WILLIAM HART

INTERESTING NOTES

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We have advanced to new and surprising levels of bafflement

Miles Vorkosigan, Komarr

Introduction

This is a collection of notes on topics that have periodically interested me. The notes are available from github at `https://github.com/will-hart/interesting-notes` under an Apache 2.0 license.

The book uses the `tufte-book` and `tufte-handout` document classes which are also under the Apache 2.0 License. Feel free to read, suggest topics or submit corrections.

Market Basics

This section deals with some basic stock market information and a brief introduction into fundamental stock analysis.

Description of Basic Securities

Bonds[1] are typically based on debt - that is bond holders lend their money to the bond issuer in return for a "fixed income". The fixed income is in the form of periodic interest payments on the amount loaned. At the end of the bond term the issuer repays the original amount. Bonds are considered low risk, where the issuer is a stable entity such as a government with a strong underlying economy. As a result of the low risk the return is also low.

Bonds can be traded on the secondary market, and are usually sold at a market price which is not the same as the face value of the bond.[2] Bonds can be "fixed coupon" — which pays a fixed (percentage) amount bi-annually — or "capital indexed" which pay interest quarterly based on a percentage of the face value adjusted for inflation.

Stocks[3] allow their owners (or holders) to purchase part of a business. The business can periodically distribute some of its profit to its share holders in the form of *dividends*. Shareholders can also make money by the change in value of stocks as they are sold on a secondary market. Stocks have higher potential returns but also higher risks.

Mutual funds allow investors to pool their money to buy a combination of bonds and shares in a single fund. Depending on the mutual fund this has the advantage of built in diversification, as well as buying a portfolio of investments (hopefully) managed by an expert, with a lower cost in time and money than researching and purchasing each of the securities individually.

[1] sometimes called debt

[2] Face Value is the amount to be repaid at maturity

[3] also called shares or equity

Description of Advanced Securities

Derivatives TODO!

Warrants TODO!

Portfolio Theory

A way to spread risk and achieve a certain balance of aggressive vs conservative is to maintain a portfolio of stock. For instance an aggressive investment portfolio may include 50% in shares, 40% in fixed income investments and 10% in cash or equivalent. A conservative portfolio may have 70% in fixed income, 20% in equities and 10% in cash and equivalents [4]. Other sources such as Graham's *Intelligent Investor* suggest a 50-50 split between debt and equity, however note that this split changes with the circumstances of both the wider market and the risk aversion of the investor[5].

Spreading the types and areas of investment is a strategy of *diversification*, which relies on various assets performing in different ways at different times. If any one class or category of security performs poorly, the idea is that other assets in a diverse portolio will maintain the value of the portfolio.

[4] investopedia.com. Investing 101. URL http://www.investopedia.com/university/beginner/

[5] Benjamin Graham. *The Intelligent Investor*. Harper Business, fourth revised edition, 1973

Bond Pricing Analysis

TODO! (http://www.mysmp.com/bonds/bond-math.html ?)

Efficient Market Hypothesis

TODO!

Quantitative Finance

This section deals with some applications of quantitative finance, both computational and theoretical.

Equity Mathematical Models

Equities and finance are heavily influenced by random events (see [6] for a detailed philosophical discussion). As such, the analysis of financial instruments lends itself to a stochastic process.

[6] Nicholas Nasim Taleb. *Fooled By Randomness*. Penguin, 2005

A *Geometric Brownian Motion* (GBM) equation (a form of stochastic differential equation, or SDE known as the Black-Sholes-Merton equation[7]) can be used to approximate a random process over time. In its continuous form this looks like [8]:

[7] Yves Hilpisch. *Python for Finance*. O'Reilly Media Inc, 2014

[8] Alonso Pena. *Advanced Quantative Finance With C++*. Packt Publishing, 2014

$$ds = \mu S dt + \sigma S dW \tag{1}$$

This breaks the movement of a stock price down into two key effects:

- a deterministic effect (the left of the plus sign)
- a stochastic effect (the right of the plus sign)

In the equation, μ is known as *drift*, σ is *volatility*, S is the stock price, dt is the change in time and dW is an increment in a *Weiner process*.

As equity markets are a discrete process, this equation must be transformed into a discrete equation.

$$S_{t+1} = S_t(1 + r\Delta t + \sigma \varepsilon_t \sqrt{\Delta t}) \tag{2}$$

Here ε is a sample from a gaussian distribution with zero mean and standard deviation of 1 (i.e. $N(0,1)$) and r is the risk free rate of return. This equation can be solved iteratively for a given time period if r, σ, ε and S_0 are provided.

