MAT 308 Notes: Laplace Transforms

These are the first notes/handout that I will make for MAT 308 (maybe the last too; we'll see how this goes). They are on Laplace Transforms. Hopefully this helps some people. **Note:** if you see any errors (no matter how small!), please let me know either via Discord (user: `will.lancer`), or by email (`william.lancer@stonybrook.edu`).

DISCLAIMER: I have literally no idea what we were taught in class 💀, so this will most likely be off-topic in some (a lot of?) areas. It should still be useful, though.

1 The Basics

Idea 1: Laplace transforms

The *Laplace transform* is a linear transform (also called an operator; same thing) that is defined to act on functions like so

$$\mathcal{L}[f(t)] = \tilde{f}(s) \equiv \int_0^\infty dt\, f(t) e^{-st}$$

Note the change of *argument* for the function $f(t)$ when transforming to $\tilde{f}(s)$[a].

[a] If you're not used to seeing $\int dt\, f(t)$ as opposed to $\int f(t)\, dt$, just know the two are equivalent, and the former notation is the standard in theoretical physics.

Remark 1: Some general comments about the Laplace transform

Some comments on Laplace transforms:

- The Laplace transform is a *linear* transform/operator, which means it obeys homogeneity and addivity. Let f, g be functions, and $\alpha, \beta \in \mathbb{R}$. For a linear transform, we would have that,

$$\begin{aligned}
\mathcal{L}[\alpha f(t) + \beta g(t)] &= \mathcal{L}[\alpha f(t)] + \mathcal{L}[\beta g(t)] && \text{(Additivity)} \\
&= \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)] && \text{(Homogeneity)} \\
&= \alpha \tilde{f}(s) + \beta \tilde{g}(s).
\end{aligned}$$

- Again, note the change in argument for the function; it goes from $t \to s$. This is non-trivial, as the function is now pulling its values from a different *domain* than before. This is incredibly useful in physics, where Laplace transforms are often used to switch between the time and frequency domains.

- Note that the Laplace transform is invertible for "nice"[a] functions. We know that an invertible map is just another word for an *isomorphism*; this tells us that, even though the functions may seem quite different on the surface of it, they actually still maintain the same *structure* as one another.

[a] I'll actually define what this means later.

Motivation 1: Why should we care about Laplace transforms?

A reasonable thing to ask is, "Why do we even care about Laplace transforms"? The reason is that, without overstating,

> *Laplace transforms are the most powerful general technique for solving ordinary differential equations there is.*

If you had the choice for only *one* method for solving differential equations, you would be best off if you chose Laplace transforms. Due to their structure, and their easy rules, they can easily deal with extremely complicated ordinary differential equations. The main strength of Lapace transforms is that *they turn differential equations into algebra.* As you know, algebra and basic integration[a] is much easier to do than solving differential equations. Another bonus for Laplace transforms is, as you will see later, the boundary conditions for the differential equation are *built into* the transform. Common Laplace transforms W.

[a] Better yet, just looking stuff up in a table

I think the best way to get a feel for this is through examples. Here are the canonical ones:

Example 1: Constants

Find the Laplace transform for 1. Generalize this to any constant.

Solution 1

We use the definition of the Laplace transform. Recall

$$\mathcal{L}[f(t)] = \tilde{f}(s) \equiv \int_0^\infty dt\, f(t)e^{-st}.$$

For $f(t) = 1$, we have, (remember that you have to take the limit of upper bound to infinity instead of actually "integrating at infinity")[a]

$$\begin{aligned}\tilde{f}(s) &= \lim_{B\to\infty} \int_0^B dt\, e^{-st} \\ &= \lim_{B\to\infty} -\left[\frac{e^{-st}}{s}\right]_0^B.\end{aligned}$$

If $s > 0$, then this integral converges. Thus, we only define our integral for $s > 0$. We would then have that,

$$\begin{aligned}\tilde{f}(s) &= \lim_{B\to\infty} -\left[\frac{e^{-st}}{s}\right]_0^B \\ &= \lim_{B\to\infty} \left[\frac{e^{-st}}{s}\right]_B^0 \\ &= \frac{1}{s} - \frac{0}{s} = \frac{1}{s}, \qquad s > 0.\end{aligned}$$

Thus our answer for the Laplace transform of $f(t) = 1$ is $\mathcal{L}(f(t)) = \tilde{f}(s) = 1/s$. To generalize this to any constant, use the *linearity* of the Laplace transform to your advantage. Let $a \in \mathbb{R}$.

$$\begin{aligned}\mathcal{L}[a] = \tilde{f}(s) &= \int_0^\infty dt\, ae^{-st} \\ &= a\int_0^\infty dt\, e^{-st} = a\mathcal{L}(1) = \frac{a}{s}, \qquad s > 0.\end{aligned}$$

Which is our answer. Notice how powerful linearity is; we will continually use this property to our advantage in the future.

