac{z}{1-z} - \frac{2^{n+1} z^{2^{n+1}}}{1-z^{2^{n+1}}}$

$= \frac{z}{1-z}.$

22. 假设存在有限划分，使得

$\mathbb{N} = S_1 \cup S_2 \cdots \cup S_k$

其中 $S_j = \{a_j, a_j+d_j, a_j+2d_j, \cdots\cdots\}$

且 $a_i \neq a_j$, $d_i \neq d_j$ $\forall i \neq j$

则 $\sum_{n=1}^{\infty} z^n = \sum_j \sum_{n \in S_j} z^n$

即 $\frac{z}{1-z} = \frac{z^{a_1}}{1-z^{d_1}} + \cdots + \frac{z^{a_k}}{1-z^{d_k}}$ （z为级数的收敛点）

两边通分得 $z(1-z^{d_1})\cdots(1-z^{d_k}) = \sum_j z^{a_j} \prod_{\substack{i=1 \\ i\neq j}}^{k} (1-z^{d_i})$

取 $d = \max\{d_k\}$，考察 $z_0 = e^{i\frac{2\pi}{d}}$ 则 z_0 为左边的零点，但非右边的零点，矛盾。$\therefore$ 假设不成立.

23. 当 $x<0$, $f'(x)=0$.

$x>0$, $f'(x)=\left(e^{-\frac{1}{x^2}}\right)'$

$$= e^{-\frac{1}{x^2}} \cdot (-1)\frac{-2x}{x^4}$$

$$= \frac{2e^{-\frac{1}{x^2}}}{x^3}$$

$\therefore$ $x=0$, $f'_-(0)=\lim\limits_{x\to 0^-}\dfrac{f(x)-f(0)}{x-0}=0$

$$f'_+(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}$$

$$=\lim_{x\to 0^+}\frac{e^{-\frac{1}{x^2}}}{x}$$

$$=\lim_{x\to 0^+}\frac{1}{xe^{\frac{1}{x^2}}}$$

$$\because \lim_{x\to 0^+} xe^{\frac{1}{x^2}}=\lim_{x\to 0^+}\frac{e^{\frac{1}{x^2}}}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{e^{\frac{1}{x^2}}\cdot\frac{-2x}{x^4}}{\frac{-1}{x^2}}$$

$$=\lim_{x\to 0^+}\frac{2e^{\frac{1}{x^2}}}{x}=\infty$$

$\therefore f'_+(0)=0.$ $\quad\therefore f'(0)=0$

$$\therefore f'(x)=\begin{cases}0, & x\le 0\\ \dfrac{2e^{-\frac{1}{x^2}}}{x^3}, & x>0\end{cases}$$

继续上述求导过程. 注意. $\lim\limits_{x\to 0^+}\dfrac{e^{-\frac{1}{x^2}}}{x^k}$ $(k\ge 1)$

$$=\lim_{x\to 0^+}\left(\frac{1}{x^2}\right)^{\frac{k}{2}}\cdot\frac{1}{e^{\frac{1}{x^2}}}$$

$$=\lim_{y\to +\infty} y^{\frac{k}{2}}\cdot\frac{1}{e^y}=0.$$

易得. $f^{(n)}(0)=0$. $\forall n\ge 1$.

24. $\int_\gamma f(z)dz$

$$=\int_a^b f(z(t))\,dz(t)=\int_a^b f(z(t))z'(t)dt$$

$$=-\int_b^a f(z(t))z'(t)dt$$

$$=-\int_{\gamma^-} f(z)dz.$$

25. (a) γ: $z=re^{i\theta}$ r固定. $0\le\theta\le 2\pi$.

$$\int_\gamma z^n dz=\int_0^{2\pi}(re^{i\theta})^n\, d\,re^{i\theta}$$

$$=\int_0^{2\pi} r^n e^{in\theta}\, re^{i\theta}\cdot i\,d\theta$$

$$=ir^{n+1}\int_0^{2\pi}e^{i(n+1)\theta}d\theta$$

$$=ir^{n+1}\left.\frac{e^{i(n+1)\theta}}{i(n+1)}\right|_0^{2\pi}$$

$$=\frac{r^{n+1}}{n+1}\left(e^{i(n+1)2\pi}-e^{0}\right)$$

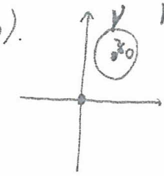
$$=0 \quad (n\ne -1).$$

若 $n=-1$ 则 $\int_\gamma \frac{1}{z}dz$

$$=\int_0^{2\pi}\frac{1}{re^{i\theta}}\cdot re^{i\theta}\cdot i\,d\theta$$

$$=2\pi i$$

(b).

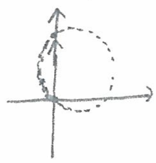


$n\ne -1$ $\int_\gamma z^n dz$

$$=\left.\frac{z^{n+1}}{n+1}\right|_{w_1}^{w_2}=0.$$

(Corollary 3.3)

$n=-1$ $\int_\gamma \frac{1}{z}dz=\ln z\Big|_{w_1}^{w_2}=0.$



若原点恰好在γ上.

(c). $\int_\gamma \dfrac{1}{(z-a)(z-b)}dz=\dfrac{1}{a-b}\int_\gamma\left(\dfrac{1}{z-a}-\dfrac{1}{z-b}\right)dz.$

$$=\frac{1}{a-b}\left[\int_\gamma\frac{1}{z-a}dz-\int_\gamma\frac{1}{z-b}dz\right]$$

$$=\frac{2\pi i}{a-b}$$ (a在γ内部. b在γ外部).

26. 设 $F'=f$, $G'=f$

则 $(F-G)'=0$

由 Corollary 3.4. $F-G\equiv C$