Another formulation for pricing European Call options gives a similar result of the discretised SDE[9]:

[9] Yves Hilpisch. *Python for Finance*. O'Reilly Media Inc, 2014

$$S_t = S_{t-\Delta t} e^{(r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t} z_t} \tag{3}$$

In this instance, z_t represents the random variable. Similar formulations can be determined for foreign exchange:

$$X_{t+1} = X_t(1 + (r_d - r_f)\Delta t + \sigma \varepsilon_t \sqrt{\Delta t}) \tag{4}$$

where r_d and r_f are the domestic and foreign fisk free rates of return.

Structural and Intensity Models

TODO! See [10] chapter 2.

[10] Alonso Pena. *Advanced Quantative Finance With C++*. Packt Publishing, 2014

Monte Carlo Simulation

Monte Carlo simulation is a method of estimating a probabilistic outcome through a high quantity of simulations. In a quantitative finance context, we may wish to estimate the future price of an equity based on use of a GBM equation simulated M times.

For instance the process could be as follows for calculating a call option derivative based on estimate the price of a stock using Monte Carlo simulation:

1. Generate M different "trajectories" for the stock using a GBM simulation from time $t = 0$ to time $t = T$. This generates a set of N price estimates for M different simulations, with the notation:

$$\{S_i^j\} \qquad i = 0...N, \qquad j = 1...M \tag{5}$$

   This produces a vector of M values for S_T,

$$\{S_T^i\} \qquad i = 0...M \tag{6}$$

2. We next compute the pay off for each stock value. This is given by

$$H(S_T^i) \qquad i = 1...M \tag{7}$$

   Where

$$H(S_T) = max(S_T - K, 0) \tag{8}$$

   and K is the actual price of the equity or premium. The expected pay off can then be computed by the average of all pay offs:

$$E[H(S_T^i)] = \frac{1}{M}\Sigma_{i=1}^{M} H(S_T^i) \tag{9}$$

3. The item should then be discounted to present value, by either applying a discount factor DF_T or

$$\pi = e^{-rT} \times E[H(S_T)] \tag{10}$$

   Where π is the value of the derivative.

Binomial Trees [11]

[11] Alonso Pena. *Advanced Quantative Finance With C++*. Packt Publishing, 2014

This approach builds a tree of possible prices. At each stage, the underlying can be assumed to go up or down by a given amount. The amount of change up (u) or down (d) is described by

$$u = e^{\sigma\sqrt{\Delta t}} \tag{11}$$

$$d = e^{-\sigma\sqrt{\Delta t}} \tag{12}$$

The probability of an equity going up p is

$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{13}$$

The probability of the underlying going down is $1 - p$. The binomial tree is then built in the following phases:

1. Construct a tree where each level corresponds to a time step in the simulation period from $t = 0$ to $t = T$. For example in two simulation steps,

   $$At = 0, S = S_0 \tag{14}$$

   $$At = t_1, S = uS_0, \qquad dS_0 \tag{15}$$

   $$At = t_2, S = u^2S_0, \qquad udS_0, \qquad udS_0, \qquad d^2S_0 \tag{16}$$

   Note that the central value at the last period is shared between adjacent nodes, hence there are only three distinct estimates after $N = 2$ steps.

   This produces a number of prices at each time step. The notation is based on the estimate number k at time T. There are N time steps

   $$\{S_T^k\} \qquad k = 1...N + 1 \tag{17}$$

2. Once the tree has been built, the payoff $H(S_T^k)$ should be calculated for each S_T^k

3. Finally the tree is traversed back up towards the root node, calculating the discounted weighted probability of each node. If we define a given node (not a leaf, i.e. $t \neq T$) it has two children, one for the up price (denoted S_T^u) and one for the down price, S_T^d. The value for the parent node is given by

   $$V_{T-1}^k = e^{-r\Delta t}[pH(S_T^u) + (1 - p)H(S_T^d)] \tag{18}$$

   in this case V_T^l is shorthand for $H(S_T^k)$

4. Once the tree has been traversed back to the top, the value of the derivative $\pi = V_1^1$

Finite Difference Method

The finite difference method is a method for discretising a differential equation.[12] In the quantitative finance approach, we want to discretise partial differential equations (PDEs). The finite differences method is based on the relationship:

[12] Alonso Pena. *Advanced Quantative Finance With C++*. Packt Publishing, 2014

$$f(x) = \frac{df}{dx} \approx \frac{\Delta f}{\Delta x} = \frac{f_{i+1} - f_i}{\Delta x} \tag{19}$$

The most important PDE in finance is the *Black-Scholes PDE* , which is given by:

$$\frac{\delta V}{\delta t} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 V}{\delta S^2} + rS\frac{\delta V}{\delta S} - rV = 0 \tag{20}$$