[a] I will be sloppy with doing this for the rest of the handout.

Example 2: Exponentials

What is the Laplace transform of e^{at} where $a \in \mathbb{C}$?

Solution 2

This is going to be one of the most important Laplace transform identities we will ever use. We're actually going to answer a *different* question instead. What is the value of

$$\mathcal{L}[e^{at}f(t)],$$

for some arbitrary function $f(t)$? Let's give it a shot. Using the definition of the Laplace transform gives us,

$$\begin{aligned}\mathcal{L}[e^{at}f(t)] = \tilde{f}(s) &= \int_0^\infty dt\, e^{at}f(t)e^{-st} \\ &= \int_0^\infty dt\, f(t)e^{-(s-a)t} \\ &= \tilde{f}(s-a).\end{aligned}$$

Which is our answer. Notice the change of variables this gives us; this is due to the additivity of exponents for multiplying exponentials. Notice that if we let $f(t) = 1$, we would be able to find the Laplace transform for e^{at}. Recall that $\mathcal{L}[1] = \tilde{f}(s) = 1/s$. Computing $\mathcal{L}[e^{at}f(t)]$ with $f(t) = 1$ gives us,

$$\mathcal{L}[e^{at}f(t)] = \mathcal{L}[e^{at}(1)] = \tilde{f}(s-a) = \frac{1}{s-a}, \qquad s > 0$$

Which is our answer. As you can see, this general method is quite powerful for computing Laplace transforms of general exponentials. This process is sometimes called *s-shift*.

Example 3: Sines and cosines

What are the Laplace transforms of $\cos(t)$ and $\sin(t)$?

Solution 3

Hmmmm. This feels difficult. Trying to integrate something like,

$$\mathcal{L}[\cos(t)] = \tilde{f}(s) = \int_0^\infty dt\, \cos(t) e^{-st}$$

seems kind of ugly. Question: is there any way we can *rewrite* $\cos(t)$ to somehow make use of our previous techniques? Preferably something to do with exponentials and constants, because they're easy. The answer is *yes*, and it comes from a familiar identity:

$$\textbf{Euler's Identity:} \qquad e^{it} = \cos(t) + i\sin(t).$$

Solving for cosine from this gives us,

$$\cos(at) = \frac{e^{iat} - e^{-iat}}{2}.$$

Well well well... we know how to solve this! We know the Laplace transform of e^{iat} is (by the previous question),

$$\mathcal{L}[e^{iat}] = \tilde{f}(s - ia) = \frac{1}{s - ia}.$$

We now have everything we need to solve our problem.

$$\begin{aligned}
\mathcal{L}[\cos(at)] &= \mathcal{L}\left[\frac{e^{iat} - e^{-iat}}{2}\right] \\
&= \frac{1}{2}\left(\mathcal{L}[e^{iat}] + \mathcal{L}[e^{-iat}]\right) \qquad \text{(Linearity)} \\
&= \frac{1}{2}\left(\frac{1}{s - ia} + \frac{1}{s + ia}\right) \\
&= \frac{1}{2}\left(\frac{2s}{s^2 + a^2}\right) \\
&= \frac{s}{s^2 + a^2}, \qquad \operatorname{Re} s > 0
\end{aligned}$$

Which is our answer. We can follow an almost identical process for sine, which the reader is encouraged to do (the following relation might help 😉)

$$\sin(at) = \frac{e^{iat} - e^{-iat}}{2i}.$$

Going through the calculations gives us (as the reader should verify),

$$\mathcal{L}[\sin(t)] = \tilde{f}(s - a) = \frac{a}{s^2 + a^2}, \qquad \operatorname{Re} s > 0$$

We can also derive the same results by taking the real and imaginary parts of e^{iat}, which is an exercise for the reader; I will put them in formal problems so you're more motivated to do it.

Problem 1. Derive the Laplace transform for $\cos(at)$ by taking the real part of the Laplace transform for e^{iat}.

Problem 2. Derive the Laplace transform for $\sin(at)$ by taking the imaginary part of the Laplace transform for e^{iat}.

I'm watching you… you better do these problems 🤨.

