Cooling zone behind shock

Equations from my WR shock notes

Assume plane-parallel flow.

No magnetic fields or body forces.

Mass: $\rho v = \Phi_0 = \rho_1 v_1$ (i)

Momentum: $\rho(a^2 + v^2) = \Pi_0 = \rho_1 a_1^2 (1 + M_1^2)$ (ii)

Energy $\frac{5}{2}\rho v a^2 \left(1 + \frac{1}{5}M^2\right) = \mathcal{E}_0 - \int L\,dx$ (iii)

where $\mathcal{E}_0 = \frac{5}{2}\rho_1 v_1 a_1^2 \left(1 + \frac{1}{5}M\right.$

Note that L should really be the net heating: $L - H$

Derive Energy equation from Shu Volume II

Isothermal versus adiabatic sound speed

I am using a to mean isothermal sound speed here, but maybe I should switch to adiabatic.

$$\int_V (\Gamma - \Lambda)\, dV = \oint_A \frac{1}{2}\rho\left(u^2 + \underbrace{3a^2}_{\text{from } \mathcal{E}} + \underbrace{2a^2}_{\text{from } P}\right)\underline{u}\cdot\hat{\underline{n}}\, dA$$

$dV = dx\,dy\,dz \quad dA = dy\,dz$

$$\int (\Gamma - \Lambda)\, dx = \frac{1}{2}\rho u\left(u^2 + 5a^2\right) - \frac{1}{2}\rho_0 u_0\left(u_0^2 + 5a_0^2\right)$$

Or if we used adiabatic sound speed instead, then we have $a_a^2 = \frac{5}{3}a_i^2 \Rightarrow 5a_i^2 = 3a_a^2$

Specific enthalpy, *h*

We can make the notation more standard by using the specific enthalpy: $h = \mathcal{E} + P/\rho = \frac{1}{\gamma - 1}a_i^2 + a_i^2 = \frac{\gamma}{\gamma-1}a_i^2 = \frac{5}{2}c$

or $h = \frac{1}{\gamma(\gamma-1)}a_a^2 + \frac{1}{\gamma}a_a^2$

$= \frac{1}{\gamma - 1}a_a^2 = \frac{3}{2}a_a^2$

$$\boxed{h = \left(\frac{\gamma}{\gamma-1}\right)\frac{P}{\rho} = \frac{3}{2}a_s^2 = \frac{5}{2}c_s}$$

$\frac{1}{10}a_a + \frac{3}{5}a_a^2 = \frac{15}{10}a_a^2$

$\frac{3}{2}\frac{3}{5} = \frac{9}{10}$

We will try and settle on the notation $a_0 = a_s$ adiabatic

$a_i = c_s$ isothermal

We need them both since c_s is best for HII regions

Rewrite energy equation in terms of specific enthalpy and adiabatic sound speed

$$\int_0^x (\Gamma - \Lambda)\, dx' = \rho u \left(\tfrac{1}{2}u^2 + h\right) - \rho_1 u_1 \left(\tfrac{1}{2}u_1^2 + h_1\right)$$

Adiabatic mach number $M^2 = \frac{u^2}{a_s^2} \Rightarrow u_1^2 = \frac{2}{3} h_1 M_1^2$

Remember $h = \frac{3}{2} a_s^2$

$$\int_0^x (\Gamma - \Lambda)\, dx' = \rho u h \left(1 + \tfrac{1}{3} M^2\right) - \rho_1 u_1 h_1 \left(1 + \tfrac{1}{3} M_1^2\right)$$

Note that ρ_1, u_1, etc are the conditions immediately post-shock, which is the starting point for the cooling zone integration.

Mass and momentum equations

$$\Phi_0 = \rho u = \rho_0 u_0$$

$$\Pi_0 = P + \rho u^2 = \frac{\rho a^2}{\gamma} + \rho u^2 = \frac{\rho a^2}{\gamma}\left(1 + \frac{\gamma u^2}{a^2}\right)$$

$$= \frac{\rho a^2}{\gamma}\left(1 + \gamma M^2\right)$$

$$= \frac{3}{5} \rho a^2 \left(1 + \tfrac{5}{3} M^2\right)$$

$$= \frac{2}{5} \rho h \left(1 + \tfrac{5}{3} M^2\right)$$

$$\frac{\Pi_0}{\Phi_0} = \frac{2}{5} \frac{h}{u} \left(1 + \tfrac{5}{3} M^2\right)$$