This is usually solved in the S and t axes, where $S \in [a, b]$ and $t \in [0, T]$. The domain of this equation is said to be

$$\Omega = \{(S, t)\forall S \in [a, b] \times t \in [0, T]\} \tag{21}$$

In other words, as the finite difference method is solving some partial differential equation in S and t, the solution space is the rectangular domain defined by the ranges of S and t. For a European call,

$$V(S, T) = max(S - K, 0) \tag{22}$$

The boundary conditions are $V(a, t) = 0$ and $V(b, t) = S$. This equation can be transformed with some variable substitution so that

$$\frac{\delta u}{\delta \tau} = \frac{\delta^2 u}{\delta x} \qquad -\infty < x < \infty, \tau > 0 \tag{23}$$

This is a dimensionless PDE with a new solution domain $\Omega = \{(x, \tau)\}$. The payoff relationship therefore becomes [13]

[13] Where $k = \frac{r}{0.5 \times \sigma^2}$

$$u(x, 0) = max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0) \tag{24}$$

Using finite differences, the return can be described as[14]

[14] Where $\alpha = \frac{\Delta\tau}{(\Delta x)^2}$

$$u_{i,j+1} = \alpha u_{i+1,j} + (1 - 2\alpha)u_{i,j} + \alpha u_{i-1,j} \tag{25}$$

This relationship can be solved iteratively, using the following steps:

1. Discretise the domain into N space divisions of dS and M time divisions of dT. Use these to determine the steps $\Delta\tau, \Delta x$.

2. Use finite differences to approximate the derivatives

3. Calculate the results of the equation iteratively for each time step

For a worked example, see [15], end of chapter 3.

[15] Alonso Pena. *Advanced Quantative Finance With C++*. Packt Publishing, 2014

Photonics

Basics

The speed of light is given by $c = 2.99 \times 10^8 m/s$ in vacuum, and slower in other medium. Light can experience **refraction** - deflection when passing from one medium to another - and **reflection** - bouncing off a surface. Detection and measurement of light energy is a field known as *radiometry. The properties of light can be described by both particle and wave analogies.*

Light as a particle

A "light particle" is called a photon, which is a particle with no mass or charge. It carries electromagnetic energy and can interact with other particles. The amount of energy E for a photon is given by

$$E = \frac{hc}{\lambda} \tag{26}$$

Where h is Planck's constant (6.25×10^{-34})*, c is the speed of light and* λ *is the light's wavelength in meters.*

The photoelectric effect gives evidence of light's particle like properties. This is the effect seen where some materials when a light is shone on them emit electrons. This behaviour reflects light as a particle because more intense radiation did not cause higher energy electrons to be emitted and electron energy was dependent on wavelength, not amplitude of the wave.

No matter how intense the light, photons below a given minimum frequency do not cause electrons to be emitted. The relationship governing the energy of emitted electrons is

$$E_{e-} = \frac{hc}{\lambda} - p \tag{27}$$

Where p is the characteristic escape energy for the given metal and E_{e-} is the energy of the escaping electron.

Light as a wave

Light also exhibits the properties of *interference and* diffraction which fit the idea of a wave model of light. Waves move energy without moving mass at a speed independent of intensity or wavelength. Light waves have an electric and magnetic field which changes at right angles to the direction of motion. The wavelength (λ) is the distance between successive peaks or troughs in a wave. The wave number ν is the inverse of the wave length. Standard mathematical properties such as period τ and frequency f can be calculated.

Light waves are not always sinusoidal in shape. In addition any particle light wave may consist of a series of waves with peaks in all different directions perpendicular to the direction of travel. The amount of energy that flows across a unit area perpendicular ot the direction of travel is called the irradiance or flux density of the wave.

Light waves may also be polarised, where the waves vibrate in specific directions perpendicular to the direction of travel. The intensity of light travelling through a linear polariser can be given by

$$I(\theta) = I_0 \cos^2(\theta) \tag{28}$$

where $I(\theta)$ is the light intensity passed by the polariser and I_0 is the incident light density.

Light waves exhibit the properties of *superposition*, *reflection*, *refraction*, *diffraction* and *interference*.

For refraction, the angle of incidence is the same as the angle of reflection, when measured from the surface normal.

Refraction occurs at the interface between surfaces due to the difference in speed of light in the particular material. The ratio of speeds for two surfaces can be given by the index of refraction:

$$\frac{n_2}{n_1} = \frac{\sin\theta}{\sin\phi} \tag{29}$$

Where n_2 and n_1 are the indices of refraction for the two media, θ is the angle of incidence (from the surface normal) and ϕ is the angle of refraction (measured from the normal).

the index of refraction can be calculated by

$$n_i = \frac{c}{v_i} \tag{30}$$

Where v_i is the velocity of light in the medium i.

The slit test proved the wave like properties of light. A slit of width d has a plane wave shone on it. Where $d < \lambda$ a very even diffuse light is shone on to a surface on the far side of the slit. Essentially small spherical waves are emitted from the slit . Where $d \approx \lambda$, then emitted wave appears to have similar properties to the plane wave.