2 Solving differential equations

Using Laplace transforms to solve differential equations is a three-step process,

$$\text{Transform} \to \text{Rearrange} \to \text{Inverse Transform}.$$

The only part you don't know yet is the inverse transform, which we'll learn now. Btw, the "rearrange" part is just algebra, as you'll see.

Idea 2: Inverse Laplace Transform

The inverse Laplace transform is just the process of going from $\tilde{f}(s) \to f(t)$. You do this by recognizing what the Laplace transform is, and then making the proper inverse transform; this is just recognizing what function $f(t)$ corresponds to $\tilde{f}(s)$.

Idea 3: Partial fractions

The main way you're going to get the Laplace transform into a form that you can easily recognize is by simplifying rational functions, which you should remember from middle/high school. If you don't remember how to do this, read this.

Remark 2: When do Laplace transforms exist?

When do Laplace transforms exist? This is a reasonable question, because *in the definition* of the Laplace transform is an integral to infinity; these kinds of integrals sometimes have the bad habit of equating to infinity. So, when are they defined? The answer is when the function you're transforming, $f(t)$, is of *exponential order*. What does this mean? It means that the scaling of $f(t)$ with t can be expressed as,

$$|f(t)| \leq Ce^{kt}.$$

For some $C, k \in \mathbb{R}$. This is equivalent to saying that

$$I = \int_0^\infty dt\, |f(t)e^{-st}| < \infty.$$

Waterman probably won't give us examples that don't converge, but it's important to know that you can't apply the Laplace transform to everything (just most things).

Example 4

Find the inverse Laplace transform of

$$\frac{12}{s}.$$

Solution 4

Recognizing this as the Laplace transform of 12 gives us our answer,

$$\frac{12}{s} = 12 \cdot \frac{1}{s} = \mathcal{L}[12] \implies \mathcal{L}^{-1}[12/s] = 12.$$

Example 5: Partial fractions

Find the inverse Laplace transform of

$$\frac{69}{s^2 + s - 6}$$

Solution 5

We must first do partial fraction decomposition. Doing so gives us,

$$\frac{69}{s^2 + s - 6} = \frac{69}{(s+3)(s-2)} \implies 69 = A(s-2) + B(s+3)$$

Solving for this gives us $A = -69/5$ and $B = 69/5$. Our problem then turns into,

$$\frac{69}{s^2 + s - 6} = \frac{-65}{5(s+3)} + \frac{69}{5(s-2)}.$$

Recognizing these as exponetials gives us our answer,

$$\frac{-65}{5(s+3)} + \frac{69}{5(s-2)} = -\frac{69}{5}\frac{1}{s+3} + \frac{69}{5}\frac{1}{s-2} = -\frac{69}{5}\mathcal{L}[e^{-3t}] + \frac{69}{5}\mathcal{L}[e^{2t}]$$
$$\implies \mathcal{L}^{-1}\left[\frac{-65}{5(s+3)} + \frac{69}{5(s-2)}\right] = -\frac{69}{5}e^{-3t} + \frac{69}{5}e^{2t}.$$

Idea 4: Laplace transform of a derivative

The Laplace transform of the time derivative of a function $f(t)$ is defined as,

$$\mathcal{L}[f'(t)] = s\tilde{f}(s) - f(0).$$

This can be recursively applied to give a formula for n time derivatives,

$$\mathcal{L}[f^{(n)}(t)] = s^n \tilde{f}(s) - \sum_{k=1}^{n} s^{n-k} f^{k-1}(0).$$

Example 6: Time derivatives

Prove the two formulas given in the idea above. **Hint:** use integration by parts.

Solution 6

I'm going to show the first one; the second one follows by induction applied to the approach you take to solve for the first. Integrating by parts gives us,

$$\begin{aligned}\mathcal{L}[f'(t)] &= \int_0^\infty dt\, f'(t)e^{-st} \\ &= [e^{-st}f(t)]_0^\infty - (-s)\int_0^\infty dt\, f(t)e^{-st} \\ &= s\tilde{f}(s) - f(0).\end{aligned}$$

The reader should prove the second part. Feel free to not even do induction; just compute the case of $\mathcal{L}[f''(t)]$ and guess a generalization from there.

Example 7: A less trivial example

Find the specific solution to,

$$x''(t) + x(t) = \cos 2t$$

with $x(0) = 0$ and $x'(0) = 1$.