$$= \frac{2 h_1}{5 u_1} \left(1 + \tfrac{5}{3} M_1^2\right)$$

$$= \frac{3}{5} \frac{a_1^2}{u_1}$$

Prandtl-Meyer relation

From Problem Set 3 Prob 4 in Shu, in our notation

$u_0 u_1 = c_*^2$ where $2c_*^2 = h + \frac{u^2}{2}$ is conserved across shoc

$h = \frac{3}{2}a^2$ so $2c_*^2 = \frac{3}{2}a_0^2 + \frac{u_0^2}{2} = \frac{1}{2}(3a_0^2+u_0^2)$

$$2c_*^2 = \frac{a_0^2}{2}(3+M_0^2) = \frac{a_1^2}{2}(3+M_1$$

so $$\frac{a_1^2}{a_0^2} = \frac{T_1}{T_0} = \frac{\left[\frac{8}{3}+\frac{10}{3}(M_0^2-1)\right]\left[\frac{8}{3}+\frac{2}{3}(M_0^2-1)\right]}{(8/3)^2 M_1^2}$$

$$= \frac{\left(\frac{10}{3}M_0^2-\frac{2}{3}\right)\left(\frac{2}{3}M_0^2-\frac{6}{3}\right)}{(8/3)^2 M_1^2} = \frac{3}{64}(10-2M_0^{-2})(2-6M_0$$

$$= \frac{3}{16}(5-M_0^{-2})(1-3M_0^{-2})$$

so $3+M_1^2 = (3+M_0^2)\frac{a_0^2}{a_1^2}$

or $M_0M_1 = \frac{u_0u_1}{a_0a_1} = \frac{c_*^2}{a_0a_1} = \frac{a_0}{4a_1}(3+M_0^2)$

$\Rightarrow M_0^2M_1^2 = \frac{a_0^2}{16a_1^2}(3+M_0^2)^2 \Rightarrow M_1^2 = \frac{a_0^2}{16a_1^2}(1+3M_0^{-2})(3+M_0^2$

$$\boxed{M_1^2 = \frac{1}{3}\frac{(1+3M_0^{-2})(3+M_0^2)}{(5-M_0^{-2})(1-3M_0^{-2})}}$$

where $(1+3M_0^{-2})(3+M_0^2) = 3+9M_0^{-2}+M_0^2+3$

Doesn't help

Different from what I had found before, but does tend to $M_1 = 1/\sqrt{5}$ as $M_0 \to \infty$

This has some extra terms in compared to previous attempt. Looks like in the previous work I wrongly canceled (1 + X) with (1 - X). Except, that the previous equation has been confirmed from multiple sources

For the shock $\rho_0 u_0 = \rho_1 u_1 = \Pi_0$

$\Phi_0 = P_0+\rho_0u_0^2 = P_1+\rho_1u_1^2 \Rightarrow \rho_0 a_0^2(\gamma^{-1}+M_0^2) = \rho_1 a_1^2(\gamma^{-1}+M$

$\frac{\Pi_0}{} \Rightarrow a_0\left(\frac{3}{5}+M_0^2\right) = a_1\left(\frac{3}{5}+M_1^2\right)$ — A

where $h = \frac{3}{}a^2$

Handwritten notes

S1

Shock-jump conditions:

Density: $$\frac{\rho_1}{\rho_0} = \frac{4M_0^2}{M_0^2+3}$$

Temperature: $$\frac{T_1}{T_0} = \frac{1}{16}(3+M^2)(5-1/M^2)$$

Velocity $$\frac{u_1}{u_0} = \frac{3+M_0^2}{4M_0^2}$$

Sound-speed (assume constant ionization): $$\frac{a_1}{a_0} = \left(\frac{T_1}{T_0}\right)^{1/2} = \frac{1}{4}(3+M^2)^{1/2}(5-1/M^2)^{1/2}$$

Mach number

$$\frac{M_1}{M_0} = \frac{u_1}{u_0}\left(\frac{a_1}{a_0}\right)^{-1} = \frac{3+M_0^2}{\cancel{4}M_0^2}\frac{\cancel{4}}{(3+M_0^2)^{1/2}(5-1/M_0^2)^{1/2}}.$$

$$\Rightarrow M_1 = \frac{(3+M_0^2)^{1/2}}{(5M_0^2-1)^{1/2}}$$

For instance,
$M_0=2 \Rightarrow M_1=\sqrt{\frac{7}{19}}$
$=0.607$

$$M_1^2 = \frac{M_0^2+3}{5M_0^2-1}\begin{cases} M_0 \to \infty : M_1 = 1/\sqrt{5} = 0.447 \\ M_0 = 1+m : M_1^2 = \frac{4+2m}{4+10m} = \left(1+\frac{m}{2}\right)(1 \end{cases}$$

$$= 1+\frac{m}{2}-\frac{5m}{2} = 1-2m$$

$$\simeq M_0^{-2}$$

$$\Rightarrow M_1 \simeq M_0^{-1}$$

This is similar to the isothermal shock, but it is for the adiabatic Mach number.