The two slit experiment in 1801 showed the two waves of light emitted through the slits interfering, resulting in a "zebra strip" sort of pattern on the far surface.

The electromagnetic spectrum

Only a small spectrum of electromagnetic radiation is visible light. Waves go from long wave to radio through infrared, visible light, ultraviolet, xrays and gamma rays in order of decreasing wave length.

White light contains a mixture of different coloured light, each of which has a different wavelength. Due to their differing wavelength, the white light can be separated when refracted through a particular medium.

Any material above absolute zero emits electromagnetic radiation, with molecules having a characteristic set of spectral lines. Atoms changing state produce visible and ultraviolet radiation, whilst molecules changing vibrational or rotational states emit infrared radiation. Liquids and solids typically have much broader spectral lines than gases as they can take on a much wider range of energy states.

At an atomic level, an atom consisting of protons and neutrons has a series of energy shells (labelled K through O) which can contain electrons. In their *grounded* state electrons have limited energy. However as energy is added they can become excited and move into higher energy shells. As they do this they absorb or emit quanta (unique amounts) of energy, with the exact nature depending on the electronic structure of the atom.

For a hydrogen atom there are 6 major energy levels, ranging from $-0.38eV$ to $-13.6eV$. This means that if an electron is in the $n = 3$ layer (which has an energy of $-1.5eV$ then it can emit a photon with

$$-1.51 - (-13.6) = 12.09eV \tag{31}$$

If an electron is freed from the atom then its energy level $E_\infty = 0$. Atoms can also absorb photons which have energy exactly matching the difference between electron energy levels. Additionally a molecule in a gas or liquid may absorb a photon where it has a vibrational or rotational energy level matching the energy of the photon.

Blackbody radiation

Blackbody radiation is the theoretical maximum radiation expected for temperature related self-radiation. That is the amount of energy radiated from a body (in various spectra) based on its temperature (in Kelvin).

The energy radiated by a black body in a given wave band is the sum of all energies radiated at the wavelengths within the band. The same holds for the power emitted, which can be calculated in watts per square meter using the Stefan-Boltzmann law:

$$W_s = \sigma_s T^4 \tag{32}$$

Where W_s is the radiated power, σ_s is the Stefan-Boltzmann constant, 5.67×10^{-8} and T is the temperature in Kelvin. W_s is the power per unit area, also known as the emitted radiant flux density. Typically graybodies do not perfectly emit and the above equation is factored down bt the emissivity (ε) of the material.

$$W_s = \varepsilon \sigma_s T^4 \tag{33}$$

Blackbodies emit radiation over a range of wavelengths. The power radiated per unit area W_λ within a given waveband $\Delta\lambda$ is given by Planck's radiation formula:

$$W_\lambda = \frac{c_1}{\lambda^5} - \frac{1}{\frac{c_2}{e\lambda T} - 1} \tag{34}$$

Where

$$c_1 = 2\pi c^2 h = 3.75 \times 10^{-16} \tag{35}$$

$$c_2 = \frac{hc}{k} = 1.44 \times 10^{-4} \tag{36}$$

$$c = 3 \times 10^8 \tag{37}$$

$$h = 6.626 \times 10^{-34} \tag{38}$$

$$k = 1.38 \times 10^{-28} \tag{39}$$

There is a maximum emission wavelength which can be given by Wien's displacement law:

$$\lambda_{max} T = 2.898 \times 10^{-3} m \cdot K \tag{40}$$

Interaction with matter

The two main interactions are *absorption* and *scattering*. Absorption has been discussed as moving an electron into a higher energy level or exciting a molecule's vibration or rotation. The spectrum of absorbed light may have missing or removed wavelengths depending on what has been absorbed. This is the basis for objects having colour.

Scattering is redirection of light based on interaction with matter. Scattered radiation may have the same or longer wavelength (reduced energy) and may have a different polarisation.

If the scatterer are significantly smaller than the wavelength λ then they may absorb and immediately re-emit photons in a different direction. Where the emitted photon has the same wavelength as the incident light it is known as *Rayleigh scattering*.

Where the wavelength of the emitted radiation is longer and the molecule is left in an excited state, it is known as *Raman scattering*. For Raman scattering, secondary photons may later be emitted when the molecule returns to the ground state.

Where the scatterer is of similar size or larger than the wavelength, all wavelengths of the incident light are equally scattered, a process known as *Mie scattering*.

Bibliography

Benjamin Graham. *The Intelligent Investor*. Harper Business, fourth revised edition, 1973.

Yves Hilpisch. *Python for Finance*. O'Reilly Media Inc, 2014.

investopedia.com. Investing 101. URL `http://www.investopedia.com/university/beginner/`.

Alonso Pena. *Advanced Quantative Finance With C++*. Packt Publishing, 2014.

Nicholas Nasim Taleb. *Fooled By Randomness*. Penguin, 2005.

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