Solution 7

Laplace transform both sides,

$$\begin{aligned}\mathcal{L}[x''(t) + x(t)] &= \mathcal{L}[\cos 2t] \\ s^2\tilde{x}(s) - sx(0) - x'(0) + \tilde{x}(s) &= \frac{s}{s^2+4}.\end{aligned}$$

Plug in the initial conditions,

$$s^2\tilde{x}(s) - 1 + \tilde{x}(s) = \frac{s}{s^2+4}.$$

Solve for $\tilde{x}(s)$ and use partial fractions,

$$\begin{aligned}\tilde{x}(s) &= \frac{s}{(s^2+1)(s^2+4)} + \frac{1}{s^2+1} \\ &= \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4} + \frac{1}{s^2+1}.\end{aligned}$$

Inverse transforming this gives us,

$$x(t) = \frac{1}{3}\cos t - \frac{1}{3}\cos(2t) + \sin t.$$

3 Convolution

I'm going to throw the definition of convolution at you, and then explain it.

Idea 5: Convolution

We define the convolution of two functions $f(t)$ and $g(t)$ as follows,

$$f * g \equiv \int_0^t du\, f(u)g(t-u).$$

Although it's not evident in the definition, convolution is commutative, so $f * g = g * f$.

Remark 3: Some convoluted comments about convolution

- Where the hell did that definition come from? To answer that question, let's answer a different question:

  Problem 3. What function $h(t)$ would I have to Laplace transform to have $\tilde{h}(s) = \tilde{f}(s)\tilde{g}(s)$? As in, what $h(t)$ gives us the following equivalence?

  $$\int_0^\infty dt\, h(t)e^{-st} = \tilde{f}(s)\tilde{g}(s).$$

  Solution. Hmmm. Let's go back to the definitions of $\tilde{f}(s)$ and $\tilde{g}(s)$ and see if anything pans out.

  $$\tilde{f}(s) \equiv \int_0^\infty du\, f(u)e^{-su} \qquad \tilde{g}(s) \equiv \int_0^\infty dv\, g(v)e^{-sv}.$$

  We let the integration variable not be t for a reason you will see in a second; it doesn't matter what it is, as it's a dummy variable. Take the product of these functions,

  $$\begin{aligned}\tilde{f}(s)\tilde{g}(s) &= \int_0^\infty du\, f(u)e^{-su} \cdot \int_0^\infty dv\, g(v)e^{-sv} \\ &= \int_0^\infty dv \int_0^\infty du\, f(u)g(v)e^{-s(u+v)}.\end{aligned}$$

  We want to make this double integral look something like a Laplace transform, so let's make a change of variables, $t = u + v$. Subbing this in gives us,

  $$\int_0^\infty dv \int_0^\infty du\, f(u)g(v)e^{-s(u+v)} = \int_0^\infty dt\, \underbrace{\int_0^t du\, f(u)g(t-u)}_{h(t)}\, e^{-st}.$$

  As you can see, if we define $h(t) \equiv \int_0^t du\, f(u)g(t-u)$, we get our desired result: $\mathcal{L}[h(t)] = \tilde{f}(s)\tilde{g}(s)$. Very cool.

- The commutativity of the convolution is clear from the secondary meaning of the convolution; $f * g$ is a function such that $\mathcal{L}[f * g] = \tilde{f}(s)\tilde{g}(s)$. As you can see, if we switch $f * g$ to $g * f$, we simply commute the terms on the RHS, and this is equivalent to the original convolution.

  $$\mathcal{L}[f * g] = \tilde{f}(s)\tilde{g}(s) = \tilde{g}(s)\tilde{f}(s) = \mathcal{L}[g * f].$$

Example 8

Find the convolution of t^2 and t.

Solution 8

Using the definition gives us,

$$\begin{aligned} f * g = t^2 * t &= \int_0^t du\, u^2(t-u) \\ &= \int_0^t du\, u^2 t - u^3 \\ &= \frac{u^3 t}{3} - \frac{u^4}{4}\bigg|_0^t = \frac{t^4}{12}. \end{aligned}$$

Example 9

Find the convolution of $f(t)$ and 1.

Solution 9

Using the definition gives us,

$$\begin{aligned} f * g = f(t) * 1 &= \int_0^t du\, f(u) \\ &= F(t) - F(0). \end{aligned}$$

4 Parting comments

Hopefully you're more comfortable with Laplace transforms now as compared to when you started this handout. I want to emphasize that Laplace transforms are *simple*; they have fairly few computational rules, and the annoying stuff can just be looked up in a table somewhere. They're one of the most powerful tools you're going to have in your differential equations toolbelt from now on; make use of it. Good luck on the MAT 308 homework 🫡.